Basic Constructive Connectives, Determinism and Matrix-based Semantics

Agata Ciabattoni[1], Ori Lahav[2,⋆], and Anna Zamansky[1]

[1] Vienna University of Technology
[2] Tel Aviv University
agata@logic.at orilahav@post.tau.ac.il annaz@logic.at

Abstract. (Non-)deterministic Kripke-style semantics is used to characterize two syntactic properties of single-conclusion canonical sequent calculi: invertibility of rules and axiom-expansion. An alternative matrix-based formulation of such semantics is introduced, which provides an algorithm for checking these properties, and also new insights into basic constructive connectives.

1 Introduction

Single-conclusion canonical systems were introduced in [3] to provide a general characterization of basic constructive connectives.[3] These systems are single-conclusion sequent calculi, which in addition to the axioms and structural rules of Gentzen's LJ calculus have only logical rules, in which exactly one occurrence of a connective is introduced and no other connective is mentioned. It was shown in [3] that every single-conclusion canonical system induces a class of Kripke frames, for which it is strongly sound and complete. The key idea behind this semantics is to relax the principle of truth-functionality, and to use *non-deterministic* semantics, in which the truth-values of the subformulas of some compound formula ψ do not always uniquely determine the truth-value of ψ. The non-deterministic Kripke-style semantics was also applied in [3] to characterize the single-conclusion canonical systems that enjoy cut-admissibility.

As shown in [3], basic constructive connectives include the standard intuitionistic connectives together with many others. Some of these connectives induce a deterministic Kripke-style semantics, while others only have a non-deterministic one. The first goal of this paper is to investigate the relationship between determinism of basic constructive connectives and two syntactic properties of their rules: invertibility and completeness of atomic axioms (axiom-expansion). Invertibility of rules is important for guiding proof search in sequent calculi and simplifies automated proofs of cut-admissibility. Axiom expansion is sometimes considered crucial when designing "well-behaved" sequent systems. Here we prove that the determinism of the underlying Kripke-style semantics is a necessary and sufficient condition for a basic constructive connective to admit axiom expansion.

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[3] See e.g. [9, 5] for alternative proposals.

The same connection also holds with the invertibility of the right introduction rule for the connective, provided that the calculus contains exactly one such rule.

A similar investigation was carried out in [2] for *multiple-conclusion* canonical calculi. This was based on their (much simpler) semantics, defined in terms of non-deterministic two-valued matrices ([1]) – a natural generalization of the standard two-valued truth-tables.

Despite the important properties discussed above, the formulation of the non-deterministic Kripke-style semantics in [3] does not provide an algorithmic approach for checking determinism of basic constructive connectives. Accordingly, the second goal of this paper is to overcome this problem by providing an alternative formulation of this semantics based on a generalization of non-deterministic two-valued matrices. These generalized matrices, used to characterize the set of frames induced by a single-conclusion canonical system, have two main advantages. First, *decidability*: there is a simple algorithm for checking whether the induced semantics is deterministic, which in turn can be used for deciding invertibility and axiom-expansion in single-conclusion canonical systems. Second, *modularity*: the semantic effect of each syntactic rule can be directly read off the corresponding matrix, therefore providing new insights into the semantic meaning of basic constructive connectives.

2 Preliminaries

In what follows $\mathcal{L}$ is a propositional language, and $Frm_{\mathcal{L}}$ is its set of wffs. We assume that the atomic formulas of $\mathcal{L}$ are $p_1, p_2, \ldots$. We use Γ, Σ, Π, E to denote finite subsets of $Frm_{\mathcal{L}}$, where E is used for sets which are either singletons or empty. A *(single-conclusion) sequent* is an expression of the form $\Gamma \Rightarrow E$. We denote sequents of the form $\Gamma \Rightarrow \{\varphi\}$ (resp. $\Gamma \Rightarrow \emptyset$) by $\Gamma \Rightarrow \varphi$ (resp. $\Gamma \Rightarrow$).

Below we shortly reproduce definitions and results from [3].

Definition 1. *A* substitution *is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, such that for every n-ary connective $\diamond$ of $\mathcal{L}$ we have: $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$. A substitution is extended to sets of formulas in the obvious way.*

Henceforth we denote by σ_{id} the substitution σ, such that $\sigma(p) = p$ for every atomic formula p. Moreover, given formulas $\psi_1, \ldots, \psi_n$, $\sigma_{\psi_1,\ldots,\psi_n}$ denotes the substitution σ such that $\sigma(p_i) = \psi_i$ for $1 \leq i \leq n$ and $\sigma(p_i) = p_i$ for $i > n$.

2.1 Single-Conclusion Canonical Systems

Definition 2. *A single-conclusion sequent calculus is called* a (single-conclusion) canonical system *iff its axioms are sequents of the form $\psi \Rightarrow \psi$ (identity axioms), cut and weakening are among its rules, and each of its other rules is either a single-conclusion canonical right rule or a single-conclusion canonical left rule, where:*

1. *A* (single-conclusion canonical) left rule *for a connective $\diamond$ of arity n is an expression of the form: $\langle \{\Pi_i \Rightarrow E_i\}_{1\leq i\leq m}, \{\Sigma_i \Rightarrow\}_{1\leq i\leq k}\rangle /\ \diamond(p_1, \ldots, p_n) \Rightarrow$,*

where $m, k \geq 0$, $\Pi_i \cup E_i \subseteq \{p_1, \ldots, p_n\}$ *for* $1 \leq i \leq m$, *and* $\Sigma_i \subseteq \{p_1, \ldots, p_n\}$ *for* $1 \leq i \leq k$. *The sequents* $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ *are called the* hard premises *of the rule, while* $\Sigma_i \Rightarrow$ $(1 \leq i \leq k)$ *are its* soft premises, *and* $\diamond(p_1, \ldots, p_n) \Rightarrow$ *is its* conclusion.
An application *of such rule is any inference step of the form:*

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1 \leq i \leq m} \quad \{\Gamma, \sigma(\Sigma_i) \Rightarrow E\}_{1 \leq i \leq k}}{\Gamma, \sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow E}$$

where $\Gamma \Rightarrow E$ *is an arbitrary sequent, and* σ *is a substitution.*

2. *A* (single-conclusion canonical) right rule *for a connective* $\diamond$ *of arity* n *is an expression of the form:* $\{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} / \ \Rightarrow \diamond(p_1, \ldots, p_n)$, *where* m, *and* $\Pi_i \Rightarrow E_i$ *are as above.* $\Pi_i \Rightarrow E_i$ $(1 \leq i \leq m)$ *are called the* premises *of the rule, and* $\Rightarrow \diamond(p_1, \ldots, p_n)$ *is its* conclusion.
An application *of such rule is any inference step of the form:*

$$\frac{\{\Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)\}_{1 \leq i \leq m}}{\Gamma \Rightarrow \sigma(\diamond(p_1, \ldots, p_n))}$$

where Γ *is a finite set of formulas and* σ *is a substitution.*

Given a canonical system **G**, *and a set of sequents* $\mathcal{S} \cup \{s\}$, *we write* $\mathcal{S} \vdash_{\mathbf{G}} s$ *iff there exists a derivation in* **G** *of* s *from* $\mathcal{S}$.

The following condition of coherence,[4] characterizing (a stronger form of) cut-admissibility in canonical systems, is an extension of the analogous condition for multiple-conclusion canonical systems (see [1]).

