Semantic Investigation of Basic Sequent Systems

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Workshop on Abstract Proof Theory, Unilog 2013

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Ori Lahav and Arnon Avron, **A Unified Semantic Framework for Fully-structural Propositional Sequent Systems**, to be published in **Transactions on Computational Logic**, 2013.

Main Contributions

- A correspondence between a wide class of proof-systems (called basic systems) and Kripke semantics.
- More precisely, a general soundness and completeness result which uniformly provides Kripke semantics for each basic system.
- Extension of the previous result to obtain semantic characterizations of crucial proot-theoretic properties of basic systems:
  - The subformula property
  - Cut-admissibility

Basic Systems: General Framework

1. Propositional sequent systems
2. Manipulate two-sided multiple-conclusion sequents
3. Fully structural :
    - Sequents are finite sets of signed formulas, e.g.

$$\psi, \varphi \Rightarrow \varphi, \psi \wedge \varphi \quad \equiv \quad \{f{:}\psi, f{:}\varphi, t{:}\varphi, t{:}(\psi \wedge \varphi)\}$$

    - Identity axiom, cut, weakening rules always present
4. The logical rules are all basic rules

Basic Rules - Examples

$$\frac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

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- Distinction between active and context formulas

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$$\frac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi} \qquad\qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

- Distinction between active and context formulas
- The structure of the active part:

$$\frac{\Rightarrow \psi}{\Rightarrow \Box\psi} \quad \rightsquigarrow \quad \Rightarrow p_1 / \Rightarrow \Box p_1 \qquad\qquad \frac{\psi \Rightarrow}{\Box\psi \Rightarrow} \quad \rightsquigarrow \quad p_1 \Rightarrow / \Box p_1 \Rightarrow$$

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- Introducing context-relations to handle the context part:

$$\frac{\Box\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} \rightsquigarrow \pi_1 = \{\langle f{:}\Box p_1, f{:}\Box p_1 \rangle\}$$

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- The final formulation:

$$\langle \Rightarrow p_1, \pi_1\rangle / \Rightarrow \Box p_1 \qquad\qquad \langle p_1 \Rightarrow, \pi_0\rangle / \Box p_1 \Rightarrow$$

Basic Rules

- A basic rule:

$$\langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / C$$

  - Premises: sequents $s_1, \ldots, s_n$
  - Corresponding context-relations: $\pi_1, \ldots, \pi_n$
  - Conclusion: sequent C

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  - Corresponding context-relations: $\pi_1, \ldots, \pi_n$
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- Its application:

$$\frac{\sigma(s_1) \cup c_1 \quad \ldots \quad \sigma(s_n) \cup c_n}{\sigma(C) \cup c'_1 \cup \ldots \cup c'_n}$$

where :

  - σ is a substitution
  - for every $1 \leq i \leq n$, $\langle c_i, c'_i \rangle$ is a π_i-instance

Basic Rules - More Examples

Basic Rule	Application
$\langle p_1 \Rightarrow, \pi_0\rangle, \langle \Rightarrow p_1, \pi_0\rangle / \Rightarrow$	$\dfrac{\Gamma_1, \psi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$
$\langle p_1 \Rightarrow p_2, \pi_0\rangle / \Rightarrow p_1 \supset p_2$	$\dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}$
$\langle p_1 \Rightarrow p_2, \pi_i\rangle / \Rightarrow p_1 \supset p_2$	$\dfrac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$
$\langle \Rightarrow p_1, \pi_{K4}\rangle / \Rightarrow \Box p_1$	$\dfrac{\Gamma_1, \Box\Gamma_2 \Rightarrow \psi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi}$

$$\pi_0 = \{\langle f{:}p_1, f{:}p_1\rangle, \langle t{:}p_1, t{:}p_1\rangle\}$$

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Basic Systems

Many useful sequent systems are basic.

