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Semantic Investigation of Proof Systems for Non-classical Logics

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by

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Abstract

Gentzen-type sequent calculi and their natural extensions (such as many-sided sequent and hypersequent calculi) provide suitable proof-theoretic frameworks for a huge variety of logics, starting from classical logic and intuitionistic logic, and including modal logics, substructural logics, many-valued logics, fuzzy logics, and paraconsistent logics. In many important cases they suggest an "algorithmic presentation" of a logic, which is particularly useful in practical applications of it, as well as for studying its properties. Thus in the last decades Gentzen-type calculi frequently arise for handling and introducing new non-classical logics. Each such calculus requires a soundness and completeness theorem with respect to its corresponding logic, and its proof-theoretic properties should be verified. Traditionally, this is done each time from scratch. In many cases the fundamental theorem of *cut-elimination* is proved. This implies the redundancy of the well-known cut rule, something which usually ensures the usefulness of the calculus. Another desirable property of Gentzen calculi is *analyticity*, namely the fact that proofs may consist only of syntactic material contained in the sequent to be proved. Often it is an immediate corollary of cut-elimination, but in various cases cut-elimination fails, and the calculus can still be shown to be analytic. This calls for an investigation of Gentzen-type calculi as mathematical objects in their own right.

This thesis aims at such a systematic general investigation of a wide variety of sequent and hypersequent calculi for many logics of different natures. Our main contribution is a *semantic* analysis of several general families of propositional Gentzen-type sequent and hypersequent calculi, that consists of the following:

1. We provide a uniform (possibly non-deterministic) semantic characterization for each calculus in the families we study. This has the form of general and modular soundness and completeness results that establish strong connections between the syntactic ingredients of a given Gentzen calculus and semantic restrictions on the corresponding set of models. The semantics provides a complementary view on Gentzen calculi, and, as we show, for certain general families of calculi it is also *effective*, naturally inducing a semantic decision procedure for the corresponding calculi.

2. We apply this semantic presentation (refining and extending it, when needed) for investigating crucial proof-theoretic properties of the calculi we study. This includes general notions of cut-admissibility, analyticity, and axiom-expansion. Indeed, an

illuminating contribution of a semantic study of proof systems is the ability to provide semantic proofs (or refutations) of syntactic properties. Even when a traditional syntactic proof exists, in many cases the semantic proofs are much simpler and easier to check. Thus we characterize these properties from a semantic point of view, providing general tools that can be applied in semantic proofs of these properties. In some of the families we study, this naturally leads to simple and decidable exact criteria for important proof-theoretic properties.

In addition, to demonstrate the applicability of our ideas and methods beyond the propositional level, we consider two particular hypersequent calculi for first-order and second-order Gödel logic. By extending the semantic analysis of propositional hypersequent calculi, we prove that these two calculi are indeed sound and complete for first-order and second-order Gödel logic (respectively), and that they enjoy cut-admissibility. In the case of the calculus for first-order Gödel logic this provides a semantic alternative account for a known result (proven syntactically in other works). In contrast, to the best of our knowledge cut-elimination was not proved before for the calculus for second-order Gödel logic.

Contents

Chapter 1

Introduction

Ever since the introduction of sequent calculi for classical and intuitionistic logic by Gentzen [56], sequent calculi have been widely applied in the fields of proof theory, mathematical logic, and automated deduction. These systems and their natural extensions (such as many-sided sequent and hypersequent calculi) provide suitable proof-theoretic frameworks for a huge variety of non-classical logics, including modal logics [96], substructural logics [55], many-valued logics [61], fuzzy logics [76], and paraconsistent logics [13]. In many important cases they suggest an "algorithmic presentation" of a logic, which is particularly useful in practical applications of it, as well as for studying its properties, such as decidability (for propositional logics), consistency, interpolation, the Herbrand theorem (for first-order logics) and others. Thus in the last decades Gentzen-type calculi frequently arise for handling and introducing new non-classical logics. Each such calculus requires a soundness and completeness theorem with respect to its corresponding logic, and its proof-theoretic properties should be verified. Traditionally, this is done each time from scratch. In many cases the fundamental theorem of *cut-elimination* is proved. This implies the redundancy of the well-known cut rule, something which usually ensures the usefulness of the calculus. Another desirable property of Gentzen calculi is *analyticity*, namely the fact that proofs may consist only of syntactic material contained in the sequent to be proved. Often it is an immediate corollary of cut-elimination, but in various cases cut-elimination fails, and the calculus can still be shown to be analytic.

This thesis aims at a systematic investigation of Gentzen-type systems as mathematical objects in their own right. We study a wide variety of sequent and hypersequent calculi for many logics of different natures. Our main contribution is a *semantic* analysis of several general families of propositional Gentzen-type sequent and hypersequent calculi, that, generally speaking, consists of the following:

1. We provide a uniform and general semantic characterization for each system in the

families we study. Thus each calculus **G** corresponds to a certain set of semantic structures $\mathcal{V}_{\mathbf{G}}$; and the consequence relation induced by $\mathcal{V}_{\mathbf{G}}$ (using an appropriate notion of when a semantic structure in $\mathcal{V}_{\mathbf{G}}$ is a *model* of a given sequent or hypersequent) is shown to be identical to $\vdash_{\mathbf{G}}$, the provability relation of **G**. For each family of calculi, we present a general uniform method for extracting the set $\mathcal{V}_{\mathbf{G}}$ for a given system **G** in this family. In many important cases the soundness and completeness of some known Gentzen-type system with respect to its usual semantics is then obtained as a particular instance of the proposed general method. The semantics provides a complementary view on Gentzen systems. In addition, we identify certain families of systems for which the obtained semantics is also *effective*, i.e. it naturally induces a semantic decision procedure for the calculi in that family. Thus we derive new general decidability results for large families of propositional Gentzen-type systems.

2. We apply this semantic presentation of calculi (and extend and refine it, when needed) for investigating crucial proof-theoretic properties of the systems we study. This includes general notions of *cut-admissibility*, *analyticity*, and *axiom-expansion*. Indeed, an illuminating contribution of a semantic study of proof systems is the ability to provide semantic proofs (or refutations) of syntactic properties. Even when a traditional syntactic proof exists, in many cases the semantic proofs are much simpler and easier to check. Thus we characterize these properties from a semantic point of view, providing general tools that can be applied in semantic proofs of these properties. In some of the families we study, these characterizations naturally lead to simple and decidable exact criteria for the aforementioned proof-theoretic properties.

Our investigation is carried out in the following five families of propositional *fully-structural* Gentzen-type systems (i.e., systems that include all the usual structural rules: exchange, contraction, and weakening):

Pure Sequent Calculi. These are sequent calculi, whose derivation rules do not enforce any limitation on the context formulas. In addition to usual two-sided sequent calculi, we include here also calculi that employ one-sided sequents or many-sided ones. This family of calculi provides a suitable proof-theoretic framework for several important propositional logics, including classical logic, many well-studied many-valued logics, and various paraconsistent logics. In the definition of this family, we do not assume any predefined set of cut rules or identity axioms, and thus handle any possible combination of these rules.

Canonical Calculi. This is a subfamily of pure sequent calculi, in which each logical rule introduces exactly one logical connective, where all formulas in the premises of a rule are immediate subformulas of the formula introduced in its conclusion. Such "well-behaved" logical rules (called: *canonical rules*), have a philosophical motivation: they naturally serve a guiding principle in the philosophy of logic, due to Gentzen [56], according to which the meanings of the connectives are determined by their derivation rules. Like in the more general case of pure sequent calculi, we again include here many-sided sequent systems with arbitrary combinations of cut rules and identity axioms. Since this family of calculi is a subfamily of pure sequent calculi, all results concerning the semantics of pure sequent calculi and the semantic characterizations of their proof-theoretic properties can be applied for canonical calculi as well. However, we show that for this more restricted family of calculi we are always able to obtain *simple and effective* semantics, as well as *decidable* characterizations of their proof-theoretic properties.

Quasi-canonical Calculi. This is another subfamily of pure sequent calculi, that extends the family of canonical calculi. Here we allow also logical rules in which unary connectives precede the connective to be introduced in conclusions of logical rules (allowing, e.g., the introduction of a formula of the form $\neg(\varphi_1 \wedge \varphi_2)$), as well as the formulas in the premises. Calculi of this family are particularly useful for many-valued logics (e.g. for the relevance logic of first degree entailment [1]) and paraconsistent logics (see, e.g., [13]). Our investigation of these calculi is not direct: instead of studying the semantics of quasi-canonical calculi, we show how to translate each quasi-canonical calculus to a canonical equivalent one, and then exploit the results concerning canonical calculi.

Basic Calculi. These are multiple-conclusion two-sided sequent calculi whose derivation rules may allow certain restrictions and manipulations on the context formulas (and for that reason they are not pure sequent calculi). Various sequent calculi that seem to have completely different natures belong to this family. Thus it includes all standard sequent calculi for modal logics, as well as the usual multiple-conclusion systems for intuitionistic logic, its dual, and bi-intuitionistic logic.

Canonical Hypersequent Calculi. We import the ideas behind canonical sequent calculi to hypersequent calculi, and define a general structure of a canonical hypersequential logical rule. Here there are many options concerning the additional hypersequential structural rules. To demonstrate our methods, we choose to study single-conclusion canonical hypersequent calculi that are based on the *communication rule*. The prototype example here is the hypersequent calculus for propositional

Gödel logic (see [22]), and thus, we call the calculi of this family *canonical Gödel hypersequent calculi*. In particular, it is possible to introduce in these calculi new non-deterministic connectives and add them to Gödel logic. Note that hypersequent calculi are now the main proof-theoretic framework for fuzzy logics [76], but Gödel logic is the only fundamental fuzzy logic that has a fully-structural hypersequent calculus (and thus falls in the scope of this work).

While the aforementioned families of calculi are all propositional, the ideas and methods for them are applicable for first-order and higher-order calculi as well. We demonstrate this in two specific calculi: the hypersequent calculus **HIF** for standard first-order Gödel logic [30], and its extension, that we call $\mathbf{HIF^2}$, for Henkin-style second-order Gödel logic. In particular, by extending the semantic methods developed for the family of (propositional) canonical Gödel hypersequent calculi, we are able to prove that the cut rule is admissible in **HIF** and $\mathbf{HIF^2}$. In the case of **HIF** this provides a semantic alternative account for the fact that **HIF** admits cut-elimination (proven syntactically in [30, 22]). In contrast, to the best of our knowledge cut-elimination was not proved for $\mathbf{HIF^2}$ before.

A crucial feature of a systematic procedure relating proof systems and semantics should be its *modularity* – the correspondence between semantics and proof systems should be based on local equivalences between semantic ingredients (requirements from the semantic structures) and their syntactic counterparts (derivation rules). Such a correspondence can allow, e.g., to predict the semantic impact of employing the same rule in different proof systems, or to provide an appropriate rule for a given semantic condition added to different logics. In particular, all semantic characterizations of cut-admissibility in each of the families of calculi listed above are based on identifying the semantic impact of the cut rule(s), and comparing the semantics of the calculi with and without the cut rule(s). These tasks are of course impossible when the proof system and its semantics are considered as a whole, and there is no possibility to separate between the different semantic effects of each particular rule. The major key to have this modularity, as well to provide semantics to *every* calculus in the families that we study, is the use of *non-deterministic semantics*. Thus, following [17, 21], we relax the principle of truth-functionality, and allow cases in which the truth value of a compound formula is not uniquely determined by the truth values of its subformulas. By allowing non truth-functional semantic structures, we are able to separately analyze the semantic effect of each component of the syntactic machinery (each derivation rule, and in fact also each ingredient of a rule). The full semantics of the calculus is then obtained by joining the semantic effects of all of its components. For this matter, we develop several frameworks of non-deterministic semantics:

Many-Valued Systems. These provide a semantic framework for specifying sets of valuations – functions assigning truth values to formulas of a given propositional language. Each many-valued system includes a set of semantic conditions, that can be easily read off the derivation rules of a pure sequent calculi, and used to restrict its corresponding set of valuations (e.g. "If φ_1 has some truth value u_1, and $\neg\varphi_1$ has some truth value u_2, then $\neg(\varphi_1 \wedge \varphi_1)$ should have the truth value u_3"). This framework generalizes the "bivaluation semantics" [34, 40], many-valued matrices [93, 61], and non-deterministic many-valued matrices [17, 21], and is used here to provide semantics for pure sequent calculi.

Partial Non-deterministic Matrices. These form a special case of many-valued systems that serve as a simpler semantic framework for canonical and quasi-canonical calculi. Thus in partial non-deterministic matrices, the semantic conditions for specifying restrictions on valuation functions can be arranged in generalized *truth tables*. Usual logical matrices are particular instances, while non-determinism is introduced as done in *non-deterministic matrices* (see [17, 21]), by possibly allowing several options in some entries of the truth tables (thus the value of $\diamond(p_1, \dots, p_n)$ is restricted, but not uniquely determined, by the values of $p_1, \dots, p_n$). However, to handle arbitrary canonical and quasi-canonical calculi we had to slightly extend the framework of non-deterministic matrices by allowing also the option of having an empty set of options in the entries of the truth tables (which intuitively mean that certain combinations of truth values are disallowed).

Non-deterministic Kripke Valuations. For basic sequent systems, we introduce a generalization of Kripke-style semantics for modal and intuitionistic logic, that we call *Kripke valuations*. As Kripke models, these semantic structures employ a set of possible worlds and accessibility relations, and certain conditions connect the truth value assigned to a formula in each world w with values assigned to other formulas in the worlds accessible from w.

We show that Kripke valuations that are based on three or four truth values can be used in semantic characterizaions of basic sequent systems with restricted cut rule and/or identity axiom (as needed e.g for characterizing cut-admissibility).

Non-deterministic Gödel Valuations. For canonical Gödel hypersequent calculi, we introduce Gödel valuations. These consist of some linearly ordered set of truth values, and a function assigning a pair of truth values from this set to each formula of a given propositional language. Intuitively, the first element in the pair of truth values assigned to some formula φ is used for occurrences of φ on the left sides of sequents, while the second element in the pair is used for occurrences of φ on the right sides. We show that the cut rule and the identity axiom "connect" these two elements: if they are both available for some formula φ, then the two elements in the pair of truth values of φ must be equal. In addition, the two values assigned to each compound formula of the form $\diamond(\varphi_1, \dots, \varphi_n)$ must lie within certain intervals whose edges are computed from the values assigned to $\varphi_1, \dots, \varphi_n$. The usual algebraic semantics of Gödel logic is a particular instance, in which all of these pairs and intervals are degenerate, and thus the value of $\diamond(\varphi_1, \dots, \varphi_n)$ is uniquely determined by the values of $\varphi_1, \dots, \varphi_n$. In turn, we provide a general construction of the functions for computing these intervals for $\diamond$-formulas given some (canonical) rules for introducing each connective $\diamond$.

Outline

The structure of this thesis is as follows. Chapter 2 is devoted to precise definitions of pure sequent calculi and their proof-theoretic properties, as well as some basic consequences of these properties. Chapter 3 introduces the semantic framework of many-valued systems, and provides a method to obtain a many-valued system for any given pure sequent calculus. Based on this semantics, in Section 3.3 we present necessary and sufficient semantic conditions for analyticity, cut-admissibility and axiom-expansion in pure calculi.

Chapter 4 discusses *canonical sequent calculi* which are defined as pure sequent calculi with additional restrictions on the structure of the logical introduction rules. In turn, in Section 4.2 we present the corresponding (effective) semantic framework of partial non-deterministic matrices, as a special restricted instance of many-valued systems. Based on the results of Chapter 3, we then show that canonical sequent calculi can be characterized by partial non-deterministic matrices, and that the aforementioned proof-theoretic properties can be easily checked using this alternative semantic presentation.

In Chapter 5 we introduce *quasi-canonical sequent calculi*, and show that each such calculus can be translated into an equivalent canonical one. In certain important cases, this translation may be used to obtain a characteristic partial non-deterministic matrix for a given quasi-canonical calculus.

In Chapter 6 we go beyond the scope of pure sequent calculi by introducing *basic sequent calculi*, in which derivation rules may include limitations on the context formulas used in their applications. Then we show that each basic calculus induces a set of generalized Kripke valuations for which it is strongly sound and complete. In Section 6.3 we derive characterization of proof-theoretic properties of basic calculi based on this Kripke semantics. Their nature is similar to the corresponding characterizations from Section 3.3. We demonstrate their applicability in various examples, including sequent calculi for modal logics and a sequent calculus for bi-intuitionistic logic.

In Chapter 7 we define and study *hypersequent Gödel calculi* from a similar angle. The semantics in this chapter is based on *Gödel valuations*, that generalize the usual many-valued semantics of propositional Gödel logic.

Chapters 8 and 9 are of a completely different nature, as each of them is devoted to one particular calculus for one particular logic. Chapter 8 discusses the hypersequent calculus **HIF** for first-order Gödel logic, and provides a semantic proof for cut-admissibility in this calculus. Chapter 9 introduces an extension of **HIF** with usual rules for second-order quantifiers, called $\mathbf{HIF}^2$. We show that $\mathbf{HIF}^2$ is sound and complete for second-order Gödel logic, and that it enjoys cut-admissibility. Note that the fact that **HIF** enjoys cut-admissibility actually follows from the fact that $\mathbf{HIF}^2$ does. Nevertheless, as a preparation and for the convenience of the reader, we provide first a full account for **HIF**, that is relatively easier to follow than the one for $\mathbf{HIF}^2$.

Finally, in Chapter 10 we conclude with a discussion of some directions for further research.

Some Related Works

Usually, the study of Gentzen-type systems is tailored to a specific logic or family of logics. Several notable exceptions include the following:

- [34] studies a general family of sequent systems, and shows that (possibly non-truth functional) bivaluation semantics can be read off the sequent rules for any given system in this family. This work is close to what we do in Chapter 3. However, the sequent systems studied in [34] are just a particular subset of the pure sequent calculi that we study here, as they all employ the usual cut rules and identity axioms. In addition, [34] does not study at all the effectiveness of this semantic framework, as well as semantic characterizations of syntactic properties of the studied calculi. Therefore besides a new look on the sequent calculus, the semantics proposed in [34] does not seem to have much practical or proof-theoretic applications.

- The introduction and first semantic investigation of canonical sequent calculi were done in [17]. That work considered only two-sided sequent calculi with arbitrary canonical rules and the usual cut rule and identity axiom. It was shown that each such calculus can be characterized by a non-deterministic matrix (Nmatrix). That Nmatrix can in turn be used to check whether the calculus is analytic and whether it enjoys cut-admissibility. Later, in [19] that work was extended to many-sided canonical sequent calculi (see also [21]). Our study of canonical calculi in Chapter 4 considers more general family of systems, with arbitrary set of “primitive rules” (these include the cut rules and the identity axioms). In fact, the main results for previously studied canonical calculi (as the characterization of analyticity and cut-admissibility) can be easily obtained from the more general theorems given in Chapter 4.

- [26] studies from a semantic point of view another general family of sequent systems, which is a proper subfamily of our canonical sequent calculi. The semantic framework employed there is truth-functional (based on usual logical many-valued matrices), and thus many sequent calculi cannot be semantically characterized in this framework. By allowing non-deterministic semantics we are able to cover much more general family of calculi, and it can be shown that for the calculi studied in [26] we practically obtain the same (deterministic) semantics.