Definition 3. *A set* $\mathcal{R}$ *of canonical rules for an* n*-ary connective* $\diamond$ *is* coherent *if* $S_1 \cup S_2 \cup S_3$ *is classically inconsistent whenever* $\mathcal{R}$ *contains both* $\langle S_1, S_2\rangle / \diamond (p_1, \ldots, p_n) \Rightarrow$ *and* $S_3 / \Rightarrow \diamond(p_1, \ldots, p_n)$. *A canonical system* **G** *is called* coherent *if for each connective* $\diamond$, *the set of rules for* $\diamond$ *in* **G** *is coherent.*

2.2 Non-deterministic Kripke-Style Semantics

Definition 4. *Let* $\mathcal{F}$ *be a set of formulas closed under subformulas. An* $\mathcal{F}$-semiframe *is a triple* $\mathcal{W} = \langle W, \leq, v\rangle$ *such that:*

1. $\langle W, \leq\rangle$ *is a nonempty partially ordered set, whose elements are called* worlds.
2. v *is a function from* $W \times \mathcal{F}$ *to* $\{t, f\}$ *obeying the persistence condition, i.e.* $v(a, \varphi) = t$ *implies* $v(b, \varphi) = t$ *for every* $b \geq a$.

When $\mathcal{F} = Frm_{\mathcal{L}}$ *then the* $\mathcal{F}$*-semiframe is called a* frame.

Definition 5. *Let* $\mathcal{W} = \langle W, \leq, v\rangle$ *be an* $\mathcal{F}$*-semiframe, and let* $a \in W$.

[4] Coherence is also equivalent to the reductivity condition of [6], which applies in presence of arbitrary structural rules.

1. *A sequent* $\Gamma \Rightarrow E$ *is* locally true *in* a *iff* $\Gamma \cup E \subseteq \mathcal{F}$ *and either* $v(a, \psi) = f$ *for some* $\psi \in \Gamma$*, or* $E = \{\varphi\}$ *and* $v(a, \varphi) = t$*.*
2. *A sequent* s *is* true *in* a *iff* s *is locally true in every* $b \geq a$*.*
3. $\mathcal{W}$ *is a* model *of a sequent* s *if* s *is locally true in every* $b \in W$*.* $\mathcal{W}$ *is a model of a set of sequents* $\mathcal{S}$ *if it is a model of every* $s \in \mathcal{S}$*.*

Definition 6. *Let* $\mathcal{W} = \langle W, \leq, v \rangle$ *be an* $\mathcal{F}$*-semiframe.*

1. *Let* σ *be a substitution, and let* $a \in W$*.*
   (a) σ (locally) satisfies *a sequent* $\Gamma \Rightarrow E$ *in* a *if* $\sigma(\Gamma) \Rightarrow \sigma(E)$ *is (locally) true in* a*.*
   (b) σ fulfils *a left rule* r *in* a *if it satisfies in* a *every hard premise of* r*, and locally satisfies in* a *every soft premise of* r*.*
   (c) σ fulfils *a right rule* r *in* a *if it satisfies in* a *every premise of* r*.*
2. *Let* r *be a canonical rule for an* n*-ary connective* $\diamond$*.* $\mathcal{W}$ respects r *if for every* $a \in W$ *and every substitution* σ*: if* σ *fulfils* r *in* a *and* $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{F}$ *then* σ *locally satisfies* r*'s conclusion in* a*.*
3. *Given a coherent canonical system* **G**, $\mathcal{W}$ *is called* **G**-legal *if it respects all the rules of* **G**.

Henceforth, when speaking of (local) trueness of sequents and fulfilment of rules by substitutions, we add "with respect to $\mathcal{W}$", whenever the (semi)frame $\mathcal{W}$ is not clear from context.

Note that a certain substitution σ may not fulfil any right rule for an n-ary connective $\diamond$ in a world a of a **G**-legal frame, and at the same time σ may not fulfil any left rule for $\diamond$ in all worlds $b \geq a$. In this case there are no restrictions on the truth-value assigned to $\sigma(\diamond(p_1, \ldots, p_n))$ in a, and the semantics of **G** is non-deterministic.

Definition 7. *Let* **G** *be a coherent canonical system, and* $\mathcal{S} \cup \{s\}$ *be a set of sequents.* $\mathcal{S} \vDash_{\mathbf{G}} s$ *iff every* **G***-legal frame which is a model of* $\mathcal{S}$ *is a model of* s*.*

Theorem 1 (7.1 in [3]). *Let* **G** *be a coherent canonical system, and* $\mathcal{F}$ *be a set of formulas closed under subformulas. If* $\mathcal{W} = \langle W, \leq, v \rangle$ *is a* **G***-legal* $\mathcal{F}$*-semiframe, then* v *can be extended to a function* v' *so that* $\mathcal{W}' = \langle W, \leq, v' \rangle$ *is a* **G***-legal frame.*

The above theorem ensures that the semantics of **G**-legal semiframes is analytic, in the sense that every **G**-legal semiframe can be extended to a **G**-legal (full) frame. This means that in order to determine whether $\mathcal{S} \vDash_{\mathbf{G}} s$, it suffices to consider semiframes, defined only on the set of all subformulas of $\mathcal{S} \cup \{s\}$.

The following theorems establish an exact correspondence between coherent canonical systems, Kripke semantics and (strong) cut-admissibility. In what follows, **G** is a coherent canonical system.

Theorem 2 (6.1 in [3]). *If* $\mathcal{S} \vdash_{\mathbf{G}} s$ *then* $\mathcal{S} \vDash_{\mathbf{G}} s$*.*

Theorem 3 (6.3 in [3]). *If $\mathcal{S} \vDash_{\mathbf{G}} s$ then $\mathcal{S} \vdash_{\mathbf{G}} s$, and moreover, there exists a proof in* $\mathbf{G}$ *of s from $\mathcal{S}$ in which all cut-formulas appear in $\mathcal{S}$.*

Theorem 3 will be strengthened in the proof of Theorem 4 below. In order to make this paper self-contained, we include here an outline of the original proof.

Proof (Outline). Assume that s does not have a proof in $\mathbf{G}$ from $\mathcal{S}$ in which all cut-formulas appear in $\mathcal{S}$ (call such a proof a *legal* proof). We construct a $\mathbf{G}$-legal frame $\mathcal{W}$ which is a model of $\mathcal{S}$ but not of s. Let $\mathcal{F}$ be the set of subformulas of $\mathcal{S} \cup \{s\}$. Given a set $E \subseteq \mathcal{F}$, which is either a singleton or empty, call a theory $\mathcal{T} \subseteq \mathcal{F}$ E*-maximal* if there is no finite subset $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow E$ has a legal proof, but every proper extension $\mathcal{T}' \subseteq \mathcal{F}$ of $\mathcal{T}$ contains such a finite subset Γ. Now let $\mathcal{W} = \langle W, \subseteq, v \rangle$, where W is the set of all E-maximal theories for some $E \subseteq \mathcal{F}$, and v is defined inductively as follows:
For atomic formulas, $v(\mathcal{T}, p) = t$ iff $p \in \mathcal{T}$. Suppose $v(\mathcal{T}, \psi_i)$ has been defined for every $\mathcal{T} \in W$ and $1 \leq i \leq n$. We let $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$ iff at least one of the following holds with respect to the semiframe at this stage:

1. There exists a right rule for $\diamond$ which is fulfilled in $\mathcal{T}$ by $\sigma_{\psi_1,\ldots,\psi_n}$.
2. $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ and there do not exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and a left rule for $\diamond$ which is fulfilled in $\mathcal{T}'$ by $\sigma_{\psi_1,\ldots,\psi_n}$.

One can now prove that $\mathcal{W}$ is a $\mathbf{G}$-legal frame. The fact that $\mathcal{W}$ is a model of $\mathcal{S}$ but not of s follows from the following properties:
For every $\mathcal{T} \in W$ and every formula $\psi \in \mathcal{F}$:

(a) If $\psi \in \mathcal{T}$ then $v(\mathcal{T}, \psi) = t$.
(b) If $\mathcal{T}$ is $\{\psi\}$-maximal then $v(\mathcal{T}, \psi) = f$.

