Cut-Admissibility as a Corollary of the Subformula Property

Ori Lahav[1] and Yoni Zohar[2]

[1] Max Planck Institute for Software Systems (MPI-SWS), Germany
[2] School of Computer Science, Tel Aviv University, Israel
orilahav@mpi-sws.org yoni.zohar@cs.tau.ac.il

Abstract. We identify two wide families of propositional sequent calculi for which cut-admissibility is a corollary of the subformula property. While the subformula property is often a simple consequence of cut-admissibility, our results shed light on the converse direction, and may be used to simplify cut-admissibility proofs in various propositional sequent calculi. In particular, the results of this paper may be used in conjunction with existing methods that establish the subformula property, to obtain that cut-admissibility holds as well.

1 Introduction

One of the major consequences of Gentzen's cut-elimination theorem for LK and LJ [16] is the subformula property: when deriving a sequent s from a set S of sequents, it suffices to consider the subformulas that occur in $S \cup \{s\}$. Other formulas may sometimes shorten derivations, but can be safely ignored when checking whether a derivation exists. Since the introduction of LK and LJ, various cut-free sequent calculi were found for important non-classical logics – e.g., modal logics [30,25], many-valued and fuzzy logics [6,22], and paraconsistent logics [9]. In all these cases, the subformula property (or some generalization of it) trivially follows from the admissibility of the cut rule.

In this paper we are interested in the converse direction: can cut-admissibility be obtained as a corollary of the subformula property?

Clearly, one cannot expect an affirmative answer to this question in the general case, as there are well-known calculi admitting the subformula property but not cut-admissibility. These include, e.g., calculi for the modal logics $S5$ and B [30,25], bi-intuitionistic logic [24], and several calculi for paraconsistent logics [8].

The main contribution of this paper is an affirmative answer to the question above for two wide families of sequent calculi. The first is a family of pure calculi [4] whose derivation rules, like those of LK, do not impose any restrictions on context formulas. The second is a family of calculi, which we call *intuitionistic calculi*, in which premises of the form $\Gamma \Rightarrow \Delta$ with $\Gamma \neq \emptyset$ in right introduction rules forbid context formulas on the right-hand side. This family includes, for example, the well-known multiple-conclusion calculus for intuitionistic logic [28], as well as the calculi for Nelson's logics N_3 and N_4 [29]. In both families, we further require a certain "directed" structure from their rules (precisely defined below), and show that it suffices to ensure that cut-admissibility follows from the subformula property.

Our result is obtained by providing two different semantics for each given calculus: one for derivations that include only subformulas of the premises and the end sequent, and another for cut-free derivations. The latter provides a sufficient semantic criterion for cut-admissibility. Then, we show that this criterion is met when the calculus enjoys the subformula property.

In order to utilize the full strength of sequent calculi, the subformula property is not enough. For example, various sequent calculi for paraconsistent logics [9] do not enjoy the subformula property, but do admit a simple generalization of it, namely: if a sequent s is derivable, then there exists a derivation of s that uses only subformulas of s and their negations. For this reason, we do not restrict ourselves to the strict subformula property, but consider a more general notion, which is based on an arbitrary "well-behaved" (precisely defined below) ordering of propositional formulas.

Besides its theoretical interest, we believe that our result can be useful in future investigation and development of sequent calculi. Proving the subformula property tends to be an easier task than proving full cut-admissibility, as it typically follows from the admissibility of non-analytic cuts (cuts on formulas that are not subformulas of the end sequent). In addition, our recent paper [20] provides a sufficient criterion for the subformula property for a wide family of pure calculi for sub-classical logics, without relying on cut-admissibility. Using the results of the current paper, we obtain the admissibility of cut in all these calculi.

The rest of this paper is organized as follows. After the definitions of pure calculi and their associated cut-admissibility property in Section 2, we introduce our generalized notion of the subformula property in Section 3. In Section 4, semantic characterizations of the different kinds of derivability in pure sequent calculi are given, and are used in Section 5 where our theorem for pure calculi is described. Finally, Section 6 provides our result for intuitionistic calculi.

Related Work

Avron and Lev [10] introduced the family of *canonical calculi*, a very restricted sub-family of pure calculi, and proved the equivalence of the subformula property and cut-admissibility in them. The proof was based on the framework of *Nmatrices* (see also [11]), a simple generalization of usual logical matrices. The present paper goes beyond canonical calculi, and Nmatrices do not suffice. Thus, our proof utilizes a more general semantic framework of Lahav and Avron [19]. In this framework, which can be seen as a generalization of Béziau's *bivaluation semantics* [13], sufficient semantic criteria for cut-admissibility and the subformula property were given. The former amounts to the ability to refine three-valued valuations into two-valued ones, while the latter amounts to the ability to extend partial two-valued valuations into full valuations. Later, [21] showed that for pure calculi, the criterion for the subformula property is also necessary. For the present paper, however, the mere ability to extend partial two-valued valuations is not enough, and a constructive extension method is introduced. Finally, in a previous work [20], we studied general conditions for the subformula property in pure calculi, while cut-admissibility was not considered at all.

2 Pure Sequent Calculi

In this section, we define the family of pure sequent calculi [4] and the notion of cut-admissibility. Several examples of well-known calculi that belong to this family are provided as well.

2.1 Preliminaries

Let $At = \{p_1, p_2, \ldots\}$ denote a fixed infinite set of propositional variables. A *propositional language* $\mathcal{L}$ is given by a set $\diamond_{\mathcal{L}}$ of connectives. *$\mathcal{L}$-formulas* are defined as usual, where *atomic $\mathcal{L}$-formulas* are the elements of At. We usually identify a propositional language with its set of formulas (e.g., when writing expressions like $\varphi \in \mathcal{L}$). For a set $\mathcal{F} \subseteq \mathcal{L}$, by *$\mathcal{F}$-formula* we mean a formula φ satisfying $\varphi \in \mathcal{F}$.

An *$\mathcal{L}$-substitution* is a function $\sigma : At \to \mathcal{L}$, naturally extended to apply on all $\mathcal{L}$-formulas and on sets of $\mathcal{L}$-formulas.

