Studying Sequent Systems via Non-deterministic Many-Valued Matrices

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"Logic"

1. A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed.
2. A binary relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying:

Reflexivity: if $A \in \Gamma$ then $\Gamma \vdash A$.
Monotonicity: if $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash A$.
Transitivity: if $\Gamma \vdash B$ and $\Gamma', B \vdash A$ then $\Gamma, \Gamma' \vdash A$.

Languages

We will only consider propositional languages, consisting of:

- Atomic formulas (we usually use $p_1, p_2, \ldots$)
- A finite set of logical connectives
- Parentheses: '(','\)'

We denote by $wff_{\mathcal{L}}$ the set of well-formed formulas of $\mathcal{L}$.

$\mathcal{L}_{cl}$ (a language for classical logic) includes the unary connective $\neg$, and the binary connectives $\wedge$, $\vee$, and $\supset$.

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The set of well-formed formulas $\mathit{wff}_{\mathcal{L}_{cl}}$:

- All atomic formulas are in $\mathit{wff}_{\mathcal{L}_{cl}}$.
- If $A, B \in \mathit{wff}_{\mathcal{L}_{cl}}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B) \in \mathit{wff}_{\mathcal{L}_{cl}}$.

Syntactic Approach to Define Logics

$\vdash$ is defined using a notion of a *derivation* in a given proof system.

For example, we can use Hilbert-style systems:

- A *Hilbert-style system* consists of: (i) a set of formulas called *axioms*, and (ii) a set of *inference rules*.
- A *derivation* of A from Γ in a Hilbert-style system $\mathbf{H}$ is a finite sequence of formulas, where the last formula is A, and each formula is: (i) an axiom of $\mathbf{H}$, (ii) a member of Γ, or (iii) obtained from previous formulas by applying some inference rule of $\mathbf{H}$.
- The consequence relation $\vdash_{\mathbf{H}}$ is defined by:

$$\Gamma \vdash_{\mathbf{H}} A \text{ if } A \text{ has a derivation from } \Gamma \text{ in } \mathbf{H}$$

The system **HCL**

Axiom schemata:

$I1$	$A \supset (B \supset A)$	$D1$	$A \supset A \vee B$
$I2$	$(A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)$	$D2$	$B \supset A \vee B$
$I3$	$((B \supset A) \supset B) \supset B$	$D3$	$(A \supset C) \supset (B \supset C) \supset (A \vee B \supset C)$
$C1$	$A \wedge B \supset A$	$N1$	$(B \supset A) \supset (B \supset \neg A) \supset \neg B$
$C2$	$A \wedge B \supset B$	$N2$	$\neg\neg A \supset A$
$C3$	$A \supset (B \supset A \wedge B)$		

Inference Rule:

$$MP \quad \frac{A \quad A \supset B}{B}$$

Definition

Classical logic = the language $\mathcal{L}_{cl}$ + the consequence relation $\vdash_{HCL}$

Gentzen-style Systems

- Hilbert-style systems operate on *formulas*.
  Gentzen-style systems operate on *sequents*.
- Sequents are objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas.
- A *Gentzen-style proof system* consists of a set of sequent rules (usually given by schemes).

Semantic Intuition for Sequents

$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ $\qquad$ $A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$

Semantic Intuition for Sequents

$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$	$A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$
$\Rightarrow B_1, \ldots, B_m$	$B_1 \vee \ldots \vee B_m$
$\Rightarrow B$	B
$A \Rightarrow$	$\neg A$
$A_1, \ldots, A_n \Rightarrow$	$\neg A_1 \vee \ldots \vee \neg A_n$
$\Rightarrow$	`False`

Gentzen-style Systems

- A *derivation* of a sequent $\Gamma \Rightarrow \Delta$ from a set of sequents Ω in **G** is a finite sequence of sequents, where the last sequent is $\Gamma \Rightarrow \Delta$, and each sequent is: (i) a member of Ω, or (ii) obtained from previous sequents in the sequence by applying some rule of **G**.
- We write $\Omega \vdash_{\mathbf{G}}^{seq} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in **G**.

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- We write $\Omega \vdash_{\mathbf{G}}^{seq} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in **G**.

A consequence relation (between formulas) is obtained by:

$$\Gamma \vdash_{\mathbf{G}} A \quad \Longleftrightarrow \quad \{ \Rightarrow B \mid B \in \Gamma \} \vdash_{\mathbf{G}}^{seq} \Rightarrow A$$

Logical Rules:

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

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$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta} \qquad (\Rightarrow \supset) \quad \frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

LK
Gentzen 1934

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$$(\wedge \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \wedge A_2, \Delta}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow A_1, A_2, \Delta}{\Gamma \Rightarrow A_1 \vee A_2, \Delta}$$

LK (cont.)

Structural Rules:

$$(W \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$(id)\quad \frac{}{A \Rightarrow A} \qquad (cut)\quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

(other structural rules are built-in when sequents are pairs of sets).

Soundness and Completeness

LK is sound and complete for classical logic, i.e. $\Gamma \vdash_{\mathbf{LK}} A$ iff $\Gamma \vdash_{HCL} A$.

For example:

$$\vdash_{\mathbf{LK}} \neg(Ob \wedge Ro) \supset \neg Ob \vee \neg Ro$$

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We show:

$$\vdash^{seq}_{\mathbf{LK}} \Rightarrow \neg(Ob \wedge Ro) \supset \neg Ob \vee \neg Ro$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Ro \Rightarrow Ro}\,(id)}{Ro \Rightarrow \neg Ob, Ro}\,(\Rightarrow W)}{\Rightarrow \neg Ob, \neg Ro, Ro}\,(\Rightarrow \neg)}{\Rightarrow \neg Ob \vee \neg Ro, Ob}\,(\Rightarrow \vee) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Ob \Rightarrow Ob}\,(id)}{Ob \Rightarrow \neg Ro, Ob}\,(\Rightarrow W)}{\Rightarrow \neg Ob, \neg Ro, Ob}\,(\Rightarrow \neg)}{\Rightarrow \neg Ob \vee \neg Ro, Ro}\,(\Rightarrow \vee)}{\dfrac{\Rightarrow \neg Ob \vee \neg Ro, Ob \wedge Ro}{\neg(Ob \wedge Ro) \Rightarrow \neg Ob \vee \neg Ro}\,(\neg \Rightarrow)}\,(\Rightarrow \wedge)}{\Rightarrow \neg(Ob \wedge Ro) \supset \neg Ob \vee \neg Ro}\,(\Rightarrow \supset)$$

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The Subformula Property

If $\Omega \vdash^{seq}_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in **LK** consisting solely of subformulas of the formulas in Ω and $\Gamma \Rightarrow \Delta$.

