Exercise 1 (20 points). Consider the program below.

\[
\begin{align*}
[x & := 1]^1; \\
[y & := 2]^2; \\
[\text{read}(z)]^3; \\
\text{if} [x \leq z \land z < y]^4 \text{ then } \{ \\
[\text{assert}(x \leq z \land z < y)]^5; \\
[\text{skip}]^6; \\
\} \\
\text{else } \{ \\
[\text{assert}(\neg(x \leq z \land z < y))]^7; \\
[z := 1]^8; \\
\} \\
[\text{print}(z)]^9
\end{align*}
\]

Recall the Constant Propagation analysis (Lecture 4, sl. 24–27) and its extension to assertions (Lecture 5, sl. 20–21). Let \( f_5 \) and \( f_7 \) be the transfer functions for blocks 5 and 7 in the program respectively. Let \( d : \text{Vars} \rightarrow (\mathbb{Z} \cup \{?\}) \) be the function for which \( d(x) = 1 \), \( d(y) = 2 \) and \( d(z) = ? \).

Give the values for \( f_5(d) \) and \( f_7(d) \).

If \( d^{LFP} = (d_1^{LFP}, d_2^{LFP}, d_3^{LFP}, d_4^{LFP}, d_5^{LFP}, d_6^{LFP}, d_7^{LFP}, d_8^{LFP}, d_9^{LFP}) \) is the least fixpoint solution to the corresponding system of dataflow equations, give the value of \( d_9^{LFP} \), that is, the information computed for the entry point of block 9.

Exercise 2 (20 points). Let \((D, \leq)\) be a complete lattice that satisfies the ascending chain condition (ACC). Show that \( \sqcup \) is a widening operator (definition in Lecture 6, sl.21).
Exercise 3 (30 points).
Recall the complete lattice \((D, \sqsubseteq)\) defined as follows:
\[
D = \{[l, h] \mid l, h \in \hat{\mathbb{Z}} \land l \leq h\} \cup \{\bot\},
\]
where \(\hat{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}\) and the relation \(\leq\) on \(\hat{\mathbb{Z}}\) extends the one on \(\mathbb{Z}\) such that \(-\infty < +\infty\) and for all \(z \in \mathbb{Z}\), we have \(z < +\infty\) and \(-\infty < z\);
\[
[l_1, h_1] \sqsubseteq [l_2, h_2] \text{ if and only if } l_2 \leq l_1 \text{ and } h_1 \leq h_2,
\]
and \(\bot \sqsubseteq d\) for every \(d \in D\).
Let us define the function \(\nabla : D \times D \to D\) as follows:
- \(\bot \nabla d := d \nabla \bot := d\) for all \(d \in D\);
- \([l_1, h_1] \nabla [l_2, h_2] := [l, h]\), where
\[
\begin{align*}
\quad l &= \begin{cases}
\quad l_1 & \text{if } l_1 \leq l_2, \\
\quad -\infty & \text{otherwise};
\end{cases} \\
\quad h &= \begin{cases}
\quad h_1 & \text{if } h_2 \leq h_1, \\
\quad +\infty & \text{otherwise}.
\end{cases}
\end{align*}
\]
a) Show that \(\nabla\) is a widening operator.
b) Consider the sequence \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq d_3 \sqsubseteq d_4 \sqsubseteq d_5\), where
\[
d_0 = \bot, d_1 = [0, 0], d_2 = [0, 1], d_3 = [-1, 1], d_4 = [-1, 2], d_5 = [-2, 2].
\]
Give the sequence \((d_i^\nabla)_{i=0,\ldots,5}\), where \(d_0^\nabla = d_0\) and \(d_{i+1}^\nabla = d_i^\nabla \nabla d_{i+1}\).
c) Now, let \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) be an arbitrary ascending chain and again \(d_0^\nabla = d_0\) and \(d_{i+1}^\nabla = d_i^\nabla \nabla d_{i+1}\) be the corresponding sequence obtained by applying widening. What can you say about the maximal size (number of different elements) of any such sequence \((d_i^\nabla)_{i=0}\)?

Exercise 4 (30 points).
Let \(f\) be a monotonic function on a complete lattice \((D, \leq)\) and \(\nabla : D \times D \to D\) be a widening operator. Recall that we defined \(f^\nabla(\bot) := \bot\) and for \(i \in \mathbb{N}\),
\[
f^{i+1}(\bot) := \begin{cases}
\quad f^\nabla(\bot) & \text{if } f(f^\nabla(\bot)) \leq f^\nabla(\bot), \\
\quad f^\nabla(\bot) \nabla f(f^\nabla(\bot)) & \text{otherwise}.
\end{cases}
\]
Then, we have that the sequence \((f^\nabla(\bot))_{i \in \mathbb{N}}\) is an ascending chain that eventually stabilizes. For \(l_\nabla := \bigsqcup \{f_i^\nabla(\bot) \mid i \in \mathbb{N}\}\) we have \(\text{lfp}(f) \leq l_\nabla\).
Show that for the sequence \(f^i(l_\nabla)\), where we define \(f^0(l_\nabla) := l_\nabla\) and \(f^{i+1}(l_\nabla) := f(f^i(l_\nabla))\) it holds that:
a) \(\text{lfp}(f) \leq f^i(l_\nabla)\) for all \(i \in \mathbb{N}\);
b) \(f^{i+1}(l_\nabla) \leq f^i(l_\nabla)\) for all \(i \in \mathbb{N}\).