(a) and **(b)** are proven together by a simultaneous induction on the complexity of ψ. For atomic formulas they easily follow from v's definition, and the fact that $p \Rightarrow p$ is an axiom. For the induction step, assume that **(a)** and **(b)** hold for $\psi_1, \ldots, \psi_n \in \mathcal{F}$. We prove **(b)** for $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$ (**(a)** is proved analogously). Assume that $\mathcal{T}$ is $\{\diamond(\psi_1, \ldots, \psi_n)\}$-maximal, but $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$. Thus $\diamond(\psi_1, \ldots, \psi_n) \notin \mathcal{T}$ (because $\diamond(\psi_1, \ldots, \psi_n) \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ is an axiom). Hence there exists a right rule, $\{\Pi_i \Rightarrow E_i\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1, \ldots, p_n)$, which is fulfilled in $\mathcal{T}$ by the substitution $\sigma = \sigma_{\psi_1,\ldots,\psi_n}$. It follows that there exists $1 \leq i_0 \leq m$ such that $\Gamma, \sigma(\Pi_{i_0}) \Rightarrow \sigma(E_{i_0})$ has no legal proof for any finite $\Gamma \subseteq \mathcal{T}$. Extend $\mathcal{T} \cup \sigma(\Pi_{i_0})$ to a $\sigma(E_{i_0})$-maximal theory $\mathcal{T}'$. The induction hypothesis implies that σ does not locally satisfy $\Pi_{i_0} \Rightarrow E_{i_0}$ in $\mathcal{T}'$. Thus contradicts our assumption. □

We now present some examples of canonical rules and their induced semantics (see [3] for more examples).

Example 1 (Implication). All the rules of Gentzen's LJ calculus for intuitionistic logic are canonical. For instance, the rules for implication are:

$$\langle \{\Rightarrow p_1\}, \{p_2 \Rightarrow\} \rangle \; / \; p_1 \supset p_2 \Rightarrow \quad \text{and} \quad \{p_1 \Rightarrow p_2\} \; / \; \Rightarrow p_1 \supset p_2$$

A frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\supset\Rightarrow)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever $v(b, \varphi) = t$ for every $b \geq a$ and $v(a, \psi) = f$. $\mathcal{W}$ respects $(\Rightarrow\supset)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$. Using the persistence condition, it is easy to see that the two rules impose the well-known Kripke semantics for intuitionistic implication ([8]).

Example 2 (Affirmation). The unary connective $\rhd$ is defined by the rules:

$$\langle \emptyset, \{p_1 \Rightarrow\}\rangle \ / \ \rhd p_1 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_1\} \ / \ \Rightarrow \rhd p_1$$

A frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\rhd\Rightarrow)$ if $v(a, \rhd \psi) = f$ whenever $v(a, \psi) = f$. It respects $(\Rightarrow\rhd)$ if $v(a, \rhd \psi) = t$ whenever $v(b, \psi) = t$ for every $b \geq a$. By the persistence condition, this means that for every $a \in W$, $v(a, \rhd \psi)$ simply equals $v(a, \psi)$.

Example 3 (Weak Affirmation). The unary connective $\blacktriangleright$ is defined by the rules:

$$\langle \{p_1 \Rightarrow\}, \emptyset\rangle \ / \ \blacktriangleright p_1 \Rightarrow \quad \text{and} \quad \{\Rightarrow p_1\} \ / \ \Rightarrow \blacktriangleright p_1$$

A frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\blacktriangleright\Rightarrow)$ if $v(a, \blacktriangleright \psi) = f$ whenever $v(b, \psi) = f$ for every $b \geq a$. It respects $(\Rightarrow\blacktriangleright)$ if $v(a, \blacktriangleright \psi) = t$ whenever $v(b, \psi) = t$ for every $b \geq a$. This implies that $v(a, \blacktriangleright \psi)$ is free to be t or f when $v(a, \psi) = f$ and $v(b, \psi) = t$ for some $b > a$ (hence this semantics is "non-deterministic"). In particular it follows that this connective cannot be expressed by the usual intuitionistic connectives.

Applications of the rules for $\supset$ are the standard ones, while those for $\rhd$ and $\blacktriangleright$ have the following forms:

$$\frac{\Gamma, \varphi \Rightarrow E}{\Gamma, \rhd \varphi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \rhd \varphi} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma, \blacktriangleright \varphi \Rightarrow E} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \blacktriangleright \varphi}$$

3 Deterministic Connectives

In general, an n-ary connective $\diamond$ is called deterministic (see e.g. [2]), if the truth-functionality principle holds for it. In other words, the truth-values assigned to $\psi_1, \ldots, \psi_n$ uniquely determine the truth-value assigned to $\diamond(\psi_1, \ldots, \psi_n)$. Adapting this property for Kripke-style semantics, one can require that the truth-values assigned to $\psi_1, \ldots, \psi_n$ *in every world of the frame* would uniquely determine whether or not the frame is a model of $\Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ (i.e. $\diamond(\psi_1, \ldots, \psi_n)$ is true in all worlds). This can be formalized as follows.

Definition 8. *Two $\mathcal{F}$-semiframes $\mathcal{W}_1$ and $\mathcal{W}_2$* agree *on some $\psi \in \mathcal{F}$, if either both are models of $\Rightarrow \psi$, or both are not models of $\Rightarrow \psi$.*

Henceforth we denote by $SF[\psi]$ and $PSF[\psi]$ the sets of subformulas of a formula ψ and proper subformulas of ψ, respectively.

Definition 9. *Let* **G** *be a coherent canonical system.*

1. *Given a formula ψ, a* **G***-legal* $PSF[\psi]$*-semiframe* $\mathcal{W}$ *is called* ψ-determined *in* **G** *if all* **G***-legal* $SF[\psi]$*-semiframes extending* $\mathcal{W}$ *agree on* ψ.
2. $\diamond$ *admits* unique analycity *in* **G** *if for every* $\psi_1, \ldots, \psi_n \in Frm_{\mathcal{L}}$*, every* **G***-legal* $PSF[\diamond(\psi_1, \ldots, \psi_n)]$*-semiframe is* $\diamond(\psi_1, \ldots, \psi_n)$*-determined in* **G**.

Example 4. Consider the coherent canonical system **G** consisting of the two rules for $\blacktriangleright$ from Example 3. Let $W = \{a, b\}$ and $\leq = \{\langle a, a\rangle, \langle b, b\rangle, \langle a, b\rangle\}$. Consider a $\{p_1\}$-semiframe $\mathcal{W} = \langle W, \leq, v\rangle$, where $v(a, p_1) = f$ and $v(b, p_1) = t$. Let $\mathcal{W}_1 = \langle W, \leq, v_1\rangle$ and $\mathcal{W}_2 = \langle W, \leq, v_2\rangle$ be two $\{p_1, \blacktriangleright p_1\}$-semiframes which extend $\mathcal{W}$, where $v_1(a, \blacktriangleright p_1) = f$ and $v_1(b, \blacktriangleright p_1) = v_2(a, \blacktriangleright p_1) = v_2(b, \blacktriangleright p_1) = t$. Following Example 3, both semiframes are **G**-legal. Clearly, $\mathcal{W}_1$ and $\mathcal{W}_2$ do not agree on $\blacktriangleright p_1$. It follows that $\blacktriangleright$ does not admit *unique analycity* in **G**.

We introduce below an alternative definition of determinism of connectives and show its equivalence with unique analycity.

Definition 10. *A* $\{p_1, \ldots, p_n\}$*-semiframe* $\mathcal{W} = \langle W, \leq, v\rangle$*, such that* $\langle W, \leq\rangle$ *has a minimum is called an* n-atomic frame. *We denote by* $\min(\mathcal{W})$ *the minimum of* $\langle W, \leq\rangle$.

