(Non-deterministic) Semantics as a Tool for Analyzing Proof Systems

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1. A formal language \( \mathcal{L} \), based on which \( \mathcal{L} \)-formulas are constructed.

2. A relation \( \vdash \) between sets of \( \mathcal{L} \)-formulas and \( \mathcal{L} \)-formulas, satisfying:

   - **Reflexivity**: if \( \psi \in \mathcal{T} \) then \( \mathcal{T} \vdash \psi \).
   - **Monotonicity**: if \( \mathcal{T} \vdash \psi \) and \( \mathcal{T} \subseteq \mathcal{T}' \), then \( \mathcal{T}' \vdash \psi \).
   - **Transitivity**: if \( \mathcal{T} \vdash \psi \) and \( \mathcal{T}', \psi \vdash \varphi \) then \( \mathcal{T}, \mathcal{T}' \vdash \varphi \).
A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed.

A relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying:

- **Reflexivity**: if $\psi \in T$ then $T \vdash \psi$.
- **Monotonicity**: if $T \vdash \psi$ and $T \subseteq T'$, then $T' \vdash \psi$.
- **Transitivity**: if $T \vdash \psi$ and $T', \psi \vdash \varphi$ then $T, T' \vdash \varphi$.

We can define logics:

- **Semantically**: $T \vdash \psi$ if every “model” of $T$ is a “model” of $\psi$.
- **Syntactically**: $T \vdash \psi$ if $\psi$ has a derivation from $T$ in a given proof system.
Motivation

Use semantics to:

- *understand* logics defined by new proof systems.
- (co-semi) *decide* such logics.
- prove (or disprove) *proof-theoretic properties* of (families of) proof systems.
  - Proof-theoretic methods are sometimes tedious and error-prone.
Use semantics to:

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- prove (or disprove) *proof-theoretic properties* of (families of) proof systems.
  - Proof-theoretic methods are sometimes tedious and error-prone.
Sequent Systems

- Sequents (here and now) are objects of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas.
Sequent Systems

- Sequents (here and now) are objects of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite *sets* of formulas.
- Semantic intuition:

  \[ \varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m \quad \iff \quad \varphi_1 \wedge \ldots \wedge \varphi_n \supset \psi_1 \vee \ldots \vee \psi_m \]
Sequent Systems

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  $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m \iff \varphi_1 \land \ldots \land \varphi_n \supset \psi_1 \lor \ldots \lor \psi_m$

- Tarskian consequence relations (logics) can obtained by:

  $\mathbf{V}: \mathcal{T} \vdash^\text{frm}_G \varphi \iff \{ \Rightarrow \psi \mid \psi \in \mathcal{T} \} \vdash_G \Rightarrow \varphi$

  $\mathbf{T}: \mathcal{T} \vdash^\text{frm}_G \varphi \iff \vdash_G \Gamma \Rightarrow \varphi \text{ for some } \Gamma \subseteq \mathcal{T}$
Sequent Systems

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- Tarskian consequence relations (logics) can obtained by:
  
  \begin{align*}
  V: \quad & \mathcal{T} \vdash_G^{\text{frm}} \varphi & \iff & \{ \Rightarrow \psi \mid \psi \in \mathcal{T} \} \vdash_G \Rightarrow \varphi \\
  T: \quad & \mathcal{T} \vdash_G^{\text{frm}} \varphi & \iff & \vdash_G \Gamma \Rightarrow \varphi \quad \text{for some } \Gamma \subseteq \mathcal{T}
  \end{align*}

  We choose $V$ because of its robustness.
Axioms:

(id) \( \varphi \Rightarrow \varphi \)

Structural Rules:

\[
(W \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \quad (\Rightarrow W) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}
\]

(cut) \[
\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}
\]

Logical Rules:

\[
(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \supset \varphi_2 \Rightarrow \Delta} \quad (\Rightarrow \supset) \quad \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}
\]

\[
(\land \Rightarrow) \quad \frac{\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta} \quad (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \land \varphi_2, \Delta}
\]
Classical Logic

The “Matrix” $M_{LK}$

- Truth-values: \{T, F\}
- Truth-tables:

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>F</th>
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<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

- An $M_{LK}$-valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v(\psi) = F$ for some $\psi \in \Gamma$ or $v(\psi) = T$ for some $\psi \in \Delta$.