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Cut-Admissibility

If $\vdash^{seq}_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ in **LK** with no applications of (cut).

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Axiom-Expansion

Atomic axioms (i.e. axioms of the form $p_i \Rightarrow p_i$) always suffice.

$\mathbf{LK}$ has many good properties

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Invertibility of Logical Rules

The premises of each logical rule can be derived from its conclusion.

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Invertibility of Logical Rules

The premises of each logical rule can be derived from its conclusion.

LK has an effective semantics.

Semantics of Classical Logic

- Two truth values: F and T
- Truth tables:

		$\supset$
F	F	T
F	T	T
T	F	F
T	T	T

		$\wedge$
F	F	F
F	T	F
T	F	F
T	T	T

. . .

- A *valuation* function assigns values to the atomic formulas, and they determine the values of the compound formulas according to the tables.

- $\Gamma \vdash A$ iff
  for every valuation v: if $v(B) = \mathrm{T}$ for every $B \in \Gamma$ then $v(A) = \mathrm{T}$.

Semantic Approach to Define Logics

$\vdash$ is defined using the notion of a *model*:

$$\Gamma \vdash A \text{ if every "model" of } \Gamma \text{ is a "model" of } A$$

For example, we can use many-valued matrices:

A many-valued matrix for a language $\mathcal{L}$ consists of:

- A set $\mathcal{V}$ of truth values.
- A subset $\mathcal{D} \subseteq \mathcal{V}$ of *designated* truth values.
- A truth table for every connective of $\mathcal{L}$, i.e. for every connective $\diamond$ we have a function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $\mathcal{V}$, where n is the arity of $\diamond$.

Many-valued Matrices

Let **M** be a many-valued matrix.

- An **M**-*valuation* is a function $v : wff_{\mathcal{L}} \to \mathcal{V}$ that respects all truth tables, i.e. for every compound formula $\diamond(A_1, \ldots, A_n)$:

$$v(\diamond(A_1, \ldots, A_n)) = \widetilde{\diamond}(v(A_1), \ldots, v(A_n))$$

- An **M**-valuation is a *model* of a formula A if $v(A) \in \mathcal{D}$.

A consequence relation is defined by:

$\Gamma \vdash_{\mathbf{M}} A$ if every model of (every formula in) Γ is a model of A.

The Matrix $\mathbf{M}_{cl}$

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- Values: $\mathcal{V} = \{\mathrm{F}, \mathrm{T}\}$
- Designated values: $\mathcal{D} = \{\mathrm{T}\}$
- Tables:

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	T
F	T	T
T	F	F
T	T	T

x_1	x_2	$\widetilde{\wedge}(x_1, x_2)$
F	F	F
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T	F	F
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Soundness and Completeness

$$\Gamma \vdash_{\mathbf{LK}} A \text{ iff } \Gamma \vdash_{\mathbf{M}_{cl}} A$$

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Soundness and Completeness

$$\Gamma \vdash_{\mathbf{LK}} A \text{ iff } \Gamma \vdash_{\mathbf{M}_{cl}} A$$

Example:

$$\vdash_{\mathbf{LK}} \neg(Ob \wedge Ro) \supset \neg Ob \vee \neg Ro$$

Indeed, every $\mathbf{M}_{cl}$-valuation is a model of this formula.

Example: Kleene Logic

The Matrix $\mathbf{M}_{kl}$

- Values: $\mathcal{V} = \{\mathrm{F}, \mathrm{T}, \mathrm{I}\}$
- Designated values: $\mathcal{D} = \{\mathrm{T}\}$
- Tables:

$\widetilde{\supset}$	F	T	I
F	T	T	T
T	F	T	I
I	I	T	I

$\widetilde{\wedge}$	F	T	I
F	F	F	F
T	F	T	I
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Idea: If there is enough information to determine the value then we put the classical value in the table. Otherwise, we put I.

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$$\not\vdash_{\mathbf{M}_{kl}} Ob \supset Ob \qquad Ob \vdash_{\mathbf{M}_{kl}} Ob$$

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If we take $\mathcal{D} = \{\mathrm{T}, \mathrm{I}\}$, we get Priest's logic of paradox.

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Idea: If there is enough information to determine the value then we put the classical value in the table. Otherwise, we put I.

$$\nvdash_{\mathbf{M}_{kl}} Ob \supset Ob \qquad Ob \vdash_{\mathbf{M}_{kl}} Ob$$

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$$\vdash_{\mathbf{M}_{pr}} Ob \supset Ob$$

Matrices for Sequents Derivability

- Recall:
  - $\Omega \vdash^{seq}_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in **LK**.
  - $\Gamma \vdash_{\mathbf{LK}} A$ if $\{ \Rightarrow B \mid B \in \Gamma\} \vdash^{seq}_{\mathbf{LK}} \Rightarrow A$.
- We would like to have semantics for $\vdash^{seq}_{\mathbf{LK}}$ as well, and not only for $\vdash_{\mathbf{LK}}$.

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- We would like to have semantics for $\vdash^{seq}_{\mathbf{LK}}$ as well, and not only for $\vdash_{\mathbf{LK}}$.
- Instead of one set of designated values $\mathcal{D}$, we have two sets:
  - $\mathcal{D}_{left}$ of *left designated values*.
  - $\mathcal{D}_{right}$ of *right designated values*.
- An **M**-valuation v is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in \mathcal{D}_{left}$ for some $A \in \Gamma$ or $v(A) \in \mathcal{D}_{right}$ for some $A \in \Delta$.

Matrices for Sequents Derivability

For $\mathbf{M}_{cl}$, we define:

$$\mathcal{D}_{left} = \{\mathrm{F}\} \quad \mathcal{D}_{right} = \{\mathrm{T}\}$$

Soundness and Completeness

$\Omega \vdash^{seq}_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ iff every $\mathbf{M}_{cl}$-valuation which is a model of Ω is also a model of $\Gamma \Rightarrow \Delta$.

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Example:

$$p_1 \Rightarrow \vdash^{seq}_{\mathbf{LK}} \; p_3 \Rightarrow p_1 \supset p_2$$

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Recall:

$$\Gamma \vdash_{\mathbf{G}} A \qquad \Longleftrightarrow \qquad \{ \Rightarrow B \mid B \in \Gamma\} \vdash^{seq}_{\mathbf{G}} \Rightarrow A$$

If $\mathbf{M}$ characterizes $\vdash^{seq}_{\mathbf{G}}$, take $\mathcal{D} = \mathcal{D}_{right}$ to obtain a matrix for $\vdash_{\mathbf{G}}$.

