SAT-based Decision Procedure for Analytic Sequent Calculi

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October 2013

Sequent Calculi

- Sequent calculi are a prominent proof-theoretic framework.
- They provide an "algorithmic presentation" of a logic.
- Suitable for a variety of logics:
  - Classical logic, intuitionistic logic
  - Modal logics, intermediate logics, bi-intuitionistic logic
  - Many-valued logics, fuzzy logics
  - Paraconsistent logics
  - Substructural logics, relevance logics
- Our goal: effectively reduce the derivability problem in a given propositional sequent calculus to SAT.

Sequents

- We take sequents to be objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas.
- Intuition:

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \quad \leftrightsquigarrow \quad A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$$

The calculus **LK** [Gentzen 1934]

Structural Rules:

$$(id)\quad \frac{}{\Gamma, A \Rightarrow A, \Delta} \qquad (cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

$$(W \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Logical Rules:

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee \Rightarrow)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\Rightarrow \vee)\quad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Example: Sequent Calculus for Propositional Primal Logic

- All usual structural rules.

Logical Rules:

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset) \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

- This multiple-conclusion calculus can be easily shown to be equivalent to the sequent-style natural deduction system in [Beklemishev,Gurevich '12].

Pure Sequent Calculi

- *Pure sequent calculi* are propositional sequent calculi that include all usual structural rules, and a finite set of pure logical rules.
- *Pure logical rules* are logical rules that allow any context [Avron '91].

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \text{but not} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Example: da Costa's Paraconsistent Logic C_1

[Avron, Konikowska, Zamansky '12]

A pure calculus for C_1 is obtained by augmenting the "positive" fragment of **LK** with the following rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta}$$

Analyticity

Definition

A calculus is *analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only subformulas of $\Gamma \cup \Delta$.

- A weaker useful notion allows to use the negations of the subformulas of $\Gamma \cup \Delta$ as well.
- If a (propositional) pure calculus is analytic then it is decidable.
- Analytic pure calculi exist for important propositional logics:
  - Propositional classical logic, propositional primal logic
  - Three and four valued logics
  - Paraconsistent logics

There is a *simple* reduction of derivability in analytic pure calculi to SAT.

Semantics for Pure Calculi

- Pure calculi can be characterized by *two-valued valuations* [Béziau '01].
- Each pure rule is translated into a semantic condition.
- By joining the semantic conditions of all rules in a calculus **G**, we obtain the set of **G**-*legal* valuations.

Soundness and Completeness

$\Gamma \Rightarrow \Delta$ is provable in **G** iff every **G**-legal valuation is a model of $\Gamma \Rightarrow \Delta$.

A valuation is a model of $\Gamma \Rightarrow \Delta$ if at least one of the following hold:

- $v(A) = \mathrm{F}$ for some $A \in \Gamma$.
- $v(A) = \mathrm{T}$ for some $A \in \Delta$.

Semantics for Pure Calculi

Example (Sequent Calculus for C_1)

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

Corresponding semantic conditions:

1. If $v(A) = \mathrm{F}$ then $v(\neg A) = \mathrm{T}$
2. If $v(A) = \mathrm{F}$ then $v(\neg\neg A) = \mathrm{F}$
3. If $v(A) = \mathrm{T}$ and $v(\neg A) = \mathrm{T}$ then $v(\neg(A \wedge \neg A)) = \mathrm{F}$
4. If $v(\neg A) = \mathrm{F}$ and $v(\neg B) = \mathrm{F}$ then $v(\neg(A \wedge B))) = \mathrm{F}$

This semantics is non-deterministic.

Reduction to SAT

- The semantic conditions are expressible in propositional classical logic.

Reduction to SAT

- Associate a variable x_A with every subformula A of $\Gamma \Rightarrow \Delta$.
- Generate a set of clauses for each semantic condition of **G** applied on the subformulas of $\Gamma \Rightarrow \Delta$.
- Generate singleton clauses x_A for every $A \in \Gamma$ and $\overline{x_A}$ for every $A \in \Delta$.

$\Gamma \Rightarrow \Delta$ is provable in **G** iff UNSAT.