An *$\mathcal{L}$-sequent* is a pair of finite sets Γ and Δ of $\mathcal{L}$-formulas, denoted $\Gamma \Rightarrow \Delta$. We employ the standard sequent notations, e.g., when writing expressions like $\Gamma, \varphi \Rightarrow \Delta$ or $\Rightarrow \varphi$. The union of two sequents $(\Gamma_1 \Rightarrow \Delta_1) \cup (\Gamma_2 \Rightarrow \Delta_2)$ is the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. We denote by $frm[\Gamma \Rightarrow \Delta]$ the set $\Gamma \cup \Delta$, and naturally extend this notation to sets of sequents. $\mathcal{L}$-substitutions are extended to apply on $\mathcal{L}$-sequents and sets of $\mathcal{L}$-sequents (by setting $\sigma(\Gamma \Rightarrow \Delta) = \sigma(\Gamma) \Rightarrow \sigma(\Delta)$ and $\sigma(S) = \{\sigma(\Gamma \Rightarrow \Delta) \mid \Gamma \Rightarrow \Delta \in S\}$).

In what follows, $\mathcal{L}$ denotes an arbitrary propositional language. When $\mathcal{L}$ can be inferred from the context, we omit the prefix "$\mathcal{L}-$" from the notions above (as well as from the ones introduced below).

2.2 Pure Sequent Calculi

Following [10], we find it technically convenient to use the object propositional language for specifying derivation rules. (One could use meta-variables and rule schemes instead.)

Definition 1. A *pure $\mathcal{L}$-rule* is a pair $\langle S, s \rangle$, denoted $S \,/\, s$, where S is a finite set of $\mathcal{L}$-sequents and s is an $\mathcal{L}$-sequent. The elements of S are called the *premises* of the rule and s is called the *conclusion* of the rule. We sometimes omit set braces around the premises, and separate them by semi-colons (e.g., when writing expressions like $\Rightarrow p_1 \,;\, \Rightarrow p_2 \,/\, \Rightarrow p_1 \wedge p_2$).

An $\mathcal{L}$-*application* of a pure $\mathcal{L}$-rule $\{s_1, \ldots, s_n\} / s$ is a pair of the form $\langle \{\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n\}, \sigma(s) \cup c_1 \cup \ldots \cup c_n \rangle$ where σ is an $\mathcal{L}$-substitution, and $c_1, \ldots, c_n$ are $\mathcal{L}$-sequents (called the *context sequents* of the application). The sequents $\sigma(s_i) \cup c_i$ are called the *premises* of the application, and the sequent $\sigma(s) \cup c_1 \cup \ldots c_n$ is called the *conclusion* of the application.

For example, the pure rules for introducing implication in classical logic are:

$$p_1 \Rightarrow p_2 \,/ \Rightarrow p_1 \supset p_2 \qquad\qquad \Rightarrow p_1 \,;\, p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow$$

Their applications take the form (respectively):

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta} \qquad\qquad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2}$$

Examples for derivation rules that *cannot* be formulated as pure rules include the following rule schemes, that are employed in intuitionistic and modal logic:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

In turn, pure calculi are simply finite sets of pure rules.

Definition 2. A *pure $\mathcal{L}$-calculus* is a finite set of pure $\mathcal{L}$-rules. A *derivation* of a sequent s from a set S of sequents (a.k.a. "assumptions" or "non-logical axioms") in a pure $\mathcal{L}$-calculus $\mathbf{G}$ is a finite sequence of sequents, where each sequent in the sequence is either one of the following: (*i*) an element of S; (*ii*) the conclusion of an application of a rule of $\mathbf{G}$, all premises of which are preceding elements of the sequence; (*iii*) the conclusion of one of the following standard structural rules,[1] again where all premises are preceding elements of the sequence:

$$\text{(ID)}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \text{(CUT)}\ \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \qquad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \text{(WEAK)}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}$$

In (CUT), φ is called the *cut formula*. We write $S \vdash_{\mathbf{G}} s$ if there is a derivation of a sequent s from a set S of sequents in $\mathbf{G}$.

In what follows, unless stated otherwise, we may refer to pure rules and pure calculi simply as *rules* and *calculi*.

The most well-studied property of sequent calculi is the admissibility of the cut rule. When cut is admissible the calculus is generally considered well-behaved, and reasoning about the calculus becomes much easier. Moreover, proof-search algorithms have no need to "guess" the cut formulas. Next, we precisely define cut-admissibility.

[1] Note that by defining sequents to be pairs of *sets* we implicitly include other standard structural rules, such as exchange and contraction.

Definition 3. A derivation of s from S in a calculus $\mathbf{G}$ is called *cut-limited* if in every application of (CUT), the cut formula is in $frm[S]$. We write $S \vdash^{\mathrm{cf}}_{\mathbf{G}} s$ if such a derivation exists. A calculus $\mathbf{G}$ enjoys *cut-admissibility* if $\vdash_{\mathbf{G}} = \vdash^{\mathrm{cf}}_{\mathbf{G}}$.

What we call here cut-admissibility is actually known as *strong* cut-admissibility, in which cuts are allowed, but they are confined to apply only on formulas that appear in the set of assumptions [5]. Usual cut-admissibility only requires that $\vdash_{\mathbf{G}} s$ iff $\vdash^{\mathrm{cf}}_{\mathbf{G}} s$ for every sequent s. For pure calculi, however, the two notions are equivalent [5]. Note that this is not the case for intuitionistic calculi, studied in Section 6.

Next, we present several examples of pure calculi (they all enjoy cut-admissibility).