One Step in the Soundness Proof

Soundness

If $\Omega \vdash^{seq}_{\mathbf{LK}} s$ then every $\mathbf{M}_{cl}$-valuation which is a model of Ω is also a model of s.

Proof by induction on the length of the derivation.

- Consider an application of the rule $(\neg \Rightarrow)$. It has the form:

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}$$

Suppose that v is a model of $\Gamma \Rightarrow A, \Delta$.
We prove that it is a model of $\Gamma, \neg A \Rightarrow \Delta$.

Recall: v is a *model* of a sequent iff $v(B) \in \mathcal{D}_{left}$ for some B on the left side or $v(B) \in \mathcal{D}_{right}$ for some B on the right side.

If $v(B) = \mathrm{F}$ for some $B \in \Gamma$, we are done.
If $v(B) = \mathrm{T}$ for some $B \in \Delta$, we are done.
Otherwise, $v(A) = \mathrm{T}$.
Since v is an $\mathbf{M}_{cl}$-valuation: $v(\neg A) = \tilde{\neg}(v(A)) = \tilde{\neg}(\mathrm{T}) = \mathrm{F}$.

Matrices for Sequents Derivability

A many-valued matrix for a language $\mathcal{L}$ consists of:

- A set $\mathcal{V}$ of truth values.
- Subsets $\mathcal{D}_{left}, \mathcal{D}_{right} \subseteq \mathcal{V}$ of *designated* truth values.
- A truth table for every connective of $\mathcal{L}$, i.e. for every connective $\diamond$ we have a function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $\mathcal{V}$, where n is the arity of $\diamond$.

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- An **M**-*valuation* is a function $v : wff_{\mathcal{L}} \to \mathcal{V}$ that respects all truth tables, i.e. for every compound formula $\diamond(A_1, \ldots, A_n)$:

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A matrix M is sound and complete for a sequent system G iff:

$\Omega \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ iff every **M**-valuation which is a model of Ω is also a model of $\Gamma \Rightarrow \Delta$.

Classical Logic

The Matrix $\mathbf{M}_{cl}$

- Values: $\mathcal{V} = \{\mathrm{F}, \mathrm{T}\}$
- Designated values: $\mathcal{D}_{left} = \{\mathrm{F}\}$ and $\mathcal{D}_{right} = \{\mathrm{T}\}$.
  - Thus, a valuation v is a *model* of a sequent iff $v(A) = \mathrm{F}$ for some A on the left side, or $v(A) = \mathrm{T}$ for some A on the right side.
- Tables:

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	T
F	T	T
T	F	F
T	T	T

x	$\widetilde{\neg}(x)$
F	T
T	F

Soundness and Completeness

$\Omega \vdash^{seq}_{\mathbf{LK}} \Gamma \Rightarrow \Delta$ iff iff every $\mathbf{M}_{cl}$-valuation which is a model of Ω is also a model of $\Gamma \Rightarrow \Delta$.

Questions

- What happens if we play with **LK**?
  e.g. add new rules, omit some rules, change some rules.
- Do we still have an effective semantics?
- What about the subformula property? cut-admissibility? axiom-expansion? invertibility of the logical rules?

Motivations

Known examples:

- If we omit ($\neg \Rightarrow$) from **LK**, we obtain a system for the paraconsistent logic CluN [Batens].
- Primal implication is defined by $\dfrac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$ instead of ($\Rightarrow\supset$) [Gurevich et al.]

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Guiding principle

The meaning of each connective is given by its introduction rules:

"... The introductions represent, as it were, the 'definition' of the symbols concerned..." [Gentzen, Investigations into logical deduction]

For example:

$$\frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \between A_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \between A_2, \Delta}$$

The system **GCLuN**

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$\mathbf{GCLuN} = \mathbf{LK} - (\neg \Rightarrow)$$

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$$\mathbf{GCLuN} = \mathbf{LK} - (\neg \Rightarrow)$$

Theorem

GCLuN *has no finite-valued characteristic matrix.*

Hint for the proof:

$$p, \neg p, \ldots, \neg^{n-1} p \nvdash_{\mathbf{GCLuN}} \neg^n p$$

"Reading off" the Semantics from Sequent Rules

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

x	$\widetilde{\neg}(x)$
F	T
T	F

"Reading off" the Semantics from Sequent Rules

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

x	$\widetilde{\neg}(x)$
F	T
T	F

Without $(\neg \Rightarrow)$:

x	$\widetilde{\neg}(x)$
F	T
T	?

Non-Deterministic Matrices - Main Idea

- Truth tables assign *sets* of truth values, and we require:

$$v(\diamond(A_1, \ldots, A_n)) \in \widetilde{\diamond}(v(A_1), \ldots, v(A_n))$$

instead of

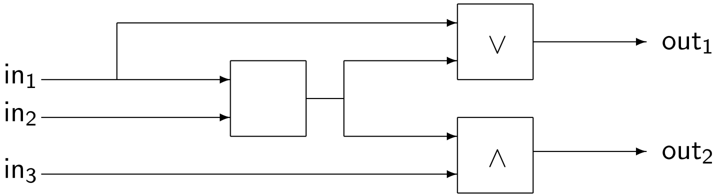
$$v(\diamond(A_1, \ldots, A_n)) = \widetilde{\diamond}(v(A_1), \ldots, v(A_n))$$

- Valuations still assign *one* value to each formula!

Non-Deterministic Matrices - Intuition

$\wedge$	F	T
F	F	F
T	F	T

$$v(\mathsf{in}_1 \wedge \mathsf{in}_2) = \widetilde{\wedge}(\mathsf{v}(\mathsf{in}_1), \mathsf{v}(\mathsf{in}_2))$$



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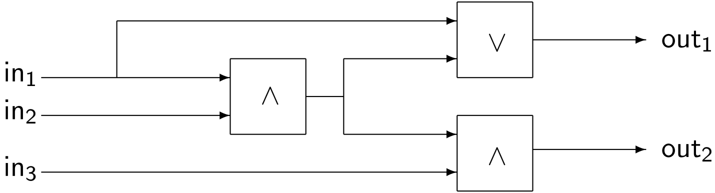

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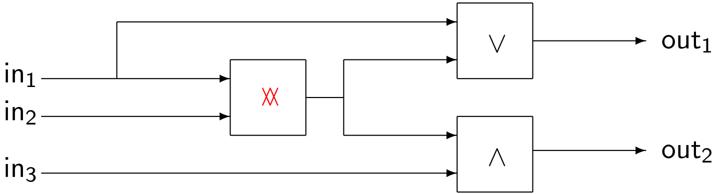