Definition 11. *An* n*-ary connective* $\diamond$ *is called* deterministic *in a coherent canonical system* **G***, if for every* n*-atomic frame* $\mathcal{W} = \langle W, \leq, v\rangle$*, either* σ_{id} *fulfils a right rule for* $\diamond$ *in* $\min(\mathcal{W})$*, or* σ_{id} *fulfils a left rule for* $\diamond$ *in some* $b \in W$.

Proposition 1. *If an* n*-ary connective* $\diamond$ *is deterministic in a coherent canonical system* **G***, then for every* **G***-legal frame* $\mathcal{W} = \langle W, \leq, v\rangle$*,* $a \in W$ *and a substitution* σ*, either* σ *fulfils a right rule for* $\diamond$ *in* a*, or* σ *fulfils a left rule for* $\diamond$ *in some* $b \in W$ *such that* $b \geq a$.

Proof. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a **G**-legal frame, let $a \in W$, and let σ be a substitution. Define an n-atomic frame $\mathcal{W}' = \langle W', \leq', v'\rangle$, where $W' = \{b \in W \mid b \geq a\}$, $\leq'$ is the restriction of $\leq$ to W', and $v'(b, p_i) = v(b, \sigma(p_i))$ for $1 \leq i \leq n$ and $b \in W'$. Note that $a = \min(\mathcal{W}')$. Since $\diamond$ is deterministic in **G**, either σ_{id} fulfils a right rule r for $\diamond$ in $\min(\mathcal{W}')$ with respect to $\mathcal{W}'$, or σ_{id} fulfils a left rule r for $\diamond$ in some $b \in W'$ such that $b \geq' a$ with respect to $\mathcal{W}'$. In the first case, it easily follows that σ fulfils r in a with respect to $\mathcal{W}$. Similarly, in the second case, it follows that since $b \geq' a$, also $b \geq a$, and so σ fulfils r in b with respect to $\mathcal{W}$. □

Proposition 2. *Let* **G** *be a coherent canonical system. A connective* $\diamond$ *is deterministic in* **G** *iff it admits unique analycity in* **G**.

Proof. $(\Rightarrow)$: Assume that $\diamond$ is deterministic in **G**. Let $\psi_1, \ldots, \psi_n \in Frm_{\mathcal{L}}$ and let $\mathcal{W} = \langle W, \leq, v\rangle$ be a **G**-legal $PSF[\diamond(\psi_1, \ldots, \psi_n)]$-semiframe. We show that $\mathcal{W}$ is $\diamond(\psi_1, \ldots, \psi_n)$-determined in **G**. Indeed, let $\mathcal{W}_1 = \langle W, \leq, v_1\rangle$ and $\mathcal{W}_2 = \langle W, \leq, v_2\rangle$ be **G**-legal $SF[\diamond(\psi_1, \ldots, \psi_n)]$-semiframes which extend $\mathcal{W}$. We show that $v_1(a, \diamond(\psi_1, \ldots, \psi_n)) = v_2(a, \diamond(\psi_1, \ldots, \psi_n))$ for every $a \in W$, and so $\mathcal{W}_1$ and $\mathcal{W}_2$ agree on ψ. Let $a \in W$. By Proposition 1, and since v_1 and v_2 are defined identically on $\psi_1, \ldots, \psi_n$ one of the following holds:

- $\sigma_{\psi_1,\ldots,\psi_n}$ fulfils a right rule for $\diamond$ in a with respect to $\mathcal{W}_1$ and to $\mathcal{W}_2$. Since they are both **G**-legal, $v_1(a, \diamond(\psi_1, \ldots, \psi_n)) = v_2(a, \diamond(\psi_1, \ldots, \psi_n)) = t$.
- $\sigma_{\psi_1,\ldots,\psi_n}$ fulfils a left rule for $\diamond$ in some $b \geq a$ with respect to $\mathcal{W}_1$ and to $\mathcal{W}_2$. Since they are both **G**-legal, $v_1(b, \diamond(\psi_1, \ldots, \psi_n)) = v_2(b, \diamond(\psi_1, \ldots, \psi_n)) = f$. By the persistence condition $v_1(a, \diamond(\psi_1, \ldots, \psi_n)) = v_2(a, \diamond(\psi_1, \ldots, \psi_n)) = f$.

($\Leftarrow$) : Assume that $\diamond$ is not deterministic in **G**. By definition, there exists an n-atomic frame, $\mathcal{W} = \langle W, \leq, v \rangle$, such that σ_{id} does not fulfil any right rule for $\diamond$ in $\min(\mathcal{W})$, and it does not fulfil any left rule for $\diamond$ in any $b \in W$. Since $\mathcal{W}$ is n-atomic frame, it is vacuously **G**-legal. Define $\mathcal{W}_1 = \langle W, \leq, v_1 \rangle$ and $\mathcal{W}_2 = \langle W, \leq, v_2 \rangle$ to be $SF[\diamond(p_1, \ldots, p_n)]$-semiframes which extend $\mathcal{W}$ such that: (1) $v_1(\min(\mathcal{W}), \diamond(p_1, \ldots, p_n)) = f$ and $v_1(b, \diamond(p_1, \ldots, p_n)) = t$ for every $b > \min(\mathcal{W})$; and (2) $v_2(b, \diamond(p_1, \ldots, p_n)) = t$ for every $b \in W$. It is easy to see that $\mathcal{W}_1$ and $\mathcal{W}_2$ are **G**-legal extensions of $\mathcal{W}$. Clearly $\mathcal{W}_1$ and $\mathcal{W}_2$ do not agree on $\diamond(p_1, \ldots, p_n)$. □

4 Axiom-Expansion

Below we show that in a coherent canonical system determinism of its connectives is equivalent to axiom expansion. We use the terms *atomic axioms* for axioms of the form $p \Rightarrow p$ (where p is an atomic formula), and *non-atomic axioms* for axioms of the form $\diamond(\psi_1, \ldots, \psi_n) \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$.

Definition 12. *An n-ary connective $\diamond$ admits* axiom-expansion *in a coherent canonical system* **G**, *if $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ has a cut-free proof in* **G** *that does not contain non-atomic axioms.*

Let **G** be a coherent canonical system in the language $\mathcal{L}$. Henceforth we denote by $\mathbf{G}^{\triangleright}$ the system **G** augmented with the rules for $\triangleright$ in Example 2 ($\mathbf{G} = \mathbf{G}^{\triangleright}$ if $\triangleright \in \mathcal{L}$). It is easy to see that $\mathbf{G}^{\triangleright}$ is coherent.

Lemma 1. *Let $\diamond$ be an n-ary connective ($n \geq 1$), and* **G** *be a coherent canonical system. If $\vdash_{\mathbf{G}^{\triangleright}} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(\triangleright\, p_1, \ldots, p_n)$ then $\diamond$ is deterministic in* **G**.

Proof. Assume that $\diamond$ is not deterministic in **G**. Thus, there exists an n-atomic frame, $\mathcal{W} = \langle W, \leq, v \rangle$, such that σ_{id} does not fulfil any right rule for $\diamond$ in $\min(\mathcal{W})$, and it does not fulfil any left rule for $\diamond$ in any $b \in W$. Let $\mathcal{F} = \{p_1, \ldots, p_n, \triangleright\, p_1, \diamond(\triangleright\, p_1, \ldots, p_n), \diamond(p_1, \ldots, p_n)\}$. Define an $\mathcal{F}$-semiframe $\mathcal{W}_1 = \langle W, \leq, v_1 \rangle$ which extends $\mathcal{W}$: (a) For every $b \in W$, $v_1(b, \diamond(p_1, \ldots, p_n)) = t$ and $v_1(b, \triangleright\, p_1) = v(b, p_1)$; (b) $v_1(\min(\mathcal{W}), \diamond(\triangleright\, p_1, \ldots, p_n)) = f$, and $v_1(b, \diamond(\triangleright\, p_1, \ldots, p_n)) = t$ for every $b > \min(\mathcal{W})$. It is easy to see that $\mathcal{W}_1$ is a $\mathbf{G}^{\triangleright}$-legal $\mathcal{F}$-semiframe. Hence by Theorem 1, it can be extended to a $\mathbf{G}^{\triangleright}$-legal frame $\mathcal{W}_1'$. $\mathcal{W}_1'$ is not a model of $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(\triangleright\, p_1, \ldots, p_n)$ (since it is not locally true in $\min(\mathcal{W})$). Therefore, $\not\vdash_{\mathbf{G}^{\triangleright}} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(\triangleright\, p_1, \ldots, p_n)$. □

Theorem 4. *Let* **G** *be a coherent canonical system. An n-ary connective $\diamond$ admits axiom-expansion in* **G** *iff $\diamond$ is deterministic in* **G**.

