Studying Sequent Systems via Non-deterministic Many-Valued Matrices

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1. A formal language $\mathcal{L}$, based on which $\mathcal{L}$-formulas are constructed.

2. A binary relation $\vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying:

   - **Reflexivity:** if $A \in \Gamma$ then $\Gamma \vdash A$.
   - **Monotonicity:** if $\Gamma \vdash A$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash A$.
   - **Transitivity:** if $\Gamma \vdash B$ and $\Gamma', B \vdash A$ then $\Gamma, \Gamma' \vdash A$.  

Languages

We will only consider **propositional** languages, consisting of:

- Atomic formulas (we usually use $p_1, p_2, \ldots$)
- A finite set of logical connectives
- Parentheses: ‘(’,‘)’

We denote by $\text{wff}_L$ the set of well-formed formulas of $L$.

$L_{cl}$ (a language for classical logic) includes the unary connective $\neg$, and the binary connectives $\land$, $\lor$, and $\supset$. 
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$L_{cl}$ (a language for classical logic) includes the unary connective $\neg$, and the binary connectives $\land, \lor, \supset$.

The set of well-formed formulas $\text{wff}_{L_{cl}}$:
- All atomic formulas are in $\text{wff}_{L_{cl}}$.
- If $A, B \in \text{wff}_{L_{cl}}$, then ($\neg A), (A \land B), (A \lor B), (A \supset B) \in \text{wff}_{L_{cl}}$. 
Syntactic Approach to Define Logics

⊢ is defined using a notion of a derivation in a given proof system.

For example, we can use Hilbert-style systems:

- A Hilbert-style system consists of: (i) a set of formulas called axioms, and (ii) a set of inference rules.

- A derivation of $A$ from $\Gamma$ in a Hilbert-style system $H$ is a finite sequence of formulas, where the last formula is $A$, and each formula is: (i) an axiom of $H$, (ii) a member of $\Gamma$, or (iii) obtained from previous formulas by applying some inference rule of $H$.

- The consequence relation $\vdash_H$ is defined by:

$$\Gamma \vdash_H A \text{ if } A \text{ has a derivation from } \Gamma \text{ in } H$$
The system **HCL**

Axiom schemata:

\[ I_1 \quad A \supset (B \supset A) \]
\[ I_2 \quad (A \supset B \supset C) \supset (A \supset B) \supset (A \supset C) \]
\[ I_3 \quad ((B \supset A) \supset B) \supset B \]
\[ C_1 \quad A \land B \supset A \]
\[ C_2 \quad A \land B \supset B \]
\[ C_3 \quad A \supset (B \supset A \land B) \]
\[ D_1 \quad A \supset A \lor B \]
\[ D_2 \quad B \supset A \lor B \]
\[ D_3 \quad (A \supset C) \supset (B \supset C) \supset (A \lor B \supset C) \]
\[ N_1 \quad (B \supset A) \supset (B \supset \neg A) \supset \neg B \]
\[ N_2 \quad \neg \neg A \supset A \]

Inference Rule:

\[
\begin{array}{c}
A \\
A \supset B \\
\hline
B
\end{array}
\]

\[ MP \]

**Definition**

Classical logic = the language \( \mathcal{L}_{cl} \) + the consequence relation \( \vdash_{HCL} \)
Hilbert-style systems operate on formulas. Gentzen-style systems operate on sequents.

Sequents are objects of the form \( \Gamma \Rightarrow \Delta \), where \( \Gamma \) and \( \Delta \) are finite sets of formulas.

A Gentzen-style proof system consists of a set of sequent rules (usually given by schemes).
Semantic Intuition for Sequents

\[ A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \]
\[ A_1 \land \ldots \land A_n \supset B_1 \lor \ldots \lor B_m \]
Semantic Intuition for Sequents

\[ A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \]

\[ \Rightarrow B \]

\[ A \Rightarrow \]

\[ A_1, \ldots, A_n \Rightarrow \]

\[ \Rightarrow \]

\[ A_1, \ldots, A_n \Rightarrow \neg A_1 \lor \ldots \lor \neg A_n \]

\[ \neg A \]

\[ B \]

\[ B_1 \lor \ldots \lor B_m \]

\[ \Rightarrow \]

\[ \neg A_1 \lor \ldots \lor \neg A_n \]

\[ B \]

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\[ \Rightarrow \]
A derivation of a sequent $\Gamma \Rightarrow \Delta$ from a set of sequents $\Omega$ in $G$ is a finite sequence of sequents, where the last sequent is $\Gamma \Rightarrow \Delta$, and each sequent is: (i) a member of $\Omega$, or (ii) obtained from previous sequents in the sequence by applying some rule of $G$.

We write $\Omega \vdash^\text{seq}_G \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in $G$. 
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We write $\Omega \vdash_{G}^{\text{seq}} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in $G$.

A consequence relation (between formulas) is obtained by:

\[ \Gamma \vdash_{G} A \iff \left\{ \Rightarrow B \mid B \in \Gamma \right\} \vdash_{G}^{\text{seq}} \Rightarrow A \]
Logical Rules:

\[(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \quad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\]
Logical Rules:

\[ (¬ \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \quad (\Rightarrow ¬) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \]

\[ (\Implies \Rightarrow) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \Implies A_2 \Rightarrow \Delta} \quad (\Rightarrow \Implies) \quad \frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \Implies A_2, \Delta} \]
Logical Rules:

\[
\begin{align*}
(-\Rightarrow) & \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \\
(\Rightarrow \neg) & \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}
\end{align*}
\]

\[
\begin{align*}
(\supset \Rightarrow) & \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma \supset A_1 \Rightarrow A_2 \Rightarrow \Delta} \\
(\Rightarrow \supset) & \quad \frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}
\end{align*}
\]

\[
\begin{align*}
(\land \Rightarrow) & \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma \Rightarrow A_1 \land A_2 \Rightarrow \Delta} \\
(\Rightarrow \land) & \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}
\end{align*}
\]

\[
\begin{align*}
(\lor \Rightarrow) & \quad \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma \Rightarrow A_1 \lor A_2 \Rightarrow \Delta} \\
(\Rightarrow \lor) & \quad \frac{\Gamma \Rightarrow A_1, A_2, \Delta}{\Gamma \Rightarrow A_1 \lor A_2, \Delta}
\end{align*}
\]
Structural Rules:

\[
(W \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad (\Rightarrow W) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}
\]

\[
(id) \quad \frac{A \Rightarrow A}{(cut)} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}
\]

(\textit{other structural rules are built-in when sequents are pairs of sets}).
Soundness and Completeness

\( \textbf{LK} \) is sound and complete for classical logic, i.e. \( \Gamma \vdash_{\text{LK}} A \) iff \( \Gamma \vdash_{\text{HCL}} A \).

For example:

\[ \vdash_{\text{LK}} \neg (Ob \land Ro) \supset \neg Ob \lor \neg Ro \]
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\[ \vdash_{\mathbf{seq}}^{\mathbf{LK}} \Rightarrow \neg (Ob \land Ro) \supset \neg Ob \lor \neg Ro \]
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For example:

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We show:

$$\vdash_{seq} LK \Rightarrow \neg (Ob \land Ro) \supset \neg Ob \lor \neg Ro$$

\[
\begin{align*}
    \frac{Ro \Rightarrow Ro}{Ro \Rightarrow \neg Ob, Ro} \quad (\Rightarrow \text{id}) & \quad \frac{Ob \Rightarrow Ob}{Ob \Rightarrow \neg Ro, Ob} \quad (\Rightarrow \text{id}) \\
    \Rightarrow \neg Ob, \neg Ro, Ro & \quad \Rightarrow \neg Ob, \neg Ro, Ob \\
    \Rightarrow \neg Ob \lor \neg Ro, Ob & \quad \Rightarrow \neg Ob \lor \neg Ro, Ro \\
    \Rightarrow \neg Ob \lor \neg Ro, Ob \land Ro & \quad \Rightarrow \neg Ob \lor \neg Ro, Ob \land Ro \\
    \Rightarrow \neg Ob \lor \neg Ro, Ob \land Ro & \quad (\neg \Rightarrow) \\
    \neg (Ob \land Ro) \Rightarrow \neg Ob \lor \neg Ro & \quad (\Rightarrow \supset) \\
    \Rightarrow \neg (Ob \land Ro) \supset \neg Ob \lor \neg Ro & \quad (\supset \Rightarrow)
\end{align*}
\]
LK has many good properties
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**The Subformula Property**

If $\Omega \vdash_{\text{seq}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in LK consisting *solely* of subformulas of the formulas in $\Omega$ and $\Gamma \Rightarrow \Delta$. 

**Cut-Admissibility**

If $\vdash_{\text{seq}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ in LK with no applications of (cut).

**Axiom-Expansion**

Atomic axioms (i.e. axioms of the form $p_i \Rightarrow p_i$) always suffice.

**Invertibility of Logical Rules**

The premises of each logical rule can be derived from its conclusion.

LK has an effective semantics.
**LK** has many good properties

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Two truth values: \( F \) and \( T \)

| \( F \) | \( F \) | \( T \) | \( F \) | \( F \) | \( F \) | \( F \) | \( T \) | \( T \) | \( F \) | \( T \) | \( F \) | \( T \) | \( T \) | \( T \) | \( T \) |
| \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) |

Truth tables:

A \textit{valuation} function assigns values to the atomic formulas, and they determine the values of the compound formulas according to the tables.

