Exercise 1 (40 points). Let \((L, \leq_L)\) and \((M, \leq_M)\) be complete lattices, with least elements \(\bot_L\) and \(\bot_M\) respectively. Let \(\alpha : L \rightarrow M\) and \(\gamma : M \rightarrow L\) be monotonic functions that form a Galois connection.

Prove the following statements.

a) \(\alpha(\bot_L) = \bot_M\)

b) \(\gamma(\top_M) = \top_L\)

c) \(\alpha \circ \gamma \circ \alpha = \alpha\)

d) \(\gamma \circ \alpha \circ \gamma = \gamma\)

Exercise 2 (30 points). Let \(k \in \mathbb{N}\) and \(k \geq 2\). Consider the complete lattices \((L, \leq_L) := (\mathcal{P}(\mathbb{Z}), \subseteq)\) and \((K, \leq_K) := (\mathcal{P}(\{0, \ldots, k-1\}), \subseteq)\).

Define monotonic functions \(\alpha_k : L \rightarrow K\) and \(\gamma_k : K \rightarrow L\) such that the image of \(\alpha_k\) is equal to \(K\), and \(\alpha_k\) and \(\gamma_k\) form a Galois connection. (You can either define the Galois connection using extraction functions, or directly define the functions \(\alpha_k\) and \(\gamma_k\) and establish conditions (G1) and (G2).)

Let \((M, \leq_M) := (\mathbb{I} \cup \{\bot\}, \sqsubseteq)\) be the interval lattice (Lecture 7, sl. 4) and \(L \xrightarrow{\gamma} M\) be the Galois connection defined in class (Lecture 7, sl. 4).

Give an example of a set \(Z \subseteq \mathbb{Z}\) such that the sets \(X = \gamma(\alpha(Z))\) and \(Y = \gamma_k(\alpha_k(Z))\) are incomparable w.r.t. \(\subseteq\) (i.e., neither \(X \subseteq Y\), nor \(Y \subseteq X\)).

Exercise 3 (30 points). Let \((L, \leq_L)\) and \((M, \leq_M)\) be complete lattices. Show that if a function \(\alpha : L \rightarrow M\) is completely additive (that is, \(\alpha(\bigsqcup L') = \bigsqcup \{\alpha(l) \mid l \in L'\}\) for all \(L' \subseteq L\)), then \(L \xrightarrow{\gamma} M\) is a Galois connection, where the function \(\gamma : M \rightarrow L\) is defined as:

\[
\gamma(m) := \bigsqcup \{l \in L \mid \alpha(l) \leq_M m\} \text{ for all } m \in M.
\]
**Bonus exercise 1 (40 extra-credit points)**. Let \((L, \leq_L) := (I \cup \{\perp\}, \sqsubseteq)\) be the interval lattice, and let \((M, \leq_M) := (\mathcal{P}\{+, 0, -\}, \subseteq)\) be the sign lattice from the lecture. Consider the function \(\gamma : M \to L\) defined as follows:

\[
\begin{align*}
\gamma(\{+, 0\}) &= [-\infty, \infty] \\
\gamma(\{-, 0\}) &= [-\infty, -1] \\
\gamma(\{0\}) &= [0, 0] \\
\gamma(\emptyset) &= \perp
\end{align*}
\]

Show that there does not exist a function \(\alpha : L \to M\) such that \(L \xleftarrow{\gamma} M \xrightarrow{\alpha} L\) is a Galois connection.

---

1The extra-credit points can only improve your score.