The Case of Propositional Primal Logic

Example (Semantics)

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\Rightarrow \vee)\quad \frac{\Gamma, \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow\supset)\quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

Semantic Reading:

1. If $v(A) = \mathrm{F}$ or $v(B) = \mathrm{F}$ then $v(A \wedge B) = \mathrm{F}$
2. If $v(A) = \mathrm{T}$ and $v(B) = \mathrm{T}$ then $v(A \wedge B) = \mathrm{T}$
3. If $v(A) = \mathrm{T}$ or $v(B) = \mathrm{T}$ then $v(A \vee B) = \mathrm{T}$
4. If $v(A) = \mathrm{T}$ and $v(B) = \mathrm{F}$ then $v(A \supset B) = \mathrm{F}$
5. If $v(B) = \mathrm{T}$ then $v(A \supset B) = \mathrm{T}$

The Case of Propositional Primal Logic

Example (Reduction to SAT)

1. If $v(A) = \mathrm{F}$ or $v(B) = \mathrm{F}$ then $v(A \wedge B) = \mathrm{F}$
2. If $v(A) = \mathrm{T}$ and $v(B) = \mathrm{T}$ then $v(A \wedge B) = \mathrm{T}$
3. If $v(A) = \mathrm{T}$ and $v(B) = \mathrm{F}$ then $v(A \supset B) = \mathrm{F}$
4. If $v(B) = \mathrm{T}$ then $v(A \supset B) = \mathrm{T}$

$\Gamma \Rightarrow \Delta$ is provable iff the following set of clauses is UNSAT:

- Three clauses for every formula $A \wedge B$ occurring in $\Gamma \Rightarrow \Delta$:
$$x_A \vee \overline{x_{A \wedge B}} \qquad x_B \vee \overline{x_{A \wedge B}} \qquad \overline{x_A} \vee \overline{x_B} \vee x_{A \wedge B}$$
- Two clauses for every formula $A \supset B$ occurring in $\Gamma \Rightarrow \Delta$:
$$\overline{x_A} \vee x_B \vee \overline{x_{A \supset B}} \qquad \overline{x_B} \vee x_{A \supset B}$$
- Singleton clauses x_A for every $A \in \Gamma$ and $\overline{x_A}$ for every $A \in \Delta$.

- In this particular case, we obtain essentially the same reduction that appears in [Beklemishev,Gurevich '12].

Semantic Analyticity

Theorem

If **G** *is analytic then every* **G***-legal partial valuation (whose domain is closed under subformulas) can be extended to a full* **G***-legal valuation.*

This property is essential for the correctness of the reduction.

- The other direction works as well.
- This provides a semantic method to prove analyticity.

Reminder:

A calculus is *analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only subformulas of $\Gamma \cup \Delta$.

Complexity of the Reduction

- Suppose that the rules in an (analytic) calculus **G** have the following natural structure:
  - Every rule contains a main formula.
  - All other formulas are subformulas of the main formula.

- Then the reduction above (for **G**) requires only linear time.
  - Use the formula parse tree [Bjorner et al., 2012], [Cotrini, Gurevich, 2013].

Reduction to HORN-SAT

For propositional primal logic the reduction produces only Horn clauses.

- This logic can be decided in linear time using a HORN-SAT solver [Beklemishev,Gurevich '12].
- A different linear-time algorithm appeared in [Gurevich,Neeman '09].

Horn Pure Calculi

In general, Horn clauses suffice if the following holds in each logical rule r:

$$\#_L(r) + \#_R(r) \leq 1$$

where

- $\#_L(r)$ is the number of premises of r whose left side is not empty.
- $\#_R(r)$ is the number of formulas on the right side of r's conclusion.

Corollary

Every analytic Horn pure calculus can be decided in linear time.