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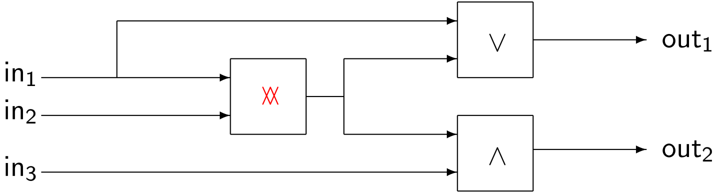
F	F	F
T	F	T

$$v(\text{in}_1 \wedge \text{in}_2) = \widetilde{\wedge}(\text{v}(\text{in}_1), \text{v}(\text{in}_2))$$

$\wedge$	F	T
F	$\{F\}$	$\{F, T\}$
T	$\{F, T\}$	$\{T\}$

$$v(\text{in}_1 ⩙ \text{in}_2) \in \widetilde{⩙}(\text{v}(\text{in}_1), \text{v}(\text{in}_2))$$



Truth-Functionality

Truth-Functionality

The value of a complex formula is uniquely determined by the values of its subformulas.

In Nmatrices we do not have truth-functionality, but a weaker property.

Non-Deterministic Matrices - Formal Definition

A non-deterministic matrix M (Nmatrix) for a language $\mathcal{L}$ consists of:

- A set $\mathcal{V}$ of truth values.
- Two subsets $\mathcal{D}_{left}, \mathcal{D}_{right} \subseteq \mathcal{V}$ of *designated* truth values.
- A non-deterministic truth table for every connective of $\mathcal{L}$, i.e. for every connective $\diamond$ we have a function $\tilde{\diamond}$ from $\mathcal{V}^n$ to $P(\mathcal{V})$, where n is the arity of $\diamond$.

- An **M**-*valuation* is a function $v : wff \rightarrow \mathcal{V}$ that respects all truth tables, i.e. $v(\diamond(A_1, \dots, A_n)) \in \tilde{\diamond}(v(A_1), \dots, v(A_n))$.
- The notion of a *model* is defined exactly as for (deterministic) matrices, i.e.:
  - An **M**-valuation v is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in \mathcal{D}_{left}$ for some $A \in \Gamma$ or $v(A) \in \mathcal{D}_{right}$ for some $A \in \Delta$.

NMatrix for **GCLuN**

The Matrix $\mathbf{M}_{CluN}$

- $\mathcal{V} = \{\mathrm{F}, \mathrm{T}\}$, $\mathcal{D}_{left} = \{\mathrm{F}\}$, $\mathcal{D}_{right} = \{\mathrm{T}\}$.
- Same tables as in $\mathbf{M}_{cl}$ (with singletons) except for:

x	$\tilde{\neg}(x)$
F	$\{\mathrm{T}\}$
T	$\{\mathrm{F}, \mathrm{T}\}$

Soundness and Completeness

$\Omega \vdash_{\mathbf{GCluN}} s$ iff every $\mathbf{M}_{CluN}$-valuation which is a model of Ω is also a model of s.

For example:

$$\Rightarrow p_1 \ , \ \Rightarrow \neg p_1 \ , \ \Rightarrow \neg\neg p_1 \nvdash^{seq}_{\mathbf{GCluN}} \Rightarrow \neg\neg\neg p_1$$

(thus $p_1, \neg p_1, \neg\neg p_1 \nvdash_{\mathbf{GCluN}} \neg\neg\neg p_1$)

Canonical Systems

Avron and Lev ('01)

A canonical system consists of:

- Structural rules as in **LK** (weakenings, (id), (cut)).
- Any finite set of *canonical* rules.

- Canonical rules are logical rule of an "ideal" form:
  - Each rule introduces exactly one connective in one side.
  - Exactly one occurrence of the introduced connective, and no other connectives are involved.
  - No restriction on context.
  - The active formulas are immediate subformulas of the principal formula.

Examples of Canonical Rules

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All logical rules of **LK** are canonical. Other examples include:

$$\frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

$$\frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \smashtimes A_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \smashtimes A_2, \Delta}$$

$$\frac{\Gamma \Rightarrow A_1, A_2, \Delta \quad \Gamma, A_3 \Rightarrow A_4, \Delta}{\Gamma \Rightarrow \diamond(A_1, A_2, A_3, A_4, A_5), \Delta}$$

Some Non-canonical Rules

$$\frac{\Gamma, A_1 \Rightarrow A_2}{\Gamma \Rightarrow A_1 \supset A_2}$$

$$\frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma \Rightarrow \neg(A_1 \supset A_2), \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \neg\neg A, \Delta}$$

Semantics of Canonical Systems

Given a canonical system G, we construct $\mathbf{M_G}$ as follows:

$\mathcal{V} = \{\mathrm{F}, \mathrm{T}\}$, $\mathcal{D}_{left} = \{\mathrm{F}\}$, $\mathcal{D}_{right} = \{\mathrm{T}\}$.

For every n-ary connective $\diamond$:

- Initialize a totally non-deterministic table.
- For every rule r for $\diamond$:
  For every $x_1, \ldots, x_n \in \mathcal{V}$:
  - If $x_1, \ldots, x_n$ satisfy the premises of r:
    - If r is a right rule, omit F from $\widetilde{\diamond}(x_1, \ldots, x_n)$.
    - If r is a left rule, omit T from $\widetilde{\diamond}(x_1, \ldots, x_n)$.

Example: Table of Implication

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$$\frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta}$$

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$$\frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	$\{\mathrm{F}, \mathrm{T}\}$
F	T	$\{\mathrm{F}, \mathrm{T}\}$
T	F	$\{\mathrm{F}, \mathrm{T}\}$
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F	F	$\{F, T\}$
F	T	$\{F, T\}$
T	F	$\{F, \not{T}\}$
T	T	$\{F, T\}$

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F	T	$\{\not{\mathrm{F}}, \mathrm{T}\}$
T	F	$\{\mathrm{F}, \not{\mathrm{T}}\}$
T	T	$\{\not{\mathrm{F}}, \mathrm{T}\}$

Example: The System **GPrim**

GPrim is obtained from **LK** by replacing the rules for $\supset$ with the following:

$$(\supset\Rightarrow)\quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$$

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Here, we obtain the following (non-deterministic) table for $\supset$:

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	$\{F, T\}$
F	T	$\{T\}$
T	F	$\{F\}$
T	T	$\{T\}$

Semantics of Canonical Systems

Soundness and Completeness

$\Omega \vdash^{seq}_{\mathbf{G}} s$ iff every $\mathbf{M_G}$-valuation which is a model of Ω is also a model of s.