- A variety of works studies the connection between syntax and semantics in sequent and hypersequent *substructural calculi*, with a focus on developing semantic and algebraic conditions for cut-admissibility in such systems (e.g. [44],[32],[91]). In this thesis we only consider *fully structural* Gentzen-type calculi, but nevertheless, some (obviously, not all) of the calculi in the families that we study fall in the scope of these works, and their semantic criteria for cut-admissibility are applicable in these cases. However, the semantic frameworks used in these works (particularly, phase semantics) is significantly more abstract and complex than the semantic frameworks that we employ.

At this point it should be noted that the idea of using non-deterministic semantics for proving cut-admissibility of a sequent system has a very long history. Indeed, in the quest to verify *Takeuti's conjecture* [89] (that was open for several years) regarding cut-admissibility in the calculus for second-order classical logic,[1] Schütte developed a three-valued non-deterministic semantics for the cut-free fragment of this calculus [85]. This provided a semantic equivalent to Takeuti's conjecture, that was verified by Tait a few years later [87], when it was shown that it is possible to extract a usual (two-valued) counter-model from every three-valued non-deterministic Schütte's counter-model. As a simple consequence, one obtains that if there is no cut-free proof of a certain sequent, then there is no proof at all (see also [58]). Basically, our semantic characterizations of cut-admissibility, as well as the cut-admissibility proofs in Chapters 7 to 9, are based on a similar (generalized) approach.

[1]More precisely, Takeuti's conjecture concerned full type-theory, namely, the completeness of the cut-free sequent calculus that includes rules for quantifiers of any finite arity. However, the proof for second-order fragment was the main breakthrough. Note that the usual syntactic arguments to prove cut-elimination dramatically fail when it comes to higher-order logic.

Finally, besides the aforementioned works on canonical calculi, we are not aware of any works aiming to study analyticity of general Gentzen-type systems, *regardless* of cut-admissibility. In many cases our criteria of analyticity turn out to be much simpler than those of cut-admissibility.

Publications Related to this Dissertation

Most of the contributions described in this thesis have first appeared in other publications. They are roughly divided as follows:

- Chapters 4-5: [68], [70], [28], [29].
- Chapter 6: [15], [73].
- Chapter 7: [71], [69].
- Chapter 8: [16], [72].

The material in Chapters 2,3 and 9 was not published before.

More details about the connections between these publications and this thesis will be given in the beginning of each chapter.

Chapter 2

Pure Sequent Calculi

In this chapter we introduce the family of *pure sequent calculi.* These will be the object of a semantic investigation in the next chapter. Roughly speaking, pure sequent calculi are propositional fully-structural sequent calculi (sequent calculi that include all the usual structural rules: exchange, contraction, and weakening), whose derivation rules do not enforce any limitation on the side formulas (following [5], the adjective *pure* stands for this requirement). This family of calculi provides a suitable proof-theoretic framework for several important propositional logics, including classical logic, important many-valued logics, and various paraconsistent logics. Our scope is broader than what is usually considered as a sequent system:

- We consider *many-sided* sequents, rather than just ordinary *two-sided* ones. This allows us to naturally capture a large family of many-valued logics (see, e.g., [67]).

- We do not presuppose that all systems include identity axioms or cut rules of a given form. This will play a major role in the semantic characterizations of proof-theoretic properties of these systems (e.g., we will be able to compare the semantics of a given system with cut, and the semantics of the same system without cut).

This chapter is organized as follows. We start by defining the notion of a propositional logic in Section 2.1. Then, we precisely formulate the framework of pure sequent calculi, and the logics they induce (Section 2.2). In Section 2.3 we introduce some fundamental proof-theoretic properties of pure sequent calculi that we will study later from a *semantic* perspective.

Publications Related to this Chapter

The material in this chapter was not published before.

2.1 Preliminaries

Definition 2.1.1. A *propositional language* $\mathcal{L}$ consists of a countably infinite set of variables $at_{\mathcal{L}} = \{p_1, p_2, ...\}$ (whose elements are called *atomic formulas*), and a finite set $\diamondsuit_{\mathcal{L}}$ of propositional connectives. Each $\diamond \in \diamondsuit_{\mathcal{L}}$ has a fixed finite arity $ar(\diamond) \geq 0$. The set of all n-ary connectives of $\mathcal{L}$ (for $n \geq 0$) is denoted by $\diamondsuit_{\mathcal{L}}^n$.

Note that propositional constants are considered as nullary connectives.

Notation 2.1.2. We shall specify proportional languages by a set of connectives, and indicate their arities in superscripts. For example, $\{\neg^1, \wedge^2\}$ denotes a language with two connectives: a unary one denoted by $\neg$, and a binary one denoted by $\wedge$.

Given a propositional language $\mathcal{L}$, $\mathcal{L}$-formulas are constructed as usual. We usually use φ, ψ as metavariables for $\mathcal{L}$-formulas, Γ, Δ for finite sets of $\mathcal{L}$-formulas, and $\mathcal{T}, \mathcal{F}, \mathcal{C}$ for (possibly infinite) sets of $\mathcal{L}$-formulas. Henceforth, $\mathcal{L}$ stands for an arbitrary propositional language. We shall usually identify the set of $\mathcal{L}$-formulas with $\mathcal{L}$ itself, e.g. when writing "$\varphi \in \mathcal{L}$" instead of "φ is an $\mathcal{L}$-formula".

Definition 2.1.3. An $\mathcal{L}$-*substitution* is a function $\sigma : at_{\mathcal{L}} \to \mathcal{L}$. It is recursively extended to $\mathcal{L}$, by $\sigma(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})) = \diamond(\sigma(\varphi_1), \dots, \sigma(\varphi_{ar(\diamond)}))$ for every $\diamond \in \diamondsuit_{\mathcal{L}}$.

We follow [21] in taking the following definitions of Tarskian consequence relations and Tarskian propositional logics:

Definition 2.1.4. A relation $\Vdash$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas is:[1]

Reflexive: if $\mathcal{T} \Vdash \varphi$ whenever $\varphi \in \mathcal{T}$.
Monotone: if $\mathcal{T}' \Vdash \varphi$ whenever $\mathcal{T} \Vdash \varphi$ and $\mathcal{T} \subseteq \mathcal{T}'$.
Transitive: if $\mathcal{T}, \mathcal{T}' \Vdash \varphi$ whenever $\mathcal{T} \Vdash \psi$ and $\mathcal{T}', \psi \Vdash \varphi$.
Structural: if $\sigma(\mathcal{T}) \Vdash \sigma(\varphi)$ for every $\mathcal{L}$-substitution σ whenever $\mathcal{T} \Vdash \varphi$.

Definition 2.1.5. A relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas which is reflexive, monotone and transitive is called a *Tarskian consequence relation (tcr)* for $\mathcal{L}$. A *(Tarskian propositional) logic* is a pair $\langle \mathcal{L}, \Vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\Vdash$ is a structural tcr for $\mathcal{L}$.

Definition 2.1.6. A logic $\langle \mathcal{L}, \Vdash \rangle$ is *finitary* if $\Gamma \Vdash \varphi$ for some finite $\Gamma \subseteq \mathcal{T}$ whenever $\mathcal{T} \Vdash \varphi$.

The most important (and popular) propositional logic is of-course *classical logic*. Its language is $\{\neg^1, \wedge^2, \vee^2, \supset^2\}$, and it is denoted below by $\mathcal{L}_{cl}$. The well-known tcr of classical logic will be denoted by $\Vdash_{cl}$ (see Example 2.2.20).

[1] We use the symbol $\Vdash$ to relate sets of formulas and formulas. The usual symbol $\vdash$ will be used to denote derivability of a sequent from a set of sequents.

2.2 Pure Calculi

Usual sequent systems are two-sided, and sequents are often written as expressions of the forms $\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m$. Dealing with many sided-sequents, we find it convenient to use a (finite) set of labels, for specifying the position(s) inside the sequent in which a certain formula occurs. Thus, in what follows $\pounds$ denotes a finite non-empty set of labels. We usually use $\mathtt{x}$ as a metavariable for a label in $\pounds$, and $\mathtt{X}$ for sets of such labels. Sequents are defined as follows:

Definition 2.2.1. An *$\pounds$-labelled $\mathcal{L}$-formula* is an ordered pair $\langle \mathtt{x}, \varphi \rangle$, denoted by $\mathtt{x}{:}\varphi$, where $\mathtt{x} \in \pounds$ and $\varphi \in \mathcal{L}$. An *$\langle \mathcal{L}, \pounds \rangle$-sequent* is a finite set of $\pounds$-labelled $\mathcal{L}$-formulas.

We usually use α, β as metavariables for labelled formulas, and s, c for sequents. Substitutions are extended to labelled formulas, sequents, sets of sequents, etc. in the obvious way. In particular, $\sigma(\emptyset) = \emptyset$.

Notation 2.2.2. For $\mathtt{X} \subseteq \pounds$ and $\varphi \in \mathcal{L}$, the expression $(\mathtt{X}{:}\varphi)$ denotes the sequent $\{\mathtt{x}{:}\varphi \mid \mathtt{x} \in \mathtt{X}\}$.

Notation 2.2.3. Usual two-sided sequents can be seen as $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequents, where $\pounds_{\mathbf{2}}$ denotes the set of labels $\{\mathtt{f}, \mathtt{t}\}$. The labels $\mathtt{f}, \mathtt{t}$ denote the "left side" and the "right side" respectively. The more usual notation $\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m$ is interpreted as $\{\mathtt{f}{:}\varphi_1, \dots, \mathtt{f}{:}\varphi_n, \mathtt{t}{:}\psi_1, \dots, \mathtt{t}{:}\psi_m\}$.

The use of $\mathtt{f}$ and $\mathtt{t}$ at this point is just a matter of tradition, as the labels should not be confused with truth values! Only in certain specific (important) cases, the truth values employed in the semantic characterization presented in Chapter 3 have one-to-one correspondence with the set $\pounds$ of labels.

Remark 2.2.4. For our purposes, we find it most convenient to define sequents using *sets*. In particular, the $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequents $\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}$, $\{\mathtt{t}{:}p_2, \mathtt{f}{:}p_1\}$, $\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}$ are all the same object. This immediately entails that the exchange rule, the contraction rule and the expansion rule (the converse of contraction) are all built-in in all sequent calculi that we study. To have a *fully-structural* system, we should only further require the presence of the weakening rules (one weakening rule for each label, as defined below).

Next, we define the form of derivation rules that are allowed in pure sequent systems.

Definition 2.2.5. A *pure $\langle \mathcal{L}, \pounds \rangle$-rule* is a pair of the form $\mathcal{S}/s$, where $\mathcal{S}$ is a finite set of $\langle \mathcal{L}, \pounds \rangle$-sequents, and s is a single $\langle \mathcal{L}, \pounds \rangle$-sequent. The elements of $\mathcal{S}$ are called the *premises* of the rule, and s is called the *conclusion* of the rule. To improve readability, we usually drop the set braces of the set of premises. An *application* of a pure $\langle \mathcal{L}, \pounds \rangle$-rule $s_1, \dots, s_n/s$ is any inference step of the following form:

$$\frac{\sigma(s_1) \cup c_1 \quad \dots \quad \sigma(s_n) \cup c_n}{\sigma(s) \cup c_1 \cup \dots \cup c_n}$$

where σ is an $\mathcal{L}$-substitution, and c_i is an $\langle \mathcal{L}, \pounds \rangle$-sequent for every $1 \leq i \leq n$. The sequents $\sigma(s_1) \cup c_1, \dots, \sigma(s_n) \cup c_n$ are called the *premises* of the application, while the sequent $\sigma(s) \cup c_1 \cup \dots \cup c_n$ is called the *conclusion* of the application. In addition, the sequents $c_1, \dots, c_n$ are called the *context sequents* (of the application).

Note that the propositional variables of the "object language" $\mathcal{L}$ are also employed in the formulation of the rules. In particular, meta-variables (which are usually used to represent derivation rules by schemes) are not used. Roughly speaking, applications of some rule are obtained by applying a substitution on the premises $s_1, \dots, s_n$ and the conclusion s of the rule, and freely adding *context* formulas.

Example 2.2.6. Suppose that $\mathcal{L}$ contains the binary "implication" connective $\supset$. The following pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-rules are usually used for introducing this connective:

$$\{\mathtt{t}{:}p_1\}, \{\mathtt{f}{:}p_2\} / \{\mathtt{f}{:}p_1 \supset p_2\} \quad \text{and} \quad \{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\} / \{\mathtt{t}{:}p_1 \supset p_2\}.$$

Their applications have (respectively) the forms:

$$\frac{\{\mathtt{t}{:}\varphi_1\} \cup c_1 \quad \{\mathtt{f}{:}\varphi_2\} \cup c_2}{\{\mathtt{f}{:}\varphi_1 \supset \varphi_2\} \cup c_1 \cup c_2} \quad \text{and} \quad \frac{\{\mathtt{f}{:}\varphi_1, \mathtt{t}{:}\varphi_2\} \cup c}{\{\mathtt{t}{:}\varphi_1 \supset \varphi_2\} \cup c}$$

In [62], a different implication connective is used, whose introduction rules can be easily formulated as pure rules. The f-rule (whose conclusion is $\{\mathtt{f}{:}p_1 \supset p_2\}$) is the same rule as above, but the t-rule has the form: $\{\mathtt{t}{:}p_2\} / \{\mathtt{t}{:}p_1 \supset p_2\}$. Its applications have the form:

$$\frac{\{\mathtt{t}{:}\varphi_2\} \cup c}{\{\mathtt{t}{:}\varphi_1 \supset \varphi_2\} \cup c}$$

Example 2.2.7. Suppose that $\mathcal{L}$ contains the binary "implication" connective $\supset$, and let $\pounds_{\mathbf{3}} = \{\mathtt{f}, \mathtt{i}, \mathtt{t}\}$. The following pure $\langle \mathcal{L}, \pounds_{\mathbf{3}} \rangle$-rules are used for introducing $\supset$ in the calculus for three-valued Łukasiewicz's logic presented in [97]:

$$\{\mathtt{t}{:}p_1\}, \{\mathtt{f}{:}p_2\} / \{\mathtt{f}{:}p_1 \supset p_2\}$$

$$(\{\mathtt{i}, \mathtt{t}\}{:}p_1), \{\mathtt{i}{:}p_1, \mathtt{i}{:}p_2\}, \{\mathtt{t}{:}p_1, \mathtt{f}{:}p_2\} / \{\mathtt{i}{:}p_1 \supset p_2\}$$

$$(\{\mathtt{f}, \mathtt{i}\}{:}p_1) \cup \{\mathtt{t}{:}p_2\}, \{\mathtt{f}{:}p_1\} \cup (\{\mathtt{i}, \mathtt{t}\}{:}p_2) / \{\mathtt{t}{:}p_1 \supset p_2\}$$

Example 2.2.8. The following rule scheme appears in a sequent system from [12] for da Costa's paraconsistent logic $\mathbf{C_1}$:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\varphi \wedge \neg\varphi) \Rightarrow \Delta}$$

This rule scheme can be formulated as the following pure $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rule (where $\neg \in \Diamond^1_{\mathcal{L}}$ and $\wedge \in \Diamond^2_{\mathcal{L}}$):

$$\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}\neg p_1\}/\{\mathtt{f}{:}\neg(p_1 \wedge \neg p_1)\}.$$

Convention 2.2.9. Obviously the names of the variables in pure $\langle\mathcal{L}, \pounds\rangle$-rules are immaterial (e.g., $\{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \supset p_2\}$ is completely equivalent to $\{\mathtt{t}{:}p_3\}/\{\mathtt{t}{:}p_5 \supset p_3\}$). To avoid further technical complications, we assume that a unique representative is chosen from every equivalence class of rules in some reasonable way, and only these representatives are considered as pure $\langle\mathcal{L}, \pounds\rangle$-rules. For example, when only one variable is involved in a rule (as in Example 2.2.8), we may suppose that this variable is p_1.

A special family of pure rules is the family of *primitive* rules. These rules are used to perform simple manipulations on the labels, and they do not mention any specific connective of the language $\mathcal{L}$. Formally they are defined as follows:

Notation 2.2.10. Given an $\pounds$-labelled $\mathcal{L}$-formula α, we denote by $frm[\alpha]$ the $\mathcal{L}$-formula appearing in α. *frm* is extended to sets of $\pounds$-labelled $\mathcal{L}$-formulas, sets of sets of $\pounds$-labelled $\mathcal{L}$-formulas, etc. in the obvious way.

Definition 2.2.11. A *primitive $\pounds$-rule* is any pure $\langle\mathcal{L}, \pounds\rangle$-sequent rule $\mathcal{S}/s$ such that $frm[\mathcal{S} \cup \{s\}] = \{p_1\}$.

By definition, all primitive $\pounds$-rules have the form $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ for some $\mathtt{X}_1, \dots, \mathtt{X}_n, \mathtt{X} \subseteq \pounds$. An application of the primitive $\pounds$-rule $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ is any inference steps of the following form:

$$\frac{(\mathtt{X}_1{:}\varphi) \cup c_1 \quad \dots \quad (\mathtt{X}_n{:}\varphi) \cup c_n}{(\mathtt{X}{:}\varphi) \cup c_1 \cup \dots \cup c_n}$$

where φ is an $\mathcal{L}$-formula, and c_i is a $\langle\mathcal{L}, \pounds\rangle$-sequent for every $1 \leq i \leq n$.

Example 2.2.12. Let $\pounds = \{\mathtt{f}, \mathtt{i}, \mathtt{t}\}$, and consider the primitive $\pounds$-rule

$$\{\mathtt{f}{:}p_1\}, \{\mathtt{i}{:}p_1\}/(\{\mathtt{i}, \mathtt{t}\}{:}p_1).$$

This rule allows to infer $(\{\mathtt{i}, \mathtt{t}\}{:}\varphi) \cup c_1 \cup c_2$ from $\{\mathtt{f}{:}\varphi\} \cup c_1$ and $\{\mathtt{i}{:}\varphi\} \cup c_2$ for every two $\langle\mathcal{L}, \pounds\rangle$-sequents c_1, c_2 and $\mathcal{L}$-formula φ.

The following primitive rules are usually present in Gentzen-type systems:

Weakening Rules For each $\mathtt{x} \in \pounds$, the weakening rule $(\mathtt{x}{:}weak)$ is the primitive $\pounds$-rule $\{\emptyset\}/\{\mathtt{x}{:}p_1\}$. Its applications have the form:

$$\frac{c}{\{\mathtt{x}{:}\varphi\} \cup c}$$

Cut Rules These are primitive $\pounds$-rules of the form $(\mathtt{X}_1{:}p_1), \ldots, (\mathtt{X}_n{:}p_1)/\emptyset$ for non-empty $\mathtt{X}_1, \ldots, \mathtt{X}_n$. An application of a cut rule of this form has the form:

$$\frac{(\mathtt{X}_1{:}\varphi) \cup c_1 \quad \ldots \quad (\mathtt{X}_n{:}\varphi) \cup c_n}{c_1 \cup \ldots \cup c_n}$$

φ is called the *cut-formula* of the application. Two-sided sequent systems usually employ $\{\mathtt{f}{:}p_1\}, \{\mathtt{t}{:}p_1\}/\emptyset$ as the only cut rule. We denote this rule by (*cut*).