Example 1 (Classical Logic). The propositional language $\mathcal{CL}$ consists of three binary connectives $\wedge$, $\vee$, $\supset$, and one unary connective $\neg$. The propositional fragment of Gentzen's fundamental sequent calculus for classical logic [16] can be directly presented as a pure $\mathcal{CL}$-calculus, denoted **LK**, that consists of the following $\mathcal{CL}$-rules:

$$
\begin{array}{rr}
\Rightarrow p_1 \,/\, \neg p_1 \Rightarrow & p_1 \Rightarrow \;/\; \Rightarrow \neg p_1 \\
p_1, p_2 \Rightarrow \;/\, p_1 \wedge p_2 \Rightarrow & \Rightarrow p_1;\; \Rightarrow p_2 \,/\; \Rightarrow p_1 \wedge p_2 \\
p_1 \Rightarrow ; p_2 \Rightarrow \;/\, p_1 \vee p_2 \Rightarrow & \Rightarrow p_1, p_2 \,/\; \Rightarrow p_1 \vee p_2 \\
\Rightarrow p_1; p_2 \Rightarrow \;/\, p_1 \supset p_2 \Rightarrow & p_1 \Rightarrow p_2 \,/\; \Rightarrow p_1 \supset p_2
\end{array}
$$

Example 2 (Paraconsistent Logics). The paper [9] provides sequent calculi for many paraconsistent logics. For example, a pure calculus for da Costa's historical paraconsistent logic C_1, which we call $\mathbf{G}_{C_1}$, consists of the rules of **LK** except for the left-introduction rule of negation, that is replaced by the following pure $\mathcal{CL}$-rules:

$$
\begin{array}{rr}
p_1 \Rightarrow \;/\, \neg\neg p_1 \Rightarrow & \\
\Rightarrow p_1;\; \Rightarrow \neg p_1 \,/\, \neg(p_1 \wedge \neg p_1) \Rightarrow & \neg p_1 \Rightarrow ; \neg p_2 \Rightarrow \;/\, \neg(p_1 \wedge p_2) \Rightarrow \\
\neg p_1 \Rightarrow ; p_2, \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow & p_1, \neg p_1 \Rightarrow ; \neg p_2 \Rightarrow \;/\, \neg(p_1 \vee p_2) \Rightarrow \\
p_1 \Rightarrow ; p_2, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow & p_1, \neg p_1 \Rightarrow ; \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow
\end{array}
$$

Similarly, a pure calculus $\mathbf{G}_{P_1}$ for the *atomic* paraconsistent logic P_1 was given in [3]. It is obtained by replacing the left-introduction rule of negation in **LK** with the following alternative rules:

$$
\begin{array}{rr}
\Rightarrow p_1;\; \Rightarrow p_2 \,/\, \neg(p_1 \wedge p_2) \Rightarrow & \Rightarrow p_1, p_2 \,/\, \neg(p_1 \vee p_2) \Rightarrow \\
p_1 \Rightarrow p_2 \,/\, \neg(p_1 \supset p_2) \Rightarrow & \Rightarrow \neg p_1 \,/\, \neg\neg p_1 \Rightarrow
\end{array}
$$

Example 3 (Many-valued Logics). The paper [6] provides pure sequent calculi for well-known many-valued logics. For example, a calculus for Łukasiewicz three-valued logic, which we call $\mathbf{G}_3$, has the following rules for implication:

$$
\begin{array}{rr}
\neg p_1 \Rightarrow ; p_2 \Rightarrow ;\; \Rightarrow p_1, \neg p_2 \,/\, p_1 \supset p_2 \Rightarrow & p_1 \Rightarrow p_2; \neg p_2 \Rightarrow \neg p_1 \,/\; \Rightarrow p_1 \supset p_2 \\
p_1, \neg p_2 \Rightarrow \;/\, \neg(p_1 \supset p_2) \Rightarrow & \Rightarrow p_1;\; \Rightarrow \neg p_2 \,/\; \Rightarrow \neg(p_1 \supset p_2)
\end{array}
$$

A pure calculus for the $\mathcal{CL}$-fragment of the logic of bilattices [2] (whose implication-free fragment coincides with the logic of first-degree entailments [1]), which we call $\mathbf{G}_4$, is obtained in a similar manner, by augmenting the positive fragment of **LK** with the following rules:

$$p_1, \neg p_2 \Rightarrow \ / \neg(p_1 \supset p_2) \Rightarrow \qquad \Rightarrow p_1;\ \Rightarrow \neg p_2 \ / \ \Rightarrow \neg(p_1 \supset p_2)$$
$$\neg p_1 \Rightarrow ; \neg p_2 \Rightarrow \ / \neg(p_1 \wedge p_2) \Rightarrow \qquad \Rightarrow \neg p_1, \neg p_2 \ / \ \Rightarrow \neg(p_1 \wedge p_2)$$
$$\neg p_1, \neg p_2 \Rightarrow \ / \neg(p_1 \vee p_2) \Rightarrow \qquad \Rightarrow \neg p_1;\ \Rightarrow \neg p_2 \ / \ \Rightarrow \neg(p_1 \vee p_2)$$
$$p_1 \Rightarrow \ / \neg\neg p_1 \Rightarrow \qquad \Rightarrow p_1 \ / \ \Rightarrow \neg\neg p_1$$

Example 4 (Logics for access control). Primal infon logic [15] was designed to efficiently reason about access control policies. The quotations-free fragment of its sequent calculus [12] can be presented as a pure calculus, which we denote by **P**. It is obtained from the positive fragment of **LK** by adding the axiomatic rules $\Rightarrow \top$ and $\bot \Rightarrow$, dismissing the left introduction rule of disjunction, and replacing the right introduction rule of implication with the following weaker rule:

$$\Rightarrow p_2 \ / \ \Rightarrow p_1 \supset p_2$$

Another security-oriented formalism that can be described as a pure calculus is the Dolev-Yao intruder deductions model from [14], where it was given as a natural deduction calculus. It is equivalent to the following pure calculus, which we denote by **DY**. Its language consists of two binary connectives: pairing and encryption. The intended meaning of $\langle p_1, p_2 \rangle$ is the ordered pair of p_1 and p_2. The intended meaning of $[p_1]_{p_2}$ is the encryption of the message p_1 using the key p_2. Accordingly, the following rules correspond to pairing, unpairing, encryption and decryption:

$$\Rightarrow p_1;\ \Rightarrow p_2 \ / \ \Rightarrow \langle p_1, p_2 \rangle \qquad p_1 \Rightarrow \ / \langle p_1, p_2 \rangle \Rightarrow \qquad p_2 \Rightarrow \ / \langle p_1, p_2 \rangle \Rightarrow$$
$$\Rightarrow p_1;\ \Rightarrow p_2 \ / \ \Rightarrow [p_1]_{p_2} \qquad \Rightarrow p_2; p_1 \Rightarrow \ / [p_1]_{p_2} \Rightarrow$$

3 Analyticity: a Generalized Subformula Property

Roughly speaking, analyticity of a propositional calculus provides a computable bound on the formulas that may appear in derivations of a sequent s from a set S of sequents. The special case of the *subformula property* is obtained when the set of subformulas of formulas of $S \cup \{s\}$ provides such a bound. Many useful calculi, however, do not admit this strict property, while still allowing some other effective bound. Here, we generalize the subformula property, by assuming a given ordering of $\mathcal{L}$-formulas, denoted $\prec$, which has to satisfy certain properties, as defined next.