Semantics of Canonical Systems

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$\Omega \vdash_{\mathbf{G}}^{seq} s$ iff every $\mathbf{M_G}$-valuation which is a model of Ω is also a model of s.

Theorem

If $\mathbf{M_G}$ is non-deterministic then there is no finite-valued (deterministic) matrix for $\mathbf{G}$.

What can go wrong?

Consider the Tonk connective [Prior] defined by:

$$\frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \circledcirc A_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma \Rightarrow A_1 \circledcirc A_2, \Delta}$$

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We obtain the table:

x_1	x_2	$\widetilde{\text{ⓣ}}(x_1, x_2)$
F	F	$\{F\}$
F	T	$\{F, T\}$
T	F	$\emptyset$
T	T	$\{T\}$

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- Soundness and completeness still hold.
- There are no $\mathbf{M}_{\mathbf{LK}+ⓣ}$-valuations!
- $\vdash_{\mathbf{LK}+ⓣ}$ is trivial.

Empty Sets in Truth Tables

Proposition

For every canonical system **G**:
if we have an empty set in a table of $\mathbf{M}_G$ then

$$\Rightarrow p_1 \;,\; p_2 \Rightarrow \vdash^{seq}_{\mathbf{G}} \Rightarrow$$

The Subformula property

Let **G** be an arbitrary canonical system.

Notation

Let $\mathcal{E}$ be a set of formulas.

- An $\mathcal{E}$-sequent is a sequent consisting solely of formulas from $\mathcal{E}$.
- $\Omega \vdash_{\mathbf{G}}^{\mathcal{E} seq} \Gamma \Rightarrow \Delta$ iff there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in a system **G** consisting solely of $\mathcal{E}$-sequents.

The Subformula Property

$$\Omega \vdash_{\mathbf{G}}^{seq} \Gamma \Rightarrow \Delta \qquad \Longrightarrow \qquad \Omega \vdash_{\mathbf{G}}^{sub[\Omega \cup \{\Gamma \Rightarrow \Delta\}]seq} \Gamma \Rightarrow \Delta$$

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We will take a “semantic approach”.

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We will take a "semantic approach".

Can we find semantics for $\vdash_{\mathbf{G}}^{\mathcal{E} seq}$?

Semantics for $\vdash_{\mathbf{G}}^{\mathcal{E}seq}$

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, set Ω of $\mathcal{E}$-sequents, and $\mathcal{E}$-sequent $\Gamma \Rightarrow \Delta$:
$\Omega \vdash_{\mathbf{G}}^{\mathcal{E}seq} \Gamma \Rightarrow \Delta$ iff every partial $\mathbf{M_G}$-valuation, defined on $\mathcal{E}$, which is a model of Ω is also a model of $\Gamma \Rightarrow \Delta$.

Semantics for $\vdash^{\mathcal{E}seq}_{\mathbf{G}}$

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For example, verify that:

$\Rightarrow \mathrm{CarStarts} \supset \mathrm{Trip}\ ,\ \ \Rightarrow \neg\mathrm{Trip} \vdash^{\mathcal{E}seq}_{\mathbf{LK}} \Rightarrow \neg\mathrm{CarStarts}$
for $\mathcal{E} = \{\mathrm{CarStarts}, \mathrm{Trip}, \neg\mathrm{Trip}, \neg\mathrm{CarStarts}, \mathrm{CarStarts} \supset \mathrm{Trip}\}$

$\Rightarrow p_1\ ,\ p_2 \Rightarrow \nvdash^{\mathcal{E}seq}_{\mathbf{LK}+ⓣ} \Rightarrow$
for $\mathcal{E} = \{p_1, p_2\}$

Semantic Proof of the Subformula Property

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Now, proving the subformula property for **G** reduces to proving that every partial $\mathbf{M_G}$-valuation (defined on a closed set of formulas) can be extended to a (full) $\mathbf{M_G}$-valuation.

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Now, proving the subformula property for **G** reduces to proving that every partial $\mathbf{M_G}$-valuation (defined on a closed set of formulas) can be extended to a (full) $\mathbf{M_G}$-valuation.

For **LK** this is trivial!
Thus **LK** has the subformula property.

The Subformula Property in Canonical Systems

Consider the following procedure:

Extension Procedure

By recursion on the build-up of formulas:

- When $v(p)$ is undefined choose it arbitrarily.
- When $v(\diamond(A_1, \ldots, A_n))$ is undefined choose it arbitrarily from $\widetilde{\diamond}(v(A_1), \ldots, v(A_n))$.

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When does it works?

If we do not have any empty sets in the tables.

Coherence

Definition

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In particular, **GCluN** and **GPrim** have the subformula property.

Understanding Coherence

- We obtain an empty set iff there exists a right rule and a left rule for the same connective, whose premises are satisfied by the same n values.
- In other words, we need that the right rules and the left rules for each connective to be contradictory.
- This does not hold for the rules of Tonk:

$$\frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \circledt A_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma \Rightarrow A_1 \circledt A_2, \Delta}$$

A Bigger Picture

- We demonstrated the semantic approach to establish the subformula property.
- In canonical systems, the subformula property is equivalent to *semantic analyticity* — the fact that every partial valuation can be extended.
- Similar approach works for many other Gentzen-type systems.
- The subformula property was proved regardless of cut-elimination.

So far

- We defined the family of canonical systems.
- We introduced the semantic framework of Nmatrices.
- We provided a method to obtain a two-valued Nmatrix for every canonical system.
- We introduced the coherence criterion – a necessary and sufficient criterion for the subformula property in canonical systems.

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What about cut-admissibility in canonical systems?

Cut-Admissibility

$$(cut) \quad \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Cut-Admissibility

$$\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta \qquad \Longrightarrow \qquad \vdash^{seq}_{\mathbf{G}-(cut)} \Gamma \Rightarrow \Delta$$

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- **LK** enjoys cut-admissibility (Gentzen, 1934).
- What about other canonical systems?

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- **LK** enjoys cut-admissibility (Gentzen, 1934).
- What about other canonical systems?

We will take a "semantic approach".

Can we find semantics for **LK** $-$ (cut)?