Proof. $(\Rightarrow)$: Assume we have a cut-free proof δ of $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ in $\mathbf{G}$ that uses only atomic axioms. By suitably modifying δ we can obtain a proof $\delta^{\triangleright}$ of $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(\triangleright p_1, \ldots, p_n)$ in the extended system $\mathbf{G}^{\triangleright}$. The claim therefore follows by Lemma 1. $\delta^{\triangleright}$ is obtained from δ as follows: as δ contains only atomic axioms, the formula $\diamond(p_1, \ldots, p_n)$ is inferred in succedents of sequents in (possibly) various nodes of δ by applications of weakening or of right rules for $\diamond$.[5] In the first case, we simply replace each application of weakening with formula $\diamond(p_1, \ldots, p_n)$, with an application of weakening with formula $\diamond(\triangleright p_1, \ldots, p_n)$. When $\diamond(p_1, \ldots, p_n)$ is inferred in a sequent $\Gamma \Rightarrow \diamond(p_1, \ldots, p_n)$ by a right rule r for $\diamond$, we consider the premise of this application. These have the form $\Gamma, \Pi, p_1 \Rightarrow E$; $\Gamma, \Pi \Rightarrow p_1$; and/or $\Gamma, \Pi \Rightarrow E$. Therefore we first apply the left and/or right rules for $\triangleright$ to infer $\Gamma, \Pi, \triangleright p_1 \Rightarrow E$ and/or $\Gamma, \Pi \Rightarrow \triangleright p_1$, and then we apply r to derive the sequent $\Gamma \Rightarrow \diamond(\triangleright p_1, \ldots, p_n)$. The rest of the proof is changed accordingly.

$(\Leftarrow)$: We first prove the following strengthening of Theorem 3: $(*)$ If $\mathcal{S} \vdash_{\mathbf{G}} s$ then there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ in which all cut-formulas appear in $\mathcal{S}$, and identity axioms of the form $\diamond(\psi_1, \ldots, \psi_n) \Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ are not used when $\diamond$ is deterministic in $\mathbf{G}$. Since every frame is a model of $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ for every n-ary connective $\diamond$, it follows that when $\diamond$ is deterministic in $\mathbf{G}$, $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ has a cut-free proof in $\mathbf{G}$ that uses only atomic axioms.

To prove $(*)$, note that the only place in which non-atomic axioms are used in the proof of Theorem 3 (see a proof outline in Section 2) is for proving property **(b)**, namely that if $\mathcal{T}$ is $\{\psi\}$-maximal then $v(\mathcal{T}, \psi) = f$.[6] More specifically, non-atomic axioms are only used to dismiss the possibility that $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{T}$ when $\mathcal{T}$ is $\{\diamond(\psi_1, \ldots, \psi_n)\}$-maximal and $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$. If $\diamond$ is deterministic in $\mathbf{G}$, we can handle this possibility as follows:

> Since $\mathcal{W}$ is $\mathbf{G}$-legal and $v(\mathcal{T}, \diamond(\psi_1, \ldots, \psi_n)) = t$, there cannot exist $\mathcal{T}' \in W$, $\mathcal{T} \subseteq \mathcal{T}'$, and a left rule for $\diamond$ which is fulfilled in $\mathcal{T}'$ by $\sigma_{\psi_1,\ldots,\psi_n}$. When $\diamond$ is deterministic in $\mathbf{G}$, by Proposition 1, there exists a right rule for $\diamond$, which is fulfilled in $\mathcal{T}$ by $\sigma_{\psi_1,\ldots,\psi_n}$. The rest of the proof proceeds as in the original proof.

□

Remark 1. [6] investigates single-conclusion systems with non-standard sets of structural rules. An algebraic semantics using phase spaces is provided for left and right introduction rules for connectives; a result similar to Theorem 4 is shown, namely a connective $\diamond$ has a "deterministic semantics"(i.e., the interpretations for the left and right rules coincide) iff $\diamond$ admits axiom expansion.

5 Invertibility of Rules

We investigate the connection between rules invertibility and determinism in coherent canonical systems.

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[5] Weakening can also be done by applying a left rule which does not have soft premises.

[6] Identity axioms are not needed to prove property **(a)** (see Theorem 6.3 in [3]).

Definition 13. *A canonical rule r is* canonically invertible *in a coherent canonical system* **G** *iff each premise of r has a proof in* **G** *from r's conclusion.*

In contrast with the multiple-conclusion case (see [2]) determinism does not guarantee invertibility of *left rules* and the latter does not imply determinism. One direction can be easily seen by considering the usual left rule for implication (Example 1): $\supset$ is deterministic but $(\supset\Rightarrow)$ is not canonically invertible. The other direction follows by the next example:

Example 5. $\blacktriangleright$ is non-deterministic in a canonical system including the two rules for $\blacktriangleright$ (see Example 4). However, the left rule for $\blacktriangleright$ is canonically invertible ($p_1 \Rightarrow$ can be easily derived from $\blacktriangleright p_1 \Rightarrow$ using an identity axiom, the rule $(\Rightarrow\blacktriangleright)$ and a cut on $\blacktriangleright p_1$).

For right rules the following theorem holds (notice that the $(\Rightarrow)$ direction needs the existence in **G** of *exactly one* right rule for $\diamond$):

Theorem 5. *If a coherent canonical system* **G** *includes exactly one right rule for an n-ary connective* $\diamond$*, then* $\diamond$ *is deterministic in* **G** *if and only if this rule is canonically invertible in* **G**.

Proof. $(\Rightarrow)$: Let r be the right rule for $\diamond$, and let s be one of its premises. We show that $\Rightarrow \diamond(p_1,\ldots,p_n) \vdash_{\mathbf{G}} s$. Canonical invertibility of r then follows by Theorem 3. For this we show that every **G**-legal frame that is not a model of s is also not a model of $\Rightarrow \diamond(p_1,\ldots,p_n)$. Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a **G**-legal frame, which is not a model of s. By definition, σ_{id} does not locally satisfy s in some $a \in W$. Hence, σ_{id} does not fulfil r in a. Since $\diamond$ is deterministic in **G** and $\diamond$ has no other right rules, Proposition 1 implies that there exists a left rule for $\diamond$, which is fulfilled by σ_{id} in some $b \geq a$. But, since $\mathcal{W}$ is **G**-legal, $\mathcal{W}$ respects this rule, and hence $v(b, \diamond(p_1,\ldots,p_n)) = f$. Hence, $\mathcal{W}$ is not a model of $\Rightarrow \diamond(p_1,\ldots,p_n)$.

