Exercise 1 (20 points).
Consider the following program
\[
\begin{align*}
[x := 1]^1; \\
\text{while } [y > 0]^2 \text{ do } \\
\quad [x := x - 1]^3 \\
\} \\
[x := 2]^4
\end{align*}
\]
Perform a Live Variables analysis (Lecture 4, sl. 5–9) to this program: give the transfer functions \(f_1, \ldots, f_4\), the system of dataflow equations, and compute the least fixpoint.

Exercise 2 (40 points).
Consider the following program
\[
\begin{align*}
[x := 1]^1; \\
[x := x - 1]^2; \\
[x := 2]^3
\end{align*}
\]
The variable \(x\) is dead (i.e., not live) at the exits of blocks 2 and 3. However, \(x\) is live at the exit of block 1, even though it is only used to compute a new value for a variable that turns out to be dead. Let us define a faint variable to be a variable that is dead or is only used to compute values for faint variables. We call a variable that is not faint a strongly live variable. In the above example, variable \(x\) is faint at the exits of blocks 1, 2, and 3.

Define a dataflow analysis that detects strongly live variables: give a complete lattice \((D, \leq)\), initial value \(i \in D\) for the final block, and the definitions of the transfer functions. (Hint: for an assignment \([v := a]^b\) the definition of \(f_b(d)\) should be by cases on whether \(v\) is in \(d\) or not.)
Exercise 3 (40 points).
Let $(D, \leq)$ be a lattice. Recall from Homework 2 that a function $f : D \to D$ is called distributive if $f(d_1 \sqcup d_2) = f(d_1) \sqcup f(d_2)$ for all $d_1, d_2 \in D$.

a) If $D = \mathcal{P}(A)$ for some finite set $A$ and $\leq$ is either $\subseteq$ or $\supseteq$, and $f_b : \mathcal{P}(A) \to \mathcal{P}(A)$ is a transfer function defined as

$$f_b(d) = (d \setminus \text{kill}(b)) \cup \text{gen}(b),$$

for all $d \in \mathcal{P}(A)$, where $\text{kill}(b) \in \mathcal{P}(A)$ and $\text{gen}(b) \in \mathcal{P}(A)$, prove that $f_b$ is distributive.

b) Consider the Constant Propagation analysis from Lecture 4 (sl. 24–27). Give an example that demonstrates that some of the transfer functions (defined on slide 26) is not distributive.

Bonus exercise 1 (60 extra-credit points\[1\]).

a) In a Detection of Signs Analysis one models all negative integers by the symbol $-$, zero by the symbol 0, and all positive integers by the symbol +. For example, the abstraction of the set $\{-2, -1, 3\}$ is modeled by the set $\{-, +\} \in \mathcal{P}(\{-, 0, +\})$. Define a forward may analysis for Detection of Signs using a lattice $(D, \leq)$ in which $D = \text{Vars} \to \mathcal{P}(\{-, 0, +\})$ is the set of functions from $\text{Vars}$ to $\mathcal{P}(\{-, 0, +\})$, where $\text{Vars}$ is the set of variables occurring in the given program.

b) Now, define a variant of the analysis from a) that uses $\mathcal{P}(\text{Vars} \to \{-, 0, +\})$ as the domain of the lattice. That is, each element of the lattice is a set of functions from $\text{Vars}$ to $\{-, 0, +\}$. Is there a difference in the precision of the two analyses? Justify your answer.

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\[1\]The extra-credit points can only improve your score.