\( \Gamma \vdash A \) iff for every valuation \( \nu \): if \( \nu(B) = T \) for every \( B \in \Gamma \) then \( \nu(A) = T \).
\( \vdash \) is defined using the notion of a \textit{model}: 

\[ \Gamma \vdash A \text{ if every "model" of } \Gamma \text{ is a "model" of } A \]

For example, we can use \textit{many-valued matrices}:

A many-valued matrix for a language \( \mathcal{L} \) consists of:

- A set \( \mathcal{V} \) of \textit{truth values}.
- A subset \( \mathcal{D} \subseteq \mathcal{V} \) of \textit{designated} truth values.
- A \textit{truth table} for every connective of \( \mathcal{L} \), i.e. for every connective \( \diamond \) we have a function \( \tilde{\diamond} \) from \( \mathcal{V}^n \) to \( \mathcal{V} \), where \( n \) is the arity of \( \diamond \).
Many-valued Matrices

Let $M$ be a many-valued matrix.

- An $M$-valuation is a function $v : \text{wff}_L \to \mathcal{V}$ that respects all truth tables, i.e. for every compound formula $\diamondsuit(A_1, \ldots, A_n)$:

  $$v(\diamondsuit(A_1, \ldots, A_n)) = \tilde{\diamondsuit}(v(A_1), \ldots, v(A_n))$$

- An $M$-valuation is a model of a formula $A$ if $v(A) \in \mathcal{D}$.

A consequence relation is defined by:

$$\Gamma \vdash^M A \text{ if every model of (every formula in) } \Gamma \text{ is a model of } A.$$
The Matrix $M_{cl}$

Values: $V = \{F, T\}$

Designated values: $D = \{T\}$

Tables:

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<th>$x_1$</th>
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<th>$\tilde{\lor}(x_1, x_2)$</th>
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Soundness and Completeness

$\Gamma \vdash_{LK} A$ iff $\Gamma \vdash_{M_{cl}} A$
The Matrix $\mathbf{M}_{cl}$

- **Values:** $\mathcal{V} = \{F, T\}$
- **Designated values:** $\mathcal{D} = \{T\}$

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**Soundness and Completeness**

$\Gamma \vdash_{LK} A$ iff $\Gamma \vdash_{M_{cl}} A$

**Example:**

$$\vdash_{LK} \neg (Ob \land Ro) \supset \neg Ob \lor \neg Ro$$

Indeed, every $\mathbf{M}_{cl}$-valuation is a model of this formula.
Example: Kleene Logic

The Matrix $M_{kl}$

- **Values:** $\mathcal{V} = \{F, T, I\}$
- **Designated values:** $\mathcal{D} = \{T\}$

\[
\begin{array}{c|ccc}
\sim & F & T & I \\
\hline
F & T & T & T \\
T & F & T & I \\
I & I & T & I \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\tilde{\land} & F & T & I \\
\hline
F & F & F & F \\
T & F & T & I \\
I & F & I & I \\
\end{array}
\]

If we take $D = \{T, I\}$, we get Priest's logic of paradox.

Idea: If there is enough information to determine the value, then we put the classical value in the table. Otherwise, we put $I$. 

$\not\vdash M_{kl}$

Ob

$\vdash M_{pr}$ Ob

17/75
Example: Kleene Logic

The Matrix $M_{kl}$

- Values: $V = \{F, T, I\}$
- Designated values: $D = \{T\}$

$$
\begin{array}{c|ccc}
\top & F & T & I \\
\hline
F & T & T & T \\
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$\vdash_{\mathbf{M}_{kl}} Ob \supset Ob \quad Ob \vdash_{\mathbf{M}_{kl}} Ob$
Example: Kleene Logic

The Matrix $M_{kl}$

- Values: $V = \{F, T, I\}$
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<td>I</td>
<td>I</td>
<td>F</td>
<td>I</td>
</tr>
</tbody>
</table>

Idea: If there is enough information to determine the value then we put the classical value in the table. Otherwise, we put $I$.

$\not\vdash_{M_{kl}} Ob \supset Ob$  $\vdash_{M_{kl}} Ob$

If we take $\mathcal{D} = \{T, I\}$, we get Priest’s logic of paradox.

$\not\vdash_{M_{pr}} Ob \supset Ob$  $\vdash_{M_{pr}} Ob$
Matrices for Sequents Derivability

- Recall:
  - $\Omega \vdash_{\text{LK}}^{\text{seq}} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in $\text{LK}$.
  - $\Gamma \vdash_{\text{LK}} A$ if $\{ \Rightarrow B \mid B \in \Gamma \} \vdash_{\text{LK}}^{\text{seq}} \Rightarrow A$.

- We would like to have semantics for $\vdash_{\text{LK}}^{\text{seq}}$ as well, and not only for $\vdash_{\text{LK}}$. 
Recall:

- $\Omega \vdash^{seq}_{LK} \Gamma \Rightarrow \Delta$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in $\text{LK}$.
- $\Gamma \vdash_{LK} A$ if \{ $\Rightarrow B \mid B \in \Gamma$ \} $\vdash^{seq}_{LK} \Rightarrow A$.

We would like to have semantics for $\vdash_{LK}^{seq}$ as well, and not only for $\vdash_{LK}$.

Instead of one set of designated values $\mathcal{D}$, we have two sets:

- $\mathcal{D}_{\text{left}}$ of left designated values.
- $\mathcal{D}_{\text{right}}$ of right designated values.

An $\mathbf{M}$-valuation $\nu$ is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $\nu(A) \in \mathcal{D}_{\text{left}}$ for some $A \in \Gamma$ or $\nu(A) \in \mathcal{D}_{\text{right}}$ for some $A \in \Delta$. 

Matrices for Sequents Derivability
Matrices for Sequents Derivability

For $\mathbf{M}_{cl}$, we define:

$D_{left} = \{ F \} \quad D_{right} = \{ T \}$

Soundness and Completeness

$\Omega \vdash_{LK}^{\text{seq}} \Gamma \Rightarrow \Delta$ iff every $\mathbf{M}_{cl}$-valuation which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$. 

Example:

$p_1 \Rightarrow \vdash_{LK} \qquad p_3 \Rightarrow p_1 \supset p_2$
Matrices for Sequents Derivability

For $\mathbf{M}_{cl}$, we define:

$$D_{\text{left}} = \{\text{F}\} \quad D_{\text{right}} = \{\text{T}\}$$

Soundness and Completeness

$$\Omega \vdash_{\text{seq}}^{\text{LK}} \Gamma \Rightarrow \Delta \text{ iff every } \mathbf{M}_{cl}\text{-valuation which is a model of } \Omega \text{ is also a model of } \Gamma \Rightarrow \Delta.$$  

Example:

$$p_1 \Rightarrow \vdash_{\text{seq}}^{\text{LK}} \quad p_3 \Rightarrow p_1 \supset p_2$$
Matrices for Sequents Derivability

For $M_{cl}$, we define:

$$D_{left} = \{ F \} \quad D_{right} = \{ T \}$$

Soundness and Completeness

$$\Omega \vdash_{LK} \Gamma \Rightarrow \Delta \text{ iff every } M_{cl}-valuation \text{ which is a model of } \Omega \text{ is also a model of } \Gamma \Rightarrow \Delta.$$ 

Example:

$$p_1 \Rightarrow \vdash_{LK} p_3 \Rightarrow p_1 \supset p_2$$

Recall:

$$\Gamma \vdash_{G} A \iff \{ \Rightarrow B \mid B \in \Gamma \} \vdash_{seq}^{G} \Rightarrow A$$

If $M$ characterizes $\vdash_{G}^{seq}$, take $D = D_{right}$ to obtain a matrix for $\vdash_{G}$. 
One Step in the Soundness Proof

Soundness

If $\Omega \vdash_{LK}^{seq} s$ then every $M_{cl}$-valuation which is a model of $\Omega$ is also a model of $s$.

Proof by induction on the length of the derivation.

- Consider an application of the rule ($\neg \Rightarrow$). It has the form:

$$
\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}
$$

Suppose that $\nu$ is a model of $\Gamma \Rightarrow A, \Delta$.

We prove that it is a model of $\Gamma, \neg A \Rightarrow \Delta$.

Recall: $\nu$ is a model of a sequent iff $\nu(B) \in D_{left}$ for some $B$ on the left side or $\nu(B) \in D_{right}$ for some $B$ on the right side.

If $\nu(B) = F$ for some $B \in \Gamma$, we are done.

If $\nu(B) = T$ for some $B \in \Delta$, we are done.

Otherwise, $\nu(A) = T$.

Since $\nu$ is an $M_{cl}$-valuation: $\nu(\neg A) = \tilde{\nu}(\nu(A)) = \tilde{\nu}(T) = F$. 

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A many-valued matrix for a language $\mathcal{L}$ consists of:

- A set $\mathcal{V}$ of truth values.
- Subsets $\mathcal{D}_{\text{left}}, \mathcal{D}_{\text{right}} \subseteq \mathcal{V}$ of designated truth values.
- A truth table for every connective of $\mathcal{L}$, i.e. for every connective $\Diamond$ we have a function $\tilde{\Diamond}$ from $\mathcal{V}^n$ to $\mathcal{V}$, where $n$ is the arity of $\Diamond$. 
Matrices for Sequents Derivability

A many-valued matrix for a language $\mathcal{L}$ consists of:

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- A truth table for every connective of $\mathcal{L}$, i.e. for every connective $\diamond$ we have a function $\tilde{\diamond}$ from $\mathcal{V}^n$ to $\mathcal{V}$, where $n$ is the arity of $\diamond$.