Identity Axioms These are primitive $\pounds$-rules of the form $\emptyset/(\mathtt{X}{:}p_1)$ for non-empty $\mathtt{X}$. An application of an identity axiom of this form has the form:

$$\frac{}{(\mathtt{X}{:}\varphi)}$$

φ is called the *id-formula* of the application. Note that applications of identity axioms do not include context formulas. However, when the weakening rules are available for every $\mathtt{x} \in \pounds$, it is possible to derive $(\mathtt{X}{:}\varphi) \cup c$ from $(\mathtt{X}{:}\varphi)$ for every sequent c. Two-sided sequent systems usually employ $\emptyset/\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_1\}$ as the only identity axiom. We denote this rule by (*id*).

Remark 2.2.13. Note that there are several useful options for cut rules and identity axioms when $|\pounds| > 2$. For example, the systems in [27] have a cut rule $\{\mathtt{x}{:}p_1\}, \{\mathtt{y}{:}p_1\}/\emptyset$ for every $\mathtt{x} \neq \mathtt{y}$ in $\pounds$, and $\emptyset/(\pounds{:}p_1)$ is their only identity axiom; while the systems in [24] employ one cut rule of the form $\{\{\mathtt{x}{:}p_1\} \mid \mathtt{x} \in \pounds\}/\emptyset$, and an identity axiom $\emptyset/(\{\mathtt{x}, \mathtt{y}\}{:}p_1)$ for every $\mathtt{x} \neq \mathtt{y}$ in $\pounds$. Other useful combinations arise when quasi-canonical systems are translated into canonical ones (see Chapter 5).

Next, we define the family of pure sequent calculi. In addition to the structural rules of contraction, exchange and expansion that are implicit in our calculi, we also require that pure sequent calculi contain all weakening rules. Thus we refer to these systems as *fully-structural*.

Definition 2.2.14. A *pure* $\langle \mathcal{L}, \pounds \rangle$*-calculus* consists of a finite set of pure $\langle \mathcal{L}, \pounds \rangle$-rules, that includes $(\mathtt{x}{:}weak)$ for every $\mathtt{x} \in \pounds$. A *proof* in a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ of an $\langle \mathcal{L}, \pounds \rangle$-sequent s from a set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents (called *assumptions*) is a finite list[2] of $\langle \mathcal{L}, \pounds \rangle$-sequents ending with s, such that every sequent in the list is either an element of $\mathcal{S}$, or a conclusion of some application of some rule of $\mathbf{G}$, provided that all premises of this application appear before. We write $\mathcal{S} \vdash_{\mathbf{G}} s$ to denote the existence of such a proof.

Convention 2.2.15. Henceforth, we assume that $\mathbf{G}$ does not include the (trivial) pure $\langle \mathcal{L}, \pounds \rangle$-rule $\emptyset/\emptyset$.

[2]Similarly, one can use finite trees or DAGs.

Notation 2.2.16. Given a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, we denote by $\mathsf{P}_{\mathbf{G}}$ and $\mathsf{R}_{\mathbf{G}}$ the set of primitive $\pounds$-rules of $\mathbf{G}$ except for the weakening rules, and the set of non-primitive rules of $\mathbf{G}$ (respectively).

The following simple observations will be useful in the sequel.

Proposition 2.2.17. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus.

1. If $\mathcal{S} \cup \{s\} \vdash_{\mathbf{G}} s'$, then $\mathcal{S} \cup \{s \cup c\} \vdash_{\mathbf{G}} s' \cup c$ for every $\langle \mathcal{L}, \pounds \rangle$-sequent c.
2. If $\mathcal{S} \vdash_{\mathbf{G}} s$ then $\sigma(\mathcal{S}) \vdash_{\mathbf{G}} \sigma(s)$ for every $\mathcal{L}$-substitution σ.

Recall that sequent calculi are a tool to characterize logics. As defined below, each pure $\langle \mathcal{L}, \pounds \rangle$-calculus naturally induces $2^{|\pounds|}$ logics, each of which is based on some subset of $\pounds$.

Definition 2.2.18. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, and $\mathtt{X} \subseteq \pounds$ a set of labels. The tcr induced by $\mathbf{G}$ and $\mathtt{X}$, denoted by $\Vdash_{\mathbf{G}}^{\mathtt{X}}$, is the relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas defined by: $\mathcal{T} \Vdash_{\mathbf{G}}^{\mathtt{X}} \varphi$ iff $\{(\mathtt{X}{:}\psi) \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} (\mathtt{X}{:}\varphi)$.

It is easy to verify that for every $\mathbf{G}$ and $\mathtt{X}$, $\Vdash_{\mathbf{G}}^{\mathtt{X}}$ is indeed a tcr (see Definition 2.1.4). In fact, we have the following:

Proposition 2.2.19. For every $\mathbf{G}$ and $\mathtt{X}$ as above, $\langle \mathcal{L}, \Vdash_{\mathbf{G}}^{\mathtt{X}} \rangle$ is a finitary logic.

Proof. The fact that $\Vdash_{\mathbf{G}}^{\mathtt{X}}$ is structural directly follows from Proposition 2.2.17. The fact that it is finitary follows from the definitions. □

Example 2.2.20. The most important sequent calculus is the fundamental Gentzen's system *LK* for classical logic [56]. Its propositional fragment can be straightforwardly presented as a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus, which we denote by **LK**. The rules of **LK** are presented in Figure 2.1. The consequence relation $\Vdash_{cl}$ of propositional classical logic is equal to $\Vdash_{\mathbf{LK}}^{\{\mathtt{t}\}}$ – the logic induced by **LK** and the set $\{\mathtt{t}\}$.

Remark 2.2.21. In the case of **LK**, there is another natural way to define the induced logic: $\mathcal{T} \Vdash_{cl} \varphi$ iff $\vdash_{\mathbf{LK}} \{\mathtt{f}{:}\psi \mid \psi \in \Gamma\} \cup \{\mathtt{t}{:}\varphi\}$ for some finite $\Gamma \subseteq \mathcal{T}$. It is easy to see that in **LK**, and actually in every pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{G}$ such that $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$, we have that $\mathcal{T} \Vdash_{\mathbf{G}}^{\{\mathtt{t}\}} \varphi$ (according to Definition 2.2.18) iff $\vdash_{\mathbf{G}} \{\mathtt{f}{:}\psi \mid \psi \in \Gamma\} \cup \{\mathtt{t}{:}\varphi\}$ for some finite $\Gamma \subseteq \mathcal{T}$. Therefore, the two alternatives to define $\Vdash_{cl}$ are equivalent. However, the formulation we gave in Definition 2.2.18 is more general, as it ensures that we obtain a logic for *every* pure calculus with arbitrary primitive rules.

($\mathtt{f}$:*weak*)	$\{\emptyset\}/\{\mathtt{f}{:}p_1\}$	($\mathtt{t}$:*weak*)	$\{\emptyset\}/\{\mathtt{t}{:}p_1\}$
(*cut*)	$\{\mathtt{f}{:}p_1\},\{\mathtt{t}{:}p_1\}/\emptyset$	(*id*)	$\emptyset/\{\mathtt{f}{:}p_1,\mathtt{t}{:}p_1\}$
($\mathtt{f}{:}\neg$)	$\{\mathtt{t}{:}p_1\}/\{\mathtt{f}{:}\neg p_1\}$	($\mathtt{t}{:}\neg$)	$\{\mathtt{f}{:}p_1\}/\{\mathtt{t}{:}\neg p_1\}$
($\mathtt{f}{:}\wedge$)	$\{\mathtt{f}{:}p_1,\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1\wedge p_2\}$	($\mathtt{t}{:}\wedge$)	$\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1\wedge p_2\}$
($\mathtt{f}{:}\vee$)	$\{\mathtt{f}{:}p_1\},\{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1\vee p_2\}$	($\mathtt{t}{:}\vee$)	$\{\mathtt{t}{:}p_1,\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1\vee p_2\}$
($\mathtt{f}{:}\supset$)	$\{\mathtt{t}{:}p_1\},\{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1\supset p_2\}$	($\mathtt{t}{:}\supset$)	$\{\mathtt{f}{:}p_1,\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1\supset p_2\}$

Figure 2.1: The pure $\langle\mathcal{L}_{cl},\mathscr{L}_{\mathbf{2}}\rangle$-calculus **LK**

In addition to **LK**, the family of *canonical calculi*, that was defined and studied in [17], falls under the definition of pure calculi. Many other previously studied useful sequent calculi can be naturally presented as pure calculi. This includes all calculi for paraconsistent logics from [12], all labelled calculi for finite valued logics from [26], and the signed calculi from [20].

Example 2.2.22. In [12] a pure $\langle\mathcal{L}_{cl},\mathscr{L}_{\mathbf{2}}\rangle$-calculus for da Costa's historical paraconsistent logic $\mathbf{C_1}$ was introduced.[3] This calculus, denoted here by $\mathbf{G_{C_1}}$, is obtained from **LK** by discarding the rule ($\mathtt{f}{:}\neg$) and adding the following rules:

($\mathtt{f}{:}\neg\neg$)	$\{\mathtt{f}{:}p_1\}/\{\mathtt{f}{:}\neg\neg p_1\}$
($\mathtt{f}{:}\neg\wedge^1$)	$\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}\neg p_1\}/\{\mathtt{f}{:}\neg(p_1\wedge\neg p_1)\}$
($\mathtt{f}{:}\neg\wedge^2$)	$\{\mathtt{f}{:}\neg p_1\},\{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1\wedge p_2)\}$
($\mathtt{f}{:}\neg\vee^1$)	$\{\mathtt{f}{:}\neg p_1\},\{\mathtt{f}{:}p_2,\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1\vee p_2)\}$
($\mathtt{f}{:}\neg\vee^2$)	$\{\mathtt{f}{:}p_1,\mathtt{f}{:}\neg p_1\},\{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1\vee p_2)\}$
($\mathtt{f}{:}\neg\supset^1$)	$\{\mathtt{f}{:}p_1\},\{\mathtt{f}{:}p_2,\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1\supset p_2)\}$
($\mathtt{f}{:}\neg\supset^2$)	$\{\mathtt{f}{:}p_1,\mathtt{f}{:}\neg p_1\},\{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1\supset p_2)\}$

Remark 2.2.23. One can choose to define sequents using *lists* (as in the original work of Gentzen) or *multisets*, and explicitly include contraction and exchange in the definition of a pure sequent calculus. Obviously, this would not affect the derivability relation $\vdash_{\mathbf{G}}$. In fact, for all aspects of proof systems studied in this thesis (semantics, cut-admissibility, analyticity, etc.) this choice is immaterial, since any result in one formulation trivially holds in the other. Of course, this might not be the case when studying other properties (like e.g. in [49]). Similarly, we formulated the applications of rules as *multiplicative* (context-independent) rather than additive (context-sharing) applications (see [57],[92]). Clearly, in the presence of all structural rules, the multiplicative version and the additive one are interderivable. Again, this decision does not affect any property we discuss below.

[3] Here and henceforth, when we say that a two-sided calculus **G** is a calculus *for a logic* **L**, we mean that $\Vdash^{\{\mathtt{t}\}}_{\mathbf{G}}$ is equal to the consequence relation of the logic **L**.

2.3 Proof-Theoretic Properties

In this section we define several important proof-theoretic properties of pure calculi. In the next chapter we will provide a semantic counterpart for each of these properties.

2.3.1 Analyticity

Analyticity is a crucial property of fully-structural propositional proof systems, as it usually implies its decidability and consistency (the fact that the empty sequent is not derivable) . Roughly speaking, a sequent calculus is *analytic* if whenever a sequent s is provable in it, then s can be proven using only the syntactic material available inside s. Now, there is more than one way to precisely define the "material available within some sequent". Usually, it is taken to consist of all subformulas occurring in the sequent, and then analyticity amounts to *the global subformula property* (i.e., if there exists a proof of a sequent s, then there exists a proof of s using only its subformulas). However, it is also possible (and sometimes necessary, see, e.g., Example 3.3.4) to consider analyticity properties that are based on different relations defining the "material available within sequents". While these substitutes might be weaker than the global subformula property, they still suffice to imply the consistency and the decidability of a proof system. Next we define a generalized analyticity property, based on an arbitrary *safe* partial order.

Definition 2.3.1. Let $\leq$ be a partial order on $\mathcal{L}$. For every formula φ, we denote by $\downarrow^{\leq}[\varphi]$ the set $\{\psi \in \mathcal{L} \mid \psi \leq \varphi\}$. This notation is extended to sets of formulas, sequents, and sets of sequents in the natural way: $\downarrow^{\leq}[\mathcal{T}] = \bigcup_{\varphi\in\mathcal{T}} \downarrow^{\leq}[\varphi]$ for a set $\mathcal{T}$ of formulas; $\downarrow^{\leq}[s] = \downarrow^{\leq}[\mathit{frm}[s]]$ for a sequent s; and $\downarrow^{\leq}[\mathcal{S}] = \bigcup_{s\in\mathcal{S}} \downarrow^{\leq}[s]$ for a set $\mathcal{S}$ of sequents. $\leq$ is called *safe* if $\downarrow^{\leq}[\varphi]$ is finite for every $\varphi \in \mathcal{L}$, and $\lambda\varphi \in \mathcal{L}.\downarrow^{\leq}[\varphi]$ is computable.

Henceforth, $\leq$ denotes an arbitrary safe partial order on $\mathcal{L}$. A particularly important one is the subformula relation (here we mean the reflexive-transitive closure of the direct subformula relation). For this relation we employ the following notation:

Notation 2.3.2. We denote by *sub* the subformula relation between formulas. In the case of *sub*, we simply write $sub[\,\cdot\,]$ instead of $\downarrow^{sub}[\,\cdot\,]$.

Definition 2.3.3. Given a set $\mathcal{F}$ of $\mathcal{L}$-formulas, a formula φ is called an *$\mathcal{F}$-formula* if $\varphi \in \mathcal{F}$. In turn, an *£-labelled $\mathcal{F}$-formula* is an £-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi$ with $\varphi \in \mathcal{F}$; and an *$\langle\mathcal{F}, £\rangle$-sequent* is an $\langle\mathcal{L}, £\rangle$-sequent consisting only of £-labelled $\mathcal{F}$-formulas.

Notation 2.3.4. For a pure $\langle\mathcal{L}, £\rangle$-calculus $\mathbf{G}$, we write $\mathcal{S} \vdash^{\mathcal{F}}_{\mathbf{G}} s$ if there is a proof in $\mathbf{G}$ of s from $\mathcal{S}$ consisting only of $\langle\mathcal{F}, £\rangle$-sequents.

Definition 2.3.5. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus **G** is $\leq$-*analytic* if for every $\langle \mathcal{L}, \pounds \rangle$-sequent s: $\vdash_{\mathbf{G}} s$ implies that $\vdash_{\mathbf{G}}^{\downarrow^{\leq}[s]} s$.

The above notion of analyticity considers only proof from empty set of assumptions (speaking only about the theorems of the system). A strong version is defined as follows:

Definition 2.3.6. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus **G** is *strongly* $\leq$-*analytic* if for every set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents and $\langle \mathcal{L}, \pounds \rangle$-sequent s: $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that $\mathcal{S} \vdash_{\mathbf{G}}^{\downarrow^{\leq}[\mathcal{S} \cup \{s\}]} s$.

In the next chapters we focus only on strong $\leq$-analyticity. Obviously, $\leq$-analyticity follows from strong $\leq$-analyticity (take $\mathcal{S} = \emptyset$). Next, we show that in the simple (and most common) case of two-sided calculi that include (*cut*) and (*id*) these two properties are actually equivalent. The main idea of this proof appeared already [6], where it was proved that cut-admissibility implies strong cut-admissibility (see definition below) for the specific case of **LK**.

Theorem 2.3.7. Let **G** be a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus, that includes (*cut*) and (*id*). If **G** is $\leq$-analytic then it is strongly $\leq$-analytic.

Proof. Suppose that **G** is $\leq$-analytic. We show that $\mathcal{S} \vdash_{\mathbf{G}} s$ implies $\mathcal{S} \vdash_{\mathbf{G}}^{\downarrow^{\leq}[\mathcal{S} \cup \{s\}]} s$. Clearly, it suffices to prove this for finite $\mathcal{S}$ (otherwise, take a finite subset $\mathcal{S}^*$ of $\mathcal{S}$ such that $\mathcal{S}^* \vdash_{\mathbf{G}} s$). We use induction on the number of $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequents in $\mathcal{S}$. The case that $\mathcal{S}$ is empty follows from our assumption. Suppose the claim holds when the number of sequents in $\mathcal{S}$ is n, and let $\mathcal{S}' = \{s_0, \dots, s_n\}$ be a set of $n+1$ $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequents, such that $\mathcal{S}' \vdash_{\mathbf{G}} s$. Proposition 2.2.17 implies that $\{s_0 \cup c, s_1, \dots, s_n\} \vdash_{\mathbf{G}} s \cup c$, for every $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequent c. In particular, for every $\pounds_{\mathbf{2}}$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi \in s_0$, $\{s_0 \cup \{\overline{\mathtt{x}}{:}\varphi\}, s_1, \dots, s_n\} \vdash_{\mathbf{G}} s \cup \{\overline{\mathtt{x}}{:}\varphi\}$, where $\overline{\mathtt{f}} = \mathtt{t}$ and $\overline{\mathtt{t}} = \mathtt{f}$. Now, for every $\mathtt{x}{:}\varphi \in s_0$, the sequent $s_0 \cup \{\overline{\mathtt{x}}{:}\varphi\}$ is derivable in **G** using only (*id*) and weakenings, and therefore we have $\{s_1, \dots, s_n\} \vdash_{\mathbf{G}} s \cup \{\overline{\mathtt{x}}{:}\varphi\}$. By the induction hypothesis we obtain that $\{s_1, \dots, s_n\} \vdash_{\mathbf{G}}^{\downarrow^{\leq}[\mathcal{S} \cup \{s\}]} s \cup \{\overline{\mathtt{x}}{:}\varphi\}$ for every $\pounds_{\mathbf{2}}$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi \in s_0$. The sequent s can then be inferred from these sequents and s_0 by $|s_0|$ applications of (*cut*) without introducing any formulas outside $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$. □

The following are three major consequences of (strong) $\leq$-analyticity.

Proposition 2.3.8 (Consistency)**.** Let **G** be pure $\langle \mathcal{L}, \pounds \rangle$-calculus, which is $\leq$-analytic for some safe partial order $\leq$. Then, $\nvdash_{\mathbf{G}} \emptyset$.

Proof. Assume that $\vdash_{\mathbf{G}} \emptyset$. Since **G** is $\leq$-analytic, there exists a proof of the empty sequent using no formulas at all. The only way to have this is using the rule $\emptyset/\emptyset$, which was disallowed in pure systems (Convention 2.2.15). □

Proposition 2.3.9 (Conservativity)**.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two propositional languages, such that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$ by some set of connectives. Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be a pure $\langle \mathcal{L}_1, \pounds \rangle$-calculus and a pure $\langle \mathcal{L}_2, \pounds \rangle$-calculus (respectively). Assume that $\mathbf{G_2}$ is obtained from $\mathbf{G_1}$ by adding to the latter rules involving connectives in $\mathcal{L}_2 \setminus \mathcal{L}_1$ (i.e., at least one connective in $\mathcal{L}_2 \setminus \mathcal{L}_1$ appears in each rule in $\mathbf{G_2} \setminus \mathbf{G_1}$). Let $\leq$ be a safe partial order on $\mathcal{L}_2$, such that $\mathcal{L}_1$ is closed under $\leq$ (i.e., $\downarrow^{\leq}[\mathcal{L}_1] = \mathcal{L}_1$). If $\mathbf{G_2}$ is strongly $\leq$-analytic, then $\mathbf{G_2}$ is a conservative extension of $\mathbf{G_1}$ (i.e., if $\mathit{frm}[\mathcal{S} \cup \{s\}] \subseteq \mathcal{L}_1$ then $\mathcal{S} \vdash_{\mathbf{G_1}} s$ iff $\mathcal{S} \vdash_{\mathbf{G_2}} s$).