Notation 1. Given a binary relation R on $\mathcal{L}$, we denote by $R[\varphi]$ the set $\{\psi \in \mathcal{L} \mid \langle \psi, \varphi \rangle \in R\}$. This notation is naturally extended to sets ($R[\Gamma] = \bigcup_{\varphi \in \Gamma} R[\varphi]$), sequents ($R[\Gamma \Rightarrow \Delta] = R[\Gamma] \cup R[\Delta]$), and sets of sequents ($R[S] = \bigcup_{s \in S} R[s]$).

Definition 4. An order relation (i.e., irreflexive and transitive relation) $\prec$ is called:

– *safe* if it is prefinite ($\prec[\varphi]$ is finite for every $\varphi \in \mathcal{L}$), and the function $\lambda\varphi \in \mathcal{L}.\prec[\varphi]$ is computable.
– *structural* if $\varphi \prec \psi$ implies $\sigma(\varphi) \prec \sigma(\psi)$ for every substitution σ.

Example 5. The usual subformula relation over $\mathcal{CL}$, which we denote by $\prec_0$, is a structural safe order relation. Another useful structural and safe order relation on $\mathcal{CL}$, denoted $\prec_1$, is given by $\varphi \prec_1 \psi$ iff $\varphi \prec_0 \psi$ or $\varphi \neq \psi$ and $\varphi = \neg\psi'$ for some $\psi' \prec_0 \psi$.

In what follows, $\prec$ denotes an arbitrary safe and structural order relation over $\mathcal{L}$.

The above definition allows us to present a generalization of the subformula property, which we call $\prec$-analyticity.

Definition 5. We call a derivation of a sequent s from a set S of sequents in a calculus $\mathbf{G}$ $\prec$-*analytic* if it consists solely of $\preceq[S \cup \{s\}]$-formulas ($\preceq$ denotes the reflexive closure of $\prec$), and write $S \vdash^{\prec}_{\mathbf{G}} s$ if there exists a $\prec$-analytic derivation of s from S in $\mathbf{G}$. A calculus $\mathbf{G}$ is called $\prec$-*analytic* if $\vdash_{\mathbf{G}} = \vdash^{\prec}_{\mathbf{G}}$.

This generalization of the subformula property inherits its most important consequence, which is decidability. Clearly, if $\prec$ is safe and S is finite, it is decidable whether $S \vdash^{\prec}_{\mathbf{G}} s$. When $\mathbf{G}$ is $\prec$-analytic, the same holds for $\vdash_{\mathbf{G}}$.

Considering the examples above, **LK**, $\mathbf{G}_{P_1}$, **P**, and **DY** are $\prec_0$-analytic; while $\mathbf{G}_{C_1}$, $\mathbf{G}_3$ and $\mathbf{G}_4$ are not $\prec_0$-analytic, but are $\prec_1$-analytic. These facts can be derived from cut-admissibility, and also directly by the method of [20]. The infinite family of calculi for weak double negations from [17], presented in the next example, goes beyond $\prec_0$ and $\prec_1$.

Example 6. In [17], Kamide provides a way of constructing sequent calculi for paraconsistent logics that admit the double negation principle as well as its weaker forms (e.g. $\neg\neg\neg\psi \leftrightarrow \neg\psi$). For this purpose, the paper investigates a hierarchy of weak double negations, by presenting an infinite set $\left\{L2^{n+2} \mid n \in \mathbb{N}\right\}$ of pure calculi, all of which admit cut-admissibility. For example, $L4$ is the calculus $\mathbf{G}_4$ from Example 3, which is $\prec_1$-analytic. Furthermore, for every n, let $\prec_n$ be the transitive closure of the relation $\lhd_n$, defined by: $\varphi \lhd_n \psi$ iff either $\psi = \neg\varphi$, or $\psi = \varphi_1 \sharp \varphi_2$ and $\varphi = \neg^m \varphi_i$ for some φ_1, φ_2, $\sharp \in \{\wedge, \vee, \supset\}$, $0 \leq m \leq n$, and $i \in \{1, 2\}$. Each $L2^{n+2}$ is $\prec_{n+1}$-analytic. Clearly, the previous definitions of $\prec_0$ and $\prec_1$ coincide with the new ones.

4 Semantics of Pure Sequent Calculi

Cut-admissibility is traditionally proved syntactically, by some form of induction on derivations. In this case, what is actually shown is *cut-elimination*: a method to eliminate cuts from derivations. However, going back at least to [26], semantic methods have also shown to be useful to prove cut-admissibility. We follow the semantic approach, and generalize the framework of bivaluations [13] to obtain semantic counterparts of $\prec$-analytic derivations and cut-limited derivations. The latter allows us to define a semantic sufficient condition for cut-admissibility, that is essential for our result. The soundness and completeness theorems of this section follow from the general result of [19].

We start by defining *trivaluations* – functions that employ three truth values: 1, -1, and 0, that intuitively correspond to "true", "false" and "undetermined", respectively.

Definition 6. An $\mathcal{L}$-*trivaluation* is a function v from $\mathcal{L}$ to $\{-1, 0, 1\}$. We say that v *satisfies a sequent* $\Gamma \Rightarrow \Delta$, denoted $v \models \Gamma \Rightarrow \Delta$, if either $v(\varphi) < 1$ for some $\varphi \in \Gamma$ or $v(\psi) > -1$ for some $\psi \in \Delta$. We say that v *satisfies a set* S *of sequents*, denoted $v \models S$, if $v \models s$ for every $s \in S$.

In order to associate a set of trivaluations to a given calculus, the following semantic reading of derivation rules is employed:

Definition 7. A trivaluation v *respects* a rule $S\,/\,s$ if $v \models \sigma(s)$ whenever $v \models \sigma(S)$ for every substitution σ. v is called **G**-*legal* for a calculus **G** if it respects all rules of **G**.

Depending on **G**, this semantics may not be truth-functional, that is, the value of a compound formula is not always uniquely determined by the values of its immediate subformulas. For this reason trivaluations are defined over the entire language rather than only over atomic formulas.