Soundness and Completeness

$\Omega \vdash_{LK} s$ iff every $M_{LK}$-valuation which is a model of every sequent in $\Omega$ is also a model of $s$. 
Subformula Property

**Notation:** \( \Omega \vdash^E_G s \) iff there exists a derivation of \( s \) from \( \Omega \) in \( G \) consisting solely of \( E \)-sequents (i.e. sequents consisting solely of formulas from \( E \)).
Subformula Property

**Notation:** \( \Omega \vdash_{G}^{\mathcal{E}} s \iff \text{there exists a derivation of } s \text{ from } \Omega \text{ in } G \text{ consisting solely of } \mathcal{E}\text{-sequents (i.e. sequents consisting solely of formulas from } \mathcal{E}). \)

\[ \Omega \vdash_{G}^{\mathcal{E}} s \implies \Omega \vdash_{G}^{\text{sub}[\Omega,s]} s \]

Q: Can we find “semantics” for \( \vdash_{L,K}^{\mathcal{E}} \)?
“Semantics” for $\vdash^E_G$

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, and set $\Omega \cup \{s\}$ of $\mathcal{E}$-sequents:
$\Omega \vdash^E_L$ $s$ iff every partial $M_{LK}$-valuation, defined on $\mathcal{E}$, which is a model of every sequent in $\Omega$ is also a model of $s$. 
“(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, and set $\Omega \cup \{s\}$ of $\mathcal{E}$-sequents:

$\Omega \vdash_{\mathcal{E}}^{\mathcal{G}} s$ iff every partial $\mathcal{M}_{\mathcal{LK}}$-valuation, defined on $\mathcal{E}$, which is a model of every sequent in $\Omega$ is also a model of $s$.

Now, proving the subformula property for $\mathcal{LK}$ reduces to proving that every partial $\mathcal{M}_{\mathcal{LK}}$-valuation (defined on a closed set of formulas) can be extended to a (full) $\mathcal{M}_{\mathcal{LK}}$-valuation.
“Semantics” for $\vdash^\mathcal{E}_G$

**Stronger) Soundness and Completeness**

For every closed set $\mathcal{E}$ of formulas, and set $\Omega \cup \{s\}$ of $\mathcal{E}$-sequents: $\Omega \vdash^\mathcal{E}_{LK} s$ iff every partial $M_{LK}$-valuation, defined on $\mathcal{E}$, which is a model of every sequent in $\Omega$ is also a model of $s$.

Now, proving the subformula property for $LK$ reduces to proving that every partial $M_{LK}$-valuation (defined on a closed set of formulas) can be extended to a (full) $M_{LK}$-valuation.

This is trivial. 😊
### Cut-Admissibility

<table>
<thead>
<tr>
<th>Cut-Admissibility</th>
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<tr>
<td>$\frac{s}{\Gamma}$</td>
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Theorem $\Gamma \Rightarrow \varphi$ does not have a finite characteristic matrix.
Cut-Admissibility

Cut-Admissibility

\[ \frac{\vdash_G s}{\vdash_{G-(cut)} s} \]

- Holds for LK (Gentzen, 1934).
Holds for \textbf{LK} (Gentzen, 1934).

Q: Can we find semantics for \textbf{LK} – \textit{(cut)}?
Cut-Admissibility

Holds for $\mathbf{LK}$ (Gentzen, 1934).

Q: Can we find semantics for $\mathbf{LK} - (cut)$?

- Does not hold in the presence of assumptions, e.g.

  \[ \Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2) \]

  \[ \Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK} - (cut)} \Rightarrow p_1 \supset (p_3 \supset p_2) \]
Cut-Admissibility

Holds for $\text{LK}$ (Gentzen, 1934).

Q: Can we find semantics for $\text{LK} - (cut)$?

- Does not hold in the presence of assumptions, e.g.

$$\Rightarrow p_1 \supset p_2 \vdash_{\text{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

$$\Rightarrow p_1 \supset p_2 \vdash_{\text{LK} - (cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

Theorem

$\vdash_{\text{LK} - (cut)}^{\text{frm}}$ does not have a finite characteristic matrix.
Non-Deterministic Matrices

- Truth-tables assign non-empty *sets* of truth-values.
- \( v(\bigcirc(\psi_1, \ldots, \psi_n)) \in \tilde{\bigcirc}(v(\psi_1), \ldots, v(\psi_n)) \) instead of
  \( v(\bigcirc(\psi_1, \ldots, \psi_n)) = \tilde{\bigcirc}(v(\psi_1), \ldots, v(\psi_n)). \)
Non-Deterministic Matrices

- Truth-tables assign non-empty \textit{sets} of truth-values.