Quotations

- DKAL employs *quotations*, e.g.

$$p \texttt{ said } A \qquad q \texttt{ said } p \texttt{ said } A \supset B$$

- These are unary modalities: $\Box_1, \Box_2, \ldots$

New Logical Rules:

$$(\wedge \Rightarrow)\ \frac{\Gamma, \vec{\Box} A, \vec{\Box} B \Rightarrow \Delta}{\Gamma, \vec{\Box}(A \wedge B) \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \vec{\Box} A, \Delta \quad \Gamma \Rightarrow \vec{\Box} B, \Delta}{\Gamma \Rightarrow \vec{\Box}(A \wedge B), \Delta}$$

$$(\Rightarrow \vee)\ \frac{\Gamma, \Rightarrow \vec{\Box} A, \vec{\Box} B, \Delta}{\Gamma \Rightarrow \vec{\Box}(A \vee B), \Delta}$$

$$(\supset \Rightarrow)\ \frac{\Gamma \Rightarrow \vec{\Box} A, \Delta \quad \Gamma, \vec{\Box} B \Rightarrow \Delta}{\Gamma, \vec{\Box}(A \supset B) \Rightarrow \Delta} \qquad (\Rightarrow \supset)\ \frac{\Gamma \Rightarrow \vec{\Box} B, \Delta}{\Gamma \Rightarrow \vec{\Box}(A \supset B), \Delta}$$

- This calculus can be easily shown to be equivalent to the Hilbert system in [Cotrini,Gurevich '13].

Alternative Calculus for Primal Logic with Quotations

Alternatively, it is possible to augment the propositional calculus with one additional rule:

$$(\mathbf{KD}!) \qquad \frac{\Gamma \Rightarrow \Delta}{\vec{\Box}\Gamma \Rightarrow \vec{\Box}\Delta}$$

A similar rule can be used for:

- $\Box$ and $\Diamond$ in the modal logic of functional Kripke models.
- *Next* in **LTL** [Kaway '87].

Proposition

For every pure calculus, adding (**KD**!) is equivalent to prefixing the non-context formulas in each rule with $\vec{\Box}$.

Pure Calculi with Quotations

Definition

A *pure calculus with quotations* is a propositional pure calculus augmented with the rule (**KD**!).

Theorem

The addition of (**KD**!) *preserves analyticity.*

- In particular, the (**KD**!) calculus for primal logic with quotations is analytic.
- The first calculus is *"locally analytic"*.

Definition (Local Formulas)

- A is local to itself.
- For every $1 \leq i \leq n$: $\vec{\Box} A_i$ is local to $\vec{\Box}(\diamond(A_1, \ldots, A_n))$.
- If A is local to B and B is local to C, then A is local to C.

Semantics for Pure Calculi with Quotations

- Pure calculi with quotations are characterized by two-valued functional Kripke models.

Definition (Functional Kripke Model)

A functional Kripke model is a triple $\langle W, \mathcal{R}, \mathcal{V} \rangle$:

- W is a set of possible worlds.
- $\mathcal{R}$ assigns a function $R_{\Box} : W \to W$ to every quotation $\Box$.
- $\mathcal{V}$ assigns a valuation $v_w : Frm_{\mathcal{L}} \to \{\mathrm{F}, \mathrm{T}\}$ to every world $w \in W$.
- For every $w \in W$, quotation $\Box$ and formula A: $v_w(\Box A) = v_{R_{\Box}(w)}(A)$.

- In **G**-legal Kripke models the semantic conditions of **G** are imposed on each function v_w.

Soundness and Completeness

$\Gamma \Rightarrow \Delta$ is provable in **G** iff every **G**-legal Kripke model is a model of $\Gamma \Rightarrow \Delta$.

Semantics for Pure Calculi with Quotations

Example (Sequent Calculus for C_1 + Quotations)

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\vec{\Box}\Gamma \Rightarrow \vec{\Box}\Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

For every $w \in W$, quotation $\Box$, and formulas A, B:

1. If $v_w(A) = \mathrm{F}$ then $v_w(\neg A) = \mathrm{T}$
2. If $v_w(A) = \mathrm{F}$ then $v_w(\neg\neg A) = \mathrm{F}$
3. If $v_w(A) = \mathrm{T}$ and $v_w(\neg A) = \mathrm{T}$ then $v_w(\neg(A \wedge \neg A)) = \mathrm{F}$
4. If $v_w(\neg A) = \mathrm{F}$ and $v_w(\neg B) = \mathrm{F}$ then $v_w(\neg(A \wedge B))) = \mathrm{F}$
5. $v_w(\Box A) = v_{R_\Box(w)}(A)$

Reduction to SAT

The reduction for pure calculi can be modified for calculi with quotations:

- Associate a variable $x_{\vec{\Box}A}$ with every formula $\vec{\Box}A$ that is local to $\Gamma \Rightarrow \Delta$.
- Generate a set of clauses for each semantic condition of **G** applied on the local formulas of $\Gamma \Rightarrow \Delta$.