Proof. Obviously, $\mathcal{S} \vdash_{\mathbf{G_1}} s$ implies $\mathcal{S} \vdash_{\mathbf{G_2}} s$. For the converse, assume that $\mathcal{S} \vdash_{\mathbf{G_2}} s$. Since $\mathbf{G_2}$ is strongly $\leq$-analytic, there exists a proof in $\mathbf{G_2}$ of s from $\mathcal{S}$ consisting of $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-formulas only. Since $\mathit{frm}[\mathcal{S} \cup \{s\}] \subseteq \mathcal{L}_1$, and $\mathcal{L}_1$ is closed under $\leq$, this is also a proof in $\mathbf{G_1}$, and so $\mathcal{S} \vdash_{\mathbf{G_1}} s$. □

Proposition 2.3.10 (Decidability)**.** Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus. Suppose that $\mathbf{G}$ is strongly $\leq$-analytic for some safe partial order $\leq$. Then, given a finite set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents and an $\langle \mathcal{L}, \pounds \rangle$-sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathbf{G}} s$ or not.

Proof. Exhaustive proof-search is possible. Since $\mathbf{G}$ is strongly $\leq$-analytic, $\mathcal{S} \vdash_{\mathbf{G}} s$ iff there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ consisting of $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-sequents only. Since $\leq$ is safe, one can construct the (finite) set $\mathcal{S}'$ of all $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-sequents. Clearly, $\mathcal{S} \vdash_{\mathbf{G}} s$ iff there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ of length less than or equal to $|\mathcal{S}'|$, consisting only of sequents from $\mathcal{S}'$. Thus one can construct all possible candidates. By definition $\mathbf{G}$ is finite, and hence it is possible to check whether a certain candidate is indeed a proof in $\mathbf{G}$ of s from $\mathcal{S}$. □

2.3.2 Cut-Admissibility

Usual two-sided sequent calculi include the rule (*cut*), which is very problematic from a proof-search perspective. The admissibility of (*cut*) (i.e. the fact that for every sequent s, $\vdash_{\mathbf{G}} s$ implies that there is a cut-free proof in $\mathbf{G}$ of s) is then desirable. However, forbidding all applications of cut rules seems to be too strong while dealing with arbitrary pure $\langle \mathcal{L}, \pounds \rangle$-calculi. Indeed, consider applications of a cut rule in which the cut-formula occurs inside the context sequents (i.e. inferring a sequent of the form $c_1 \cup ... \cup c_n$ from the sequents $(\mathtt{X}_1{:}\varphi) \cup c_1, \ldots, (\mathtt{X}_n{:}\varphi) \cup c_n$, where $\varphi \in \mathit{frm}[c_1 \cup ... \cup c_n]$). Such applications are not harmful for proof-search, as every formula in the conclusion of the application also occurs (as is) in one of its premises. These considerations lead to the following formulation of cut admissibility in pure calculi:

Definition 2.3.11. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ enjoys *cut-admissibility* if for every $\langle \mathcal{L}, \pounds \rangle$-sequent s: $\vdash_{\mathbf{G}} s$ implies that there exists proof P of s in $\mathbf{G}$ such that the cut-formula of every application of a cut rule in P occurs in one of the context sequents of that application.

As for analyticity, this notion of cut-admissibility refers only to proofs without assumptions (i.e., proofs from $\emptyset$). Obviously, we cannot expect full cut-admissibility when the set of assumptions is not empty (in usual calculi, the only way to derive the empty sequent from $\{\mathtt{f}{:}p_1\}$ and $\{\mathtt{t}{:}p_1\}$ is using (cut)). Thus we consider the property called *strong* cut-admissibility in [6], which is formulated as follows in our framework:

Definition 2.3.12. A pure calculus $\mathbf{G}$ enjoys *strong cut-admissibility* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that there exists a proof P of s from $\mathcal{S}$ in $\mathbf{G}$ such that the cut-formula of every application of a cut rule in P occurs either in one of the context sequents of that application or in $frm[\mathcal{S}]$.

Obviously, cut-admissibility follows from strong cut-admissibility (take $\mathcal{S} = \emptyset$).

Equivalent definition of cut-admissibility and strong cut-admissibility are obtained by considering an enrichment of $\mathbf{G}$ with *non-cut* rules, so that all applications of the cut rules in which the cut-formula occurs in the context can be replaced by applications of the new rules. This is done as follows:

Definition 2.3.13. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus. $s(\mathbf{G})$ denotes the pure $\langle \mathcal{L}, \pounds \rangle$-calculus obtained by augmenting $\mathbf{G}$ with the primitive $\pounds$-rules $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\{\mathtt{x}{:}p_1\}$ for every cut rule $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\emptyset$ of $\mathbf{G}$ and $\mathtt{x} \in \pounds$ such that $\{\mathtt{x}\} \notin \{\mathtt{X}_1, \dots, \mathtt{X}_n\}$.

Example 2.3.14. Let $\pounds = \{\mathtt{f}, \mathtt{i}, \mathtt{t}\}$ and suppose that is $\{\mathtt{f}{:}p_1\}, \{\mathtt{i}{:}p_1\}/\emptyset$ is the only cut rule of $\mathbf{G}$. $s(\mathbf{G})$ is obtained by adding to $\mathbf{G}$ the primitive rule $\{\mathtt{f}{:}p_1\}, \{\mathtt{i}{:}p_1\}/\{\mathtt{t}{:}p_1\}$.

Note that for a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{G}$ whose only cut rule is (cut), we have that $s(\mathbf{G}) = \mathbf{G}$.

Proposition 2.3.15. $\vdash_{s(\mathbf{G})} = \vdash_{\mathbf{G}}$ for every pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$.

Proof. All applications of the new rules in $s(\mathbf{G})$ can be simulated in $\mathbf{G}$ by applications of the corresponding cut rule, followed by an application of a weakening rule. □

Notation 2.3.16. Given a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, we denote by $\mathbf{G}_{cf}$ the calculus obtained from $\mathbf{G}$ by discarding all cut rules of $\mathbf{G}$. In particular, $s(\mathbf{G})_{cf}$ is the calculus obtained from $\mathbf{G}$ by replacing every cut rule of the form $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\emptyset$ with all rules of the form $(\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\{\mathtt{x}{:}p_1\}$ such that $\mathtt{x} \in \pounds$ and $\{\mathtt{x}\} \notin \{\mathtt{X}_1, \dots, \mathtt{X}_n\}$.

Notation 2.3.17. Given a set $\mathcal{C} \subseteq \mathcal{L}$, a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, a set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents, and an $\langle \mathcal{L}, \pounds \rangle$-sequent s, we write $\mathcal{S} \vdash_{\mathbf{G}}^{cuts:\mathcal{C}} s$ if there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ in which the cut-formula of every application of a cut rule is an element of $\mathcal{C}$.

Proposition 2.3.18. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus.

1. $\mathbf{G}$ enjoys cut-admissibility iff $\vdash_{s(\mathbf{G})_{cf}} s$ whenever $\vdash_{s(\mathbf{G})} s$.
2. $\mathbf{G}$ enjoys strong cut-admissibility iff $\mathcal{S} \vdash_{s(\mathbf{G})}^{cuts:frm[\mathcal{S}]} s$ whenever $\mathcal{S} \vdash_{s(\mathbf{G})} s$.

Proof. Note that every application of a cut rule in $\mathbf{G}$ in which the cut-formula occurs in the context sequents can be simulated in $s(\mathbf{G})_{cf}$ (by using its new primitive rules or weakening). Similarly, every application of a new primitive $\pounds$-rule in $s(\mathbf{G})_{cf}$ can be simulated in $\mathbf{G}$ by applying weakening and the corresponding cut rule where the cut-formula occurs in the context sequents. The claims then follow from the definitions. □

It follows that a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{G}$ whose only cut rule is (cut) (and thus $s(\mathbf{G}) = \mathbf{G}$) enjoys cut-admissibility iff $\vdash_{\mathbf{G}} \subseteq \vdash_{\mathbf{G}_{cf}}$. Such a calculus enjoys strong cut-admissibility iff $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that there exists a proof of s from $\mathcal{S}$ in $\mathbf{G}$ such that the cut-formula of every application of a cut rule is an element of $frm[\mathcal{S}]$. Hence Definition 2.3.11 and Definition 2.3.12 indeed generalize the known notions for ordinary two-sided sequent calculi. In addition, in ordinary two-sided calculi that include (cut) and (id) cut-admissibility is equivalent to strong cut-admissibility (like in the case of analyticity, see Theorem 2.3.7).

Theorem 2.3.19. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus, that includes (cut) and (id). If $\mathbf{G}$ enjoys cut-admissibility then $\mathbf{G}$ enjoys strong cut-admissibility.

Proof. Similar to the proof of Theorem 2.3.7. □

2.3.3 Axiom-Expansion

Another property which is often studied in two-sided sequent calculi (that include (id)) is the property of axiom-expansion [44]. This property means that non-atomic applications of (id) (deriving sequents of the form $\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$ where φ is not atomic) are redundant.[4] In our broader context it can be formulated as follows:

Notation 2.3.20. Given a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, $\mathbf{G}_{if}$ denotes the pure $\langle \mathcal{L}, \pounds \rangle$-calculus obtained from $\mathbf{G}$ by discarding all identity axioms of $\mathbf{G}$.

[4]The term "axiom-expansion" is commonly used, but it is somewhat unfortunate. In fact, this property concerns the *reducibility* of arbitrary axioms to atomic ones.

Definition 2.3.21. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ enjoys *axiom-expansion* if

$$\{(\mathtt{X}{:}p) \mid \emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}, p \in at_{\mathcal{L}}\} \vdash_{\mathbf{G}_{if}} (\mathtt{Y}{:}\varphi)$$

for every $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $\varphi \in \mathcal{L}$.

It is easy to see that we have the following:

Proposition 2.3.22. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ enjoys *axiom-expansion* if

$$\{(\mathtt{X}{:}p_i) \mid \emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}, 1 \leq i \leq ar(\diamond)\} \vdash_{\mathbf{G}_{if}} (\mathtt{Y}{:} \diamond (p_1, \dots, p_{ar(\diamond)}))$$

for every $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $\diamond \in \Diamond_{\mathcal{L}}$.

Proof. A simple inductive argument (using Proposition 2.2.17) suffices. □

Thus, following [44], we define this property for a *given connective* as follows:

Definition 2.3.23. A connective $\diamond \in \Diamond_{\mathcal{L}}$ admits *axiom-expansion* in a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ if $\{(\mathtt{X}{:}p_i) \mid \emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}, 1 \leq i \leq ar(\diamond)\} \vdash_{\mathbf{G}_{if}} (\mathtt{Y}{:} \diamond (p_1, \dots, p_{ar(\diamond)}))$ for every $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$.

Remark 2.3.24. Unlike [44], we do not require that there exists a *cut-free* proof of $(\mathtt{Y}{:} \diamond (p_1, \dots, p_{ar(\diamond)}))$.

Note that a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ enjoys axiom-expansion iff each connective of $\mathcal{L}$ admits axiom-expansion in $\mathbf{G}$.

Example 2.3.25. It is easy to see that each connective of $\mathcal{L}_{cl}$ admits axiom-expansion in **LK**, and thus **LK** enjoys axiom-expansion.

Chapter 3

Semantics for Pure Sequent Calculi

In this chapter we introduce a method for providing (possibly non-deterministic) many-valued semantics for any given pure calculus. The semantics is then used to characterize the proof-theoretic properties of pure calculi that were introduced in the previous chapter. It should be noted that while dealing with the general framework of pure calculi, one cannot expect to obtain effective semantics in all cases. Indeed, the proposed semantics is quite close to the input proof system. However, it provides a complementary presentation of pure calculi, sheds light on their syntactic properties, reveals deep useful connections between semantics and proof theory, and turns out to be useful for proving these properties in particular examples. In addition, applying the tools of this chapter for narrower families of pure calculi (as done in the next chapter for *canonical* calculi) leads to effective semantics and decidable criteria for proof-theoretic properties.[1]

Publications Related to this Chapter

The material in this chapter was not published before.

3.1 The Semantic Framework

The proposed semantic framework is a denotational semantics, based on *valuations*, which are simply functions whose domain is $\mathcal{L}$:

[1]This inherent limitation of an investigation of such general frameworks was summarized by J.Y. Béziau in [34] as follows:

> "What is involved in this paper is mainly general abstract nonsense. The main difficulty of our results is rather conceptual. But what we show is that when this general abstract nonsense is rightly organized we can get meaningful results with a lot of powerful applications."

Definition 3.1.1. An $\mathcal{L}$*-valuation* is a function v from $\mathcal{L}$ to some set $\mathcal{U}$ of *truth values.* We denote by Im_v the image of v.

Note that we do not specify what are truth values, and in fact anything can serve as a truth value. To define restrictions on $\mathcal{L}$-valuations we introduce $\mathcal{L}$*-semantic conditions.*

Definition 3.1.2. Let $\mathcal{U}$ be some set of truth values.

1. An $\mathcal{L}$*-semantic disjunction* over $\mathcal{U}$ is a finite set of pairs $\langle \varphi, u \rangle$, denoted by $\varphi \doteqdot u$, where $u \in \mathcal{U}$ and $\varphi \in \mathcal{L}$. $\mathcal{L}$-substitutions are naturally extended to $\mathcal{L}$-semantic disjunctions by $\sigma(I) = \{\sigma(\varphi) \doteqdot u \mid \varphi \doteqdot u \in I\}$.
2. An $\mathcal{L}$*-semantic condition* over $\mathcal{U}$ is a pair $\langle \mathcal{I}, I \rangle$, denoted by $\mathcal{I} \Rightarrow I$, where $\mathcal{I}$ is a finite set of $\mathcal{L}$-semantic disjunctions over $\mathcal{U}$, and I is a single $\mathcal{L}$-semantic disjunction over $\mathcal{U}$.
3. An $\mathcal{L}$-valuation v with $Im_v \subseteq \mathcal{U}$ *satisfies*:
   (a) an $\mathcal{L}$*-semantic disjunction* I over $\mathcal{U}$ if $\varphi \doteqdot v(\varphi) \in I$ for some $\varphi \in \mathcal{L}$.
   (b) an $\mathcal{L}$*-semantic condition* $\mathcal{I} \Rightarrow I$ over $\mathcal{U}$ if for every $\mathcal{L}$-substitution σ, v satisfies $\sigma(I)$ whenever it satisfies $\sigma(I')$ for every $I' \in \mathcal{I}$.
   (c) a *set* Λ *of* $\mathcal{L}$*-semantic conditions* over $\mathcal{U}$ if it satisfies every $\mathcal{I} \Rightarrow I \in \Lambda$.

We write $v \models X$ to denote that v satisfies X, where X is either an $\mathcal{L}$-semantic disjunction, an $\mathcal{L}$-semantic condition, or a set of $\mathcal{L}$-semantic conditions.[2]

Example 3.1.3. Suppose that $\diamond, \triangleright \in \diamondsuit^2_{\mathcal{L}}$, and let $\mathcal{U} = \{u_1, u_2\}$. Consider the semantic disjunctions: $I_1 = \{p_1 \doteqdot u_1, p_2 \doteqdot u_2\}$, and $I_2 = \{p_1 \diamond p_2 \doteqdot u_1, p_2 \triangleright p_1 \doteqdot u_1\}$. Let v be an $\mathcal{L}$-valuation with $Im_v \subseteq \mathcal{U}$. Then, $v \models I_1$ iff $v(p_1) = u_1$ or $v(p_2) = u_2$. $v \models I_2$ iff $v(p_1 \diamond p_2) = u_1$ or $v(p_2 \triangleright p_1) = u_1$. v satisfies the $\mathcal{L}$-semantic condition $\{I_1\} \Rightarrow I_2$ iff for every $\varphi_1, \varphi_2 \in \mathcal{L}$: if $v(\varphi_1) = u_1$ or $v(\varphi_2) = u_2$, then $v(\varphi_1 \diamond \varphi_2) = u_1$ or $v(\varphi_2 \triangleright \varphi_1) = u_1$.

Example 3.1.4. Obviously, restrictions arising from "truth tables" can be represented as semantic conditions. For example, to capture the classical truth table of implication, we use the following conditions over $\{f, t\}$: $\{\{p_1 \doteqdot t\}, \{p_2 \doteqdot f\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot f\}$ and $\{\{p_1 \doteqdot f, p_2 \doteqdot t\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot t\}$. An $\mathcal{L}$-valuation satisfies these two $\mathcal{L}$-semantic conditions iff it respects the usual truth table of $\supset$.

To obtain semantic characterizations of logics we introduce a class of structures called *many-valued systems*, that generalizes the usual notion of a many-values matrix (see, e.g., [93]), by allowing arbitrary semantic conditions.

[2]Note that $v \models \emptyset$ is ambiguous: it holds for the empty set of $\mathcal{L}$-semantic conditions, and does not hold for the empty $\mathcal{L}$-semantic disjunction.

Definition 3.1.5. A *many-valued system* $\mathbf{M}$ for $\mathcal{L}$ consists of:

1. A set $\mathcal{V}_{\mathbf{M}}$ of truth values.
2. A subset $\mathcal{U}_{\mathbf{M}} \subseteq \mathcal{V}_{\mathbf{M}}$ of *legal* truth values.
3. A subset $\mathcal{D}_{\mathbf{M}} \subseteq \mathcal{V}_{\mathbf{M}}$ of *designated* truth values.
4. A set $\Lambda_{\mathbf{M}}$ of $\mathcal{L}$-semantic conditions over $\mathcal{V}_{\mathbf{M}}$.

$\mathbf{M}$ is called *finite* if so are $\mathcal{V}_{\mathbf{M}}$ and $\Lambda_{\mathbf{M}}$.

Definition 3.1.6. Let $\mathbf{M}$ be a many-valued system for $\mathcal{L}$.

1. An $\mathcal{L}$-valuation v is called $\mathbf{M}$-*legal* if $Im_v \subseteq \mathcal{U}_{\mathbf{M}}$ and $v \models \Lambda_{\mathbf{M}}$.
2. An $\mathcal{L}$-valuation v is said to be a *model* (with respect to $\mathbf{M}$) of:
   (a) an *$\mathcal{L}$-formula* φ, written $v \models^{\mathbf{M}} \varphi$, if $v(\varphi) \in \mathcal{D}_{\mathbf{M}}$.
   (b) a *set $\mathcal{T}$ of $\mathcal{L}$-formulas*, written $v \models^{\mathbf{M}} \mathcal{T}$, if $v \models^{\mathbf{M}} \varphi$ for every $\varphi \in \mathcal{T}$.
3. An $\mathcal{L}$-formula φ *follows* from a set $\mathcal{T}$ of $\mathcal{L}$-formulas with respect to $\mathbf{M}$ (denoted by: $\mathcal{T} \Vdash_{\mathbf{M}} \varphi$) if for every $\mathbf{M}$-legal $\mathcal{L}$-valuation v: $v \models^{\mathbf{M}} \varphi$ whenever $v \models^{\mathbf{M}} \mathcal{T}$.