Semantics for **LK** $-$ (*cut*)

- Not the same semantics as for **LK**!
- Cut-admissibility does not hold in the presence of assumptions, e.g.

$$\Rightarrow p_1 \ , \ p_1 \Rightarrow \vdash^{seq}_{\mathbf{LK}} \ \Rightarrow$$

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Theorem

$\vdash_{\mathbf{LK}-(cut)}$ *does not have a finite characteristic matrix.*

Semantics for **LK** − (*cut*)

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Semantics for **LK** $-$ (*cut*)

$$\Rightarrow p_1 \;,\; p_1 \Rightarrow \nvdash^{seq}_{\mathbf{LK}-(cut)} \;\Rightarrow$$

- Without cut, there should be a valuation which is both a model of $p_1 \Rightarrow$ and of $\Rightarrow p_1$.

Semantics for $\mathbf{LK} - (cut)$

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- Without cut, there should be a valuation which is both a model of $p_1 \Rightarrow$ and of $\Rightarrow p_1$.
- Recall: An **M**-valuation v is a *model* of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in \mathcal{D}_{left}$ for some $A \in \Gamma$ or $v(A) \in \mathcal{D}_{right}$ for some $A \in \Delta$.

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- $v(p_1)$ should be both in $\mathcal{D}_{left}$ and in $\mathcal{D}_{right}$.

- We will add a third value $\top$: $\mathcal{V} = \{\mathrm{F}, \mathrm{T}, \top\}$.
- $\top$ makes a sequent true on both sides:

$$\mathcal{D}_{left} = \{\mathrm{F}, \top\} \quad \mathcal{D}_{right} = \{\mathrm{T}, \top\}$$

- *The construction of the tables is done using the same method used for canonical systems.*

The NMatrix $\mathbf{M}_{\mathbf{LK}-(cut)}$

$$\mathcal{V} = \{\text{F}, \text{T}, \top\} \qquad \mathcal{D}_{left} = \{\text{F}, \top\} \quad \mathcal{D}_{right} = \{\text{T}, \top\}$$

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	$\{\text{T}, \top\}$
F	T	$\{\text{T}, \top\}$
T	F	$\{\text{F}, \top\}$
T	T	$\{\text{T}, \top\}$
F	$\top$	$\{\text{T}, \top\}$
T	$\top$	$\{\top\}$
$\top$	F	$\{\top\}$
$\top$	T	$\{\text{T}, \top\}$
$\top$	$\top$	$\{\top\}$

x	$\widetilde{\neg}(x)$
F	$\{\text{T}, \top\}$
T	$\{\text{F}, \top\}$
$\top$	$\{\top\}$

Semantics for **LK** $-$ (cut)

Soundness and Completeness - sequents

$\Omega \vdash^{seq}_{\mathbf{LK}-(cut)} s$ iff every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is a model of Ω is also a model of s.

Soundness and Completeness - formulas

$\Gamma \vdash_{\mathbf{LK}-(cut)} A$ iff $\Gamma \vdash_{\mathbf{M}_{\mathbf{LK}-(cut)}} A$ (i.e. every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is a model of Γ is also a model of A). *(where $\mathcal{D} = \mathcal{D}_{right}$)*

For example, verify that:

$$\text{CarStarts} \supset \text{Trip}, \neg\text{Trip} \nvdash_{\mathbf{LK}-(cut)} \neg\text{CarStarts}$$

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For example, verify that:

$$\text{CarStarts} \supset \text{Trip}, \neg\text{Trip} \not\vdash_{\mathbf{LK}-(cut)} \neg\text{CarStarts}$$

$\hookrightarrow$ New formulation of results of Schütte (1960) and Girard (1987).

Example: Construction of a table for $\wedge$

- Initialize a totally non-deterministic table.
- For every rule r for $\diamond$:

  For every $x_1, \ldots, x_n \in \mathcal{V}$:
  - If $x_1, \ldots, x_n$ satisfy the premises of r:
    - If r is a right rule, omit F from $\widetilde{\diamond}(x_1, \ldots, x_n)$.
    - If r is a left rule, omit T from $\widetilde{\diamond}(x_1, \ldots, x_n)$.

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \wedge A_2, \Delta}$$

$\widetilde{\wedge}$	F	T	$\top$
F	$\{\mathrm{F}, \mathrm{T}, \top\}$	$\{\mathrm{F}, \mathrm{T}, \top\}$	$\{\mathrm{F}, \mathrm{T}, \top\}$
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- All usual connectives have non-deterministic semantics.
- Non-determinism is a result of the missing cut rule.

Canonical Systems without (*cut*)

- The same construction works for every canonical system without (*cut*).
- $\top$ is included in every entry in every table.
  - Thus, all canonical systems without (*cut*) have the subformula property. (This is obvious from a syntactic point of view.)
- The $\{\mathrm{F}, \mathrm{T}\}$-entries of the tables for the system without cut are equal to those of the system with cut, except for the addition of $\top$.

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Observation

Every $\mathbf{M}_{\mathbf{G}-(cut)}$-valuation over $\{\mathrm{F}, \mathrm{T}\}$ is an $\mathbf{M}_{\mathbf{G}}$-valuation.

Cut-Admissibility for **LK**

$$\vdash_{\mathbf{LK}} \Gamma \Rightarrow \Delta \quad \Longrightarrow \quad \vdash_{\mathbf{LK}-(cut)} \Gamma \Rightarrow \Delta$$

Semantic Equivalent

If every $\mathbf{M}_{\mathbf{LK}}$-valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ then every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation is a model of $\Gamma \Rightarrow \Delta$.

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- To prove cut-admissibility for **LK**, we have to prove:
  For every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is not a model of some sequent $\Gamma \Rightarrow \Delta$, there exists an $\mathbf{M}_{\mathbf{LK}}$-valuation which is not a model of $\Gamma \Rightarrow \Delta$.
- Using the previous observation, it suffices to show:
  For every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation which is not a model of some sequent $\Gamma \Rightarrow \Delta$, there exists an $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation over $\{\mathrm{F}, \mathrm{T}\}$ which is not a model of $\Gamma \Rightarrow \Delta$.
- It suffices to show:
  For every $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation v there exists an $\mathbf{M}_{\mathbf{LK}-(cut)}$-valuation v' over $\{\mathrm{F}, \mathrm{T}\}$ such that $v'(A) = v(A)$ whenever $v(A) \in \{\mathrm{F}, \mathrm{T}\}$.

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Refinement Procedure

By recursion on the build-up of formulas:

- If $v(A) \in \{\mathrm{F}, \mathrm{T}\}$: $v'(A) := v(A)$.
- Otherwise:
  - If A is atomic, choose $v'(A)$ to be either F or T arbitrarily.
  - If $A = \diamond(A_1, \ldots, A_n)$, choose $v'(A)$ to be either F or T arbitrarily from $\widetilde{\diamond}(v(A_1), \ldots, v(A_n))$.