$(\Leftarrow)$: Let r be a right rule for $\diamond$. Assume that $\diamond$ is not deterministic in **G**. Thus, there exists an n-atomic frame, $\mathcal{W} = \langle W, \leq, v\rangle$, such that σ_{id} does not fulfil r in $\min(\mathcal{W})$, and it does not fulfil any left rule for $\diamond$ in any $b \in W$. In particular, there exists a premise s of r, which is not satisfied in $\min(\mathcal{W})$ by σ_{id}. Define an extension of $\mathcal{W}$, $\mathcal{W}_1 = \langle W, \leq, v_1\rangle$, which is an $SF[\diamond(p_1,\ldots,p_n)]$-semiframe, such that for every $a \in W$, $v_1(a, \diamond(p_1,\ldots,p_n)) = t$. It is easy to see that $\mathcal{W}_1$ is a **G**-legal $SF[\diamond(p_1,\ldots,p_n)]$-semiframe. Thus, by Theorem 1, it can be extended to a **G**-legal frame, $\mathcal{W}_1' = \langle W, \leq, v_1'\rangle$. $\mathcal{W}_1'$ is a model of $\Rightarrow \diamond(p_1,\ldots,p_n)$, but it is not a model of s. By Theorem 3, $\Rightarrow \diamond(p_1,\ldots,p_n) \not\vdash_{\mathbf{G}} s$. $\square$

6 Matrix-based (Kripke) Semantics

The formulation of the Kripke-style semantics presented in Section 2.2 is too abstract to provide a constructive method for checking determinism of connectives in canonical systems. In this section we introduce an alternative formulation of this semantics, which is a generalization of the two-valued non-deterministic matrices used in [1, 2] to characterize multiple-conclusion canonical systems. The new formulation can be constructively extracted from the rules of a canonical calculus, and it provides an *algorithmic* and natural way of checking determinism of logical connectives.

For an intuitive motivation of this approach, recall that in a (standard or non-deterministic) two-valued matrix, the interpretation of an n-ary connective $\diamond$ is a function applied to n-ary vectors of truth-values. Thus, the truth-value of $\diamond(\psi_1, \ldots, \psi_n)$ depends on (although is not necessarily uniquely determined by) the truth-values assigned to $\psi_1, \ldots, \psi_n$. In the context of Kripke-style frames, however, the interpretation is more complex: the truth-value assigned to $\diamond(\psi_1, \ldots, \psi_n)$ in a world a depends, in addition to the truth-values assigned to $\psi_1, \ldots, \psi_n$ in a, also on the truth-values assigned to these formulas in all worlds $b \geq a$. However, which truth-values are assigned to $\psi_1, \ldots, \psi_n$ in which world is immaterial, what matters is their distribution[7] $D = \{\langle v_1^b, \ldots, v_n^b \rangle |\ b \geq a\}$, where v_i^b is the truth-value assigned to ψ_i in the world b. This information can be captured by an n-ary distribution vector of the form $\langle \langle v_1^a, \ldots, v_n^a \rangle, D \rangle$. Note that since $\geq$ is reflexive, $\langle v_1^a, \ldots, v_n^a \rangle \in D$ for all frames. Moreover, a formula assigned t in some world a remains true also in all accessible worlds $b \geq a$. This can be formalized as follows:

Definition 14. *For $n \geq 1$, an n-ary distribution vector V is a pair of the form $\langle \langle x_1, \ldots, x_n \rangle, D \rangle$ where $x_1, \ldots, x_n \in \{t, f\}$, $D \subseteq \{t, f\}^n$, and which satisfies: (i) $\langle x_1, \ldots, x_n \rangle \in D$, and (ii) if $x_i = t$ then $y_i = t$ for all $\langle y_1, \ldots, y_n \rangle \in D$. We denote the set of n-ary distribution vectors by $\mathbf{V}_n$.*

Definition 15. *A* two-valued distribution Nmatrix *(*2Nmatrix*) $\mathcal{M}$ for $\mathcal{L}$ is a set of (two-valued) interpretations, such that for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{M}$ includes an interpretation function $\tilde{\diamond}_{\mathcal{M}} : \mathbf{V}_n \to P^+(\{t, f\})$.*

Definition 16. *Let $\langle x_1, \ldots, x_n \rangle \in \{t, f\}^n$. A sequent $\Pi \Rightarrow E$ over $\{p_1, \ldots, p_n\}$ is* compatible with *$\langle x_1, \ldots, x_n \rangle$ if any two-valued valuation v, such that $v(p_i) = x_i$ satisfies $\Pi \Rightarrow E$ (i.e., there is some $1 \leq i \leq n$, such that either $p_i \in \Pi$ and $v(p_i) = f$, or $p_i \in E$ and $v(p_i) = t$).*

Definition 17. *Let $V = \langle \overline{x}, D \rangle$ be any n-ary distribution vector.*

1. *A right rule r is V-*valid *if every premise of r is compatible with every $\overline{y} \in D$.*
2. *A left rule of r is V-*valid *if every hard premise of r is compatible with every $\overline{y} \in D$, and every soft premise of r is compatible with $\overline{x}$.*

Definition 18. *Let $\mathbf{G}$ be a coherent canonical system. The 2Nmatrix $\mathcal{M}_G$ induced by $\mathbf{G}$ is defined as follows. For every n-ary connective $\diamond$ and every $V \in \mathbf{V}_n$:*

$$\tilde{\diamond}_{\mathcal{M}_G}(V) = \begin{cases} \{t\} & \mathbf{G} \text{ has a } V\text{-valid right rule for } \diamond \\ \{f\} & \mathbf{G} \text{ has a } V\text{-valid left rule for } \diamond \\ \{t, f\} & \text{otherwise} \end{cases}$$

[7] A "distribution-based approach" is usually used to interpret quantifiers in many-valued matrices (see, e.g. [11]). For instance, the classical interpretation of $\forall$ is a function $\tilde{\forall} : P^+(\{t, f\}) \to \{t, f\}$. Given a structure with a set of elements D, we compute $\tilde{\forall}(\{v(\psi\{\overline{a}/x\})|\ a \in D\})$, where $\overline{a}$ is an individual constant denoting a for every $a \in D$.

It is easy to see that checking V-validity of rules is constructive. The following proposition guarantees that $\tilde{\diamond}_{\mathcal{M}_G}$ is well-defined:

Proposition 3. *A coherent canonical system* $\mathbf{G}$ *has no pair of a left and a right rules for the same n-ary connective* $\diamond$*, which are both V-valid for some* $V \in \mathbf{V}_n$*.*

Proof. Suppose by contradiction that there are a right rule r_r and a left rule r_l for $\diamond$ in $\mathbf{G}$, which are both V-valid for some $V = \langle \overline{x}, D \rangle \in \mathbf{V}_n$. Since $\overline{x} \in D$, all the premises of r_r and r_l are compatible with $\overline{x}$. Thus these premises are all satisfiable by a classical two-valued valuation, and so are classically consistent, in contradiction to the coherence of $\mathbf{G}$. □

It is important to note that given a coherent canonical system $\mathbf{G}$, its associated 2Nmatrix $\mathcal{M}_G$ does not yet faithfully represent the meaning of the connectives of $\mathbf{G}$, as $\mathcal{M}_G$ might contain some options forbidden by the persistence condition.

Example 6. Let $\mathbf{G}$ be the canonical system consisting only of the right rule for implication and $\mathbf{G}'$ be the system obtained by adding to $\mathbf{G}$ the left rule for implication (see Example 1). The induced 2Nmatrices $\mathcal{M}_G$ and $\mathcal{M}_{G'}$ are displayed in the table below (columns $\tilde{\supset}_{\mathcal{M}_G}$ and $\tilde{\supset}_{\mathcal{M}_{G'}}$, respectively). Note that $\tilde{\supset}_{\mathcal{M}_{G'}}$ contains some non-deterministic choices, although the semantics for implication given in Example 1 is completely deterministic.