\[
\nu(\Diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\Diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)) \]

\[
\nu(\Diamond(\psi_1, \ldots, \psi_n)) = \tilde{\Diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).
\]
Non-Deterministic Matrices

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- \( v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) \) instead of \( v(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) \).
- Particularly useful to handle *syntactic underspecification*. 

\[
\begin{array}{c|c|c|c}
\tilde{\wedge} & T & F \\
\hline
T  & T & F \\
F  & F & F \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\tilde{\wedge} & T & F \\
\hline
T  & \{T\} & \{F\} \\
F  & \{F\} & \{F\} \\
\end{array}
\]
Non-Deterministic Matrices

- Truth-tables assign non-empty sets of truth-values.

\[ \nu(\diamond (\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)) \] instead of \[ \nu(\diamond (\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)). \]

- Particularly useful to handle syntactic underspecification.

\[
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T & T & F \\
F & F & F \\
\end{array}
\quad
\begin{array}{c|c|c}
\tilde{\wedge} & T & F \\
T & \{T\} & \{F\} \\
F & \{F\} & \{F\} \\
\end{array}
\]

\[
\frac{(\wedge \Rightarrow)}{(\Rightarrow \wedge)} \quad \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}
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Non-Deterministic Matrices

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- \( \nu(\wedge(\psi_1, \ldots, \psi_n)) \in \tilde{\wedge}(\nu(\psi_1), \ldots, \nu(\psi_n)) \) instead of \( \nu(\wedge(\psi_1, \ldots, \psi_n)) = \tilde{\wedge}(\nu(\psi_1), \ldots, \nu(\psi_n)) \).
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\[(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta} \quad (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \land \varphi_2, \Delta} \]
Semantics for $\mathbf{LK} − \text{(cut)}$

\[(\text{cut}) \quad \frac{\varphi \Rightarrow \ \Rightarrow \varphi}{\Rightarrow} \]
Semantics for $\textbf{LK} - (\text{cut})$

$\text{(cut)} \quad \frac{\varphi \Rightarrow \Rightarrow}{\Rightarrow}$

The “NMatrix” $\mathbf{M}_{\text{LK} - (\text{cut})}$

- **Truth-values:** $\{\langle F, F \rangle, \langle T, T \rangle, \langle F, T \rangle\}$
- **Truth-tables:**

<table>
<thead>
<tr>
<th>$\tilde{\wedge}$</th>
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<td>${\langle F, F \rangle, \langle F, T \rangle}$</td>
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</table>

- **An $\mathbf{M}_{\text{LK} - (\text{cut})}$-valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v_{l}(\psi) = F$ for some $\psi \in \Gamma$ or $v_{r}(\psi) = T$ for some $\psi \in \Delta$.**
The “NMatrix” $M_{LK-(cut)}$

\[
\begin{array}{c}
\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta \\
\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma \Rightarrow \varphi_1, \Delta \\
\Gamma \Rightarrow \varphi_2, \Delta
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\sim & \langle T, T \rangle & \langle F, F \rangle & \langle F, T \rangle \\
\hline
\langle T, T \rangle & \{\langle T, T \rangle, \langle F, T \rangle\} & \{\langle F, F \rangle, \langle F, T \rangle\} & \{\langle F, T \rangle\} \\
\langle F, F \rangle & \{\langle F, F \rangle, \langle F, T \rangle\} & \{\langle F, F \rangle, \langle F, T \rangle\} & \{\langle F, F \rangle, \langle F, T \rangle\} \\
\langle F, T \rangle & \{\langle F, T \rangle\} & \{\langle F, F \rangle, \langle F, T \rangle\} & \{\langle F, T \rangle\}
\end{array}
\]
The “NMatrix” $M_{LK-\text{(cut)}}$

\[
\begin{array}{c}
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\left( \wedge \Rightarrow \right) \\
\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta \\
\hline
\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta
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\]

\[
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\end{array}
\end{array}
\]

\begin{array}{|c|c|c|c|}
\hline
\tilde{\wedge} & \langle T, T \rangle & \langle F, F \rangle & \langle F, T \rangle \\
\hline
\langle T, T \rangle & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\hline
\langle F, F \rangle & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} \\
\hline
\langle F, T \rangle & \{ \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\hline
\end{array}
The “NMatrix” $\textbf{M}_{\text{LK}}^{\text{(cut)}}$

$$(\land \Rightarrow) \quad \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta} \quad (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \land \varphi_2, \Delta}$$