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Why does it work?

In the tables of $\mathbf{M}_{\mathbf{LK}-(cut)}$, $\{\mathrm{F}, \mathrm{T}\}$-entries always include F or T in addition to $\top$.

Cut-Admissibility for Canonical Systems

Theorem

LK *enjoys cut-admissibility.*

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What about canonical systems in general?

- To have cut-admissibility, we should not have $\widetilde{\diamond}(x_1, \ldots, x_n) = \{\top\}$ for $x_1, \ldots, x_n \in \{\mathrm{F}, \mathrm{T}\}$.
- Recall: The $\{\mathrm{F}, \mathrm{T}\}$-entries of the tables for the system without cut are equal to those of the system with cut, except for the addition of $\top$.
- Thus $\widetilde{\diamond}(x_1, \ldots, x_n) = \{\top\}$ for $x_1, \ldots, x_n \in \{\mathrm{F}, \mathrm{T}\}$ only if $\widetilde{\diamond}(x_1, \ldots, x_n) = \emptyset$ in the tables for the same system with cut.

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Theorem

Every coherent canonical system enjoys cut-admissibility.

Triple Correspondence

Note that if a system is not coherent then it does not enjoy cut-admissibility (since $\Rightarrow p_1$, $p_2 \Rightarrow \vdash^{seq}_{\mathbf{G}} \Rightarrow$).

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Corollary

For every canonical system **G**, *the following are equivalent:*

- **G** *is coherent.*
- **G** *has the subformula property.*
- **G** *enjoys cut-admissibility.*

A Bigger Picture

- We demonstrated the "semantic approach" to prove cut-admissibility.
- We had three steps:
  - Find semantics 1 for the system with cut.
  - Find semantics 2 for the system without cut.
  - Show that every non-model of some sequent $\Gamma \Rightarrow \Delta$ in 2 can be turned into a non-model of $\Gamma \Rightarrow \Delta$ in 1.

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  - Safer and less tedious.
  - Better understanding of the meaning of cut.
  - Easier to generalize.
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- In comparison to the syntactic approach:
  - Safer and less tedious.
  - Better understanding of the meaning of cut.
  - Easier to generalize.
  - The method can be adapted to higher-order logics.
- On the other hand:
  - We only have *cut-admissibility* and not *cut-elimination*.
  - If it does not work then it does not easily lead to counter example.

Axiom-Expansion

- (id) is the rule allowing to derive all sequents of the form $A \Rightarrow A$ (with no premises).
- Atomic applications of (id) derive sequents of the form $p \Rightarrow p$, where p is an atomic formula.

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Axiom-Expansion

If $\Omega \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ from Ω in **G** in which all applications of (id) are atomic.

Axiom-Expansion

Equivalent Formulation

For every *n*-ary connective:

$\{p_i \Rightarrow p_i \mid i \geq 1\} \vdash_{\mathbf{G}-(id)} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$

Axiom-Expansion

Equivalent Formulation

For every *n*-ary connective:
$\{p_i \Rightarrow p_i \mid i \geq 1\} \vdash_{\mathbf{G}-(id)} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$

LK admits axiom-expansion. For example:

$$\dfrac{\dfrac{p_1 \Rightarrow p_1 \quad p_2 \Rightarrow p_2}{p_1, p_1 \supset p_2 \Rightarrow p_2}}{p_1 \supset p_2 \Rightarrow p_1 \supset p_2}$$

Again, we would like to obtain a semantic equivalent of this property.
What is the semantics of canonical systems without (id) ? In particular, of **LK** $- (id)$?

Semantics for $\mathbf{LK} - (id)$

Theorem

$\vdash_{\mathbf{LK}-(id)}$ *does not have a finite characteristic matrix.*

Semantics for **LK** $-$ (*id*)

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- Without id, there should be a valuation which is not a model of $p \Rightarrow p$.
- Thus, $v(p)$ should be neither in $\mathcal{D}_{left}$ nor in $\mathcal{D}_{right}$.

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Theorem

$\vdash_{\mathbf{LK}-(id)}$ *does not have a finite characteristic matrix.*

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- Without id, there should be a valuation which is not a model of $p \Rightarrow p$.
- Thus, $v(p)$ should be neither in $\mathcal{D}_{left}$ nor in $\mathcal{D}_{right}$.
- We will add a third value $\perp$: $\mathcal{V} = \{\mathrm{F}, \mathrm{T}, \perp\}$.
- $\perp$ never makes a sequent true:

$$\mathcal{D}_{left} = \{\mathrm{F}\} \quad \mathcal{D}_{right} = \{\mathrm{T}\}$$

- The construction of the tables is almost the same.

Example: Construction of a table for $\wedge$

- Initialize a totally non-deterministic table.
- For every rule r for $\diamond$:
  For every $x_1, \ldots, x_n \in \mathcal{V}$:
  - If $x_1, \ldots, x_n$ satisfy the premises of r:
    - If r is a right rule, omit F and $\bot$ from $\widetilde{\diamond}(x_1, \ldots, x_n)$.
    - If r is a left rule, omit T and $\bot$ from $\widetilde{\diamond}(x_1, \ldots, x_n)$.

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \wedge A_2, \Delta}$$

$\widetilde{\wedge}$	F	T	$\bot$
F	$\{\mathrm{F}, \mathrm{T}, \bot\}$	$\{\mathrm{F}, \mathrm{T}, \bot\}$	$\{\mathrm{F}, \mathrm{T}, \bot\}$
T	$\{\mathrm{F}, \mathrm{T}, \bot\}$	$\{\mathrm{F}, \mathrm{T}, \bot\}$	$\{\mathrm{F}, \mathrm{T}, \bot\}$
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F	$\{\mathrm{F}\}$	$\{\mathrm{F}\}$	$\{\mathrm{F}\}$
T	$\{\mathrm{F}\}$	$\{\mathrm{T}\}$	$\{\mathrm{F}, \mathrm{T}, \perp\}$
$\perp$	$\{\mathrm{F}\}$	$\{\mathrm{F}, \mathrm{T}, \perp\}$	$\{\mathrm{F}, \mathrm{T}, \perp\}$

- All usual connectives have non-deterministic semantics.
- Non-determinism is a result of the missing identity axiom.