This includes systems for (the propositional fragments of):

- Classical logic
- Intuitionistic logic, its dual, and bi-intuitionistic logic
- Variety of modal logics
- Intuitionistic modal logics
- Many-valued logics
- Variety of paraconsistent logics

Kripke Semantics in General

Definition

A Kripke frame consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times \mathit{wff} \to \{\mathsf{T}, \mathsf{F}\}$

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To obtain Kripke semantics for a basic system **G**, we identify a set of **G**-legal frames for which **G** is sound and complete, i.e.
$\vdash_{\mathbf{G}}$ *s* iff every **G**-legal frame is a model of *s*.

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$\vdash_{\mathbf{G}} s$ iff every **G**-legal frame is a model of s.

- A frame is a model of a sequent s if s true in every world
- A sequent s is true in a world w if s contains at least one signed formula which is true in w
- A signed formula $\mathsf{X}{:}\psi$ is true in a world w if $v(w, \psi) = \mathsf{X}$

Kripke Semantics for Basic Systems

For a basic system **G**:

- Each context-relation in **G** and each basic rule of **G** imposes a constraint on the set of frames.
- Joining all of these constraints, we obtain the set of **G**-legal frames.

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- Each context-relation in **G** and each basic rule of **G** imposes a constraint on the set of frames.
- Joining all of these constraints, we obtain the set of **G**-legal frames.

It might produce non-deterministic semantics.

G-legal Frames

- For every context-relation π in **G** there is a corresponding accessibility relation R_π, where R_{π_0} is the identity relation.
- The constraint imposed by the context-relation π:
  if $wR_\pi u$ then for every π-instance $\langle \text{X}{:}\psi, \text{Y}{:}\varphi \rangle$, either $v(u, \psi) \neq \text{X}$ or $v(w, \varphi) = \text{Y}$.
- The constraint imposed by the basic rule $\langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / C$:
  For every world w, substitution σ, if for every $1 \leq i \leq n$, $\sigma(s_i)$ is true in every u such that $wR_{\pi_i}u$, then $\sigma(C)$ is true in w.

G-legal Frames

Reminder: $\pi_0 = \{\langle f{:}p_1, f{:}p_1\rangle, \langle t{:}p_1, t{:}p_1\rangle\}$

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Example

$$\langle \Rightarrow p_1, \pi_K \rangle / \Rightarrow \Box p_1$$

$$\pi_K = \{\langle f{:}p_1, f{:}\Box p_1 \rangle\}$$

$$\frac{\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$$

In legal frames:

- An accessibility relation $R_{\pi_K} \in \mathcal{R}$.
- If $wR_{\pi_K}u$ then for every ψ, either $v(w, \Box\psi) = \mathsf{F}$ or $v(u, \psi) \neq \mathsf{F}$, i.e. if $v(w, \Box\psi) = \mathsf{T}$, then $v(u, \psi) = \mathsf{T}$ for every u such that $wR_{\pi_K}u$.
- If $v(u, \psi) = \mathsf{T}$ for every u such that $wR_{\pi_K}u$, then $v(w, \Box\psi) = \mathsf{T}$.

Example - Primal Implication [Gurevich et al.]

$$\pi_0 = \{\langle f{:}p_1, f{:}p_1\rangle, \langle t{:}p_1, t{:}p_1\rangle\}$$

$$\pi_i = \{\langle f{:}p_1, f{:}p_1\rangle\}$$

$$\langle \Rightarrow p_2, \pi_i\rangle / \Rightarrow p_1 \rightsquigarrow p_2 \qquad \langle \Rightarrow p_1, \pi_0\rangle, \langle p_2 \Rightarrow, \pi_0\rangle / p_1 \rightsquigarrow p_2 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \rightsquigarrow \varphi} \qquad \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \rightsquigarrow \varphi \Rightarrow \Delta_1, \Delta_2}$$

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In legal frames:

- A accessibility relation $R_{\pi_i} \in \mathcal{R}$.
- If $wR_{\pi_i}u$ and $v(w, \psi) = \mathsf{T}$ then $v(u, \psi) = \mathsf{T}$.
- If $v(w, \varphi) = \mathsf{T}$ then $v(w, \psi \rightsquigarrow \varphi) = \mathsf{T}$.
- If $v(w, \psi) = \mathsf{T}$ and $v(w, \varphi) = \mathsf{F}$ then $v(w, \psi \rightsquigarrow \varphi) = \mathsf{F}$.