	D	$\tilde{\supset}_{\mathcal{M}_G}$	$\tilde{\supset}_{\mathcal{M}_{G'}}$	$\mathsf{R}(\tilde{\supset}_{\mathcal{M}_{G'}})$
$\langle t,t\rangle$	$\{\langle t,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle t,f\rangle$	$\{\langle t,f\rangle\}$	$\{t,f\}$	$\{f\}$	$\{f\}$
$\langle t,f\rangle$	$\{\langle t,f\rangle,\langle t,t\rangle\}$	$\{t,f\}$	$\{f\}$	$\{f\}$
$\langle f,t\rangle$	$\{\langle f,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,t\rangle$	$\{\langle f,t\rangle,\langle t,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle f,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle\}$	$\{t,f\}$	$\{t,f\}$	$\{f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,t\rangle,\langle f,t\rangle\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,f\rangle\}$	$\{t,f\}$	$\{t,f\}$	$\{f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,t\rangle\}$	$\{t,f\}$	$\{t,f\}$	$\{f\}$
$\langle f,f\rangle$	$\{\langle f,f\rangle,\langle t,f\rangle,\langle f,t\rangle,\langle t,t\rangle\}$	$\{t,f\}$	$\{t,f\}$	$\{f\}$

We now formulate a procedure for removing the illegal options. As shown below, its application to any 2Nmatrix $\mathcal{M}_G$ leads to a matrix-based representation which faithfully reflects the semantics from Section 2.2.

Definition 19. *Let* $\tilde{\diamond} : \mathbf{V}_n \to P^+(\{t,f\})$ *be an interpretation of an n-ary connective* $\diamond$*. The reduced interpretation* $\mathsf{R}(\tilde{\diamond})$ *is obtained by the following algorithm:*

- $L_0 \leftarrow \tilde{\diamond}$ *and* $i \leftarrow 0$*.*
Repeat

– $i \leftarrow i+1$ *and* $L_i \leftarrow L_{i-1}$.
– *Let* $V = \langle \overline{x}, D\rangle$, *such that* $L_{i-1}(V) = \{t, f\}$. *If there is some* $\overline{y} \in D$, *such that for every* $D' \subseteq D$, *such that* $\langle \overline{y}, D'\rangle \in \mathbf{V}_n$: $L_{i-1}(\langle \overline{y}, D'\rangle) = \{f\}$, *then* $L_i(V) \leftarrow \{f\}$.

Until $L_i = L_{i-1}$

Example 7. By applying the algorithm to the 2Nmatrix $\mathcal{M}_{G'}$ in Example 6, we obtain the reduced 2Nmatrix displayed in the last column of the table above (denoted by $\mathsf{R}(\tilde{\supset}_{\mathcal{M}_{G'}})$). Note that $\mathsf{R}(\tilde{\supset}_{\mathcal{M}_{G'}})$ does not codify a particular Kripke frame, as in the matrix-based semantics for intuitionistic logic described in [10]; $\mathsf{R}(\tilde{\supset}_{\mathcal{M}_{G'}})$ represents instead the "semantic meaning" of intuitionistic implication.

Below we show that for any coherent canonical system $\mathbf{G}$, the determinism of $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})$ is equivalent to the determinism of $\diamond$ in $\mathbf{G}$ (in the sense of Definition 11), thus obtaining an algorithm for checking the latter.

Definition 20. *Given an* n*-atomic frame* $\mathcal{W} = \langle W, \leq, v\rangle$, *the distribution vector* $\mathsf{V}_{\mathcal{W}}$ *induced by* $\mathcal{W}$ *is defined as follows:* $\mathsf{V}_{\mathcal{W}} = \langle\langle v(a,p_1), \ldots, v(a,p_n)\rangle, \mathsf{D}_{\mathcal{W}}\rangle$, *where* $a = \min(\mathcal{W})$ *and* $\mathsf{D}_{\mathcal{W}} = \{\langle v(b,p_1), \ldots, v(b,p_n)\rangle \mid b \in W\}$.

Lemma 2. *Let* $\mathcal{W}$ *be an* n*-atomic frame.* σ_{id} *fulfils a canonical rule* r *in* $min(\mathcal{W})$ *with respect to* $\mathcal{W}$ *iff* r *is* $\mathsf{V}_{\mathcal{W}}$*-valid.*

Lemma 3. *Let* $\mathbf{G}$ *be a coherent canonical system for* $\mathcal{L}$ *and* $\diamond$ *an* n*-ary connective of* $\mathcal{L}$. *If* $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V) = \{f\}$, *then for every* n*-atomic frame* $\mathcal{W}$ *inducing* V, σ_{id} *fulfils a left rule for* $\diamond$ *of* $\mathbf{G}$ *in some world* $b \geq \min(\mathcal{W})$ *with respect to* $\mathcal{W}$.

Proof. We prove by induction on i that the claim holds for every L_i as defined in Definition 19. It follows that the claim holds for $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})$. For $i = 0$, $L_i = \tilde{\diamond}_{\mathcal{M}_G}$, and hence the claim follows from Lemma 2 by the definition of $\mathcal{M}_G$. Suppose that the claim holds for all $i < k$ and let $i = k$. Let $\mathcal{W}$ be an n-atomic frame inducing $V \in \mathbf{V}_n$. If $L_{k-1}(V) = \{f\}$, the claim holds by the induction hypothesis. Otherwise $L_{k-1}(V) = \{t, f\}$, and there is some $\overline{y} \in D$, such that for every $D' \subseteq D$ for which $\langle \overline{y}, D'\rangle \in \mathbf{V}_n$: $L_{k-1}(\langle \overline{y}, D'\rangle) = \{f\}$. Let $b \geq \min(\mathcal{W})$ be a world such that $v(b,p_i) = y_i$ (it exists since $\overline{y} \in D$). Let $D_0 = \{\langle v(c,p_1), \ldots, v(c,p_n)\rangle \mid c \geq b\}$. Since $D_0 \subseteq D$, $L_{k-1}(V_0) = \{f\}$, where $V_0 = \langle \overline{y}, D_0\rangle$. Let $\mathcal{W}_0$ be the subframe of $\mathcal{W}$, such that $\min(\mathcal{W}_0) = b$. Since $\mathcal{W}_0$ induces V_0, by the induction hypothesis, there is some $c \geq b$, in which σ_{id} fulfils a left rule r in $\mathbf{G}$ for $\diamond$ with respect to $\mathcal{W}_0$. It easily follows that σ_{id} fulfils r in $c \geq b \geq \min(\mathcal{W})$ with respect to $\mathcal{W}$. □

Theorem 6. *Let* $\mathbf{G}$ *be a coherent canonical system. An* n*-ary connective* $\diamond$ *is deterministic in* $\mathbf{G}$ *if and only if* $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})$ *is deterministic (i.e.* $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V)$ *is either* $\{t\}$ *or* $\{f\}$ *for every* $V \in \mathbf{V}_n$*).*

Proof. ($\Leftarrow$): Denote by R_r and R_l the sets of right and left rules for $\diamond$ in $\mathbf{G}$ (respectively). Suppose that $\diamond$ is deterministic in $\mathbf{G}$, and assume by contradiction that $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V)$ is not deterministic. Define a partial order on n-ary vectors over