- An $\mathbf{M}$-valuation is a function $\nu : \text{wff} \mathcal{L} \to \mathcal{V}$ that respects all truth tables, i.e. for every compound formula $\diamond(A_1, \ldots, A_n)$:
  \[
  \nu(\diamond(A_1, \ldots, A_n)) = \tilde{\diamond}(\nu(A_1), \ldots, \nu(A_n))
  \]

- An $\mathbf{M}$-valuation $\nu$ is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $\nu(A) \in \mathcal{D}_{\text{left}}$ for some $A \in \Gamma$ or $\nu(A) \in \mathcal{D}_{\text{right}}$ for some $A \in \Delta$. 

A matrix $\mathbf{M}$ is sound and complete for a sequent system $\mathcal{G}$ iff:

$\Omega \vdash_{\text{seq}} \mathcal{G} \Gamma \Rightarrow \Delta$ iff every $\mathbf{M}$-valuation which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$. 
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$$\nu(\diamond(A_1, \ldots, A_n)) = \tilde{\diamond}(\nu(A_1), \ldots, \nu(A_n))$$

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A matrix $\mathbf{M}$ is sound and complete for a sequent system $\mathbf{G}$ iff:

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Classical Logic

The Matrix $\text{M}_{cl}$

- Values: $\mathcal{V} = \{F, T\}$
- Designated values: $\mathcal{D}_{left} = \{F\}$ and $\mathcal{D}_{right} = \{T\}$.
  - Thus, a valuation $\nu$ is a model of a sequent iff $\nu(A) = F$ for some $A$ on the left side, or $\nu(A) = T$ for some $A$ on the right side.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\sim(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sim(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Soundness and Completeness

$\Omega \vdash_{LK}^{seq} \Gamma \Rightarrow \Delta$ iff iff every $\text{M}_{cl}$-valuation which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$. 
What happens if we play with LK?
e.g. add new rules, omit some rules, change some rules.

Do we still have an effective semantics?

What about the subformula property? cut-admissibility? axiom-expansion? invertibility of the logical rules?
Motivations

Known examples:

- If we omit (¬ ⇒) from LK, we obtain a system for the paraconsistent logic CluN [Batens].

- Primal implication is defined by \( \Gamma \Rightarrow A_2, \Delta \) instead of \( (\Rightarrow \supset) \) [Gurevich et al.]

Motivations

Known examples:
- If we omit $\neg \Rightarrow$ from $\textbf{LK}$, we obtain a system for the paraconsistent logic CluN [Batens].

- Primal implication is defined by

  \[ \frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta} \]

  instead of $\Rightarrow \supset$

  [Gurevich et al.]

Guiding principle

The meaning of each connective is given by its introduction rules:

“... The introductions represent, as it were, the ‘definition’ of the symbols concerned...” [Gentzen, Investigations into logical deduction]

For example:

\[
\frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \& A_2 \Rightarrow \Delta}
\]

\[
\frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \& A_2, \Delta}
\]
The system **GCLuN**

\[
(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \quad (\Rightarrow \neg) \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}
\]

\[\text{GCLuN} = \text{LK} - (\neg \Rightarrow)\]
The system **GCLuN**

\[
\begin{align*}
(\neg \Rightarrow) & \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \\
(\Rightarrow \neg) & \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}
\end{align*}
\]

\[\text{GCLuN} = \text{LK} - (\neg \Rightarrow)\]

**Theorem**

**GCLuN** has no finite-valued characteristic matrix.

**Hint for the proof:**

\[p, \neg p, \ldots, \neg^{n-1} p \vdash_{\text{GCLuN}} \neg^n p\]
“Reading off” the Semantics from Sequent Rules

(¬ ⇒) \[ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \]

(⇒ ¬) \[ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \]

\[ \begin{array}{c|c|c|c}
 x & \neg (x) & \neg (x) \\
 \hline
 F & T & F \\
 T & T & F \\
 \end{array} \]
“Reading off” the Semantics from Sequent Rules

\[
\begin{align*}
(- \Rightarrow) & \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} & \quad \Rightarrow \neg & \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}
\end{align*}
\]

Without \((- \Rightarrow)\):

\[
\begin{array}{c|c|c}
    x & \sim(x) & \text{?} \\
    \hline
    F & T & \text{?} \\
    T & F & \text{?}
\end{array}
\]
Truth tables assign *sets* of truth values, and we require:

\[ \nu(\diamond(A_1, \ldots, A_n)) \in \sim(\nu(A_1), \ldots, \nu(A_n)) \]

instead of

\[ \nu(\diamond(A_1, \ldots, A_n)) = \sim(\nu(A_1), \ldots, \nu(A_n)) \]

Valuations still assign *one* value to each formula!
Non-Deterministic Matrices - Intuition

\[
\begin{array}{c|cc}
\wedge & F & T \\
\hline
F & F & F \\
T & F & T \\
\end{array}
\]

\[\nu(\text{in}_1 \wedge \text{in}_2) = \tilde{\wedge}(\nu(\text{in}_1), \nu(\text{in}_2))\]
Non-Deterministic Matrices - Intuition

\[
\begin{array}{c|cc}
\wedge & F & T \\
F & F & F \\
T & F & T \\
\end{array}
\]

\[v(in_1 \wedge in_2) = \wedge(v(in_1), v(in_2))\]
Non-Deterministic Matrices - Intuition

\[
\begin{array}{c|cc}
\land & F & T \\
F & F & F \\
T & F & T \\
\end{array}
\]

\[
v(in_1 \land in_2) = \tilde{\land}(v(in_1), v(in_2))
\]
Non-Deterministic Matrices - Intuition

\[
\begin{array}{|c|c|c|}
\hline
\wedge & F & T \\
\hline
F & F & F \\
T & F & T \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\wedge & F & T \\
\hline
F & \{F\} & \{F, T\} \\
T & \{F, T\} & \{T\} \\
\hline
\end{array}
\]

\[v(in_1 \wedge in_2) = \wedge(v(in_1), v(in_2))\]

\[v(in_1 \times in_2) \in \times(v(in_1), v(in_2))\]
The value of a complex formula is uniquely determined by the values of its subformulas.

In Nmatrices we do not have truth-functionality, but a weaker property.
A non-deterministic matrix $M$ (Nmatrix) for a language $L$ consists of:

- A set $V$ of truth values.
- Two subsets $D_{\text{left}}, D_{\text{right}} \subseteq V$ of designated truth values.
- A non-deterministic truth table for every connective of $L$, i.e. for every connective $\diamond$ we have a function $\tilde{\diamond}$ from $V^n$ to $P(V)$, where $n$ is the arity of $\diamond$.

- An $M$-valuation is a function $v : \text{wff} \rightarrow V$ that respects all truth tables, i.e. $v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n))$.

- The notion of a model is defined exactly as for (deterministic) matrices, i.e.:
  - An $M$-valuation $v$ is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in D_{\text{left}}$ for some $A \in \Gamma$ or $v(A) \in D_{\text{right}}$ for some $A \in \Delta$. 
The Matrix $M_{CluN}$

- $\mathcal{V} = \{F, T\}$, $\mathcal{D}_{left} = \{F\}$, $\mathcal{D}_{right} = \{T\}$.
- Same tables as in $M_{cl}$ (with singletons) except for:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\tilde{x}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>${T}$</td>
</tr>
<tr>
<td>$T$</td>
<td>${F, T}$</td>
</tr>
</tbody>
</table>

Soundness and Completeness

$\Omega \vdash_{GCluN} s$ iff every $M_{CluN}$-valuation which is a model of $\Omega$ is also a model of $s$.

For example:

$$\Rightarrow p_1, \Rightarrow \neg p_1, \Rightarrow \neg \neg p_1 \vdash_{GCluN}^{seq} \Rightarrow \neg \neg \neg \neg p_1$$

(thus $p_1, \neg p_1, \neg \neg p_1 \vdash_{GCluN} \neg \neg \neg p_1$)
A canonical system consists of:

- Structural rules as in **LK** (weakenings, (id), (cut)).
- Any finite set of *canonical* rules.

**Canonical rules** are logical rule of an “ideal” form:

- Each rule introduces *exactly one connective in one side*.
- Exactly one occurrence of the introduced connective, and *no other connectives are involved*.
- No restriction on context.
- The active formulas are *immediate subformulas* of the principal formula.
Examples of Canonical Rules

- **Canonical rules** are logical rules of an “ideal” form:
  - Each rule introduces exactly one connective in one side.
  - Exactly one occurrence of the introduced connective, and no other connectives are involved.
  - No restriction on context.
  - The active formulas are immediate subformulas of the principal formula.

All logical rules of LK are canonical. Other examples include:

\[
\frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}
\]

\[
\frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \& A_2 \Rightarrow \Delta}
\]

\[
\frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \& A_2, \Delta}
\]

\[
\frac{\Gamma \Rightarrow A_1, A_2, \Delta \quad \Gamma, A_3 \Rightarrow A_4, \Delta}{\Gamma \Rightarrow \Diamond(A_1, A_2, A_3, A_4, A_5), \Delta}
\]
Some Non-canonical Rules

\[
\Gamma, A_1 \Rightarrow A_2 \\
\Gamma \Rightarrow A_1 \supset A_2
\]

\[
\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta \\
\Gamma \Rightarrow \neg(A_1 \supset A_2), \Delta
\]

\[
\Gamma \Rightarrow A, \Delta \\
\Gamma \Rightarrow \neg\neg A, \Delta
\]
Semantics of Canonical Systems

Given a canonical system $G$, we construct $M_G$ as follows:

$\mathcal{V} = \{ F, T \}$, $\mathcal{D}_{left} = \{ F \}$, $\mathcal{D}_{right} = \{ T \}$.

For every $n$-ary connective $\diamond$:

- Initialize a totally non-deterministic table.

- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\diamond}(x_1, \ldots, x_n)$. 
Initialize a totally non-deterministic table.