If one is interested in all possible derivations in a pure calculus (without any restrictions on formulas that may appear in the derivation or serve as cut formulas), the third value 0 is redundant, and an equivalent semantics could be defined using only $\{1, -1\}$. For the cases of $\prec$-analytic and cut-limited derivations, some restrictions apply for when this value can and cannot be used. These restrictions are defined using the following notion of the *support* of trivaluations.

Definition 8. The *support* of a trivaluation v, denoted $\mathrm{supp}(v)$, is the set $\{\varphi \in \mathcal{L} \mid v(\varphi) \neq 0\}$. v is called:

- $\mathcal{F}$-*determined* (for $\mathcal{F} \subseteq \mathcal{L}$) if $\mathcal{F} \subseteq \mathrm{supp}(v)$; and
- *fully determined* if it is $\mathcal{L}$-determined.

The semantic reading of rules as constraints on trivaluations, together with different restrictions on the usage of 0 as a truth value, provide an equivalent semantic view of derivations:

Theorem 1 (Soundness and Completeness).

1. *$S \vdash_{\mathbf{G}} s$ iff $v \models S$ implies $v \models s$ for every fully determined* **G***-legal trivaluation v.*
2. *$S \vdash^{\preceq}_{\mathbf{G}} s$ iff $v \models S$ implies $v \models s$ for every $\preceq[S \cup \{s\}]$-determined* **G***-legal trivaluation v.*

3. $S \vdash^{\text{cf}}_{\mathbf{G}} s$ *iff* $v \models S$ *implies* $v \models s$ *for every* $frm[S]$*-determined* **G***-legal trivaluation* v.

Roughly speaking, in the case of $\prec$-analytic derivations, the values -1 and 1 are associated with the formulas that are allowed to be used in derivations. Thus, when semantically describing the existence of a $\prec$-analytic derivation of a sequent s from a set S of sequents in a calculus **G**, all formulas that are allowed to appear in such a derivation must be assigned either 1 or -1. These are exactly the formulas in $\preceq [S \cup \{s\}]$. Similarly, in the case of cut-limited derivations, cut formulas must be assigned either 1 or -1, and thus cut-limited derivations of s from S are tied to trivaluations in which $frm[S]$-formulas are never assigned 0. Intuitively, if φ cannot serve as a cut formula, we may need a trivaluation v that satisfies $\Rightarrow \varphi$ and $\varphi \Rightarrow$, which is possible iff $v(\varphi) = 0$. Obviously, when allowing all formulas to serve as cut formulas, or when there is no restriction on the formulas that may be used in derivations, all formulas must be assigned either 1 or -1.

Example 7 (Semantics of Classical Logic). It is easy to see that a fully determined $\mathcal{CL}$-trivaluation v is **LK**-legal iff it respects the classical truth tables. For example, the first line of the truth table for conjunction is obtained as follows: Suppose $v(p_1) = v(p_2) = 1$. Then $v \models \{\Rightarrow p_1, \Rightarrow p_2\}$, and since v is **LK**-legal, $v \models \Rightarrow p_1 \wedge p_2$, and so $v(p_1 \wedge p_2) = 1$. In addition, the three valued semantics for the cut-limited fragment of **LK** that is obtained from Theorem 1 is equivalent to the Nmatrix semantics in [18].

Example 8 (Alternative Semantics of Łukasiewicz three-valued logic). $\mathbf{G}_3$-legal fully determined trivaluations provide an alternative semantics to Łukasiewicz three-valued logic (Example 3). This semantics is two-valued (as only *fully determined* trivaluations are considered), but not truth-functional. Another two-valued semantics for this logic was presented in [27], and was then used to construct a different calculus for it in [13].

Theorem 1 gives rise to a sufficient semantic criterion for cut-admissibility, which is based on the following notion of *determination*:

Definition 9. We say that a trivaluation v' is a *determination* of a trivaluation v (alternatively, we say that v' *determines* v) if $v(\varphi) = v'(\varphi)$ for every $\varphi \in \text{supp}(v)$. v' is called an $\mathcal{F}$*-determination of* v if, in addition, it is $\mathcal{F}$-determined. If v' is fully determined we call it a *full determination* of v.

It immediately follows from our definitions that:

Proposition 1. *Suppose that* v' *determines* v*. Then for every sequent* s*, if* $v' \models s$ *then* $v \models s$*. The converse holds as well when* v *is* $frm[s]$*-determined.*

A sufficient semantic criterion for cut-admissibility is given in the following corollary:

Corollary 1. *If every* **G***-legal trivaluation has a* **G***-legal full determination, then* **G** *enjoys cut-admissibility.*

Proof. Suppose $S \nvdash^{cf}_{\mathbf{G}} s$. By Theorem 1, there exists some $frm[S]$-determined **G**-legal trivaluation v such that $v \models S$ and $v \not\models s$. Let v' be a **G**-legal full determination of v. By Proposition 1, $v' \models S$ and $v' \not\models s$, and by Theorem 1, we have $S \nvdash_{\mathbf{G}} s$. □

Remark 1. We note that [19] connects $\prec$-analytic derivations to *partial* two-valued valuations, that are defined over a subset of the language. This subset corresponds to the support of the trivaluations that are used here. For the characterization of cut-limited derivations in [19], three-valued valuations were employed. In the current paper, where the connection between $\prec$-analyticity and cut-admissibility is the main subject, we find it more natural to use a three-valued semantics both for $\prec$-analytic and cut-limited derivations.

5 From Analyticity to Cut-admissibility

For many calculi, including all calculi presented above, all rules except (CUT) are "$\prec$-ordered": in every application of the rule, every formula φ that appears in the premises satisfies $\varphi \preceq \psi$ for some formula ψ that appears in the conclusion. For such calculi, cut-admissibility immediately entails $\prec$-analyticity, as every cut-limited derivation is $\prec$-analytic. Whether or not the converse holds is the subject of this section.