Proposition 3.1.7. For every many-valued system $\mathbf{M}$ for $\mathcal{L}$, $\langle \mathcal{L}, \Vdash_{\mathbf{M}} \rangle$ is a logic.

Proof. Easily follows from the definitions. To prove that $\Vdash_{\mathbf{M}}$ is structural, note that if v is an $\mathbf{M}$-legal $\mathcal{L}$-valuation, then so is $v \circ \sigma$ for every $\mathcal{L}$-substitution σ. □

Example 3.1.8. Classical logic $\langle \mathcal{L}_{cl}, \Vdash_{cl} \rangle$ is obtained by taking a many-valued system $\mathbf{M}_{cl}$ with $\mathcal{V}_{\mathbf{M}_{cl}} = \mathcal{U}_{\mathbf{M}_{cl}} = \{f, t\}$, $\mathcal{D}_{\mathbf{M}_{cl}} = \{t\}$, and $\Lambda_{\mathbf{M}_{cl}}$ consist of the semantic conditions over $\{f, t\}$ that correspond to the classical truth tables (e.g. as in Example 3.1.4).

Example 3.1.9. Many-valued systems generalize the notion of a logical many-valued matrix [93, 61]. Thus any many-valued logic that is defined by such a matrix is captured in this general framework. In addition, non-deterministic many-valued matrices [17, 21] can be easily presented as particular cases of many-valued systems. This will be discussed in Chapter 4.

Example 3.1.10. The framework of *bivaluations* [34, 40] corresponds to *two*-valued systems (that is, many-valued systems with $|\mathcal{V}_{\mathbf{M}}| = |\mathcal{U}_{\mathbf{M}}| = 2$ and $|\mathcal{D}_{\mathbf{M}}| = 1$. In addition, the dyadic semantics of [39] is also a subclass of two-valued many-valued systems.

In many cases in the sequel, we need a many-valued system $\mathbf{M}$ just to specify a set of ($\mathbf{M}$-legal) valuations, rather than to define a logic. The set $\mathcal{D}_{\mathbf{M}}$ of designated truth values is redundant in these cases, and can be discarded. The obtained structures will be called *many-valued pre-systems*:

Definition 3.1.11. A *many-valued pre-system* $\mathbf{M}$ for $\mathcal{L}$ is defined exactly like a many-valued system (Definition 3.1.5), except that we exclude the set $\mathcal{D}_{\mathbf{M}}$ of designated truth values.

Remark 3.1.12. Evidently, it suffices to consider many-valued systems with $\mathcal{U}_{\mathbf{M}} = \mathcal{V}_{\mathbf{M}}$. Indeed, given a many-valued system $\mathbf{M}$, we can always define a many-valued system $\mathbf{M}'$ by $\mathcal{V}_{\mathbf{M}'} = \mathcal{U}_{\mathbf{M}'} = \mathcal{U}_{\mathbf{M}}$, $\mathcal{D}_{\mathbf{M}'} = \mathcal{D}_{\mathbf{M}} \cap \mathcal{U}_{\mathbf{M}}$, and $\Lambda_{\mathbf{M}'}$ is obtained from $\Lambda_{\mathbf{M}}$ by discarding all occurrences of pairs $\varphi \div u$ with $u \notin \mathcal{U}_{\mathbf{M}}$ from the semantic conditions. Clearly, $\mathbf{M}$-legal $\mathcal{L}$-valuations are exactly $\mathbf{M}'$-legal $\mathcal{L}$-valuations, and we also have $\Vdash_{\mathbf{M}} = \Vdash_{\mathbf{M}'}$. However, we find the distinction between $\mathcal{U}_{\mathbf{M}}$ and $\mathcal{V}_{\mathbf{M}}$ technically convenient, as it allows us to change the set of legal truth values in many-valued (pre-) systems, without changing any of its other components. This is mainly beneficial for the modularity of the constructions below.

3.1.1 Partial Valuations and Semantic Analyticity

An important attractive property that we would like a semantic framework to have is *effectiveness*, namely the fact that it can be used to provide a semantic decision procedure for the logics it induces. The framework of many-valued systems is too wide to have this property in general. In this section we identify a sufficient condition for the the effectiveness of a many-valued system. This condition will also play a main role below for characterizing $\leq$-analyticity in pure sequent calculi.

Generally speaking, the naive approach to check whether $\Gamma \Vdash_{\mathbf{M}} \varphi$ for a many-valued system $\mathbf{M}$ (given a finite set of formulas Γ and a formula φ) would be to consider one by one all possible $\mathbf{M}$-legal $\mathcal{L}$-valuations, and return "true" iff none of them is a *counter-model* – a model of Γ but not of φ (with respect to $\mathbf{M}$). Obviously, this cannot serve as a decision procedure since there are infinitely many $\mathcal{L}$-valuations to check, and each of them is infinite. Thus, as is usually done in decision procedures based on denotational semantic frameworks, one has to consider partial valuations defined only on the syntactic material included in Γ and φ. This, however, requires that the existence of a counter-model in the form of a partial valuation always indicates the existence of an (infinite) full counter-model. Obviously, this requirement holds when every partial valuation can be extended to a full one. Next, we define partial valuations, and precisely formulate these observations.

Definition 3.1.13. A *partial $\mathcal{L}$-valuation* is a function v from some set $Dom_v \subseteq \mathcal{L}$ to some set $\mathcal{U}$ of truth values. We denote by Im_v the image of v.

The previous notions for $\mathcal{L}$-valuations are adapted to partial $\mathcal{L}$-valuations as follows:

Definition 3.1.14. A partial $\mathcal{L}$-valuation v *satisfies*:

1. an *$\mathcal{L}$-semantic disjunction* I if $\varphi \doteqdot v(\varphi) \in I$ for some $\varphi \in Dom_v$.

2. an *$\mathcal{L}$-semantic condition* $\mathcal{I} \Rightarrow I$ if for every $\mathcal{L}$-substitution σ such that $\sigma(\varphi) \in Dom_v$ and for every φ that occurs in $\mathcal{I} \Rightarrow I$, v satisfies $\sigma(I)$ whenever it satisfies $\sigma(I')$ for every $I' \in \mathcal{I}$.

3. a *set* Λ *of $\mathcal{L}$-semantic conditions* if it satisfies every $\mathcal{I} \Rightarrow I \in \Lambda$.

We write $v \models X$ to denote that v satisfies X, where X is either an $\mathcal{L}$-semantic disjunction, an $\mathcal{L}$-semantic condition, or a set of $\mathcal{L}$-semantic conditions.

Given a many-valued pre-system $\mathbf{M}$ for $\mathcal{L}$, $\mathbf{M}$-legal partial $\mathcal{L}$-valuations are defined exactly as $\mathbf{M}$-legal (full) $\mathcal{L}$-valuations (i.e. $Im_v \subseteq \mathcal{U}_\mathbf{M}$ and $v \models \Lambda_\mathbf{M}$). Note that Definitions 3.1.13 and 3.1.14 generalize the corresponding notions defined above for (full) $\mathcal{L}$-valuations. Indeed, by taking $Dom_v = \mathcal{L}$, we obtain exactly the definitions for $\mathcal{L}$-valuations.

Definition 3.1.15. Let v and v' be two partial $\mathcal{L}$-valuations. We say that v' *extends* v if $Dom_v \subseteq Dom_{v'}$ and $v'(\varphi) = v(\varphi)$ for every $\varphi \in Dom_v$.

Definition 3.1.16. Let $\leq$ be a partial order on $\mathcal{L}$. A many-valued (pre-) system $\mathbf{M}$ is called *$\leq$-analytic* if any $\mathbf{M}$-legal partial $\mathcal{L}$-valuation whose domain is finite and closed under $\leq$ can be extended to an $\mathbf{M}$-legal (full) $\mathcal{L}$-valuation.

Example 3.1.17. Revisiting the many-valued system $\mathbf{M}_{cl}$ from Example 3.1.8 for classical logic, we note that this system is *sub*-analytic. Indeed, $\mathbf{M}_{cl}$-legal partial $\mathcal{L}_{cl}$-valuation, whose domain is finite and closed under $\leq$, are usual classical partial valuations which can be obviously extended to full classical valuations (i.e. $\mathbf{M}_{cl}$-legal $\mathcal{L}_{cl}$-valuations).

Note that we use the same term "$\leq$-analytic" in two different contexts. When referring to many-valued (pre-) systems as $\leq$-analytic we mean the semantic extension property defined above (the term "analyticity" was used to describe a similar property in previous works, see e.g. [21]). On the other hand, we call a pure sequent system $\leq$-analytic if it satisfies the syntactic property given in Definition 2.3.5. In Theorem 3.3.3 below we establish a correspondence between these two notions of analyticity.

Next, we prove that $\leq$-analyticity (for a safe relation $\leq$) suffices for the effectiveness of a given many-valued system.

Theorem 3.1.18. Let $\mathbf{M}$ be a finite many-valued system for $\mathcal{L}$. Suppose that $\mathbf{M}$ is $\leq$-analytic for some safe partial order $\leq$ on $\mathcal{L}$. Given a finite set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash_\mathbf{M} \varphi$ or not.

Proof. Since $\leq$ is safe, $\downarrow^{\leq}[\Gamma \cup \{\varphi\}]$ is finite. Thus, to decide whether $\Gamma \Vdash_{\mathbf{M}} \varphi$ one can enumerate all partial $\mathcal{L}$-valuations v with $Dom_v = \downarrow^{\leq}[\Gamma \cup \{\varphi\}]$ and $Im_v = \mathcal{U}_{\mathbf{M}}$, and check if one of them satisfies the following three conditions: (1) v is $\mathbf{M}$-legal; (2) $v \models^{\mathbf{M}} \Gamma$; and (3) $v \not\models^{\mathbf{M}} \varphi$. Each of these conditions is obviously decidable for a given v. We claim that $\Gamma \Vdash_{\mathbf{M}} \varphi$ iff such a function is not found. To see this, note that if $\Gamma \nVdash_{\mathbf{M}} \varphi$, then by definition there exists an $\mathbf{M}$-legal $\mathcal{L}$-valuation v' such that $v' \models^{\mathbf{M}} \Gamma$ but $v' \not\models^{\mathbf{M}} \varphi$. Its restriction to $\downarrow^{\leq}[\Gamma \cup \{\varphi\}]$ is a function $v : \downarrow^{\leq}[\Gamma \cup \{\varphi\}] \to \mathcal{U}_{\mathbf{M}}$ satisfying the conditions above. On the other hand, if there exists such a function v, then since $\mathbf{M}$ is $\leq$-analytic, v can be extended to an $\mathbf{M}$-legal (full) $\mathcal{L}$-valuation v'. Clearly, $v' \models^{\mathbf{M}} \Gamma$ but $v' \not\models^{\mathbf{M}} \varphi$. Consequently, $\Gamma \nVdash_{\mathbf{M}} \varphi$ in this case. □

Examining the proof above, we are able to provide a slightly weaker requirement:

Theorem 3.1.19. Let $\mathbf{M}$ be a finite many-valued system for $\mathcal{L}$, and $\leq$ a safe partial order on $\mathcal{L}$. Suppose that given an $\mathbf{M}$-legal partial $\mathcal{L}$-valuation v, whose domain is finite and closed under $\leq$, it is decidable whether v can be extended to an $\mathbf{M}$-legal (full) $\mathcal{L}$-valuation or not. Then, given a finite set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash_{\mathbf{M}} \varphi$ or not.

Proof. The proof goes as the proof of Theorem 3.1.18, with the addition of a forth condition: (4) v can be extended to an $\mathbf{M}$-legal $\mathcal{L}$-valuation. □

In Chapters 4 and 5 we will use this theorem to prove the decidability of a large family of logics induced by many-valued systems of a certain restricted form (of which many-valued matrices and their non-deterministic counterparts are particular instances). The decidability of important subfamilies of pure sequent calculi will be obtained as a consequence.

Remark 3.1.20. In the literature of non-deterministic matrices (see, e.g., [9]) effectiveness is usually identified with (semantic) analyticity. However, the observations above show that this property is not a necessary condition for decidability. To guarantee the latter, instead of requiring that *all* partial valuations are extendable, it is sufficient to have an algorithm that establishes which of them are.

3.2 Semantics for Pure Sequent Calculi

In this section we show that the logics induced by pure calculi can be semantically characterised by finite many-valued systems. Thus, our goal is to construct a finite many-valued system $\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}$ for a given pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ and a set of labels $\mathtt{X} \subseteq \pounds$, for which we would have $\Vdash^{\mathtt{X}}_{\mathbf{G}} = \Vdash_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}}$. To obtain this, we begin with a construction of a many-valued *pre*-system $\mathbf{M_G}$ for a given pure $\langle\mathcal{L}, \pounds\rangle$-calculus $\mathbf{G}$, together with a definition of when an $\mathbf{M_G}$-legal $\mathcal{L}$-valuation v is a model of an $\langle\mathcal{L}, \pounds\rangle$-sequent s. Then we show that $\mathbf{M_G}$ induces the same consequence relation between *sequents* that is induced by the calculus $\mathbf{G}$. Our construction is *modular*: each syntactic ingredient corresponds to a certain semantic component, and the semantics of the whole calculus is obtained by joining all semantic components. We start with precise definitions of each component.

Truth Values Intuitively, the truth value assigned to an $\mathcal{L}$-formula φ should carry enough information to determine for which labels $\mathtt{x} \in \pounds$ the $\pounds$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi$ is "true". In general, there can be $2^{|\pounds|}$ options for that. Thus we take the truth values in the many-valued system for a given pure $\langle\mathcal{L}, \pounds\rangle$-calculus $\mathbf{G}$ to be the subsets of $\pounds$ (that is, $\mathcal{V}_{\mathbf{M_G}} = 2^{\pounds}$), and use the following definition:

Definition 3.2.1. An $\mathcal{L}$-valuation v with $Im_v \subseteq 2^{\pounds}$ is said to be a *model* of an $\pounds$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi$, written $v \models \mathtt{x}{:}\varphi$, if $\mathtt{x} \in v(\varphi)$.

Note that the last definition concerns an $\mathcal{L}$-valuation v with $Im_v \subseteq 2^{\pounds}$ regardless of a many-valued system. Clearly, given such an $\mathcal{L}$-valuation v and a formula φ, $v(\varphi) = \{\mathtt{x} \in \pounds \mid v \models \mathtt{x}{:}\varphi\}$. Thus if we have a many-valued system $\mathbf{M}$ for $\mathcal{L}$ with $\mathcal{V}_{\mathbf{M}} = 2^{\pounds}$, then $v \models^{\mathbf{M}} \varphi$ (according to Definition 3.1.6) iff the set $\{\mathtt{x} \in \pounds \mid v \models \mathtt{x}{:}\varphi\}$ is in $\mathcal{D}_{\mathbf{M}}$. In turn, sequents are intuitively interpreted as disjunctions of labelled formulas, and sets of sequents (that constitute the sets of assumptions) are conjunctions of sequents.

Definition 3.2.2. An $\mathcal{L}$-valuation v with $Im_v \subseteq 2^{\pounds}$ is said to be a *model* of:

1. an $\langle\mathcal{L}, \pounds\rangle$*-sequent* s, written $v \models s$, if $v \models \alpha$ for some $\alpha \in s$.
2. a *set* $\mathcal{S}$ *of* $\langle\mathcal{L}, \pounds\rangle$*-sequents*, written $v \models \mathcal{S}$, if $v \models s$ for every $s \in \mathcal{S}$.

These notions naturally lead to the following definition of the consequence relation between sequents induced by a many-valued (pre-) system with $\mathcal{V}_{\mathbf{M}} = 2^{\pounds}$:

Definition 3.2.3. Let $\mathbf{M}$ be a many-valued (pre-) system for $\mathcal{L}$ with $\mathcal{V}_{\mathbf{M}} = 2^{\pounds}$. An $\langle\mathcal{L}, \pounds\rangle$-sequent s *follows* from a set $\mathcal{S}$ of $\langle\mathcal{L}, \pounds\rangle$-sequents with respect to $\mathbf{M}$ (denoted by: $\mathcal{S} \vdash_{\mathbf{M}} s$) if every $\mathbf{M}$-legal $\mathcal{L}$-valuation which is a model of $\mathcal{S}$ is also a model of s.

Remark 3.2.4. The general framework of many-valued (pre-) systems presented above allows anything to serve as a truth value. However, a semantic consequence relation $\vdash_{\mathbf{M}}$ between sequents is defined here only for many-valued (pre-) systems whose truth values consist of sets of labels ($\mathcal{V}_{\mathbf{M}} = 2^{\pounds}$).

The soundness of the weakening rules directly follows from the definitions:

Proposition 3.2.5. $\{s\} \vdash_{\mathbf{M}} s \cup \{\alpha\}$ for every many-valued (pre-) system $\mathbf{M}$ for $\mathcal{L}$ with $\mathcal{V}_{\mathbf{M}} = 2^{\pounds}$, $\langle\mathcal{L}, \pounds\rangle$-sequent s, and $\pounds$-labelled $\mathcal{L}$-formulas α.

Legal Truth Values While $\mathcal{V}_{\mathbf{M_G}}$ is taken to be all subsets of $\pounds$, the set $\mathcal{U}_{\mathbf{M_G}}$ of legal truth values (those that are actually allowed to be used in $\mathbf{M_G}$-legal valuations) is determined according to the primitive rules of $\mathbf{G}$. Indeed, each primitive rule of $\mathbf{G}$ forbid some of the $2^{|\pounds|}$ options. For example, say that $\mathtt{f}, \mathtt{i} \in \pounds$, and consider the primitive rule $\{\mathtt{f}{:}p_1\}/\{\mathtt{i}{:}p_1\}$. Semantically, this rule means that an $\pounds$-labelled $\mathcal{L}$-formula of the form $\mathtt{i}{:}\varphi$ should be "true" whenever $\mathtt{f}{:}\varphi$ is "true". This semantic requirement can be easily reflected by disallowing truth values that include the label $\mathtt{f}$ but not $\mathtt{i}$. We will denote by $\pounds(r)$ the set of subsets of $\pounds$ that are not forbidden by the primitive rule r. Formally, $\pounds(r)$ is defined as follows:

Definition 3.2.6. Let $r = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ be a primitive $\pounds$-rule. Then:

$$\pounds(r) = \{\mathtt{Y} \subseteq \pounds \mid \mathtt{X}_i \cap \mathtt{Y} = \emptyset \textit{ for some } 1 \leq i \leq n \textit{ or } \mathtt{X} \cap \mathtt{Y} \neq \emptyset\}.$$

This definition is naturally extended to sets R of primitive $\pounds$-rules by: $\pounds(R) = \bigcap_{r \in R} \pounds(r)$.

Example 3.2.7. Let $\pounds = \{\mathtt{f}, \mathtt{i}, \mathtt{t}\}$. For a primitive $\pounds$-rule $r = (\{\mathtt{f}, \mathtt{t}\}{:}p_1), \{\mathtt{i}{:}p_1\}/\{\mathtt{t}{:}p_1\}$, $\pounds(r)$ consists of all subsets of $\pounds$ except for $\{\mathtt{f}, \mathtt{i}\}$.