Kripke Semantics for Basic Systems

Theorem

Every basic system **G** *is sound and complete with respect to the semantics of* **G***-legal frames.*

Kripke Semantics for Basic Systems

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Every basic system **G** *is sound and complete with respect to the semantics of* **G***-legal frames.*

- General and uniform:
  - Various known soundness and completeness results are specific cases of this general theorem
- Modular

The Subformula Property

- A basic system has the subformula property if $\vdash_{\mathbf{G}} s$ implies that there exists a proof of s in **G** consisting only of subformulas of the formulas in s.
- In basic systems the subformula property implies decidability and consistency.
- Q: What is the semantic meaning of the subformula property?

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- In basic systems the subformula property implies decidability and consistency.
- Q: What is the semantic meaning of the subformula property?

Next, we strengthen the soundness and completeness theorem to characterize proofs containing only formulas from a given set $\mathcal{F}$.

For this we introduce $\mathcal{F}$-semiframes.

Frames

Definition

A frame consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times wff \to \{\mathsf{T}, \mathsf{F}\}$

Theorem

There exists a proof in **G** *of s*

if and only if

every **G**-*legal frame is a model of s.*

Semiframes

Definition

An $\mathcal{F}$-semiframe consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times \mathcal{F} \rightarrow \{\mathsf{T}, \mathsf{F}\}$

Theorem

There exists a proof in **G** *of s containing only formulas from* $\mathcal{F}$

if and only if

every **G***-legal* $\mathcal{F}$*-semiframe is a model of s.*

Semantic Characterization of the Subformula Property

- The last theorem leads to a semantic decision procedure for basic systems that have the subformula property (just check all possible semiframes).
- Semantic sufficient condition for the subformula property: If every **G**-legal $\mathcal{F}$-semiframe can be extended to a **G**-legal frame for every set $\mathcal{F}$ of formulas closed under subformulas, then **G** has the subformula property.
- This criterion is applicable for many interesting basic systems.

Cut-Admissibility

To characterize cut-admissibility in basic systems, we provide another soundness and completeness theorem for cut-free proofs.

Cut-Admissibility

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Intuition

An application of cut: $$\frac{\psi \Rightarrow \quad \Rightarrow \psi}{\Rightarrow}$$

If cut is forbidden, we need a frame which is a model of $\psi \Rightarrow$ and $\Rightarrow \psi$.

Quasiframes

Definition

A frame consists of:

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Quasiframes

Definition

A quasiframe consists of:

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A sequent s is true in a world w if at least one of the following hold:

- $v(w, \psi) = \mathsf{F}$ for some ψ on the left side of s
- $v(w, \psi) = \mathsf{T}$ for some ψ on the right side of s
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- $v(w, \psi) = \mathfrak{i}$ for some ψ in s

If $v(w, \psi) = \mathfrak{i}$, then both $\{f{:}\psi\}$ and $\{t{:}\psi\}$ are true in w.

Semantic Characterization of Cut-Admissibility

Theorem

There exists a cut-free proof in **G** *of s*

if and only if

every **G***-legal quasiframe is a model of s.*

Semantic Characterization of Cut-Admissibility

Theorem

There exists a cut-free proof in **G** *of s*

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every **G***-legal quasiframe is a model of s.*

- Semantic sufficient condition for cut-admissibility:
  If every **G**-legal quasiframe can be refined into a **G**-legal frame, then **G** enjoys cut-admissibility
  (by refinement, we mean changing all $\mathfrak{i}$'s to T's or F's).
- Provides a uniform basis for semantic proofs of cut-admissibility in basic systems.

Extensions

Similar method is applicable to:

- Provide semantics when cut is allowed only on some formulas (to characterize *strong* cut-admissibility).
- Provide semantics when the identity axiom is available only for some formulas (to characterize *axiom-expansion*).

Thank you!