First, note that (even for "$\prec$-ordered" calculi), $\prec$-analyticity may not imply cut-admissibility:

Example 9. Consider the calculus $\mathbf{LK}_{AX}$, that consists of the following axiomatic rules:

$\emptyset / p_1, p_2 \Rightarrow p_1 \wedge p_2$	$\emptyset / p_1 \wedge p_2 \Rightarrow p_1$	$\emptyset / p_1 \wedge p_2 \Rightarrow p_2$
$\emptyset / p_1 \vee p_2 \Rightarrow p_1, p_2$	$\emptyset / p_1 \Rightarrow p_1 \vee p_2$	$\emptyset / p_2 \Rightarrow p_1 \vee p_2$
$\emptyset / p_2 \Rightarrow p_1 \supset p_2$	$\emptyset / \Rightarrow p_1, p_1 \supset p_2$	$\emptyset / p_1, p_1 \supset p_2 \Rightarrow p_2$
$\emptyset / \Rightarrow p_1, \neg p_1$	$\emptyset / p_1, \neg p_1 \Rightarrow$	

It can be easily shown that $\mathbf{LK}_{AX}$ is $\prec_0$-analytic (since **LK** is $\prec_0$-analytic). However, it does not admit cut-admissibility (for instance, the sequent $p_1 \wedge p_2 \Rightarrow p_1 \vee p_2$ has no derivation without cut).

Next, we identify a family of calculi in which analyticity does imply cut-admissibility.

Definition 10. A rule S/s is called *$\prec$-directed* if $frm[S] \subseteq \prec [s]$, and s has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$ for some formula φ. A calculus **G** is called *$\prec$-directed* if all its rules are $\prec$-directed.

The calculi **LK**, $\mathbf{G}_{P_1}$, **P**, and **DY** are $\prec_0$-directed, $\mathbf{G}_{C_1}$, $\mathbf{G}_3$ and $\mathbf{G}_4$ are $\prec_1$-directed, and for every n, $L2^{n+2}$ is $\prec_{n+1}$-directed. In contrast, $\mathbf{LK}_{AX}$ (Example 9) is not $\prec$-directed for any $\prec$, as its conclusions include several formulas.

Our first main result is that $\prec$-analyticity guarantees cut-admissibility in the family of $\prec$-directed pure calculi.

Theorem 2. *Every $\prec$-analytic $\prec$-directed pure calculus enjoys cut-admissibility.*

The proof of Theorem 2 goes through Corollary 1: given a pure calculus **G** that is $\prec$-analytic and $\prec$-directed, we show that every **G**-legal trivaluation has a **G**-legal full determination. This is done by iteratively extending the support of a given **G**-legal trivaluation v by a single formula φ that is not in $\mathrm{supp}(v)$, but $\prec[\varphi] \subseteq \mathrm{supp}(v)$. The value of φ is determined as follows:

$$v'(\varphi) = \begin{cases} 1 & \nvdash_{\mathbf{G}} \Gamma_v, \varphi \Rightarrow \Delta_v \\ -1 & \textit{otherwise} \end{cases}$$

where $\Gamma_v = \{\psi \in \prec[\varphi] \mid v(\psi) = 1\}$ and $\Delta_v = \{\psi \in \prec[\varphi] \mid v(\psi) = -1\}$. The correctness of this construction follows from the fact that **G** is $\prec$-directed and $\prec$-analytic. By enumerating the formulas while respecting $\prec$, we inductively determine all the formulas that are assigned 0 by v.

For all the calculi mentioned above (except $\mathbf{LK}_{AX}$), this theorem allows one to obtain cut-admissibility as a consequence of $\prec$-analyticity for some (structural and safe) order $\prec$.

6 Intuitionistic Calculi

For various important non-classical logics, there is no known cut-free pure calculus. In particular, Gentzen's original calculus for intuitionistic logic, LJ, is not pure, as it manipulates *single-conclusion sequents*, in which the right-hand side includes at most one formula. An equivalent cut-free sequent calculus, which we call $\mathbf{LJ}'$, was presented in [28]. This calculus employs multiple-conclusion sequents, and restricts only the right introduction rules of implication and negation to apply on single-conclusion sequents. In other words, $\mathbf{LJ}'$ is obtained from **LK** by adding the requirement that applications of $p_1 \Rightarrow p_2 \,/ \Rightarrow p_1 \supset p_2$ and $p_1 \Rightarrow \,/\, \Rightarrow \neg p_1$ have the forms:

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$$

Put differently, $\mathbf{LJ}'$ is obtained from **LK** by forbidding right context formulas in all premises of the form $\Gamma \Rightarrow \Delta$ with $\Gamma \neq \emptyset$ of right-introduction rules (rules that introduce some formula on the right-hand side).

Another well-known calculus that follows this pattern, which we call $\mathbf{G}_4'$, is obtained by extending the positive fragment of $\mathbf{LJ}'$ with the rules for negation of $\mathbf{G}_4$ (see Example 3). $\mathbf{G}_4'$, investigated in [7,29], is sound and complete for Nelson's paraconsistent constructive logic $N4$ [23].

Next, we define a general family of calculi, which we call *intuitionistic calculi*, of which $\mathbf{LJ}'$ and $\mathbf{G}_4'$ are particular examples. For them, we show that cut-admissibility is a consequence of $\prec$-analyticity.

Definition 11. A pure rule is called *positive* if its conclusion has the form $\Gamma \Rightarrow \Delta$ for some $\Delta \neq \emptyset$. A derivation in a pure calculus $\mathbf{G}$ is called *intuitionistic* if in every application $\langle \{\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n\}, \sigma(s_0) \cup c_1 \cup \ldots \cup c_n \rangle$ of a positive rule $s_1, \ldots, s_n \,/\, s_0$, for every $1 \leq i \leq n$ we have that if s_i has the form $\Gamma_i \Rightarrow \Delta_i$ with $\Gamma_i \neq \emptyset$, then c_i has the form $\Gamma_i' \Rightarrow$.

Derivability, cut-admissibility and $\prec$-analyticity are adopted to intuitionistic derivations in the obvious way:

Definition 12. For a pure calculus $\mathbf{G}$, we write $S \vdash_{\mathbf{G_{Int}}} s$ if there is an intuitionistic derivation of a sequent s from a set S of sequents in $\mathbf{G}$. We write $S \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}} s$ if there is such a derivation which is also cut-limited, and $S \vdash^{\prec}_{\mathbf{G_{Int}}} s$ if there is such a derivation which is $\prec$-analytic (see Definitions 3 and 5). We say that $\mathbf{G}$ enjoys *Int-cut-admissibility* if $\vdash_{\mathbf{G_{Int}}} = \vdash^{\mathrm{cf}}_{\mathbf{G_{Int}}}$, and is *Int-$\prec$-analytic* if $\vdash_{\mathbf{G_{Int}}} = \vdash^{\prec}_{\mathbf{G_{Int}}}$.