<table>
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<tr>
<th>$\tilde{\land}$</th>
<th>$\langle T, T \rangle$</th>
<th>$\langle F, F \rangle$</th>
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The "NMatrix" $\mathbf{M}_{\text{LK}-(\text{cut})}$

($\land \Rightarrow$) \[ \frac{\Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta} \]

($\Leftrightarrow \land$) \[ \frac{\Gamma \Rightarrow \varphi_1, \Delta \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \land \varphi_2, \Delta} \]

($\lor \Rightarrow$) \[ \frac{\Gamma \Rightarrow \varphi_1, \Delta \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \lor \varphi_2, \Delta} \]

($\Rightarrow \lor$) \[ \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \lor \varphi_2, \Delta} \]

\[\begin{array}{c|c|c|c}
\wedge & \langle T, T \rangle & \langle F, F \rangle & \langle F, T \rangle \\
\hline
\langle T, T \rangle & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\langle F, F \rangle & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} \\
\langle F, T \rangle & \{ \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\end{array}\]

\[\begin{array}{c|c|c|c}
\lor & \langle T, T \rangle & \langle F, F \rangle & \langle F, T \rangle \\
\hline
\langle T, T \rangle & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle F, F \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\langle F, F \rangle & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle T, T \rangle, \langle F, T \rangle \} \\
\langle F, T \rangle & \{ \langle T, T \rangle, \langle F, T \rangle \} & \{ \langle F, T \rangle \} & \{ \langle F, T \rangle \} \\
\end{array}\]
Semantics for $\text{LK} - (cut)$

### Soundness and Completeness

$$\Omega \vdash_{\text{LK} - (cut)} s \iff \text{every } M_{\text{LK} - (cut)}\text{-valuation which is a model of every sequent in } \Omega \text{ is also a model of } s.$$  

→ New formulation of results of Schütte (1960) and Girard (1987).
Proving Cut-Admissibility for \( \text{LK} \)

**Cut-Admissibility for \( \text{LK} \)**

\[
\vdash_{\text{LK}} S \quad \iff \quad \vdash_{\text{LK} - (\text{cut})} S
\]
**Proving Cut-Admissibility for LK**

<table>
<thead>
<tr>
<th>Cut-Admissibility for LK</th>
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<tr>
<td>( \vdash_{LK} s \quad \iff \quad \vdash_{LK-(cut)} s )</td>
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- Reduces to proving that for every \( M_{LK-(cut)} \)-valuation which is not a model of some sequent \( s \), there exists an \( M_{LK} \)-valuation which is not a model of \( s \).

- Simply, by induction on the build-up of formulas.
Similar ideas can be used to study:

- Systems without \((id)\) (in fact, any rule except for weakening, contraction and exchange).
- Concrete *proof-specifications*, specifying which formulas:
  - Are allowed to appear in derivations on each side of the sequent.
  - Are allowed to serve as active formulas of each derivation rule.
These methods can be applied in broad families of proof systems:

**Canonical Systems**

\[
\Gamma \Rightarrow \varphi_2, \Delta \\
\Gamma \Rightarrow \varphi_1 \sim \varphi_2, \Delta
\]

**Labelled Systems**

\[
s \cup \{a : \varphi_1\} \\
\Gamma \Rightarrow \varphi_2, \Delta \\
\Gamma \Rightarrow \varphi_1 \sim \varphi_2, \Delta \\
\Gamma \Rightarrow \varphi_1 \bowtie \varphi_2 \\
\Gamma \Rightarrow \varphi
\]

**Basic Systems**

\[
s \cup \{b : \varphi_2\} \\
\Gamma \Rightarrow \varphi_2, \Delta \\
\Gamma \Rightarrow \varphi_1 \sim \varphi_2, \Delta \\
\Gamma \Rightarrow \varphi_1 \bowtie \varphi_2 \\
\Gamma \Rightarrow \varphi
\]

**Canonical Gödel Systems**
The System **HIF**

Manipulates single-conclusion hypersequents.