For every rule $r$ for $\Diamond$:

For every $x_1, \ldots, x_n \in V$:

- If $x_1, \ldots, x_n$ satisfy the premises of $r$:
  - If $r$ is a right rule, omit $F$ from $\vec{\Diamond}(x_1, \ldots, x_n)$.
  - If $r$ is a left rule, omit $T$ from $\vec{\Diamond}(x_1, \ldots, x_n)$.

\[
\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta \quad \Gamma, A_1 \supset A_2 \Rightarrow \Delta
\]

\[
\begin{align*}
\Gamma, A_1 \Rightarrow A_2, \Delta \\
\Gamma \Rightarrow A_1 \supset A_2, \Delta
\end{align*}
\]
Example: Table of Implication

- Initialize a totally non-deterministic table.
- For every rule \( r \) for \( \diamond \):
  - For every \( x_1, \ldots, x_n \in \mathcal{V} \):
    - If \( x_1, \ldots, x_n \) satisfy the premises of \( r \):
      - If \( r \) is a right rule, omit \( F \) from \( \tilde{\diamond}(x_1, \ldots, x_n) \).
      - If \( r \) is a left rule, omit \( T \) from \( \tilde{\diamond}(x_1, \ldots, x_n) \).

\[
\begin{array}{c|c|c}
\Gamma \Rightarrow A_1, \Delta & \Gamma, A_2 \Rightarrow \Delta & \tilde{\bowtie}(x_1, x_2) \\
\hline
F & F & \{F, T\} \\
F & T & \{F, T\} \\
T & F & \{F, T\} \\
T & T & \{F, T\}
\end{array}
\]
Initialize a totally non-deterministic table.

For every rule $r$ for $\diamond$:
For every $x_1, \ldots, x_n \in V$:

- If $x_1, \ldots, x_n$ satisfy the premises of $r$:
  - If $r$ is a right rule, omit $F$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
  - If $r$ is a left rule, omit $T$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\tilde{\diamond}(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>${F, T}$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>${F, T}$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>${F, T}$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>${F, T}$</td>
</tr>
</tbody>
</table>
Example: Table of Implication

- Initialize a totally non-deterministic table.
- For every rule \( r \) for \( \diamond \):
  - For every \( x_1, \ldots, x_n \in \mathcal{V} \):
    - If \( x_1, \ldots, x_n \) satisfy the premises of \( r \):
      - If \( r \) is a right rule, omit \( \mathbb{F} \) from \( \widetilde{\diamond}(x_1, \ldots, x_n) \).
      - If \( r \) is a left rule, omit \( \mathbb{T} \) from \( \widetilde{\diamond}(x_1, \ldots, x_n) \).

\[
\begin{array}{c|c|c|c}
\hline
x_1 & x_2 & \widetilde{\circ}(x_1, x_2) \\
\hline
F & F & \{F, T\} \\
F & T & \{F, T\} \\
T & F & \{F, T\} \\
T & T & \{F, T\} \\
\hline
\end{array}
\]

\[
\begin{align*}
\Gamma & \Rightarrow A_1, \Delta & \Gamma, A_2 \Rightarrow \Delta \\
\hline
\Gamma, A_1 \supset A_2 & \Rightarrow \Delta \\
\hline
\Gamma, A_1 \Rightarrow A_2, \Delta \\
\hline
\Gamma & \Rightarrow A_1 \supset A_2, \Delta
\end{align*}
\]
Example: Table of Implication

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\Diamond$:
  For every $x_1, \ldots, x_n \in V$:
  - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
    - If $r$ is a right rule, omit $F$ from $\bar{\Diamond}(x_1, \ldots, x_n)$.
    - If $r$ is a left rule, omit $T$ from $\bar{\Diamond}(x_1, \ldots, x_n)$.

\[
\begin{array}{c|c|c}
  \bar{\Diamond} & (x_1, x_2) & \bar{\Diamond}(x_1, x_2) \\
  \hline
  F & F & \{F, T\} \\
  F & T & \{F, T\} \\
  T & F & \{F, T\} \\
  T & T & \{F, T\}
\end{array}
\]
Example: Table of Implication

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\Diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ from $\tilde{\Diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\Diamond}(x_1, \ldots, x_n)$.

\[
\begin{array}{c|c|c|c}
\hline
x_1 & x_2 & \tilde{\Diamond}(x_1, x_2) \\
\hline
F & F & \{F, T\} \\
F & T & \{F, T\} \\
T & F & \{F, T\} \\
T & T & \{F, T\} \\
\hline
\end{array}
\]
Example: Table of Implication

- Initialize a totally non-deterministic table.
- For every rule $r$ for ♦:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.

\[
\begin{array}{c|c|c}
  x_1 & x_2 & \tilde{\diamond}(x_1, x_2) \\
  \hline
  F & F & \{F, T\} \\
  F & T & \{F, T\} \\
  T & F & \{F, T\} \\
  T & T & \{F, T\}
\end{array}
\]
Example: The System **GPrim**

**GPrim** is obtained from **LK** by replacing the rules for $\supset$ with the following:

\[
\begin{align*}
(\supset \Rightarrow) & \quad \frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta} & (\Rightarrow \supset) & \frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}
\end{align*}
\]
**Example: The System **\( \text{GPrim} \)**

\( \text{GPrim} \) is obtained from \( \text{LK} \) by replacing the rules for \( \supset \) with the following:

\[
\begin{array}{c}
(\supset \Rightarrow) \\
\frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma, A_1 \supset A_2 \Rightarrow \Delta}
\end{array}
\quad
\begin{array}{c}
(\Rightarrow \supset) \\
\frac{\Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta}
\end{array}
\]

Here, we obtain the following (non-deterministic) table for \( \supset \):

\[
\begin{array}{c|c|c}
\hline
x_1 & x_2 & \supset (x_1, x_2) \\
\hline
F & F & \{F, T\} \\
F & T & \{T\} \\
T & F & \{F\} \\
T & T & \{T\} \\
\hline
\end{array}
\]
Semantics of Canonical Systems

Soundness and Completeness

\[ \Omega \vdash^\text{seq}_G s \iff \text{every } M_G\text{-valuation which is a model of } \Omega \text{ is also a model of } s. \]
Semantics of Canonical Systems

Soundness and Completeness

\[ \Omega \vdash_{G}^{seq} s \iff \text{every } M_{G}-\text{valuation which is a model of } \Omega \text{ is also a model of } s. \]

Theorem

*If* \( M_{G} \) *is non-deterministic then there is no finite-valued (deterministic) matrix for* \( G \).
What can go wrong?

Consider the **Tonk connective** [Prior] defined by:

\[
\begin{align*}
&\Gamma, A_2 \Rightarrow \Delta \\
&\Gamma, A_1 \oplus A_2 \Rightarrow \Delta \\
&\Gamma \Rightarrow A_1, \Delta \\
&\Gamma \Rightarrow A_1 \oplus A_2, \Delta
\end{align*}
\]

Soundness and completeness still hold. There are no \( M_{LK}^+ \)-valuations! \( \vdash LK^+ \) is trivial.
What can go wrong?

Consider the **Tonk connective** [Prior] defined by:

\[
\begin{align*}
\Gamma, A_2 & \Rightarrow \Delta \\
\Gamma, A_1 \Downarrow A_2 & \Rightarrow \Delta
\end{align*}
\]

We obtain the table:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\Downarrow (x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>{F}</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>{F, T}</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>\emptyset</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>{T}</td>
</tr>
</tbody>
</table>

Soundness and completeness still hold. There are no $\mathcal{LK}_t$-valuations! $\Gamma \vdash \mathcal{LK}_t$ is trivial.
What can go wrong?

Consider the **Tonk connective** [Prior] defined by:

\[
\begin{align*}
\Gamma, A_2 & \Rightarrow \Delta \\
\Gamma & \Rightarrow A_1, \Delta \\
\Gamma & \Rightarrow A_1 \oplus A_2, \Delta
\end{align*}
\]

We obtain the table:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\oplus (x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>{F}</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>{F, T}</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>{T}</td>
</tr>
</tbody>
</table>

- **Soundness and completeness still hold.**
- **There are no $M_{LK+\oplus}$ -valuations!**
- $\vdash_{LK+\oplus}$ is trivial.
Empty Sets in Truth Tables

Proposition
For every canonical system $G$: if we have an empty set in a table of $M_G$ then

$$\Rightarrow p_1, p_2 \Rightarrow \vdash_G^{\text{seq}} \Rightarrow$$
The Subformula property

Let $G$ be an arbitrary canonical system.

**Notation**

Let $E$ be a set of formulas.

- An $E$-sequent is a sequent consisting solely of formulas from $E$.
- $\Omega \vdash_{G}^{\mathcal{E}seq} \Gamma \Rightarrow \Delta$ iff there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in a system $G$ consisting solely of $E$-sequents.

**The Subformula Property**

$$\Omega \vdash_{G}^{\mathcal{E}seq} \Gamma \Rightarrow \Delta \implies \Omega \vdash_{G}^{\mathcal{sub}[\Omega \cup \{\Gamma \Rightarrow \Delta\}]_{seq}} \Gamma \Rightarrow \Delta$$
Let $G$ be an arbitrary canonical system.

### Notation

Let $E$ be a set of formulas.

- An $E$-sequent is a sequent consisting solely of formulas from $E$.
- $\Omega \vdash^E_{G} \Gamma \Rightarrow \Delta$ iff there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in a system $G$ consisting solely of $E$-sequents.

### The Subformula Property

$$\Omega \vdash^E_{G} \Gamma \Rightarrow \Delta \implies \Omega \vdash^{\text{sub}[\Omega \cup \{\Gamma \Rightarrow \Delta\}]}_{G} \Gamma \Rightarrow \Delta$$

Syntactic proofs are possible (as a consequence of cut-elimination).
We will take a “semantic approach”.
Let $G$ be an arbitrary canonical system.