The difference between pure and intuitionistic calculi is not in the *rules*, but rather in *applications* that are allowed to appear in derivations. Thus, any pure calculus has an intuitionistic counterpart, obtained by considering only intuitionistic derivations. In particular, derivations in $\mathbf{LJ}'$ are exactly intuitionistic derivations of $\mathbf{LK}$. Indeed, for a finite set Γ of formulas and a formula φ, φ follows from Γ in intuitionistic logic iff $\vdash_{\mathbf{LK_{Int}}} \Gamma \Rightarrow \varphi$. In contrast, φ follows from Γ in classical logic iff $\vdash_{\mathbf{LK}} \Gamma \Rightarrow \varphi$.

Theorem 3. *Every Int-$\prec$-analytic $\prec$-directed pure calculus enjoys Int-cut-admissibility.*

The proof of Theorem 3 has a similar general structure to the proof for pure calculi, but is more challenging, because simple valuation functions do not suffice to characterize the calculi of this family. Instead, a more complex semantic interpretation is employed, which is based on Kripke models. The description of this extended semantics, as well as its role in the proof of Theorem 3, are left for an extended version of this paper.

Theorem 3 allows one to derive the fact that cut is admissible in $\mathbf{LJ}'$ from the fact that $\mathbf{LJ}'$ enjoys the subformula property. More precisely, Int-cut-admissibility of $\mathbf{LK}$ follows from its Int-$\prec_0$-analyticity. Such entailment also holds for the pure calculi presented in the examples above, as well as for the calculi of the next example.

Example 10 (Constructive Negations). The paper [7] includes sequent calculi for logics that replace classical negation with several non-classical negations. One of the families investigated there consists of calculi that are obtained from the positive fragment of $\mathbf{LJ}'$ by augmenting it with pure rules for negation. All calculi of this family, except those described in Example 11 below, allow only intuitionistic derivations, and are $\prec_1$-directed and Int-$\prec_1$-analytic. From these facts, Theorem 3 allows us to conclude that cut is admissible in them. These calculi include a calculus for Nelson's constructive logic N_3 [23], as well as the calculus $\mathbf{G}_4'$ presented above for its paraconsistent variant N_4.

Intuitionistic derivations disallow right context formulas in premises of positive rules (Definition 11), in which the left-hand side is not empty. A natural question that arises regarding Theorem 3 is: Does it still hold if we allow right context formulas for certain premises of a right introduction rule with a non-empty left-hand side, and forbid them in others? The answer is negative as the next example demonstrates.

Example 11 (Beyond Intuitionistic Derivations). Following Example 10, we note that [7,8] investigate also several calculi that include *both* the single-conclusion right-introduction rule of implication and the multiple-conclusion right-introduction rule of negation. The former conforms with the restriction to intuitionistic derivations, as right context formulas are forbidden. The latter allows for non-intuitionistic derivations, as it allows right context formulas in a premise that has a non-empty left side. Such calculi are therefore left out from the scope of Theorems 2 and 3. And indeed, as was shown in [8], all of them are $\prec_1$-analytic, but none of them enjoys cut-admissibility.

7 Conclusion

We identified two general families of propositional sequent calculi, in which a generalized subformula property is equivalent to cut-admissibility. The first is the family of pure calculi that are $\prec$-directed for some safe and structural order $\prec$. The second is the family of "$\prec$-directed intuitionistic calculi", obtained by considering intuitionistic derivations in $\prec$-directed pure calculi.

This result sheds light on the relation between these two fundamental properties. Furthermore, we believe that it may be useful in obtaining simpler cut-admissibility proofs:

1. Theorems 2 and 3 reduce the burden in proving cut-admissibility to establishing only *analytic cut-admissibility.* An application of (CUT) in a derivation of s from S is called a *$\prec$-analytic cut* if the cut formula is in $\prec[S \cup \{s\}]$. In turn, $\prec$-analytic cut-admissibility concerns only the admissibility of non-$\prec$-analytic cuts. Proving this property is often easier than showing full cut-admissibility. For example, when $\prec_0$-analytic cuts are allowed, it is straightforward to prove that **LK** is complete for the classical truth tables. Indeed, assuming $S \nvdash_{\mathbf{LK}} \Gamma \Rightarrow \Delta$, one extends $\Gamma \Rightarrow \Delta$ to a maximal underivable

sequent $\Gamma^* \Rightarrow \Delta^*$ that consists solely of $\prec_0 [S \cup \{\Gamma \Rightarrow \Delta\}]$-formulas. Then, a countermodel v can be defined simply by setting $v(\varphi) = 1$ for every $\varphi \in \Gamma^*$ and $v(\psi) = -1$ for every $\psi \in \Delta^*$. Using $\prec_0$-analytic cuts, it immediately follows that $\Gamma^* \cup \Delta^* = \prec_0 [S \cup \{\Gamma \cup \Delta\}]$, which makes it easy to prove that v respects the classical truth tables, and can therefore be extended to a full classical assignment. By Theorem 2, we may conclude that **LK** enjoys (full) cut-admissibility.

2. The results of this paper are useful in combination with our recent paper [20], where we provided a general method for proving $\prec_n$-analyticity (see Example 6 for the definition of $\prec_n$) in a wide family of pure calculi. Concretely, we showed that the $\prec_n$-analyticity of a $\prec_n$-directed calculus **G** is guaranteed if the following property holds:

   For every two rules of **G** of the forms $S_1 \,/ \Rightarrow \varphi_1$ and $S_2 \,/\, \varphi_2 \Rightarrow$, and substitutions σ_1, σ_2 such that $\sigma_1(\varphi_1) = \sigma_2(\varphi_2)$, the empty sequent is derivable from $\sigma(S_1) \cup \sigma(S_2)$ using only (CUT).

   Then, Theorem 2 ensures that these calculi are not only $\prec_n$-analytic, but they also admit cut-admissibility.

