Program Analysis, WS 2016/2017
Homework 1

Dr. Rayna Dimitrova (lecturer) Filip Niksic (TA)
posted: 25.10.2016, **due: 03.11.2016, 13:30**

**Exercise 1 (20 points).**
Which of the following structures are (complete) lattices?

<table>
<thead>
<tr>
<th>structure</th>
<th>not lattice</th>
<th>lattice</th>
<th>complete</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="structure1" /></td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td><img src="image2" alt="structure2" /></td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td><img src="image3" alt="structure3" /></td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td><img src="image4" alt="structure4" /></td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
</tbody>
</table>

**Exercise 2 (5 points).**
Consider the partial order \((\mathcal{P}(\{a, b, c, d\}), \subseteq)\), where \(\mathcal{P}(S) = \{U \mid U \subseteq S\}\).
Let \(X = \{\{a, b\}, \{a, c\}\}\). List all the lower bounds of \(X\), all the upper bounds of \(X\), and give the least upper bound \(\bigcup X\) and the greatest lower bound \(\bigcap X\).
Exercise 3 (75 points).

a) Show that for every set $S$, $(\mathcal{P}(S), \subseteq)$ is a complete lattice.

b) If $(D_1, \leq_1)$ and $(D_2, \leq_2)$ are complete lattices, show that $(D_1 \times D_2, \leq)$, is also a complete lattice, where $D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1, d_2 \in D_2\}$ and $(d_1, d_2) \leq (d'_1, d'_2)$ if and only if $d_1 \leq_1 d'_1$ and $d_2 \leq_2 d'_2$.

c) Prove that every finite lattice is complete.

d) Let $(D, \leq)$ be a complete lattice. Prove that $\bigcup \emptyset = \bigcap D$ and $\bigcap \emptyset = \bigcup D$.

e) Let $I := \{[a, b] \mid a, b \in \mathbb{R}\}$ be the set of closed real intervals, where $[a, a] = \{a\}$, and if $b < a$, then $[a, b] = \emptyset$.

Show that $(I, \subseteq)$ is a lattice by defining $d \sqcup e$ and $d \sqcap e$ for $d, e \in I$. Is $(I, \subseteq)$ a complete lattice?

Bonus exercise 1 (30 extra-credit points).^{1}

We say that a partially ordered set $(D, \leq)$ satisfies

(1) the **minimality condition** if $D$ is well founded, that is, if every non-empty subset of $D$ has at least one minimal element;

(2) the **descending chain condition** if every strictly descending chain in $D$ is finite;

(3) the **induction property** if for any property $P$ the following holds: assume that for every element $d \in D$ if $P$ holds for all elements strictly less than $d$ (i.e., all $d' \neq d$ where $d' \leq d$), then $P$ holds for $d$; then every element of $D$ has the property $P$.

Prove that all the conditions (1)-(3) are equivalent.

---

^{1}The extra-credit points can only improve your score.