Example 3.2.8. For a cut rule $r = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\emptyset$,

$$\pounds(r) = \{\mathtt{Y} \subseteq \pounds \mid \mathtt{X}_i \cap \mathtt{Y} = \emptyset \textit{ for some } 1 \leq i \leq n\}.$$

For an identity axiom $r = \emptyset/(\mathtt{X}{:}p_1)$,

$$\pounds(r) = \{\mathtt{Y} \subseteq \pounds \mid \mathtt{X} \cap \mathtt{Y} \neq \emptyset\}.$$

In particular, $\pounds(\emptyset/(\pounds{:}p_1)) = 2^{\pounds} \setminus \{\emptyset\}$. Note that if a pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ has at least one cut rule then $\pounds \notin \pounds(\mathsf{P}_{\mathbf{G}})$, and similarly, if $\mathbf{G}$ has at least one identity axiom then $\emptyset \notin \pounds(\mathsf{P}_{\mathbf{G}})$.

For a given pure $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, we will take $\mathcal{U}_{\mathbf{M_G}} = \pounds(\mathsf{P}_{\mathbf{G}})$ (recall that $\mathsf{P}_{\mathbf{G}}$ denotes the set of primitive rules of $\mathbf{G}$).

Example 3.2.9. For a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{G}$ with $\mathsf{P}_{\mathbf{G}} \subseteq \{(cut), (id)\}$ we have:

$$\pounds(\mathsf{P}_{\mathbf{G}}) = \begin{cases} \{\{\mathtt{f}\}, \{\mathtt{t}\}\} & \mathsf{P}_{\mathbf{G}} = \{(cut), (id)\} \\ \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}\} & \mathsf{P}_{\mathbf{G}} = \{(cut)\} \\ \{\{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\} & \mathsf{P}_{\mathbf{G}} = \{(id)\} \\ \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\} & \mathsf{P}_{\mathbf{G}} = \emptyset \end{cases}$$

Thus for an ordinary pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus with $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$ we get a two-valued semantics; for pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus with $\mathsf{P}_{\mathbf{G}} = \{(id)\}$ or $\mathsf{P}_{\mathbf{G}} = \{(cut)\}$ we get a three-valued semantics; and if $\mathsf{P}_{\mathbf{G}} = \emptyset$ we obtain a four-valued semantics.

Semantic Conditions The semantic conditions in $\Lambda_{\mathbf{M_G}}$ are straightforwardly derived from the rules in $\mathsf{R}_{\mathbf{G}}$ according to the next definitions (recall that $\mathsf{R}_{\mathbf{G}}$ denotes the set of non-primitive rules of $\mathbf{G}$):

Definition 3.2.10. $I(\cdot)$, the $\mathcal{L}$-semantic disjunction over $2^{\pounds}$ induced by:

1. an *$\pounds$-labelled $\mathcal{L}$-formula* $\mathtt{x}{:}\varphi$, is defined by $I(\mathtt{x}{:}\varphi) = \{\varphi \doteqdot \mathtt{X} \mid \{\mathtt{x}\} \subseteq \mathtt{X} \subseteq \pounds\}$.

2. an *$\langle\mathcal{L}, \pounds\rangle$-sequent* s, is defined by $I(s) = \bigcup_{\alpha\in s} I(\alpha)$.

Definition 3.2.11. The $\mathcal{L}$-semantic condition over $2^{\pounds}$ induced by a pure $\langle\mathcal{L}, \pounds\rangle$-rule $\mathcal{S}/s$, denoted by $\mathbf{Sem}(\mathcal{S}/s)$, is defined by:

$$\mathbf{Sem}(\mathcal{S}/s) = \{I(s') \mid s' \in \mathcal{S}\} \Rightarrow I(s).$$

This definition is extended to sets R of pure $\langle\mathcal{L}, \pounds\rangle$-rules in the obvious way:

$$\mathbf{Sem}(R) = \{\mathbf{Sem}(r) \mid r \in R\}.$$

Example 3.2.12. Suppose that $\supset\in \diamondsuit^2_{\mathcal{L}}$, and consider the usual $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rules for $\supset$: $(\mathtt{f}{:}\supset) = \{\mathtt{t}{:}p_1\}, \{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1 \supset p_2\}$ and $(\mathtt{t}{:}\supset) = \{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \supset p_2\}$ (see Example 2.2.6). Then:

$$\mathbf{Sem}((\mathtt{f}{:}\supset)) = \{\{p_1 \doteqdot \{\mathtt{t}\}, p_1 \doteqdot \{\mathtt{f},\mathtt{t}\}\}, \{p_2 \doteqdot \{\mathtt{f}\}, p_2 \doteqdot \{\mathtt{f},\mathtt{t}\}\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot \{\mathtt{f}\}, p_1 \supset p_2 \doteqdot \{\mathtt{f},\mathtt{t}\}\}.$$

$$\mathbf{Sem}((\mathtt{t}{:}\supset)) = \{\{p_1 \doteqdot \{\mathtt{f}\}, p_1 \doteqdot \{\mathtt{f},\mathtt{t}\}, p_2 \doteqdot \{\mathtt{t}\}, p_2 \doteqdot \{\mathtt{f},\mathtt{t}\}\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot \{\mathtt{t}\}, p_1 \supset p_2 \doteqdot \{\mathtt{f},\mathtt{t}\}\}.$$

Note that an $\mathcal{L}$-valuation v (with $Im_v \subseteq 2^{\pounds}$) satisfies these two semantic $\mathcal{L}$-conditions iff for every $\varphi_1, \varphi_2 \in \mathcal{L}$: (1) if $\mathtt{t} \in v(\varphi_1)$ and $\mathtt{f} \in v(\varphi_2)$, then $\mathtt{f} \in v(\varphi_1 \supset \varphi_2)$; and (2) if $\mathtt{f} \in v(\varphi_1)$ or $\mathtt{t} \in v(\varphi_2)$, then $\mathtt{t} \in v(\varphi_1 \supset \varphi_2)$.

Example 3.2.13. Suppose that $\wedge \in \diamondsuit^2_{\mathcal{L}}$ and $\neg \in \diamondsuit^1_{\mathcal{L}}$. For the $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rule from Example 2.2.8, $r = \{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}\neg p_1\}/\{\mathtt{f}{:}\neg(p_1 \wedge \neg p_1)\}$, $\mathbf{Sem}(r)$ is

$$\{\{p_1 \doteqdot \{\mathtt{t}\}, p_1 \doteqdot \{\mathtt{f},\mathtt{t}\}\}, \{\neg p_1 \doteqdot \{\mathtt{t}\}, \neg p_1 \doteqdot \{\mathtt{f},\mathtt{t}\}\}\} \Rightarrow \{\neg(p_1 \wedge \neg p_1) \doteqdot \{\mathtt{f}\}, \neg(p_1 \wedge \neg p_1) \doteqdot \{\mathtt{f},\mathtt{t}\}\}.$$

Note that an $\mathcal{L}$-valuation v (with $Im_v \subseteq 2^{\pounds}$) satisfies this semantic $\mathcal{L}$-conditios iff for every $\varphi \in \mathcal{L}$: if $\mathtt{t} \in v(\varphi)$ and $\mathtt{t} \in v(\neg\varphi)$, then $\mathtt{f} \in v\neg(\varphi \wedge \neg\varphi))$.

To conclude, $\mathbf{M_G}$ is defined as follows:

Definition 3.2.14. Let $\mathbf{G}$ be a pure $\langle\mathcal{L}, \pounds\rangle$-calculus. The many-valued pre-system $\mathbf{M_G}$ is defined by: $\mathcal{V}_{\mathbf{M_G}} = 2^{\pounds}$, $\mathcal{U}_{\mathbf{M_G}} = \pounds(\mathsf{P}_{\mathbf{G}})$, and $\Lambda_{\mathbf{M_G}} = \mathbf{Sem}(\mathsf{R}_{\mathbf{G}})$.

The following theorem establishes the connection between pure calculi and their corresponding many-valued pre-systems. Its proof is given in Section 3.4.

Theorem 3.2.15. Let $\mathbf{G}$ be a pure $\langle\mathcal{L}, \pounds\rangle$-calculus. Then $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M_G}}$. In other words: there exists a proof in $\mathbf{G}$ of an $\langle\mathcal{L}, \pounds\rangle$-sequent s from a set $\mathcal{S}$ of $\langle\mathcal{L}, \pounds\rangle$-sequents, iff every $\mathbf{M_G}$-legal $\mathcal{L}$-valuation which is a model of $\mathcal{S}$ is also a model of s.

Corollary 3.2.16. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus. and $\leq$ a safe partial order on $\mathcal{L}$. Suppose that given an $\mathbf{M_G}$-legal partial $\mathcal{L}$-valuation v, whose domain is finite and closed under $\leq$, it is decidable whether v can be extended to an $\mathbf{M_G}$-legal (full) $\mathcal{L}$-valuation or not. Then, given a finite set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents and a single $\langle \mathcal{L}, \pounds \rangle$-sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathbf{G}} s$ or not.

Proof. Construct $\mathbf{M_G}$ according to the definitions above. Then, enumerate all partial $\mathcal{L}$-valuations $v : \downarrow^{\leq}[\mathcal{S} \cup \{s\}] \to \mathcal{U}_{\mathbf{M_G}}$, and check if one of them is $\mathbf{M_G}$-legal, can be extended to an $\mathbf{M_G}$-legal (full) $\mathcal{L}$-valuation, and is model of $\mathcal{S}$ but not of s. As in Theorem 3.1.19, we have that $\mathcal{S} \vdash_{\mathbf{M_G}} s$ iff such a partial $\mathcal{L}$-valuation is not found. □

Now, to obtain a many-valued system for the *logic* induced by $\mathbf{G}$ and a set $\mathtt{X}$ of labels, one should take the designated truth values to be subsets of $\pounds$ that contain at least one label from $\mathtt{X}$:

Definition 3.2.17. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, and let $\mathtt{X} \subseteq \pounds$. The many-valued system $\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}$ is obtained by augmenting the many-valued pre-system $\mathbf{M_G}$ with $\mathcal{D}_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} = \{\mathtt{X}' \subseteq \pounds \mid \mathtt{X}' \cap \mathtt{X} \neq \emptyset\}$.

Corollary 3.2.18. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, and $\mathtt{X} \subseteq \pounds$ a set of labels. Then $\Vdash^{\mathtt{X}}_{\mathbf{G}} = \Vdash_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}}$.

Proof. Note that an $\mathbf{M_G}$-legal valuation v is a model of a sequent of the form $(\mathtt{X}{:}\varphi)$ iff it is a model of φ with respect to $\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}$. Therefore, the claim is an immediate corollary of Theorem 3.2.15. For the convenience of the reader, we prove one direction with all details: Suppose that $\mathcal{T} \Vdash^{\mathtt{X}}_{\mathbf{G}} \varphi$. Then, by Definition 2.2.18, $\{(\mathtt{X}{:}\psi) \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} (\mathtt{X}{:}\varphi)$. By Theorem 3.2.15, we have that every $\mathbf{M_G}$-legal $\mathcal{L}$-valuation which is a model of $\{(\mathtt{X}{:}\psi) \mid \psi \in \mathcal{T}\}$, is also a model of $(\mathtt{X}{:}\varphi)$. We prove that $\mathcal{T} \Vdash_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} \varphi$. Thus, by Definition 3.1.6, we should show that for every $\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation v: $v \models^{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} \varphi$ whenever $v \models^{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} \mathcal{T}$. Let v be an $\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}$-legal $\mathcal{L}$-valuation such that $v \models^{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} \mathcal{T}$. By definition, $v(\psi) \in \mathcal{D}_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}}$ for every $\psi \in \mathcal{T}$. Now, the definition of $\mathcal{D}_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}}$ entails that $v(\psi) \cap \mathtt{X} \neq \emptyset$ for every $\psi \in \mathcal{T}$. It follows, according to Definitions 3.2.1 and 3.2.2, that $v \models (\mathtt{X}{:}\psi)$ for every $\psi \in \mathcal{T}$. Consequently, $v \models \{(\mathtt{X}{:}\psi) \mid \psi \in \mathcal{T}\}$, and so $v \models (\mathtt{X}{:}\varphi)$. Thus $v(\varphi) \cap \mathtt{X} \neq \emptyset$, and so $v(\varphi) \in \mathcal{D}_{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}}$. It follows that $v \models^{\mathbf{M}^{\mathtt{X}}_{\mathbf{G}}} \varphi$. □

Example 3.2.19. As a particular instance we obtain the soundness and completeness of $\mathbf{LK}$ and $\{\mathtt{t}\}$ for classical logic. Indeed, consider the pure $\langle \mathcal{L}_{cl}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{LK}$ from Example 2.2.20. By Corollary 3.2.18, we have $\Vdash^{\{\mathtt{t}\}}_{\mathbf{LK}} = \Vdash_{\mathbf{M}^{\{\mathtt{t}\}}_{\mathbf{LK}}}$. It is straightforward to verify that $\mathbf{M_{LK}}$-legal valuations are practically classical two-valued valuations. Indeed, since $\mathbf{LK}$ includes both (cut) and (id), we have $\mathcal{U}_{\mathbf{M_{LK}}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$. The semantic $\mathcal{L}_{cl}$-conditions arising from the non-primitive rules of $\mathbf{LK}$ provide the usual definition of truth values of compound formulas in classical logic (the rules for each connective enforce its usual truth table, see, e.g., Example 3.2.12).

Example 3.2.20. By applying Corollary 3.2.18 to the pure $\langle \mathcal{L}_{cl}, \mathscr{L}_{\mathbf{2}} \rangle$-calculus $\mathbf{G_{C_1}}$ from Example 2.2.22 and $\mathtt{X} = \{\mathtt{t}\}$, we obtain effective semantics for da Costa's paraconsistent logic $\mathbf{C_1}$. The many-valued system $\mathbf{M} = \mathbf{M}^{\{\mathtt{t}\}}_{\mathbf{G_{C_1}}}$ is given by: $\mathcal{V}_{\mathbf{M}} = \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$; $\mathcal{U}_{\mathbf{M}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$; $\mathcal{D}_{\mathbf{M}} = \{\{\mathtt{t}\}\}$; and $\Lambda_{\mathbf{M}}$ consists of one $\mathcal{L}_{cl}$-semantic condition over $\mathcal{V}_{\mathbf{M}}$ for each non-primitive rule of $\mathbf{G_{C_1}}$. For example, for the rule $(\mathtt{f}{:}\neg\wedge^1)$ we have the semantic condition given in Example 3.2.13. Similarly, $(\mathtt{f}{:}\neg\neg)$ and $(\mathtt{f}{:}\neg\wedge^2)$ yield the following $\mathcal{L}_{cl}$-semantic conditions:

$$\mathbf{Sem}((\mathtt{f}{:}\neg\neg)) = \{\{p_1 \doteqdot \{\mathtt{f}\}, p_1 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}\} \Rightarrow \{\neg\neg p_1 \doteqdot \{\mathtt{f}\}, \neg\neg p_1 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}$$

$$\mathbf{Sem}((\mathtt{f}{:}\neg\wedge^2)) = \{\{\neg p_1 \doteqdot \{\mathtt{f}\}, \neg p_1 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}, \{\neg p_2 \doteqdot \{\mathtt{f}\}, \neg p_2 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}\} \Rightarrow \{\neg(p_1 \wedge p_2) \doteqdot \{\mathtt{f}\}, \neg(p_1 \wedge p_2) \doteqdot \{\mathtt{f}, \mathtt{t}\}\}$$

It is easy to see that the conditions in $\Lambda_{\mathbf{M}}$ (including the ones mentioned above) dictate the following requirements from $\mathbf{M}$-legal $\mathcal{L}_{cl}$-valuations v for every formula ψ:[3]

$(\mathtt{t}{:}\neg)$ If $\psi = \neg\varphi$ and $\mathtt{f} \in v(\varphi)$, then $\mathtt{t} \in v(\psi)$.
$(\mathtt{f}{:}\wedge)$ If $\psi = \varphi_1 \wedge \varphi_2$ and ($\mathtt{f} \in v(\varphi_1)$ or $\mathtt{f} \in v(\varphi_2)$), then $\mathtt{f} \in v(\psi)$.
$(\mathtt{t}{:}\wedge)$ If $\psi = \varphi_1 \wedge \varphi_2$, $\mathtt{t} \in v(\varphi_1)$ and $\mathtt{t} \in v(\varphi_2)$, then $\mathtt{t} \in v(\psi)$.
$(\mathtt{f}{:}\vee)$ If $\psi = \varphi_1 \vee \varphi_2$, $\mathtt{f} \in v(\varphi_1)$ and $\mathtt{f} \in v(\varphi_2)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{t}{:}\vee)$ If $\psi = \varphi_1 \vee \varphi_2$ and ($\mathtt{t} \in v(\varphi_1)$ or $\mathtt{t} \in v(\varphi_2)$), then $\mathtt{t} \in v(\psi)$.
$(\mathtt{f}{:}\supset)$ If $\psi = \varphi_1 \supset \varphi_2$, $\mathtt{t} \in v(\varphi_1)$ and $\mathtt{f} \in v(\varphi_2)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{t}{:}\supset)$ If $\psi = \varphi_1 \supset \varphi_2$ and ($\mathtt{f} \in v(\varphi_1)$ or $\mathtt{t} \in v(\varphi_2)$), then $\mathtt{t} \in v(\psi)$.
$(\mathtt{f}{:}\neg\neg)$ If $\psi = \neg\neg\varphi$ and $\mathtt{f} \in v(\varphi)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\wedge^1)$ If $\psi = \neg(\varphi \wedge \neg\varphi)$, $\mathtt{t} \in v(\varphi)$ and $\mathtt{t} \in v(\neg\varphi)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\wedge^2)$ If $\psi = \neg(\varphi_1 \wedge \varphi_2)$, $\mathtt{f} \in v(\neg\varphi_1)$ and $\mathtt{f} \in v(\neg\varphi_2)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\vee^1)$ If $\psi = \neg(\varphi_1 \vee \varphi_2)$, $\mathtt{f} \in v(\neg\varphi_1)$ and ($\mathtt{f} \in v(\varphi_2)$ or $\mathtt{f} \in v(\neg\varphi_2)$), then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\vee^2)$ If $\psi = \neg(\varphi_1 \vee \varphi_2)$, ($\mathtt{f} \in v(\varphi_1)$ or $\mathtt{f} \in v(\neg\varphi_1)$) and $\mathtt{f} \in v(\neg\varphi_2)$, then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\supset^1)$ If $\psi = \neg(\varphi_1 \supset \varphi_2)$, $\mathtt{f} \in v(\varphi_1)$ and ($\mathtt{f} \in v(\varphi_2)$ or $\mathtt{f} \in v(\neg\varphi_2)$), then $\mathtt{f} \in v(\psi)$.
$(\mathtt{f}{:}\neg\supset^2)$ If $\psi = \neg(\varphi_1 \supset \varphi_2)$, ($\mathtt{f} \in v(\varphi_1)$ or $\mathtt{f} \in v(\neg\varphi_1)$) and $\mathtt{f} \in v(\neg\varphi_2)$, then $\mathtt{f} \in v(\psi)$.

It is easy to verify that these requirements correspond exactly to the conditions on C_1-bivaluations described in [34].

Now, $\mathbf{M}$ is not *sub*-analytic (see Definition 3.1.16). Indeed, consider the partial $\mathcal{L}_{cl}$-valuation v with $Dom_v = \{p_1, p_2, \neg p_1, \neg p_2, \neg\neg p_1, \neg p_1 \wedge \neg p_2, \neg(\neg p_1 \wedge \neg p_2)\}$, and:

[3] Note that since $\mathcal{U}_{\mathbf{M}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$, we can write $v(\varphi) = \{\mathtt{x}\}$ instead of $\mathtt{x} \in v(\varphi)$ (for $\mathtt{x} \in \{\mathtt{f}, \mathtt{t}\}$). We prefer the latter since we will reuse this list of conditions in Example 3.3.10 where $\{\mathtt{f}, \mathtt{t}\}$ is also included in $\mathcal{U}_{\mathbf{M}}$.