We propose two particular directions for future research. First, our approach should be further developed for more expressible languages, which include quantifiers and modalities. For the former, the three-valued semantics should be elevated to three-valued first-order structures. For the latter, we prospect that the Kripke semantics used here for intuitionistic calculi could be adapted for calculi with modalities. We note, however, that such an approach is expected to have certain limitations, as some analytic calculi for modal logics (e.g., $S5$ and B [30,25]) do not admit cut-admissibility.

Second, the following questions regarding the relations between derivations and intuitionistic derivations are currently left open: Does $\prec$-analyticity imply Int-$\prec$-analyticity? Does cut-admissibility imply Int-cut-admissibility? Do either of the converses hold?

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References

1. A. R. Anderson and N. D. Belnap. *Entailment: The Logic of Relevance and Necessity, Vol.I.* Princeton University Press, 1975.
2. Ofer Arieli and Arnon Avron. The value of the four values. *Artificial Intelligence*, 102(1):97–141, 1998.
3. Ofer Arieli and Arnon Avron. Three-valued paraconsistent propositional logics. In Jean-Yves Beziau, Mihir Chakraborty, and Soma Dutta, editors, *New Directions in Paraconsistent Logic: 5th WCP, Kolkata, India, February 2014*, pages 91–129. Springer India, New Delhi, 2015.

4. Arnon Avron. Simple consequence relations. *Information and Computation*, 92(1):105–139, may 1991.
5. Arnon Avron. Gentzen-type systems, resolution and tableaux. *Journal of Automated Reasoning*, 10(2):265–281, 1993.
6. Arnon Avron. Classical Gentzen-type methods in propositional many-valued logics. In Melvin Fitting and Ewa Orłowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 117–155. Physica-Verlag HD, 2003.
7. Arnon Avron. A non-deterministic view on non-classical negations. *Studia Logica: An International Journal for Symbolic Logic*, 80(2/3):159–194, 2005.
8. Arnon Avron. Non-deterministic semantics for families of paraconsistent logics. *Handbook of Paraconsistency*, 9:285–320, 2007.
9. Arnon Avron, Beata Konikowska, and Anna Zamansky. Modular construction of cut-free sequent calculi for paraconsistent logics. In *Proceedings of the 27th Annual IEEE/ACM Symposium on Logic in Computer Science*, LICS '12, pages 85–94. IEEE Computer Society, 2012.
10. Arnon. Avron and Iddo Lev. Non-deterministic multi-valued structures. *Journal of Logic and Computation*, 15:241–261, 2005. Conference version: A. Avron and I. Lev. Canonical Propositional Gentzen-Type Systems. In *International Joint Conference on Automated Reasoning, IJCAR 2001. Proceedings, LNAI* 2083, 529–544. Springer, 2001.
11. Arnon Avron and Anna Zamansky. Non-deterministic semantics for logical systems. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 16, pages 227–304. Springer Netherlands, 2011.
12. Lev Beklemishev and Yuri Gurevich. Propositional primal logic with disjunction. *Journal of Logic and Computation*, 24(1):257–282, 2014.
13. Jean-Yves Béziau. Sequents and bivaluations. *Logique et Analyse*, 44(176):373–394, 2001.
14. Hubert Comon-Lundh and Vitaly Shmatikov. Intruder deductions, constraint solving and insecurity decision in presence of exclusive or. In *Logic in Computer Science, 2003. Proceedings. 18th Annual IEEE Symposium on*, pages 271–280, June 2003.
15. Carlos Cotrini and Yuri Gurevich. Basic primal infon logic. *Journal of Logic and Computation*, 26(1):117–141, 2016.
16. Gerhard Gentzen. Investigations into logical deduction, 1934. In German. An English translation appears in 'The Collected Works of Gerhard Gentzen', edited by M. E. Szabo, North-Holland, 1969.
17. Norihiro Kamide. A hierarchy of weak double negations. *Studia Logica*, 101(6):1277–1297, 2013.
18. Ori Lahav. Studying sequent systems *via* non-deterministic multiple-valued matrices. *Multiple-Valued Logic and Soft Computing*, 21(5-6):575–595, 2013.
19. Ori Lahav and Arnon Avron. A unified semantic framework for fully structural propositional sequent systems. *ACM Transactions on Computational Logic*, 14(4):271–273, November 2013.
20. Ori Lahav and Yoni Zohar. On the construction of analytic sequent calculi for sub-classical logics. In U. Kohlenbach, P. Barcel, and R. de Queiroz, editors, *Logic, Language, Information, and Computation*, volume 8652 of *Lecture Notes in Computer Science*, pages 206–220. Springer Berlin Heidelberg, 2014.
21. Ori Lahav and Yoni Zohar. SAT-based decision procedure for analytic pure sequent calculi. In S. Demri, D. Kapur, and C. Weidenbach, editors, *Automated Reason-*

ing, volume 8562 of *Lecture Notes in Computer Science*, pages 76–90. Springer International Publishing, 2014.
22. George Metcalfe, Nicola Olivetti, and Dov M. Gabbay. *Proof theory for fuzzy logics*, volume 36. Springer Science & Business Media, 2008.
23. David Nelson. Constructible falsity. *Journal of Symbolic Logic*, 14(1):16–26, 005 1949.
24. Luís Pinto and Tarmo Uustalu. Proof search and counter-model construction for bi-intuitionistic propositional logic with labelled sequents. In Martin Giese and Arild Waaler, editors, *Automated Reasoning with Analytic Tableaux and Related Methods: 18th International Conference, TABLEAUX 2009, Oslo, Norway, July 6-10, 2009. Proceedings*, pages 295–309. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.
25. Francesca Poggiolesi. *Gentzen calculi for modal propositional logic*, volume 32 of *Trends in Logic*. Springer Science & Business Media, 2010.
26. Kurt Schütte. *Beweistheorie*. Springer-Verlag, Berlin, 1960.
27. Roman Suszko. Remarks on Łukasiewicz's three-valued logic. *Bulletin of the Section of Logic*, 4(3):87–90, 1975.
28. Gaisi Takeuti. *Proof Theory*. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1975.
29. Heinrich Wansing. *The Logic of Information Structures*, volume 681 of *Lecture Notes in Computer Science*. Springer, 1993.
30. Heinrich Wansing. Sequent systems for modal logics. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic, 2nd edition*, volume 8, pages 61–145. Springer, 2002.