**Notation**

Let $E$ be a set of formulas.

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**The Subformula Property**

\[ \Omega \vdash_G^{\text{seq}} \Gamma \Rightarrow \Delta \implies \Omega \vdash_G^{\text{sub}[\Omega \cup \{ \Gamma \Rightarrow \Delta \}]\text{-seq}} \Gamma \Rightarrow \Delta \]

Syntactic proofs are possible (as a consequence of cut-elimination). We will take a “semantic approach”.

Can we find semantics for $\vdash_G^{E\text{-seq}}$?
Semantics for $\vdash_{Eseq}^G$

(Stronger) Soundness and Completeness

For every closed set $E$ of formulas, set $\Omega$ of $E$-sequents, and $E$-sequent $\Gamma \Rightarrow \Delta$:

$\Omega \vdash_{Eseq}^G \Gamma \Rightarrow \Delta$ iff every partial $M_G$-valuation, defined on $E$, which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$. 
Semantics for $\vdash_{G}^{\mathcal{E}\text{seq}}$

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, set $\Omega$ of $\mathcal{E}$-sequents, and $\mathcal{E}$-sequent $\Gamma \Rightarrow \Delta$:

$\Omega \vdash_{G}^{\mathcal{E}\text{seq}} \Gamma \Rightarrow \Delta$ iff every partial $M_{G}$-valuation, defined on $\mathcal{E}$, which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$.

For example, verify that:

$\Rightarrow \text{CarStarts} \supset \text{Trip} \, , \, \Rightarrow \neg \text{Trip} \vdash_{\text{LK}}^{\mathcal{E}\text{seq}} \Rightarrow \neg \text{CarStarts}$

for $\mathcal{E} = \{ \text{CarStarts}, \text{Trip}, \neg \text{Trip}, \neg \text{CarStarts}, \text{CarStarts} \supset \text{Trip} \}$

$\Rightarrow p_{1} \, , \, p_{2} \Rightarrow \vdash_{\text{LK}+ \; \Box}^{\mathcal{E}\text{seq}} \Rightarrow$

for $\mathcal{E} = \{ p_{1}, p_{2} \}$
(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, set $\Omega$ of $\mathcal{E}$-sequents, and $\mathcal{E}$-sequent $\Gamma \Rightarrow \Delta$:

$\Omega \vdash_{\mathcal{E}}^{\text{seq}} \Gamma \Rightarrow \Delta$ iff every partial $M_G$-valuation, defined on $\mathcal{E}$, which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$. 
Semantic Proof of the Subformula Property

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, set $\Omega$ of $\mathcal{E}$-sequents, and $\mathcal{E}$-sequent $\Gamma \Rightarrow \Delta$:

$\Omega \vdash_{\mathcal{E}\text{seq}}^G \Gamma \Rightarrow \Delta$ iff every partial $M^G$-valuation, defined on $\mathcal{E}$, which is a model of $\Omega$ is also a model of $\Gamma \Rightarrow \Delta$.

Now, proving the subformula property for $G$ reduces to proving that every partial $M^G$-valuation (defined on a closed set of formulas) can be extended to a (full) $M^G$-valuation.
Semantic Proof of the Subformula Property

(Stronger) Soundness and Completeness

For every closed set $\mathcal{E}$ of formulas, set $\Omega$ of $\mathcal{E}$-sequents, and $\mathcal{E}$-sequent $\Gamma \Rightarrow \Delta$:

$$\Omega \models_{\mathcal{E} \text{seq}}^G \Gamma \Rightarrow \Delta \text{ iff every partial } M_G \text{-valuation, defined on } \mathcal{E}, \text{ which is a model of } \Omega \text{ is also a model of } \Gamma \Rightarrow \Delta.$$  

Now, proving the subformula property for $G$ reduces to proving that every partial $M_G$-valuation (defined on a closed set of formulas) can be extended to a (full) $M_G$-valuation.

For $LK$ this is trivial!
Thus $LK$ has the subformula property.
Consider the following procedure:

**Extension Procedure**

By recursion on the build-up of formulas:

- When \( \nu(p) \) is undefined choose it arbitrarily.
- When \( \nu(\Diamond(A_1, \ldots, A_n)) \) is undefined choose it arbitrarily from \( \tilde{\Diamond}(\nu(A_1), \ldots, \nu(A_n)) \).
The Subformula Property in Canonical Systems

Consider the following procedure:

**Extension Procedure**

By recursion on the build-up of formulas:

- When $\nu(p)$ is undefined choose it arbitrarily.
- When $\nu(\diamondsuit(A_1, \ldots, A_n))$ is undefined choose it arbitrarily from $\tilde{\diamondsuit}(\nu(A_1), \ldots, \nu(A_n))$.

When does it works?
Consider the following procedure:

**Extension Procedure**

By recursion on the build-up of formulas:

- When $\nu(p)$ is undefined choose it arbitrarily.
- When $\nu(\Big(A_1, \ldots, A_n\Big))$ is undefined choose it arbitrarily from $\tilde{\nu}(\nu(A_1), \ldots, \nu(A_n))$.

When does it work?

If we do not have any empty sets in the tables.
A canonical system $G$ is called "coherent" if there are no empty sets in the tables of $M_G$. 

Theorem

A canonical system has the subformula property iff it is coherent. In particular, $GCluN$ and $GPrim$ have the subformula property.
Coherence

Definition
A canonical system $G$ is called *coherent* if there are no empty sets in the tables of $M_G$.

Theorem
*A canonical system has the subformula property iff it is coherent.*
Coherence

**Definition**

A canonical system $G$ is called *coherent* if there are no empty sets in the tables of $M_G$.

**Theorem**

*A canonical system has the subformula property iff it is coherent.*

In particular, $G\text{CluN}$ and $G\text{Prim}$ have the subformula property.
We obtain an empty set iff there exists a right rule and a left rule for the same connective, whose premises are satisfied by the same \( n \) values.

In other words, we need that the right rules and the left rules for each connective to be \textit{contradictory}.

This does not hold for the rules of Tonk:

\[
\frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \uplus A_2 \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma \Rightarrow A_1 \uplus A_2, \Delta}
\]
We demonstrated the semantic approach to establish the subformula property.

In canonical systems, the subformula property is equivalent to *semantic analyticity* — the fact that every partial valuation can be extended.

Similar approach works for many other Gentzen-type systems.

The subformula property was proved regardless of cut-elimination.
So far

- We defined the family of canonical systems.
- We introduced the semantic framework of Nmatrices.
- We provided a method to obtain a two-valued Nmatrix for every canonical system.
- We introduced the coherence criterion – a necessary and sufficient criterion for the subformula property in canonical systems.
So far

- We defined the family of **canonical systems**.
- We introduced the semantic framework of **Nmatrices**.
- We provided a **method** to obtain a **two-valued Nmatrix** for every canonical system.
- We introduced the **coherence criterion** – a necessary and sufficient criterion for the subformula property in canonical systems.

What about cut-admissibility in canonical systems?
Cut-Admissibility

\[(\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \]

LK enjoys cut-admissibility (Gentzen, 1934). What about other canonical systems? We will take a "semantic approach". Can we find semantics for LK−(cut)?
Cut-Admissibility

\[(\text{cut}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Gamma \Rightarrow \Delta} \quad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\]

- **LK** enjoys cut-admissibility (Gentzen, 1934).
- What about other canonical systems?
Cut-Admissibility

\[(cut) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, \Gamma \Rightarrow \Delta} \Rightarrow \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\]

- **LK** enjoys cut-admissibility (Gentzen, 1934).
- What about other canonical systems?

We will take a “semantic approach”.

Can we find semantics for **LK** − (cut)?
Semantics for $\textbf{LK} - (cut)$

- Not the same semantics as for $\textbf{LK}$!
- Cut-admissibility does not hold in the presence of assumptions, e.g.

$$\Rightarrow p_1, p_1 \Rightarrow \vdash_{\text{LK}} \Rightarrow$$

$$\Rightarrow p_1, p_1 \Rightarrow \vdash_{\text{LK} - (cut)} \Rightarrow$$
Not the same semantics as for $\textbf{LK}$!

Cut-admissibility does not hold in the presence of assumptions, e.g.

\[ \Rightarrow p_1 \land p_1 \Rightarrow \vdash_{\text{seq}}^{\text{LK}} \Rightarrow \]

\[ \Rightarrow p_1 \land p_1 \Rightarrow \vdash_{\text{seq}}^{\text{LK}-(cut)} \Rightarrow \]

**Theorem**

$\vdash_{\text{LK}-(cut)}$ does not have a finite characteristic matrix.
Semantics for $\text{LK} - (\text{cut})$

\[ \Rightarrow p_1, p_1 \Rightarrow \vdash_{\text{LK}}^{\text{seq}}(\text{cut}) \Rightarrow \]
Semantics for $\mathbf{LK} - (cut)$

\[ \Rightarrow p_1, p_1 \Rightarrow \vdash_{\mathbf{LK}}^{\text{seq}} (cut) \Rightarrow \]

- Without cut, there should be a valuation which is both a model of $p_1 \Rightarrow$ and of $\Rightarrow p_1$. 

Recall: An $M$-valuation $v$ is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in D_{\text{left}}$ for some $A \in \Gamma$ or $v(A) \in D_{\text{right}}$ for some $A \in \Delta$.

$v(p_1)$ should be both in $D_{\text{left}}$ and in $D_{\text{right}}$.

We will add a third value $\top$: $V = \{f, t, \top\}$.