$\{t, f\}$ as follows: $\overline{x} <_n \overline{y}$ if for every $1 \leq i \leq n$: either $x_i = y_i$ or $x_i = f$ and $y_i = t$. Choose $V = \langle \overline{x}, D \rangle \in \mathbf{V}_n$ to be such that $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V) = \{t, f\}$ and $\overline{x}$ is maximal with respect to $<_n$. We construct an n-atomic frame $\mathcal{W}$, such that σ_{id} does not fulfil any $r \in R_r$ in $\min(\mathcal{W})$, and any $r \in R_l$ in any $b \geq \min(\mathcal{W})$ (with respect to $\mathcal{W}$). If $D = \{\overline{x}\}$, then let $\mathcal{W}$ be the n-atomic frame with one world a, such that $v(a, p_i) = x_i$. Since $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V) = \{t, f\}$, it must be the case that $\tilde{\diamond}_{\mathcal{M}_G}(V) = \{t, f\}$. Then by definition of $\mathcal{M}_G$, there is no V-valid rule in $R_r \cup R_l$. By Lemma 2, σ_{id} does not fulfil any $r \in R_r \cup R_l$ in a with respect to $\mathcal{W}$. Otherwise, $D = \{\overline{x}, \overline{y}^1, \ldots, \overline{y}^m\}$. Let $\mathcal{W}$ be the n-atomic frame $\mathcal{W} = \langle W, \leq, v \rangle$, such that $W = \{a, a_1, \ldots, a_m\}$, where $a < a_j$ for all $1 \leq j \leq m$, $v(a, p_i) = x_i$, and $v(a_j, p_i) = y_i^j$. Like in the above case, it can be shown that σ_{id} does not fulfil any $r \in R_r \cup R_l$ in $a = \min(\mathcal{W})$. It remains to show that no $r \in R_l$ is fulfilled in a_j. Suppose by contradiction that this is the case for some $r \in R_l$ and a_j. Let $\mathcal{W}'$ be the subframe of $\mathcal{W}$ such that $\min(\mathcal{W}') = a_j$. The distribution vector induced by $\mathcal{W}'$ is $V_j = \langle \overline{y}^j, \{\overline{y}^j\} \rangle$. By Lemma 2, r is V_j-valid, and so $\tilde{\diamond}_{\mathcal{M}}(V_j) = \{f\}$. Hence also $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V_j) = \{f\}$. One of the following cases holds:

- For all $D_0 \subseteq D$, such that $V' = \langle \overline{y}^j, D_0 \rangle \in \mathbf{V}_n$, $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V') = \{f\}$. But then $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V) = \{f\}$, contradicting our assumption.
- There is some $D_0 \subseteq D$, such that $V' = \langle \overline{y}^j, D_0 \rangle \in \mathbf{V}_n$ and $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V') = \{t\}$, and so also $\tilde{\diamond}_{\mathcal{M}}(V') = \{t\}$. This means that there exists some V'-valid $r \in R_r$. By definition, r is $\langle \overline{y}^j, D' \rangle$-valid for every $D' \subseteq D_0$. It follows that $\tilde{\diamond}_{\mathcal{M}}(\langle \overline{y}^j, \{\overline{y}^j\} \rangle) = \{t\}$, in contradiction to our assumption.
- There is some $D_0 \subseteq D$, such that $V' = \langle \overline{y}^j, D_0 \rangle \in \mathbf{V}_n$ and $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(V') = \{t, f\}$. But since $\overline{x} <_n \overline{y}^j$, this is in contradiction to the maximality of $\overline{x}$.

Thus it cannot be the case that σ_{id} fulfils a rule from R_l in some $a_j \geq \min(\mathcal{W})$, hence $\diamond$ is not deterministic in $\mathbf{G}$, in contradiction to our assumption.

($\Rightarrow$): Suppose that $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})$ is deterministic and assume by contradiction that there is some n-atomic frame $\mathcal{W}$, such that σ_{id} does not fulfil any right rule of $\mathbf{G}$ for $\diamond$ in $\min(\mathcal{W})$, and any left rule of $\mathbf{G}$ for $\diamond$ in any $b \geq \mathcal{W}$. Let V be the distribution vector induced by $\mathcal{W}$. By Lemma 2, there is no V-valid right rule for $\diamond$ in $\mathbf{G}$, and so by definition of $\mathcal{M}_G$, $\tilde{\diamond}_{\mathcal{M}}(\mathsf{V}_{\mathcal{W}}) \neq \{t\}$, and so also $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(\mathsf{V}_{\mathcal{W}}) \neq \{t\}$. Since $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})$ is deterministic, it must be the case that $\mathsf{R}(\tilde{\diamond}_{\mathcal{M}_G})(\mathsf{V}_{\mathcal{W}}) = \{f\}$. But then by Lemma 3, for every n-atomic frame $\mathcal{W}'$ inducing V, σ_{id} fulfils a left rule for $\diamond$ in $\mathbf{G}$ in some $b \geq \min(\mathcal{W}')$. In particular, this holds for $\mathcal{W}$, in contradiction to our assumption. □

Corollary 1. *For a coherent canonical system* $\mathbf{G}$*, the following questions are decidable: (i) Is* $\diamond$ *deterministic in* $\mathbf{G}$*? (ii) Does* $\mathbf{G}$ *admit axiom-expansion? (iii) (If* $\mathbf{G}$ *has exactly one right rule* r *for* $\diamond$*) is* r *invertible?.*

Finally, we establish the equivalence between the new matrix-based semantics and the non-deterministic Kripke-style semantics of [3].

Theorem 7. *Let* **G** *be a coherent canonical system. A frame* $\mathcal{W} = \langle W, \leq, v\rangle$ *is* **G***-legal (see Definition 6) iff for every* $a \in W$ *and every formula* $\diamond(\psi_1, \ldots, \psi_n)$, $v(a, \diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}_{\mathcal{M}_G}(\langle \overline{x}, D\rangle)$*, where* $\overline{x} = \langle v(a, \psi_1), \ldots, v(a, \psi_n)\rangle$ *and* D *is the set* $\{\langle v(b, \psi_1), \ldots, v(b, \psi_n)\rangle \mid b \geq a\}$.

Proof. ($\Rightarrow$): Suppose that $\mathcal{W}$ is **G**-legal. Let $a \in W$ and $\diamond(\psi_1, \ldots, \psi_n) \in Frm_{\mathcal{L}}$. Suppose that $\tilde{\diamond}_{\mathcal{M}_G}(\langle \overline{x}, D\rangle) = \{t\}$. By definition of $\mathcal{M}_G$, there is some right rule r for $\diamond$ in **G**, such that every premise of r is compatible with every $\overline{y} \in D$. It is easy to see that this implies that $\sigma_{\psi_1,\ldots,\psi_n}$ fulfils r in a. Since $\mathcal{W}$ is **G**-legal, it respects r, and so $\Rightarrow \diamond(\psi_1, \ldots, \psi_n)$ is locally true in a. It follows that $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$. The case when $\tilde{\diamond}_{\mathcal{M}_G}(\langle \overline{x}, D\rangle) = \{f\}$ is handled similarly.

($\Leftarrow$): Suppose that for every $a \in W$ and every formula $\varphi = \diamond(\psi_1, \ldots, \psi_n)$, $v(a, \varphi) \in \tilde{\diamond}_{\mathcal{M}_G}(\langle\langle v(a, \psi_1), \ldots, v(a, \psi_n)\rangle, \{\langle v(b, \psi_1), \ldots, v(b, \psi_n)\rangle \mid b \geq a\}\rangle)$. Let r be a right rule in **G** for an n-ary connective $\diamond$ (left rules are handled similarly). We prove that $\mathcal{W}$ respects r. Suppose that a substitution σ fulfils r in some $a \in W$. Hence, it locally satisfies every premise of r in every $b \geq a$. Let $x_i^b = v(b, \sigma(p_i))$ for every $b \geq a$. It is easy to see that every premise of r is compatible with $\langle x_1^b, \ldots, x_n^b\rangle$ for every $b \geq a$. By definition of $\mathcal{M}_G$, $\tilde{\diamond}(\langle \overline{x}, D\rangle) = \{t\}$, where $\overline{x} = \langle x_1^a, \ldots, x_n^a\rangle$ and $D = \{\langle x_1^b, \ldots, x_n^b\rangle \mid b \geq a\}$. Therefore, $v(a, \sigma(\diamond(p_1, \ldots, p_n))) \in \{t\}$. Hence σ locally satisfies r's conclusion in a. ◻

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