The NMatrix $\mathbf{M}_{\mathbf{LK}-(id)}$

$$\mathcal{V} = \{\mathrm{F}, \mathrm{T}, \bot\} \qquad \mathcal{D}_{left} = \{\mathrm{F}\} \quad \mathcal{D} = \mathcal{D}_{right} = \{\mathrm{T}\}$$

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	$\{\mathrm{T}\}$
F	T	$\{\mathrm{T}\}$
T	F	$\{\mathrm{F}\}$
T	T	$\{\mathrm{T}\}$
F	$\bot$	$\{\mathrm{T}\}$
T	$\bot$	$\{\mathrm{F}, \mathrm{T}, \bot\}$
$\bot$	F	$\{\mathrm{F}, \mathrm{T}, \bot\}$
$\bot$	T	$\{\mathrm{T}\}$
$\bot$	$\bot$	$\{\mathrm{F}, \mathrm{T}, \bot\}$

x	$\widetilde{\neg}(x)$
F	$\{\mathrm{T}\}$
T	$\{\mathrm{F}\}$
$\bot$	$\{\mathrm{F}, \mathrm{T}, \bot\}$

Semantics for **LK** $-$ (id)

Soundness and Completeness - sequents

$\Omega \vdash^{seq}_{\mathbf{LK}-(id)} s$ iff every $\mathbf{M}_{\mathbf{LK}-(id)}$-valuation which is a model of Ω is also a model of s.

Soundness and Completeness - formulas

$\Gamma \vdash_{\mathbf{LK}-(id)} A$ iff $\Gamma \vdash_{\mathbf{M}_{\mathbf{LK}-(id)}} A$ (i.e. every $\mathbf{M}_{\mathbf{LK}-(id)}$-valuation which is a model of Γ is also a model of A). *(where $\mathcal{D} = \mathcal{D}_{right}$)*

For example, verify that:

$$\text{CarStarts} \supset \text{Trip}, \neg\text{Trip} \nvdash_{\mathbf{LK}-(id)} \neg\text{CarStarts}$$

$\hookrightarrow$ New formulation of results of Hösli and Jäger (1994).

Semantics for Canonical Systems without (*id*)

- The same construction works for every canonical system **G**.

Axiom-Expansion

For every n-ary connective:

$$\{p_i \Rightarrow p_i \mid i \geq 1\} \vdash_{\mathbf{G}-(id)} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$$

- In other words: Whenever $v(p_i) \in \{\mathrm{F}, \mathrm{T}\}$ for every $i \geq 1$, we also have $v(\diamond(p_1, \ldots, p_n)) \in \{\mathrm{F}, \mathrm{T}\}$ for every connective $\diamond$.
- Thus, we have axiom expansion iff for every connective $\diamond$: $\bot \notin \widetilde{\diamond}(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in \{\mathrm{F}, \mathrm{T}\}$.

Axiom-Expansion for **LK**

LK admits axiom-expansion.

x_1	x_2	$\widetilde{\supset}(x_1, x_2)$
F	F	$\{T\}$
F	T	$\{T\}$
T	F	$\{F\}$
T	T	$\{T\}$
F	$\bot$	$\{T\}$
T	$\bot$	$\{F, T, \bot\}$
$\bot$	F	$\{F, T, \bot\}$
$\bot$	T	$\{T\}$
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x	$\widetilde{\neg}(x)$
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Axiom-Expansion for Canonical Systems

- We have axiom expansion iff for every connective $\diamond$: $\bot \notin \widetilde{\diamond}(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in \{\mathrm{F}, \mathrm{T}\}$.
- This means that we did at least one deletion in every $\{\mathrm{F}, \mathrm{T}\}$-entry.
- Equivalently, the tables for the system with (id) are deterministic.

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- Equivalently, the tables for the system with (id) are deterministic.

Theorem

A canonical system **G** *admits axiom-expansion iff* $\mathbf{M_G}$ *is deterministic.*

In particular, **GCluN** and **GPrim** do not admit axiom-expansion.

Invertibility of Logical Rules

- A canonical rule is called *invertible* in a system **G** if each of its premises can be derived from its conclusion in **G**.
- (Formally, this should hold for every instantiation of Γ,Δ and $A_1, A_2, \ldots$.)

$\dfrac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}$ is invertible in **LK**:

$$\dfrac{\dfrac{\Gamma \Rightarrow A_1 \supset A_2, \Delta}{\Gamma, A_1 \Rightarrow A_1 \supset A_2, A_2, \Delta}\,(w) \qquad \dfrac{\dfrac{\dfrac{}{A_1 \Rightarrow A_1}\,(id)}{\Gamma, A_1 \Rightarrow A_1, A_2, \Delta}\,(w) \quad \dfrac{\dfrac{}{A_2 \Rightarrow A_2}\,(id)}{\Gamma, A_1, A_2 \Rightarrow A_2, \Delta}\,(w)}{\Gamma, A_1, A_1 \supset A_2 \Rightarrow A_2, \Delta}\,(\supset\Rightarrow)}{\Gamma, A_1 \Rightarrow A_2, \Delta}\,(cut)$$

- A canonical right rule for $\diamond$ is invertible in **G**: if for every $\mathbf{M_G}$-valuation v, if $v(\diamond(A_1, \ldots, A_n)) = \mathrm{T}$ then the premises of the rule are satisfied by v.
- Equivalently, when $\mathrm{T} \in \widetilde{\diamond}(x_1, \ldots, x_n)$ then $x_1, \ldots, x_n$ satisfy the premises of the rule.

Semantic View of Invertibility of Logical Rules
Informal discussion

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F	F	$\{\mathrm{T}\}$
F	T	$\{\mathrm{T}\}$
T	F	$\{\mathrm{F}\}$
T	T	$\{\mathrm{T}\}$

In case we have only one right rule r for $\diamond$:

- In the construction of $\widetilde{\diamond}$, when r's premises are satisfied, we delete F.
- r is invertible in **G** iff there are no $\{\mathrm{F}, \mathrm{T}\}$'s in $\widetilde{\diamond}$.

Triple Correspondence

Corollary

For every canonical system $\mathbf{G}$*, the following are equivalent:*

- $\mathbf{M_G}$ *is deterministic.*
- $\mathbf{G}$ *admits axiom-expansion.*
- *If every connective has exactly one left rule and one right rule, then all logical rules are invertible.*

Final Remarks

- Non-deterministic semantics is a useful tool for understanding and investigating proof-theoretic properties of formal calculi.
- The semantic tools complement the usual proof-theoretic ones.
- Interesting cases arise when the "semantic approach" is applied for
  - Single-conclusion sequent systems
  - Sequent systems for modal logics
  - Many-sided sequent systems
  - Hypersequent systems
  - Sub-structural systems ??

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Thank you for your attention!
You are welcome to ask, suggest and discuss.

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