- $v(p_1) = v(p_2) = v(\neg\neg p_1) = \{\mathtt{f}\}$,
- $v(\neg p_1) = v(\neg p_2) = v(\neg p_1 \wedge \neg p_2) = v(\neg(\neg p_1 \wedge \neg p_2)) = \{\mathtt{t}\}$.

Dom_v is finite and closed under subformulas, and v is **M**-legal. Now, assume for a contradiction that there is an **M**-legal (full) $\mathcal{L}$-valuation v' that extends v. ($\mathtt{f}$:$\neg\neg$) enforces that $v'(\neg\neg p_2) = \{\mathtt{f}\}$. On the other hand, if $v'(\neg\neg p_2) = \{\mathtt{f}\}$, then ($\mathtt{f}$:$\neg\wedge^2$) enforces that $v'(\neg(\neg p_1 \wedge \neg p_2)) = \{\mathtt{f}\}$. But, $v'(\neg(\neg p_1 \wedge \neg p_2)) = v(\neg(\neg p_1 \wedge \neg p_2)) = \{\mathtt{t}\}$.

Nevertheless, it can be shown that **M** is *nsub*-analytic, where *nsub* denotes the transitive closure of $sub \cup \{\langle \neg\varphi_i, \neg(\varphi_1 \diamond \varphi_2)\rangle \mid \varphi_1, \varphi_2 \in \mathcal{L}, \diamond \in \{\wedge, \vee, \supset\}, i = 1, 2\}$. Indeed, let v be an **M**-legal partial $\mathcal{L}_{cl}$-valuation, whose domain is finite and closed under *nsub*. We construct an **M**-legal (full) $\mathcal{L}_{cl}$-valuation v' that extends v. Let $\psi_1, \psi_2, ...$ be an enumeration of all $\mathcal{L}_{cl}$ formulas such that: $i \leq j$ whenever $\langle \psi_i, \psi_j \rangle \in nsub$. We recursively construct v'. Let $i \geq 1$, and suppose that $v'(\psi_j)$ was defined for every $j < i$. $v'(\psi_i)$ is defined as follows. First, if $\psi_i \in Dom_v$ then $v'(\psi_i) = v(\psi_i)$. Otherwise, if ψ_i is an atomic formula $v'(\psi_i) = \{\mathtt{f}\}$ (say). Otherwise, ψ_i is a compound formula and then $v'(\psi_i)$ is set to be either $\{\mathtt{f}\}$ or $\{\mathtt{t}\}$ based on "classical logic reasoning" using the subformulas of ψ_i (for example, if $\psi_i = \neg\psi_j$ then $v'(\psi_i) = \{\mathtt{f}\}$ if $v'(\psi_j) = \{\mathtt{t}\}$, and otherwise $v'(\psi_i) = \{\mathtt{t}\}$). Obviously, v' extends v. It remains to show that v' is **M**-legal. For that we prove by induction on i that all the properties above hold for v' and $\psi = \psi_i$. Suppose they hold for ψ_j for every $j < i$. We do here several cases (the others are similar):

($\mathtt{t}$:$\neg$) Suppose that $\psi = \neg\varphi$ and $v'(\varphi) = \{\mathtt{f}\}$. If $\psi \in Dom_v$, then $\varphi \in Dom_v$ as well, and $v'(\psi) = \{\mathtt{t}\}$ follows since v is **M**-legal. Otherwise, $v'(\psi) = \{\mathtt{t}\}$ as well, but this time because of the classical truth tables.

($\mathtt{f}$:$\neg\neg$) Suppose that $\psi = \neg\neg\varphi$ and $v'(\varphi) = \{\mathtt{f}\}$. If $\psi \in Dom_v$, then $\varphi \in Dom_v$ as well, and $v'(\psi) = \{\mathtt{f}\}$ follows since v is **M**-legal. Otherwise, $v'(\neg\varphi) = \{\mathtt{t}\}$ (by the induction hypothesis since $\neg\varphi = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\neg$)), and thus $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

($\mathtt{f}$:$\neg\wedge^1$) Suppose that $\psi = \neg(\varphi \wedge \neg\varphi)$, $v'(\varphi) = \{\mathtt{t}\}$ and $v'(\neg\varphi) = \{\mathtt{t}\}$. If $\psi \in Dom_v$, then $\varphi, \neg\varphi \in Dom_v$ as well, and $v'(\psi) = \{\mathtt{f}\}$ follows since v is **M**-legal. Otherwise, $v'(\varphi \wedge \neg\varphi) = \{\mathtt{t}\}$ (by the induction hypothesis since $\varphi \wedge \neg\varphi = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\wedge$)), and so $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

($\mathtt{f}$:$\neg\wedge^2$) Suppose that $\psi = \neg(\varphi_1 \wedge \varphi_2)$, $v'(\neg\varphi_1) = \{\mathtt{f}\}$ and $v'(\neg\varphi_2) = \{\mathtt{f}\}$. If $\psi \in Dom_v$, then $\neg\varphi_1, \neg\varphi_2 \in Dom_v$ as well, and $v'(\psi) = \{\mathtt{f}\}$ follows since v is **M**-legal. Otherwise, $v'(\varphi_1) = v'(\varphi_2) = \{\mathtt{t}\}$ according to the classical truth tables (by the induction hypothesis since $\neg\varphi_1 = \psi_{j_1}$ and $\neg\varphi_2 = \psi_{j_2}$ for some $j_1, j_2 < i$, and the condition ($\mathtt{t}$:$\neg$)). Thus $v'(\varphi_1 \wedge \varphi_2) = \{\mathtt{t}\}$ (by the induction hypothesis since $\varphi_1 \wedge \varphi_2 = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\wedge$)), and so $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

Note that *nsub* is safe, and thus it follows that **M** provides an *effective* semantics for the logic da Costa's paraconsistent logic $\mathbf{C_1}$. We note that a semantic decision for this logic was included in [47]. While its formulation is completely different than ours, the procedure in [47] is based on similar ideas. In particular, a notion equivalent to *nsub*-analyticity plays a major role there as well.

3.3 Characterization of Proof-Theoretic Properties

In this section we use the above general soundness and completeness theorem (and provide some extensions of it) for deriving semantic characterizations of the proof-theoretic properties of pure calculi discussed in Section 2.3.

3.3.1 Strong Analyticity

Analyticity for a given calculus is traditionally obtained as a corollary of cut-admissibility (this was the case in the seminal work of Gentzen [56]). Indeed, if all rules in a pure calculus system (except for (*cut*)) admit *the local subformula property* (i.e., the premises of each rule consist only of subformulas of the formulas its conclusion), then cut-admissibility implies *sub*-analyticity.[4] However, there are many cases in which a calculus does not enjoy cut-admissibility, and it is analytic nevertheless. Thus we provide a semantic characterization of strong analyticity which is independent of cut-admissibility. To do so, we need to identify semantics for proofs in which only some formulas may appear. This can be easily done by considering *partial* valuations (see Definition 3.1.13), whose domain consists of all formulas that may be used in proofs.

First, Definitions 3.2.1 and 3.2.2 are adapted to partial $\mathcal{L}$-valuations as follows:

Definition 3.3.1. A partial $\mathcal{L}$-valuation v with $Im_v \subseteq 2^{\pounds}$ is said to be a *model* of:

1. an *£-labelled $\mathcal{L}$-formula* $\mathtt{x}{:}\varphi$ if $\varphi \in Dom_v$ and $\mathtt{x} \in v(\varphi)$.
2. an *$\langle \mathcal{L}, \pounds \rangle$-sequent* s if s is a $\langle Dom_v, \pounds \rangle$-sequent and v is a model of some $\alpha \in s$.
3. a *set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents* if v is a model of every $\langle Dom_v, \pounds \rangle$-sequent $s \in \mathcal{S}$.

We write $v \models X$ to denote that v is a model of X, where X is either an £-labelled $\mathcal{L}$-formula, an $\langle \mathcal{L}, \pounds \rangle$-sequent, or a set of $\langle \mathcal{L}, \pounds \rangle$-sequents.

[4] Quoting Ono [79]:

> "The most important proof-theoretic property is the subformula property, and the most convenient way of showing the subformula property is to show the cut elimination theorem."

Generally, we agree with the first observation (though, we believe that other notions of analyticity based on different relation than "subformula" have similar importance). However, we aim to show that in many cases cut elimination is not necessarily "the most convenient" technique.

Obviously, Definitions 3.2.1 and 3.2.2 are obtained from Definition 3.3.1 by taking $Dom_v = \mathcal{L}$. Note also that in Item 2 a partial $\mathcal{L}$-valuation v can only be a model of sequents consisting solely of formulas in Dom_v. Nevertheless, it can be a model of a *set* of sequents containing formulas which are not in Dom_v, because only $\langle Dom_v, \pounds\rangle$-sequents are considered in Item 3.

Now, the following theorem strengthens Theorem 3.2.15, by showing that proofs that consist only of formulas from a set $\mathcal{F}$ precisely correspond to the semantics given by partial valuations whose domain is $\mathcal{F}$.

Theorem 3.3.2. Let $\mathbf{G}$ be a pure $\langle\mathcal{L}, \pounds\rangle$-calculus, $\mathcal{F}$ a set of $\mathcal{L}$-formulas, $\mathcal{S}$ a set of $\langle\mathcal{L}, \pounds\rangle$-sequents, and s an $\langle\mathcal{F}, \pounds\rangle$-sequent. Then, $\mathcal{S} \vdash_{\mathbf{G}}^{\mathcal{F}} s$ (i.e. there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ consisting only of $\langle\mathcal{F}, \pounds\rangle$-sequents, see Notation 2.3.4) iff for every $\mathbf{M_G}$-legal partial $\mathcal{L}$-valuation v with $Dom_v = \mathcal{F}$: if v is a model of $\mathcal{S}$ then it is also a model of s.

The proof is given in Section 3.4. We can now establish the connection between the (syntactic) strong $\leq$-analyticity of $\mathbf{G}$ and the (semantic) $\leq$-analyticity of $\mathbf{M_G}$ (see Definition 3.1.16).

Theorem 3.3.3. A pure $\langle\mathcal{L}, \pounds\rangle$-calculus $\mathbf{G}$ is strongly $\leq$-analytic iff $\mathbf{M_G}$ is $\leq$-analytic.

Proof. ($\Rightarrow$) Suppose that $\mathbf{M_G}$ is not $\leq$-analytic. Let v be an $\mathbf{M_G}$-legal partial $\mathcal{L}$-valuation whose domain is finite and closed under $\leq$, but there does not exist an $\mathbf{M_G}$-legal $\mathcal{L}$-valuation that extends v. Let $\mathcal{S}$ and s be the set of $\langle\mathcal{L}, \pounds\rangle$-sequents and the $\langle\mathcal{L}, \pounds\rangle$-sequent defined by:

$$\mathcal{S} = \{\{\mathtt{x}{:}\varphi\} \mid \varphi \in Dom_v, \mathtt{x} \in v(\varphi)\}, \qquad s = \{\mathtt{x}{:}\varphi \mid \varphi \in Dom_v, \mathtt{x} \notin v(\varphi)\}.$$

Then, by definition $v \models \mathcal{S}$ and $v \not\models s$. By Theorem 3.3.2 we have $\mathcal{S} \not\vdash_{\mathbf{G}}^{Dom_v} s$. We show that $\mathcal{S} \vdash_{\mathbf{G}} s$. Since Dom_v is closed under $\leq$, $\downarrow^{\leq}[\mathcal{S} \cup \{s\}] = Dom_v$, and it would follow that $\mathbf{G}$ is not strongly $\leq$-analytic. Let v' be an $\mathbf{M_G}$-legal $\mathcal{L}$-valuation. Our assumption entails that v' does not extend v. Therefore there is some $\varphi \in Dom_v$ such that $v'(\varphi) \neq v(\varphi)$. Thus, at least one of the following holds: (i) there is some $\mathtt{x} \in \pounds$, such that $\mathtt{x} \in v'(\varphi)$ and $\mathtt{x} \notin v(\varphi)$; ($ii$) there is some $\mathtt{x} \in \pounds$, such that $\mathtt{x} \in v(\varphi)$ and $\mathtt{x} \notin v'(\varphi)$. If ($i$) holds, then $v' \models s$. If (ii) holds, then $v' \not\models \{\mathtt{x}{:}\varphi\}$, and thus $v' \not\models \mathcal{S}$. It follows that v' is either a model of s, or not a model of $\mathcal{S}$. Consequently, every $\mathbf{M_G}$-legal $\mathcal{L}$-valuation which is a model of $\mathcal{S}$ is also a model of s. Hence Theorem 3.2.15 implies that $\mathcal{S} \vdash_{\mathbf{G}} s$.

($\Leftarrow$) Suppose that $\mathbf{M_G}$ is $\leq$-analytic. We show that $\mathbf{G}$ is strongly $\leq$-analytic. Let $\mathcal{S} \cup \{s\}$ be a set of $\langle\mathcal{L}, \pounds\rangle$-sequents. Suppose that $\mathcal{S} \not\vdash_{\mathbf{G}}^{\mathcal{F}} s$ for $\mathcal{F} = \downarrow^{\leq}[\mathcal{S} \cup \{s\}]$. We show that $\mathcal{S} \not\vdash_{\mathbf{G}} s$. It suffices to show that for every finite subset $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{S}' \not\vdash_{\mathbf{G}} s$. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a finite subset. Obviously, $\mathcal{S}' \not\vdash_{\mathbf{G}}^{\mathcal{F}'} s$ for $\mathcal{F}' = \downarrow^{\leq}[\mathcal{S}' \cup \{s\}]$. By Theorem 3.3.2 (note that s is an $\langle\mathcal{F}', \pounds\rangle$-sequent), there is some $\mathbf{M_G}$-legal partial $\mathcal{L}$-valuation v with

$Dom_v = \mathcal{F}'$, which is a model of $\mathcal{S}'$ but not of s. $\mathcal{F}'$ is finite and closed under $\leq$, and thus our assumption entails that there exists an $\mathbf{M_G}$-legal $\mathcal{L}$-valuation v' that extends v. By definition, v' is a model of some $\langle Dom_v, \pounds\rangle$-sequent s' iff v is a model of s'. It follows that v' is a model of $\mathcal{S}'$ but not of s. Hence Theorem 3.2.15 implies that $\mathcal{S}' \not\vdash_{\mathbf{G}} s$. $\square$

Example 3.3.4. Consider again the pure $\langle \mathcal{L}_{cl}, \pounds_{\mathbf{2}}\rangle$-calculus $\mathbf{G_{C_1}}$ from Example 2.2.22. Following Example 3.2.20, the many-valued pre-system $\mathbf{M_{G_{C_1}}}$ is not *sub*-analytic. Thus, by Theorem 3.3.3, $\mathbf{G_{C_1}}$ is not strongly *sub*-analytic. However, in the same example we showed that $\mathbf{M_{G_{C_1}}}$ is *nsub*-analytic for the (safe) partial order *nsub* defined there. Therefore Theorem 3.3.3 implies that $\mathbf{G_{C_1}}$ is strongly *nsub*-analytic. Note that we prove below that $\mathbf{G_{C_1}}$ enjoys strong cut-admissibility (using another semantic characterization, see Example 3.3.10). The fact that it is strongly *nsub*-analytic follows from this proof too since all rules of $\mathbf{G_{C_1}}$ except for (cut) are closed under *nsub* (that is, for each formula φ in a premise of a rule, there is some ψ in its conclusion such that $\langle \varphi, \psi\rangle \in nsub$).

Example 3.3.5. Suppose that $\mathcal{L}$ consists of one binary connective $\between$. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}}\rangle$-calculus, with $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$, and the following pure $\langle \mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rules:

$$(\mathtt{f}{:}\between) \quad \{\mathtt{f}{:}p_1\}, \{\mathtt{f}{:}p_2\} / \{\mathtt{f}{:}p_1 \between p_2\} \qquad (\mathtt{t}{:}\between) \quad \{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\} / \{\mathtt{t}{:}p_1 \between p_2\}$$

$$(sym) \quad \emptyset / \{\mathtt{f}{:}p_1 \between p_2, \mathtt{t}{:}p_2 \between p_1\}$$

Then partial $\mathcal{L}$-valuations (whose domain is closed under subformulas) are $\mathbf{M_G}$-legal iff they satisfy the following conditions:

- $v(\varphi) \in \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$ for every $\varphi \in Dom_v$.
- If $\varphi_1 \between \varphi_2 \in Dom_v$ for some $\varphi_1, \varphi_2 \in \mathcal{L}$, then:
  - If $v(\varphi_1) = v(\varphi_2) = \{\mathtt{f}\}$ then $v(\varphi_1 \between \varphi_2) = \{\mathtt{f}\}$.
  - If $v(\varphi_1) = v(\varphi_2) = \{\mathtt{t}\}$ then $v(\varphi_1 \between \varphi_2) = \{\mathtt{t}\}$.
  - If $\varphi_2 \between \varphi_1 \in Dom_v$ then $v(\varphi_1 \between \varphi_2) = v(\varphi_2 \between \varphi_1)$.

To see the reason for the last condition, note that

$$\mathbf{Sem}((sym)) = \emptyset \Rightarrow \{p_1 \between p_2 \doteqdot \{\mathtt{f}\}, p_1 \between p_2 \doteqdot \{\mathtt{f},\mathtt{t}\}, p_2 \between p_1 \doteqdot \{\mathtt{t}\}, p_2 \between p_1 \doteqdot \{\mathtt{f},\mathtt{t}\}\}.$$

A partial $\mathcal{L}$-valuation v with $Im_v \subseteq \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$ satisfies $\mathbf{Sem}((sym))$ if for every $\mathcal{L}$-substitution σ such that $\{\sigma(p_1 \between p_2), \sigma(p_2 \between p_1)\} \subseteq Dom_v$, we have that $\mathtt{f} \in v(\sigma(p_1 \between p_2))$ or $\mathtt{t} \in v(\sigma(p_2 \between p_1))$. Equivalently, v satisfies $\mathbf{Sem}((sym))$ if for every $\mathcal{L}$-formulas φ_1 and φ_2 such that $\{\varphi_1 \between \varphi_2, \varphi_2 \between \varphi_1\} \subseteq Dom_v$, we have that $v(\varphi_1 \between \varphi_2) = \{\mathtt{f}\}$ or $v(\varphi_2 \between \varphi_1) = \{\mathtt{t}\}$. By "switching the roles of φ_1 and φ_2", we obtain that v satisfies $\mathbf{Sem}((sym))$ if for every $\mathcal{L}$-formulas φ_1 and φ_2 such that $\{\varphi_1 \between \varphi_2, \varphi_2 \between \varphi_1\} \subseteq Dom_v$, we have that $(v(\varphi_1 \between \varphi_2) = \{\mathtt{f}\}$ or $v(\varphi_2 \between \varphi_1) = \{\mathtt{t}\})$ and $(v(\varphi_2 \between \varphi_1) = \{\mathtt{f}\}$ or $v(\varphi_1 \between \varphi_2) = \{\mathtt{t}\})$. The last condition above is equivalent to this requirement.