- The reduction can still be done in linear time.
- Correctness is proved by showing that a Kripke counter-model can be constructed from a satisfying assignment (using the fact that the underlying propositional calculus is analytic).

Corollary

1. *Analytic pure calculi with quotations can be decided using a SAT solver.*
2. *Analytic Horn pure calculi with quotations can be decided in linear time using a HORN-SAT solver.*

Primal Logic with Quotations

Example (Reduction to SAT)

- Three clauses for every formula $\vec{\Box}(A \wedge B)$ local to $\Gamma \Rightarrow \Delta$:

$$x_{\vec{\Box}A} \vee \overline{x_{\vec{\Box}(A \wedge B)}} \qquad x_{\vec{\Box}B} \vee \overline{x_{\vec{\Box}(A \wedge B)}} \qquad \overline{x_{\vec{\Box}A}} \vee \overline{x_{\vec{\Box}B}} \vee x_{\vec{\Box}(A \wedge B)}$$

- Two clauses for every formula $\vec{\Box}(A \supset B)$ local to $\Gamma \Rightarrow \Delta$:

$$\overline{x_{\vec{\Box}A}} \vee x_{\vec{\Box}B} \vee \overline{x_{\vec{\Box}(A \supset B)}} \qquad \overline{x_{\vec{\Box}B}} \vee x_{\vec{\Box}(A \supset B)}$$

- Singleton clauses x_A for every $A \in \Gamma$ and $\overline{x_A}$ for every $A \in \Delta$.

- In the particular case of propositional primal logic with quotations, this reduction to HORN-SAT is practically equivalent to the reduction to Datalog from [Blass, Gurevich, 2010] and [Bjorner et al., 2012].

Extensions of Primal Logic

- It is possible to extend the calculus for primal logic (with quotations) with additional axiom schemes, e.g.:
  - $\Rightarrow A \supset A$
  - $\Rightarrow B \supset (A \supset B)$
  - $\Rightarrow (A \wedge B) \supset A$
  - $\Rightarrow (A \wedge B) \supset B$
  - $A \vee A \Rightarrow A$
  - $A \vee (A \wedge B) \Rightarrow A$
  - $(A \wedge B) \vee A \Rightarrow A$
  - $A \vee B \Rightarrow B \vee A$
- This will bring us a bit closer to classical logic (still in linear time).
- Analyticity has to be verified for each extension.

Theorem

- *If $A \Rightarrow B$ is provable in primal logic then the addition of the axiom scheme $\Rightarrow A \supset B$ to primal logic preserves analyticity.*
- *If $A \Rightarrow C$ and $B \Rightarrow C$ are both provable in primal logic (where C is a subformula of A or B) then the addition of the axiom scheme $A \vee B \Rightarrow C$ to primal logic preserves analyticity.*

Extensions of Primal Logic with $\bot$

- It is possible to extend primal logic (with quotations) with a bottom connective:

$$\frac{}{\bot \Rightarrow}$$

- Simple interactions between $\bot$, $\supset$ and $\vee$ can be recovered, using the axiom schemes:

$$\frac{}{\Rightarrow \bot \supset A} \qquad \frac{}{\bot \vee A \Rightarrow A} \qquad \frac{}{A \vee \bot \Rightarrow A}$$

- These extensions still allow the above linear time decision procedure.

Further Work

- Allow weaker notions of analyticity, as needed in many calculi for paraconsistent logics.
- Are there other useful logics that can be reduced to polynomial SAT fragments?
- Allow variables as in "Primal infon logic with variables".
- Study generalized connectives as $\bigwedge_{A \in \mathcal{A}} A$ and $\bigvee_{A \in \mathcal{A}} A$.

Thank you!