$\top$ makes a sequent true on both sides: $D_{\text{left}} = \{f, \top\}$ $D_{\text{right}} = \{t, \top\}$.

The construction of the tables is done using the same method used for canonical systems.
Semantics for $\textbf{LK} - (\textit{cut})$

\[ \implies p_1, \ p_1 \implies \vdash_{\text{seq}}^{\text{LK-(cut)}} \implies \]

- Without cut, there should be a valuation which is both a model of $p_1 \implies$ and of $\implies p_1$.
- Recall: An $\mathcal{M}$-valuation $\nu$ is a \textit{model} of a sequent $\Gamma \implies \Delta$ iff $\nu(A) \in D_{left}$ for some $A \in \Gamma$ or $\nu(A) \in D_{right}$ for some $A \in \Delta$. 
Semantics for $\textbf{LK} - (cut)$

$$\Rightarrow p_1 \ , \ p_1 \Rightarrow \vdash_{\text{seq}}^\text{LK- (cut)} \Rightarrow$$

- Without cut, there should be a valuation which is both a model of $p_1 \Rightarrow$ and of $\Rightarrow p_1$.
- Recall: An $\mathbf{M}$-valuation $v$ is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v(A) \in D_{\text{left}}$ for some $A \in \Gamma$ or $v(A) \in D_{\text{right}}$ for some $A \in \Delta$.
- $v(p_1)$ should be both in $D_{\text{left}}$ and in $D_{\text{right}}$. 

We will add a third value $\top$: $V = \{ f, t, \top \}$. $\top$ makes a sequent true on both sides: $D_{\text{left}} = \{ f, \top \}$ and $D_{\text{right}} = \{ t, \top \}$. The construction of the tables is done using the same method used for canonical systems.
Semantics for $\textbf{LK} - (\text{cut})$

\[ \Rightarrow p_1 \text{ , } p_1 \Rightarrow \vdash^{\text{seq}}_{\text{LK} - (\text{cut})} \Rightarrow \]

- Without cut, there should be a valuation which is both a model of $p_1 \Rightarrow$ and of $\Rightarrow p_1$.
- Recall: An $\mathcal{M}$-valuation $\nu$ is a \textit{model} of a sequent $\Gamma \Rightarrow \Delta$ iff $\nu(A) \in \mathcal{D}_{\text{left}}$ for some $A \in \Gamma$ or $\nu(A) \in \mathcal{D}_{\text{right}}$ for some $A \in \Delta$.
- $\nu(p_1)$ should be both in $\mathcal{D}_{\text{left}}$ and in $\mathcal{D}_{\text{right}}$.

- We will add a third value $\top$: $\mathcal{V} = \{\text{F, T, } \top\}$.
- $\top$ makes a sequent true on both sides:

$$\mathcal{D}_{\text{left}} = \{\text{F, } \top\} \quad \mathcal{D}_{\text{right}} = \{\text{T, } \top\}$$

- \textit{The construction of the tables is done using the same method used for canonical systems.}
The NMatrix $\mathbf{M}_{\mathbf{LK}}-(\text{cut})$

\[ \mathcal{V} = \{F, T, \top\} \quad \mathcal{D}_{\text{left}} = \{F, \top\} \quad \mathcal{D}_{\text{right}} = \{T, \top\} \]

<table>
<thead>
<tr>
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<th>$x_2$</th>
<th>$\tilde{\cup}(x_1, x_2)$</th>
</tr>
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<tbody>
<tr>
<td>F</td>
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<td>F</td>
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<table>
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<th>$\tilde{\cup}(x_1, x_2)$</th>
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</thead>
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</tr>
<tr>
<td>T</td>
<td>{\top}</td>
</tr>
</tbody>
</table>

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Semantics for $\text{LK} - (\text{cut})$

**Soundness and Completeness - sequents**

$\Omega \vdash_{\text{LK}-(\text{cut})}^{\text{seq}} s$ iff every $M_{\text{LK}-(\text{cut})}$-valuation which is a model of $\Omega$ is also a model of $s$.

**Soundness and Completeness - formulas**

$\Gamma \vdash_{\text{LK}-(\text{cut})} A$ iff $\Gamma \vdash_{M_{\text{LK}-(\text{cut})}} A$ (i.e. every $M_{\text{LK}-(\text{cut})}$-valuation which is a model of $\Gamma$ is also a model of $A$). *(where $D = D_{\text{right}}$)*

For example, verify that:

$$\text{CarStarts} \supset \text{Trip}, \neg \text{Trip} \not\vdash_{\text{LK}-(\text{cut})} \neg \text{CarStarts}$$
Semantics for $\textbf{LK} - (cut)$

Soundness and Completeness - sequents

$\Omega \vdash_{\text{seq}}^{\text{LK}-(cut)} s$ iff every $M_{\text{LK}-(cut)}$-valuation which is a model of $\Omega$ is also a model of $s$.

Soundness and Completeness - formulas

$\Gamma \vdash_{\text{LK}-(cut)} A$ iff $\Gamma \vdash_{M_{\text{LK}-(cut)}} A$ (i.e. every $M_{\text{LK}-(cut)}$-valuation which is a model of $\Gamma$ is also a model of $A$). (where $D = D_{\text{right}}$)

For example, verify that:

$\text{CarStarts} \supset \text{Trip}, \neg \text{Trip} \not\vdash_{\text{LK}-(cut)} \neg \text{CarStarts}$

$\rightarrow$ New formulation of results of Schütte (1960) and Girard (1987).
Example: Construction of a table for $\land$

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\Diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $\mathcal{F}$ from $\tilde{\Diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $\mathcal{T}$ from $\tilde{\Diamond}(x_1, \ldots, x_n)$.

$\left(\land \Rightarrow\right)$

\[
\frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta}
\]

$(\Rightarrow \land)$

\[
\frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}
\]

<table>
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<th>$\mathcal{T}$</th>
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<tr>
<td>$\mathcal{T}$</td>
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<tr>
<td>$\mathcal{T}$</td>
<td>{F, T, \top}</td>
<td>{F, T, \top}</td>
<td>{F, T, \top}</td>
</tr>
</tbody>
</table>
Example: Construction of a table for $\land$

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in V$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.

\[
\begin{align*}
(\land \Rightarrow) & \quad & (\Rightarrow \land) \\
\frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} & & \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}
\end{align*}
\]

<table>
<thead>
<tr>
<th>$\land$</th>
<th>F</th>
<th>T</th>
<th>$\top$</th>
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</thead>
<tbody>
<tr>
<td>F</td>
<td>${F, T, \top}$</td>
<td>${F, T, \top}$</td>
<td>${F, T, \top}$</td>
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<tr>
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</tr>
<tr>
<td>$\top$</td>
<td>${F, T, \top}$</td>
<td>${F, T, \top}$</td>
<td>${F, T, \top}$</td>
</tr>
</tbody>
</table>
Example: Construction of a table for \( \land \)

- Initialize a totally non-deterministic table.
- For every rule \( r \) for \( \Diamond \):
  - For every \( x_1, \ldots, x_n \in \mathcal{V} \):
    - If \( x_1, \ldots, x_n \) satisfy the premises of \( r \):
      - If \( r \) is a right rule, omit \( F \) from \( \tilde{\circ}(x_1, \ldots, x_n) \).
      - If \( r \) is a left rule, omit \( T \) from \( \tilde{\circ}(x_1, \ldots, x_n) \).

\[
(\land \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta}
\]

\[
(\Rightarrow \land) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}
\]

\[
\begin{array}{|c||c|c|c|}
\hline
\tilde{\land} & F & T & T \\
\hline
F & \{F, T, T\} & \{F, T, T\} & \{F, T, T\} \\
T & \{F, T, T\} & \{F, T, T\} & \{F, T, T\} \\
T & \{F, T, T\} & \{F, T, T\} & \{F, T, T\} \\
\hline
\end{array}
\]
Example: Construction of a table for $\land$

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.

$$(\land \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta}$$

$$(\Rightarrow \land) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}$$

<table>
<thead>
<tr>
<th>$\tilde{\land}$</th>
<th>$F$</th>
<th>$T$</th>
<th>$T$</th>
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</thead>
<tbody>
<tr>
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<td>{F, T, T}</td>
<td>{F, T, T}</td>
<td>{F, T, T}</td>
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<tr>
<td>$T$</td>
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- Initialize a totally non-deterministic table.
- For every rule $r$ for $\Diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
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      - If $r$ is a right rule, omit $F$ from $\tilde{\Box}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ from $\tilde{\Box}(x_1, \ldots, x_n)$.

$$(\land \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} \quad (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}$$

<table>
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<tr>
<td>$F$</td>
<td>${F, T, T}$</td>
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</tr>
<tr>
<td>$T$</td>
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Example: Construction of a table for $\land$

- Initialize a totally non-deterministic table.

- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $\bot$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $\top$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.


<table>
<thead>
<tr>
<th>$\land$</th>
<th>$\bot$</th>
<th>$\top$</th>
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<td>$\bot$</td>
<td>${\bot, \top}$</td>
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<td>$\top$</td>
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<td>${\top}$</td>
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</tbody>
</table>

- All usual connectives have **non-deterministic** semantics.

- Non-determinism is a result of the missing cut rule.
The same construction works for every canonical system without \((cut)\).

\(\top\) is included in every entry in every table.

Thus, all canonical systems without \((cut)\) have the subformula property.

(This is obvious from a syntactic point of view.)

