Problem 1  WS1S

1. Convert the following WS1S formula to equivalent WS1S$_0$ formula.

\[ \exists x \exists y (x < 3 \land y < x) \]

2. Determine the validity/satisfiability of the following formulas using the decision procedure shown in the lecture (by automaton construction).

\[ \forall x. \exists P. x \in P \]
\[ \neg \exists y. S(y, x) \]

Problem 2  Array Property Fragment

1. Apply the decision procedure for the array property fragment of the theory of arrays $T_A$ in order to decide validity of the following $\Sigma_A$-formulae:

- $\exists j. a(i \triangleleft e)[j] = e$
- $\forall j. a(i \triangleleft e)[j] = e$

2. Apply the decision procedure for the array property fragment of the theory of integer-indexed arrays $T_A^Z$ in order to decide validity of the following $\Sigma_A^Z$-formula:

\[ \text{sorted}(a, l, k) \land \text{sorted}(a, k, u) \rightarrow \text{sorted}(a, l, u) \]

Recall the definition of the $\text{sorted}$ predicate:

\[ \text{sorted}(a, l, u) := \forall i, j. l \leq i \leq j \leq u \rightarrow a[i] \leq a[j] \]

Hint: Substitute $\text{sorted}$ in the formula by its definition.
Problem 3  Bitvectors

Consider the following bitvector formula defined over variables $x_{[2]}$, $y_{[4]}$, and $z_{[4]}$.

$$(x_{[2]} \circ y_{[4]}[3 : 3])[1 : 2] \circ (z_{[4]} \circ y_{[4]} \circ 0_{[2]})[0 : 3] = (x_{[2]} \circ z_{[4]})[4 : 5] \circ y_{[4]}[0 : 2] \circ 0_{[1]}$$

Check the satisfiability of the above formulas, and, if satisfiable, construct the solution using:

- the naïve method introduced in the lecture
- the improved method introduced in the lecture

Problem 4  Nelson-Oppen Combination Procedure

Using the Nelson-Oppen combination method check the satisfiability of the following formulas:

- $x + y = 0 \land f(x) + f(-y) = 1$ (theories: $T_Z$ and $T_E$)
- $g(x+y, z) = f(g(x, y)) \land x + z = y \land z \geq 0 \land x \geq y \land g(x, x) = z \land f(z) \neq g(2x, 0)$ (theories: $T_Q$ and $T_E$)
- $c + d = e \land f(e) = c + d \land f(f(c + d)) \neq e$ (theories: $T_Z$ and $T_E$)
Problem 5  DPLL(T)

Use DPLL(T) to check the satisfiability of the following formulas. In all the assignments, when you use TheoryPropagation, explain why your use of TheoryPropagation is correct. If you terminate with a satisfying assignment, make sure your assignment is T-consistent. Finally, when you use the Backjump rule, show how you computed the Backjump clause.

- Check the satisfiability of the following set of clauses in the theory of equality with uninterpreted functions:

  \[ C_1 : \{a = b, c \neq b\} \]
  \[ C_2 : \{f(a) \neq f(c), g(b) \neq f(a), a = b\} \]
  \[ C_3 : \{g(g(a)) \neq g(g(b)), q\} \]
  \[ C_4 : \{\neg q, c = b\} \]
  \[ C_5 : \{g(a) \neq g(b), a = c\} \]

- Check the satisfiability of the following set of clauses belonging to the difference logic:

  \[ C_1 : \{y - z \leq 1, z - u \leq 1\} \]
  \[ C_2 : \{u - y \leq -3, z - u \leq 1\} \]
  \[ C_3 : \{u - z \leq 0, w - y \leq -2\} \]
  \[ C_4 : \{w - u \leq -2, u - w \leq -2\} \]
  \[ C_5 : \{y - w \leq 1\} \]
  \[ C_6 : \{z - u \leq -2, z - y \leq -4\} \]