**Axioms:**

\[ \varphi \Rightarrow \varphi \]

**Structural Rules:**

\[
\frac{H \vdash \Gamma \Rightarrow E}{H \mid \Gamma, \varphi \Rightarrow E} \quad \text{\( (IW \Rightarrow) \)} \\
\frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi} \quad \text{\( (\Rightarrow IW) \)} \\
\frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow E} \quad \text{\( (EW) \)}
\]

\[
\frac{H \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \quad H \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}{H \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \Gamma_2, \Gamma_1 \Rightarrow E_2} \quad \text{\( (com) \)}
\]

\[
\frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2 \Rightarrow E} \quad \text{\( (\Rightarrow \varnothing) \)} \\
\frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \cup \varphi_2} \quad \text{\( (\Rightarrow \cup) \)}
\]

**Logical Rules:**

\[
\frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E} \quad \text{\( (\Rightarrow \wedge) \)} \\
\frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \wedge \varphi_2} \quad \text{\( (\Rightarrow \wedge) \)}
\]
Semantics - Gödel logic

The “Matrix” \textbf{M}_{\text{HIF}}

- Truth-values: \([0, 1]\)
- Truth-tables:

\[
\tilde{\supset}(x, y) = \begin{cases} 
1 & x \leq y \\
y & x > y
\end{cases} \\
\tilde{\wedge}(x, y) = \min(x, y)
\]

An \textbf{M}_{\text{HIF}}\text{-valuation is a model:}

- of a sequent \(\Gamma \Rightarrow E\) iff \(\min\{v(\psi) \mid \psi \in \Gamma\} \leq \max\{v(\psi) \mid \psi \in E\}\).
- of a hypersequent \(H\) iff it is a model of some \(s \in H\).

Soundness and Completeness

\(\mathcal{H} \vdash_{\text{HIF}} H\) iff every \textbf{M}_{\text{HIF}}\text{-valuation which is a model of every hypersequent in } \mathcal{H} \text{ is also a model of } H\).
Semantics for **HIF** — *(cut)*

(cut)  \[
\frac{\varphi \Rightarrow}{\Rightarrow \varphi}
\]
Semantics for $\textbf{HIF} - (\text{cut})$

$\text{(cut)} \quad \varphi \Rightarrow \Rightarrow \varphi$

### The “NMatrix” $\textbf{M}_{\textbf{HIF} - (\text{cut})}$

- **Truth-values:** $\{ \langle x, y \rangle \in [0, 1] \times [0, 1] \mid x \leq y \}$
- **Truth-tables:**

$$\widetilde{\wedge}(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = \left[ 0, \left\{ \begin{array}{ll} 1 & y_1 \leq x_2 \\ x_2 & y_1 > x_2 \end{array} \right. \right] \times \left[ \begin{array}{ll} 1 & x_1 \leq y_2 \\ y_2 & x_1 > y_2 \end{array} \right]$$

$$\widetilde{\sim}(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = [0, \min(x_1, x_2)] \times [\min(y_1, y_2), 1]$$

- An $\textbf{M}_{\textbf{HIF} - (\text{cut})}$-valuation is a *model*:
  - of a sequent $\Gamma \Rightarrow E$ iff $\min\{ v_i(\psi) \mid \psi \in \Gamma \} \leq \max\{ v_r(\psi) \mid \psi \in E \}$.
  - of a hypersequent $H$ iff it is a model of some $s \in H$. 
Soundness and Completeness

$\mathcal{H} \vdash_{\text{HIF} - (cut)} H$ iff every $M_{\text{HIF} - (cut)}$-valuation which is a model of every hypersequent in $\mathcal{H}$ is also a model of $H$.

- Proving cut-admissibility for HIF reduces to proving that for every $M_{\text{HIF} - (cut)}$-valuation which is not a model of some hypersequent $H$, there exists an $M_{\text{HIF}}$-valuation which is not a model of $H$. 

Dual construction for $\text{HIF} - (\text{id})$.

This method can be generalized for arbitrary canonical derivation rules added to HIF.
Soundness and Completeness

\[ \mathcal{H} \vdash_{\text{HIF}-(cut)} H \text{ iff every } M_{\text{HIF}-(cut)}-\text{valuation which is a model of every hypersequent in } \mathcal{H} \text{ is also a model of } H. \]

- Proving cut-admissibility for HIF reduces to proving that for every \( M_{\text{HIF}-(cut)} \)-valuation which is not a model of some hypersequent \( H \), there exists an \( M_{\text{HIF}} \)-valuation which is not a model of \( H \).

- Dual construction for HIF \( - (id) \).

- This method can be generalized for arbitrary canonical derivation rules added to HIF.
Conclusions

- Non-deterministic semantics is a useful tool for investigating proof-theoretic properties of logical calculi.
- The semantic tools should complement the usual proof-theoretic ones.

Further Research

- Extensions for first order logics
- Sub-structural calculi
Thank you!