The \(\{F, T\}\)-entries of the tables for the system without cut are equal to those of the system with cut, except for the addition of \(\top\).

\[
\begin{array}{c|c|c|c}
\wedge & F & T & \top \\
\hline
F & \{F, \top\} & \{F, \top\} & \{F, \top\} \\
T & \{F, \top\} & \{T, \top\} & \{\top\} \\
\top & \{F, \top\} & \{\top\} & \{\top\}
\end{array}
\]

Why? since we do exactly the same deletions, but we begin with \(\{F, T, \top\}\).
The same construction works for every canonical system without \((cut)\).

- \(\top\) is included in every entry in every table.
  - Thus, all canonical systems without \((cut)\) have the subformula property.
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\hline
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T & \{F, T\} & \{T, T\} & \{T\} \\
\top & \{F, T\} & \{T\} & \{T\} \\
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- Why? since we do exactly the same deletions, but we begin with \(\{F, T, \top\}\).

**Observation**

Every \(M_{G-(cut)}\)-valuation over \(\{F, T\}\) is an \(M_G\)-valuation.
Cut-Admissibility for \( \text{LK} \)

\[
\vdash_{\text{LK}} \Gamma \Rightarrow \Delta \quad \iff 
\vdash_{\text{LK}-(\text{cut})} \Gamma \Rightarrow \Delta
\]

Semantic Equivalent

If every \( \mathbf{M}_{\text{LK}} \)-valuation is a model of a sequent \( \Gamma \Rightarrow \Delta \) then every \( \mathbf{M}_{\text{LK}-(\text{cut})} \)-valuation is a model of \( \Gamma \Rightarrow \Delta \).
Cut-Admissibility for \( \mathbf{LK} \)

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Semantic Equivalent

If every \( M_{\mathbf{LK}} \)-valuation is a model of a sequent \( \Gamma \Rightarrow \Delta \) then every \( M_{\mathbf{LK}-(cut)} \)-valuation is a model of \( \Gamma \Rightarrow \Delta \).

- To prove cut-admissibility for \( \mathbf{LK} \), we have to prove:
  For every \( M_{\mathbf{LK}-(cut)} \)-valuation which is not a model of some sequent \( \Gamma \Rightarrow \Delta \), there exists an \( M_{\mathbf{LK}} \)-valuation which is not a model of \( \Gamma \Rightarrow \Delta \).

- Using the previous observation, it suffices to show:
  For every \( M_{\mathbf{LK}-(cut)} \)-valuation which is not a model of some sequent \( \Gamma \Rightarrow \Delta \), there exists an \( M_{\mathbf{LK}-(cut)} \)-valuation over \( \{F, T\} \) which is not a model of \( \Gamma \Rightarrow \Delta \).

- It suffices to show:
  For every \( M_{\mathbf{LK}-(cut)} \)-valuation \( \nu \) there exists an \( M_{\mathbf{LK}-(cut)} \)-valuation \( \nu' \) over \( \{F, T\} \) such that \( \nu'(A) = \nu(A) \) whenever \( \nu(A) \in \{F, T\} \).
Cut-Admissibility for $\text{LK}$

**GOAL:** For every $\text{M}_{\text{LK}}-(\text{cut})$-valuation $v$ there exists an $\text{M}_{\text{LK}}-(\text{cut})$-valuation $v'$ over $\{F, T\}$ such that $v'(A) = v(A)$ whenever $v(A) \in \{F, T\}$.

### Refinement Procedure

By recursion on the build-up of formulas:

- **If** $v(A) \in \{F, T\}$: $v'(A) := v(A)$.
- **Otherwise:**
  - If $A$ is atomic, choose $v'(A)$ to be either $F$ or $T$ arbitrarily.
  - If $A = \Diamond(A_1, \ldots, A_n)$, choose $v'(A)$ to be either $F$ or $T$ arbitrarily from $\tilde{\Diamond}(v(A_1), \ldots, v(A_n))$. 

**Why does it work?**

In the tables of $\text{M}_{\text{LK}}-(\text{cut})$, $\{F, T\}$-entries always include $F$ or $T$ in addition to $\top$. 

---

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GOAL: For every $M_{LK-\text{(cut)}}$-valuation $v$ there exists an $M_{LK-\text{(cut)}}$-valuation $v'$ over $\{F, T\}$ such that $v'(A) = v(A)$ whenever $v(A) \in \{F, T\}$.

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Why does it work?
Cut-Admissibility for LK

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Theorem

**LK** enjoys *cut-admissibility*.

What about canonical systems in general?
Cut-Admissibility for Canonical Systems

**Theorem**

LK enjoys cut-admissibility.

What about canonical systems in general?

- To have cut-admissibility, we should not have $\tilde{\diamond}(x_1, \ldots, x_n) = \{\top\}$ for $x_1, \ldots, x_n \in \{\text{F, T}\}$.

- Recall: The $\{\text{F, T}\}$-entries of the tables for the system without cut are equal to those of the system with cut, except for the addition of $\top$.

- Thus $\tilde{\diamond}(x_1, \ldots, x_n) = \{\top\}$ for $x_1, \ldots, x_n \in \{\text{F, T}\}$ only if $\tilde{\diamond}(x_1, \ldots, x_n) = \emptyset$ in the tables for the same system with cut.
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**Theorem**

Every coherent canonical system enjoys cut-admissibility.

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Note that if a system is not coherent then it does not enjoy cut-admissibility (since $\Rightarrow p_1, p_2 \Rightarrow \vdash_{G}^{\text{seq}} \Rightarrow$).
Note that if a system is not coherent then it does not enjoy cut-admissibility (since $\Rightarrow p_1 \land p_2 \Rightarrow \vdash_{\mathcal{G}} seq \Rightarrow$).

**Corollary**

For every canonical system $\mathcal{G}$, the following are equivalent:

- $\mathcal{G}$ is coherent.
- $\mathcal{G}$ has the subformula property.
- $\mathcal{G}$ enjoys cut-admissibility.
We demonstrated the "semantic approach" to prove cut-admissibility. We had three steps:

- Find semantics 1 for the system with cut.
- Find semantics 2 for the system without cut.
- Show that every non-model of some sequent $\Gamma \Rightarrow \Delta$ in 2 can be turned into a non-model of $\Gamma \Rightarrow \Delta$ in 1.

In comparison to the syntactic approach:

- Safer and less tedious.
- Better understanding of the meaning of cut.
- Easier to generalize.
- The method can be adapted to higher-order logics.

On the other hand:

- We only have cut-admissibility and not cut-elimination.
- If it does not work then it does not easily lead to counter example.
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Axiom-Expansion

- \((id)\) is the rule allowing to derive all sequents of the form \(A \Rightarrow A\) (with no premises).
- Atomic applications of \((id)\) derive sequents of the form \(p \Rightarrow p\), where \(p\) is an atomic formula.
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Atomic applications of (id) derive sequents of the form $p \Rightarrow p$, where $p$ is an atomic formula.

If $\Omega \vdash_{G}^{\text{seq}} \Gamma \Rightarrow \Delta$ then there exists a derivation of $\Gamma \Rightarrow \Delta$ from $\Omega$ in $G$ in which all applications of (id) are atomic.
Axiom-Expansion

Equivalent Formulation

For every $n$-ary connective:
\[
\{ p_i \Rightarrow p_i \mid i \geq 1 \} \vdash_{\mathcal{G}-(id)} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)
\]
**Axiom-Expansion**

**Equivalent Formulation**

For every $n$-ary connective:

$$\{p_i \Rightarrow p_i \mid i \geq 1\} \vdash_{G-(id)} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$$

**LK** admits axiom-expansion. For example:

\[
\begin{align*}
p_1 \Rightarrow p_1 & \quad p_2 \Rightarrow p_2 \\
p_1, p_1 \supset p_2 & \Rightarrow p_2 \\
p_1 \supset p_2 & \Rightarrow p_1 \supset p_2
\end{align*}
\]

Again, we would like to obtain a semantic equivalent of this property. What is the semantics of canonical systems without $(id)$? In particular, of $LK - (id)$?
Semantics for $\mathbf{LK} - (id)$

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vdash_{\mathbf{LK} - (id)}$ does not have a finite characteristic matrix.</td>
</tr>
</tbody>
</table>
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Theorem

$\vdash_{\textbf{LK}-(id)}$ does not have a finite characteristic matrix.

$\forall \vdash_{\textbf{LK}-(id)} p \Rightarrow p$
Semantics for $\text{LK} - (id)$

**Theorem**

$\vdash \text{LK} - (id)$ does not have a finite characteristic matrix.

$\forall \text{LK} - (id) \ p \Rightarrow p$

- Without id, there should be a valuation which is **not a model of** $p \Rightarrow p$.
- Thus, $\nu(p)$ should be **neither in** $D_{left}$ nor in $D_{right}$.
Semantics for $\text{LK} - (id)$

**Theorem**

$\vdash_{\text{LK} - (id)} \text{does not have a finite characteristic matrix.}$

\[
\vdash_{\text{LK} - (id)} p \Rightarrow p
\]

- Without id, there should be a valuation which is not a model of $p \Rightarrow p$.
- Thus, $v(p)$ should be neither in $D_{left}$ nor in $D_{right}$.

