Exercise 1 (30 points). Let $\mathcal{P}(D_0) \xleftarrow{\gamma_1} \mathcal{P}(D_1)$ and $\mathcal{P}(D_1) \xrightarrow{\gamma_2} \mathcal{P}(D_2)$ be Galois connections defined by extraction functions $\eta_1 : D_0 \rightarrow D_1$ and $\eta_2 : D_1 \rightarrow D_2$ respectively. Show that the sequential composition $\mathcal{P}(D_0) \xrightarrow{\alpha \eta_2 \circ \alpha \eta_1} \mathcal{P}(D_2)$ of these Galois connections is given by the extraction function $\eta_2 \circ \eta_1$, that is, show that $\alpha \eta_2 \circ \alpha \eta_1 = \alpha \eta_2 \circ \eta_1$ and $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \eta_1$.

Exercise 2 (30 points). Consider the Galois connection $\mathcal{P}(\mathbb{Z}) \xleftarrow{\text{sign}} \mathcal{P}(\{+, 0, -\})$ defined in Lecture 7 (sl. 14).

Let $n \in \mathbb{Z}$ and let us define the functions $f^+_n : \mathcal{P}(\mathbb{Z}) \rightarrow \mathcal{P}(\mathbb{Z})$ and $f^*_n : \mathcal{P}(\mathbb{Z}) \rightarrow \mathcal{P}(\mathbb{Z})$ such that for $Z \subseteq \mathbb{Z}$ we let

$$f^+_n(Z) := \{z + n \mid z \in Z\}$$
$$f^*_n(Z) := \{z * n \mid z \in Z\}.$$

Define functions $f^+_n : \mathcal{P}(\{+, 0, -\}) \rightarrow \mathcal{P}(\{+, 0, -\})$ and $f^*_n : \mathcal{P}(\{+, 0, -\}) \rightarrow \mathcal{P}(\{+, 0, -\})$ that are the most precise upper approximations to the functions $f^+_n$ and $f^*_n$ respectively, with respect to this Galois connection.

Exercise 3 (40 points). Let $(L, \leq_L)$ and $(M, \leq_M)$ be complete lattices, and let $\alpha : L \rightarrow M$ and $\gamma : M \rightarrow L$ be monotonic functions such that we have a Galois connection $L \xleftarrow{\alpha} M$. Let $\nabla_M : M \times M \rightarrow M$ be a widening operator. Show that if the lattice $(M, \leq_M)$ satisfies the Ascending Chain Condition, then the function $\nabla_L : L \times L \rightarrow L$ defined by $l_1 \nabla_L l_2 := \gamma(\alpha(l_1) \nabla_M \alpha(l_2))$ is a widening operator.