- We will add a **third** value $\bot$: $\mathcal{V} = \{ F, T, \bot \}$.
- $\bot$ never makes a sequent true:

\[
D_{left} = \{ F \} \quad D_{right} = \{ T \}
\]

- The construction of the tables is almost the same.
Example: Construction of a table for $\land$

- Initialize a totally non-deterministic table.
- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in V$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ and $\bot$ from $\overline{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ and $\bot$ from $\overline{\diamond}(x_1, \ldots, x_n)$.

$$(\land \Rightarrow) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} \quad (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}$$

<table>
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<tr>
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  For every $x_1, \ldots, x_n \in V$:
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    - If $r$ is a right rule, omit $F$ and $\bot$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.
    - If $r$ is a left rule, omit $T$ and $\bot$ from $\tilde{\diamond}(x_1, \ldots, x_n)$.

\[
\begin{array}{c|c|c|c}
\land & F & T & \bot \\
\hline
F & \{F, T, \bot\} & \{F, T, \bot\} & \{F, T, \bot\} \\
T & \{F, T, \bot\} & \{F, T, \bot\} & \{F, T, \bot\} \\
\bot & \{F, T, \bot\} & \{F, T, \bot\} & \{F, T, \bot\} \\
\end{array}
\]

\[\]

$$\frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}$$
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- Initialize a totally non-deterministic table.
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$$\left(\land \Rightarrow\right) \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} \quad \left(\Rightarrow \land\right) \quad \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}$$
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      - If $r$ is a right rule, omit $\bot$ and $\bot$ from $\checkmark(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $\top$ and $\bot$ from $\checkmark(x_1, \ldots, x_n)$.

\[
\begin{align*}
\wedge \Rightarrow & \quad \frac{\Gamma, A_1, A_2 \Rightarrow \Delta}{\Gamma, A_1 \land A_2 \Rightarrow \Delta} & (\Rightarrow \land) & \frac{\Gamma \Rightarrow A_1, \Delta \quad \Gamma \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \land A_2, \Delta}
\end{align*}
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- For every rule $r$ for $\diamond$:
  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
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      - If $r$ is a right rule, omit $F$ and $\top$ from $\widehat{\diamond}(x_1, \ldots, x_n)$.
      - If $r$ is a left rule, omit $T$ and $\top$ from $\widehat{\diamond}(x_1, \ldots, x_n)$.

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  - For every $x_1, \ldots, x_n \in \mathcal{V}$:
    - If $x_1, \ldots, x_n$ satisfy the premises of $r$:
      - If $r$ is a right rule, omit $F$ and $\bot$ from $\neg(r(x_1, \ldots, x_n))$.
      - If $r$ is a left rule, omit $T$ and $\bot$ from $\neg(r(x_1, \ldots, x_n))$.

<table>
<thead>
<tr>
<th>$\sim\neg$</th>
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<tr>
<td>$\bot$</td>
<td>${F}$</td>
<td>${F, T, \bot}$</td>
<td>${F, T, \bot}$</td>
</tr>
</tbody>
</table>

- All usual connectives have non-deterministic semantics.
- Non-determinism is a result of the missing identity axiom.
The NMatrix $M_{LK} - (id)$

$\mathcal{V} = \{F, T, \bot\}$ \hspace{1cm} $\mathcal{D}_{left} = \{F\}$ \hspace{1cm} $\mathcal{D} = \mathcal{D}_{right} = \{T\}$

\[ x_1 \quad x_2 \quad \tilde{\circlearrowleft}(x_1, x_2) \]

\begin{center}
\begin{tabular}{c|c|c}
F & F & \{T\} \\
F & T & \{T\} \\
T & F & \{F\} \\
T & T & \{T\} \\
F & \bot & \{T\} \\
F & \bot & \{T\} \\
T & \bot & \{F, T, \bot\} \\
\bot & F & \{F, T, \bot\} \\
\bot & T & \{T\} \\
\bot & \bot & \{F, T, \bot\} \\
\end{tabular}
\end{center}

\[ x \quad \tilde{\circlearrowleft}(x) \]

\begin{center}
\begin{tabular}{c|c}
F & \{T\} \\
f & \{F\} \\
T & \{F\} \\
\bot & \{F, T, \bot\} \\
\end{tabular}
\end{center}
Semantics for $\textbf{LK} - (id)$

**Soundness and Completeness - sequents**

$\Omega \vdash_{\text{seq}}^{\text{LK}-(id)} s$ iff every $M_{\text{LK}-(id)}$-valuation which is a model of $\Omega$ is also a model of $s$.

**Soundness and Completeness - formulas**

$\Gamma \vdash_{\text{LK}-(id)} A$ iff $\Gamma \vdash_{M_{\text{LK}-(id)}} A$ (i.e. every $M_{\text{LK}-(id)}$-valuation which is a model of $\Gamma$ is also a model of $A$). (where $D = D_{\text{right}}$)

For example, verify that:

CarStarts $\supset$ Trip, $\neg$Trip $\not\vdash_{\text{LK}-(id)} \neg$CarStarts

$\hookrightarrow$ New formulation of results of Hösli and Jäger (1994).
The same construction works for every canonical system $G$.

**Axiom-Expansion**

For every $n$-ary connective:

$$\{p_i \Rightarrow p_i \mid i \geq 1\} \vdash_{G-(id)} \Box(p_1, \ldots, p_n) \Rightarrow \Box(p_1, \ldots, p_n)$$

- In other words: Whenever $\nu(p_i) \in \{F, T\}$ for every $i \geq 1$, we also have $\nu(\Box(p_1, \ldots, p_n)) \in \{F, T\}$ for every connective $\Box$.
- Thus, we have axiom expansion iff for every connective $\Box$:
  $$\bot \not\in \Box(x_1, \ldots, x_n) \text{ for every } x_1, \ldots, x_n \in \{F, T\}.$$
**Axiom-Expansion for LK**

**LK** admits axiom-expansion.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\supset (x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>{T}</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>{T}</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>{F}</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>{T}</td>
</tr>
<tr>
<td>F</td>
<td>$\bot$</td>
<td>{T}</td>
</tr>
<tr>
<td>T</td>
<td>$\bot$</td>
<td>{F, T, $\bot$}</td>
</tr>
<tr>
<td>$\bot$</td>
<td>F</td>
<td>{F, T, $\bot$}</td>
</tr>
<tr>
<td>$\bot$</td>
<td>T</td>
<td>{T}</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$\bot$</td>
<td>{F, T, $\bot$}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\neg(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>{T}</td>
</tr>
<tr>
<td>T</td>
<td>{F}</td>
</tr>
<tr>
<td>$\bot$</td>
<td>{F, T, $\bot$}</td>
</tr>
</tbody>
</table>
Axiom-Expansion for Canonical Systems

- We have axiom expansion iff for every connective $\diamond$: $\bot \notin \tilde{\diamond}(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in \{F, T\}$.
- This means that we did at least one deletion in every $\{F, T\}$-entry.
- Equivalently, the tables for the system with $(id)$ are deterministic.
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- Equivalently, the tables for the system with $(id)$ are deterministic.

Theorem

A canonical system $G$ admits axiom-expansion iff $M_G$ is deterministic.

In particular, $GCluN$ and $GPrim$ do not admit axiom-expansion.
A canonical rule is called *invertible* in a system $G$ if each of its premises can be derived from its conclusion in $G$.

(Formally, this should hold for every instantiation of $\Gamma, \Delta$ and $A_1, A_2, \ldots$)
A canonical right rule for $\diamond$ is invertible in $\mathbf{G}$: if for every $\mathbf{M}_G$-valuation $\nu$, if $\nu(\diamond(A_1, \ldots, A_n)) = T$ then the premises of the rule are satisfied by $\nu$.

Equivalently, when $T \in \tilde{\diamond}(x_1, \ldots, x_n)$ then $x_1, \ldots, x_n$ satisfy the premises of the rule.
A canonical right rule for $\diamond$ is invertible in $G$: if for every $M_G$-valuation $v$, if $v(\diamond(A_1, \ldots, A_n)) = T$ then the premises of the rule are satisfied by $v$.

Equivalently, when $T \in \tilde{\diamond}(x_1, \ldots, x_n)$ then $x_1, \ldots, x_n$ satisfy the premises of the rule.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\tilde{\diamond}(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>${T}$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>${T}$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>${F}$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>${T}$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Gamma \Rightarrow A_1, \Delta & \quad \Gamma, A_2 \Rightarrow \Delta \\
\hline
\Gamma, A_1 \supset A_2 \Rightarrow \Delta \\
\Gamma, A_1 \Rightarrow A_2, \Delta & \quad \Gamma \Rightarrow A_1 \supset A_2, \Delta
\end{align*}
\]
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Equivalently, when $T \in \tilde{\diamond}(x_1, \ldots, x_n)$ then $x_1, \ldots, x_n$ satisfy the premises of the rule.

$$
\begin{array}{c}
\Gamma \Rightarrow A_1, \Delta \quad \Gamma, A_2 \Rightarrow \Delta \\
\hline
\Gamma, A_1 \supset A_2 \Rightarrow \Delta \\
\end{array}
\begin{array}{c|c|c}
\hline
x_1 & x_2 & \tilde{\diamond}(x_1, x_2) \\
\hline
F & F & \{T\} \\
F & T & \{T\} \\
T & F & \{F\} \\
T & T & \{T\} \\
\hline
\end{array}
$$

In case we have only one right rule $r$ for $\diamond$:

- In the construction of $\tilde{\diamond}$, when $r$’s premises are satisfied, we delete $F$.
- $r$ is invertible in $G$ iff there are no $\{F, T\}$’s in $\tilde{\diamond}$.
Corollary

For every canonical system $G$, the following are equivalent:

- $M_G$ is deterministic.
- $G$ admits axiom-expansion.
- If every connective has exactly one left rule and one right rule, then all logical rules are invertible.
Final Remarks

- **Non-deterministic** semantics is a useful tool for understanding and investigating proof-theoretic properties of formal calculi.

- The semantic tools complement the usual proof-theoretic ones.

- Interesting cases arise when the “semantic approach” is applied for
  - Single-conclusion sequent systems
  - Sequent systems for modal logics
  - Many-sided sequent systems
  - Hypersequent systems
  - Sub-structural systems ??
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Thank you for your attention!
You are welcome to ask, suggest and discuss.
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orilahav@gmail.com