Using Theorem 3.3.3, it easily follows that **G** is strongly *sub*-analytic. Roughly speaking, given an $\mathbf{M_G}$-legal partial $\mathcal{L}$-valuation v (whose domain is closed under subformulas), we can recursively extend v to an $\mathcal{L}$-valuation by setting $v(\varphi_1 \between \varphi_2) = v(\varphi_1)$ if $v(\varphi_1) = v(\varphi_2)$; otherwise $v(\varphi_1 \between \varphi_2) = v(\varphi_2 \between \varphi_1)$ if $v(\varphi_2 \between \varphi_1)$ was defined before; and $v(\varphi_1 \between \varphi_2) = \{\mathtt{f}\}$ (say) otherwise.

3.3.2 Strong Cut-Admissibility

To obtain a simple semantic characterization of strong cut-admissibility, we slightly extended Theorem 3.2.15 by: (*a*) considering "extended sequents" that may be infinite; and (*b*) restricting the truth values of certain formulas (those on which cut is allowed) to a certain subset of $\mathcal{V}_\mathbf{M}$.

Definition 3.3.6. An *extended* $\langle \mathcal{L}, \pounds \rangle$*-sequent* is a (possibly infinite) set of $\pounds$-labelled $\mathcal{L}$-formulas. An $\mathcal{L}$-valuation v with $Im_v \subseteq 2^{\pounds}$ is said to be a *model* of an extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ, written $v \models \mu$, if $v \models \alpha$ for some $\alpha \in \mu$.

Definition 3.3.7. Let **M** be a many-valued pre-system for $\mathcal{L}$, $\mathcal{U} \subseteq \mathcal{V}_\mathbf{M}$ a set of truth values, and $\mathcal{C}$ a set of $\mathcal{L}$-formulas. An **M**-legal $\mathcal{L}$-valuation v is called $\langle \mathcal{U}, \mathcal{C} \rangle$*-restricted* $v(\varphi) \in \mathcal{U}$ for every $\varphi \in \mathcal{C}$.

Theorem 3.3.8. The following are equivalent for every pure $\langle \mathcal{L}, \pounds \rangle$-calculus **G**, set $\mathcal{C}$ of $\mathcal{L}$-formulas, set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents, and extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ:

- $\mathcal{S} \vdash_{\mathbf{G}}^{cuts:\mathcal{C}} s$ for some $\langle \mathcal{L}, \pounds \rangle$-sequent $s \subseteq \mu$ (recall that $\mathcal{S} \vdash_{\mathbf{G}}^{cuts:\mathcal{C}} s$ denotes that there exists a proof in **G** of s from $\mathcal{S}$ in which the cut-formula of every application of a cut rule is an element of $\mathcal{C}$).
- Every $\langle \mathcal{U}_{\mathbf{M_G}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal $\mathcal{L}$-valuation which is a model of $\mathcal{S}$ is also a model of μ (recall that $\mathbf{G}_{cf}$ is the calculus obtained from **G** by discarding all cut rules).

The proof is given in Section 3.4. Using this soundness and completeness theorem, we obtain the following semantic characterization of strong cut-admissibility in pure calculi.

Theorem 3.3.9. A pure $\langle \mathcal{L}, \pounds \rangle$-calculus **G** enjoys strong cut-admissibility iff for every $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v, there exists an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v' such that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$.[5]

[5]Recall that for a pure $\langle \mathcal{L}, \pounds_\mathbf{2} \rangle$-calculus **G** whose only cut rule is (*cut*), we have that $s(\mathbf{G}) = \mathbf{G}$ (see Definition 2.3.13). In this case $\mathbf{M}_{s(\mathbf{G})_{cf}}$ can be replaced by $\mathbf{M}_{\mathbf{G}_{cf}}$.

Before proving this theorem, we demonstrate its usefulness in the particular case of $\mathbf{G_{C_1}}$.

Example 3.3.10. Consider again the pure $\langle \mathcal{L}_{cl}, \mathscr{L}_{\mathbf{2}} \rangle$-calculus $\mathbf{G_{C_1}}$ from Examples 2.2.22 and 3.3.4. In [12] it was shown that $\mathbf{G_{C_1}}$ enjoys cut-admissibility (and thus, by Theorem 2.3.19 it enjoys strong cut-admissibility as well). We use Theorem 3.3.9 to show that this fact can be obtained by our semantic criterion above. The only cut rule of $\mathbf{G_{C_1}}$ is (cut), and thus $s(\mathbf{G_{C_1}}) = \mathbf{G_{C_1}}$. The many-valued pre-system $\mathbf{M} = \mathbf{M}_{\mathbf{G_{C_1}}^{cf}}$ has $\mathcal{U}_{\mathbf{M}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$. The conditions in $\Lambda_{\mathbf{M}} = \mathbf{Sem}(\mathsf{R}_{\mathbf{G_{C_1}}})$ are those described in Example 3.2.20. Now, let v be an $\mathbf{M}$-legal $\mathcal{L}_{cl}$-valuation. We construct an $\mathbf{M}$-legal $\mathcal{L}_{cl}$-valuation v' that satisfies the condition in Theorem 3.3.9. Let $\psi_1, \psi_2, ...$ be an enumeration of all $\mathcal{L}_{cl}$ formulas such that $i \leq j$ whenever $\langle \psi_i, \psi_j \rangle \in nsub$ (see Example 3.2.20 for the definition of $nsub$). We recursively construct v'. Let $i \geq 1$, and suppose that $v'(\psi_j)$ was defined for every $j < i$. $v'(\psi_i)$ is defined as follows. First, if $v(\psi_i) \neq \{\mathtt{f}, \mathtt{t}\}$, then $v'(\psi_i) = v(\psi_i)$. Otherwise, if ψ_i is an atomic formula $v'(\psi_i) = \{\mathtt{f}\}$ (say). Otherwise, ψ_i is a compound formula and then $v'(\psi_i)$ is set to be either $\{\mathtt{f}\}$ or $\{\mathtt{t}\}$ based on "classical logic reasoning" using the subformulas of ψ_i (for example, if $\psi_i = \neg\psi_j$ then $v'(\psi_i) = \{\mathtt{f}\}$ if $v'(\psi_j) = \{\mathtt{t}\}$, and otherwise $v'(\psi_i) = \{\mathtt{t}\}$). Obviously, $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \{\mathtt{f}, \mathtt{t}\}$. It remains to show that v' is $\mathbf{M}$-legal. For that we prove by induction on i that all the requirements of $\Lambda_{\mathbf{M}}$ (listed in Example 3.2.20) hold for v' and $\psi = \psi_i$. Suppose they hold for ψ_j for every $j < i$.

($\mathtt{t}$:$\neg$) Suppose that $\psi = \neg\varphi$ and $\mathtt{f} \in v'(\varphi)$ (i.e. $v'(\varphi) = \{\mathtt{f}\}$). Then $\mathtt{f} \in v(\varphi)$, and since v is $\mathbf{M}$-legal, $v(\psi)$ is either $\{\mathtt{t}\}$ or $\{\mathtt{f}, \mathtt{t}\}$. In the first, case we have $v'(\psi) = v(\psi) = \{\mathtt{t}\}$. In the latter, $v'(\psi) = \{\mathtt{t}\}$ as well, but this time because of the classical truth tables.

($\mathtt{f}$:$\neg\neg$) Suppose that $\psi = \neg\neg\varphi$ and $\mathtt{f} \in v'(\varphi)$ (i.e. $v'(\varphi) = \{\mathtt{f}\}$). Then $\mathtt{f} \in v(\varphi)$, and since v is $\mathbf{M}$-legal, $v(\psi)$ is either $\{\mathtt{f}\}$ or $\{\mathtt{f}, \mathtt{t}\}$. In the first, case we have $v'(\psi) = v(\psi) = \{\mathtt{f}\}$. In the latter, we have $v'(\neg\varphi) = \{\mathtt{t}\}$ (by the induction hypothesis since $\neg\varphi = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\neg$)), and thus $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

($\mathtt{f}$:$\neg\wedge^1$) Suppose that $\psi = \neg(\varphi \wedge \neg\varphi)$, $\mathtt{t} \in v'(\varphi)$ and $\mathtt{t} \in v'(\neg\varphi)$ (i.e. $v'(\varphi) = \{\mathtt{t}\}$ and $v'(\neg\varphi) = \{\mathtt{t}\}$). Then $\mathtt{t} \in v(\varphi)$ and $\mathtt{t} \in v(\neg\varphi)$. Since v is $\mathbf{M}$-legal, $v(\psi)$ is either $\{\mathtt{f}\}$ or $\{\mathtt{f}, \mathtt{t}\}$. In the first, case we have $v'(\psi) = v(\psi) = \{\mathtt{f}\}$. In the latter, $v'(\varphi \wedge \neg\varphi) = \{\mathtt{t}\}$ (by the induction hypothesis since $\varphi \wedge \neg\varphi = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\wedge$)), and so $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

($\mathtt{f}$:$\neg\wedge^2$) Suppose that $\psi = \neg(\varphi_1 \wedge \varphi_2)$, $\mathtt{f} \in v'(\neg\varphi_1)$ and $\mathtt{f} \in v'(\neg\varphi_2)$ (i.e. $v'(\neg\varphi_1) = \{\mathtt{f}\}$ and $v'(\neg\varphi_2) = \{\mathtt{f}\}$). Then $\mathtt{f} \in v(\neg\varphi_1)$ and $\mathtt{f} \in v(\neg\varphi_2)$. Since v is $\mathbf{M}$-legal, $v(\psi)$

is either $\{\mathtt{f}\}$ or $\{\mathtt{f},\mathtt{t}\}$. In the first, case we have $v'(\psi) = v(\psi) = \{\mathtt{f}\}$. In the latter, $v'(\varphi_1) = \{\mathtt{t}\}$ and $v'(\varphi_2) = \{\mathtt{t}\}$ according to the classical truth tables (by the induction hypothesis since $\neg\varphi_1 = \psi_{j_1}$ and $\neg\varphi_2 = \psi_{j_2}$ for some $j_1, j_2 < i$, and the condition ($\mathtt{t}$:$\neg$)). Thus $v'(\varphi_1 \wedge \varphi_2) = \{\mathtt{t}\}$ (by the induction hypothesis since $\varphi_1 \wedge \varphi_2 = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\wedge$)), and so $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

($\mathtt{f}$:$\neg\vee^1$) Suppose that $\psi = \neg(\varphi_1 \vee \varphi_2)$, $\mathtt{f} \in v'(\neg\varphi_1)$ (i.e. $v'(\neg\varphi_1) = \{\mathtt{f}\}$) and ($\mathtt{f} \in v'(\varphi_2)$ or $\mathtt{f} \in v'(\neg\varphi_2)$). Then $\mathtt{f} \in v(\neg\varphi_1)$ and ($\mathtt{f} \in v(\varphi_2)$ or $\mathtt{f} \in v(\neg\varphi_2)$). Since v is **M**-legal, $v(\psi)$ is either $\{\mathtt{f}\}$ or $\{\mathtt{f},\mathtt{t}\}$. In the first, case we have $v'(\psi) = v(\psi) = \{\mathtt{f}\}$. In the latter, $v'(\varphi_1) = \{\mathtt{t}\}$ (by the induction hypothesis since $\neg\varphi_1 = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\neg$)). Thus $v'(\varphi_1 \vee \varphi_2) = \{\mathtt{t}\}$ (by the induction hypothesis since $\varphi_1 \vee \varphi_2 = \psi_j$ for some $j < i$, and the condition ($\mathtt{t}$:$\vee$)), and so $v'(\psi) = \{\mathtt{f}\}$ according to the classical truth tables.

The other cases are similar. It follows that $\mathbf{G}_{\mathbf{C_1}}$ enjoys strong cut-admissibility.

To prove Theorem 3.3.9, we use the following lemma.

Lemma 3.3.11. Let **G** be a pure $\langle\mathcal{L}, \pounds\rangle$-calculus.

1. $\mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}} = \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}} \cup \{\pounds\}$.
2. An $\mathcal{L}$-valuation v is $\mathbf{M}_{s(\mathbf{G})}$-legal iff it is $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal and $v'(\varphi) \neq \pounds$ for every $\varphi \in \mathcal{L}$.

Proof. 2 directly follows from 1 since the only difference between $\mathbf{M}_{s(\mathbf{G})}$ and $\mathbf{M}_{s(\mathbf{G})_{cf}}$ is in the set of legal truth values. We prove 1. Since $\mathsf{P}_{s(\mathbf{G})_{cf}} \subseteq \mathsf{P}_{s(\mathbf{G})}$, we obviously have $\mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}} \subseteq \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$. In addition, since there are no cut rules in $s(\mathbf{G})_{cf}$, $\pounds \in \pounds(r)$ for every $r \in \mathsf{P}_{s(\mathbf{G})_{cf}}$, and thus $\pounds \in \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$. Now, let $\mathtt{X} \in \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$, and suppose that $\mathtt{X} \notin \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}$. We show that $\mathtt{X} = \pounds$. By definition, $\mathtt{X} \subseteq \pounds$. To show that $\pounds \subseteq \mathtt{X}$, let $\mathtt{x} \in \pounds$. Since $\mathtt{X} \notin \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}$, $\mathtt{X} \notin \pounds(r)$ for some $r \in \mathsf{P}_{s(\mathbf{G})}$. The fact that $\mathtt{X} \in \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$ entails that r must be a cut rule, namely $r = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\emptyset$ for some $\mathtt{X}_1, \dots, \mathtt{X}_n \subseteq \pounds$. Since $\mathtt{X} \notin \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}$, we have $\mathtt{X}_i \cap \mathtt{X} \neq \emptyset$ for every $1 \leq i \leq n$. Now, if $\{\mathtt{x}\} \in \{\mathtt{X}_1, \dots, \mathtt{X}_n\}$, then we have that $\mathtt{x} \in \mathtt{X}$ and we are done. Otherwise, by definition, $s(\mathbf{G})_{cf}$ includes the primitive $\pounds$-rule $r_{\mathtt{x}} = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/\{\mathtt{x}{:}p_1\}$. The fact that $\mathtt{X} \in \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$ then entails that $\mathtt{X} \in \pounds(r_{\mathtt{x}})$. It follows that $\{\mathtt{x}\} \cap \mathtt{X} \neq \emptyset$, and thus $\mathtt{x} \in \mathtt{X}$. □

Proof of Theorem 3.3.9. ($\Rightarrow$) Suppose that v is an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation, and there does not exist an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v' such that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$. Let $\mathcal{S}$ and μ be the set of $\langle\mathcal{L}, \pounds\rangle$-sequents and the extended $\langle\mathcal{L}, \pounds\rangle$-sequent defined by:

$$\mathcal{S} = \{\{\mathtt{x}{:}\varphi\} \mid \varphi \in \mathcal{L}, v(\varphi) \neq \pounds, \mathtt{x} \in v(\varphi)\} \quad \text{and} \quad \mu = \{\mathtt{x}{:}\varphi \mid \varphi \in \mathcal{L}, \mathtt{x} \notin v(\varphi)\}.$$

Then, by definition $v \models \mathcal{S}$ and $v \not\models \mu$. Now, $frm[\mathcal{S}] \subseteq \{\varphi \in \mathcal{L} \mid v(\varphi) \neq \pounds\}$, and thus v is $\langle \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}, frm[\mathcal{S}]\rangle$-restricted (using Lemma 3.3.11). By Theorem 3.3.8 we have $\mathcal{S} \nvdash_{s(\mathbf{G})}^{cuts:frm[\mathcal{S}]} s$ for every $\langle \mathcal{L}, \pounds\rangle$-sequent $s \subseteq \mu$. We show that $\mathcal{S} \vdash_{s(\mathbf{G})} s$ for some $\langle \mathcal{L}, \pounds\rangle$-sequent $s \subseteq \mu$. By Proposition 2.3.18, it would then follow that $\mathbf{G}$ does not enjoy cut-admissibility. Let v' be an $\mathbf{M}_{s(\mathbf{G})}$-legal $\mathcal{L}$-valuation. By Lemma 3.3.11, it is $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal and $v'(\varphi) \neq \pounds$ for every $\varphi \in \mathcal{L}$. Thus our assumption entails that there is some $\varphi \in \mathcal{L}$ such that $v'(\varphi) \neq v(\varphi)$ and $v(\varphi) \neq \pounds$. Thus, at least one of the following holds: (i) there is some $\mathtt{x} \in \pounds$, such that $\mathtt{x} \in v'(\varphi)$ and $\mathtt{x} \notin v(\varphi)$; (ii) there is some $\mathtt{x} \in \pounds$, such that $\mathtt{x} \in v(\varphi)$ and $\mathtt{x} \notin v'(\varphi)$. If (i) holds, then $v' \models \mu$. If (ii) holds, then $v' \not\models \{\mathtt{x}{:}\varphi\}$, and thus $v' \not\models \mathcal{S}$. It follows that v' is either a model of μ, or not a model of $\mathcal{S}$. Consequently, every $\mathbf{M}_{s(\mathbf{G})}$-legal $\mathcal{L}$-valuation which is a model of $\mathcal{S}$ is also a model of μ. Theorem 3.3.8 implies that $\mathcal{S} \vdash_{s(\mathbf{G})} s$ for some $\langle \mathcal{L}, \pounds\rangle$-sequent $s \subseteq \mu$.

($\Leftarrow$) If $\mathbf{G}$ does not have any cut rule, then the claim is trivially true. Assume otherwise. Suppose that for every $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v, there exists an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v' such that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$. We prove that for every set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds\rangle$-sequents and $\langle \mathcal{L}, \pounds\rangle$-sequent s, $\mathcal{S} \vdash_{s(\mathbf{G})}^{cuts:frm[\mathcal{S}]} s$ whenever $\mathcal{S} \vdash_{s(\mathbf{G})} s$. By Proposition 2.3.18 it would follow that $\mathbf{G}$ enjoys strong cut-admissibility. Suppose that $\mathcal{S} \nvdash_{s(\mathbf{G})}^{cuts:frm[\mathcal{S}]} s$. Let $\mathcal{C} = frm[\mathcal{S}]$. By Theorem 3.3.8, there is a $\langle \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}, \mathcal{C}\rangle$-restricted $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v which is a model of $\mathcal{S}$, but not of s. Our assumption entails that there exists an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v' such that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$. Note that in particular, we must have $v'(\varphi) \neq \pounds$ for every $\varphi \in \mathcal{L}$, and thus, by Lemma 3.3.11, v' is $\mathbf{M}_{s(\mathbf{G})}$-legal. Now, since $\mathbf{G}$ has at least one cut rule, $\pounds \notin \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}$. Since v is $\langle \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}}, \mathcal{C}\rangle$-restricted, $v(\varphi) \neq \pounds$ for every $\varphi \in frm[\mathcal{S}]$. Thus $v'(\varphi) = v(\varphi)$ for every $\varphi \in frm[\mathcal{S}]$, and so $v' \models \mathcal{S}$ as well. Since $v'(\varphi) \subseteq v(\varphi)$ for every $\varphi \in \mathcal{L}$, we have $v' \not\models s$. By Theorem 3.2.15, $\mathcal{S} \nvdash_{s(\mathbf{G})} s$. □

3.3.3 Axiom-Expansion

Using the soundness and completeness theorem above (Theorem 3.2.15), we automatically obtain a semantic characterization of axiom-expansion for a given connective in a given pure calculus.

Corollary 3.3.12. A connective $\diamond \in \diamondsuit_{\mathcal{L}}$ admits axiom-expansion in a pure $\langle \mathcal{L}, \pounds\rangle$-calculus $\mathbf{G}$ iff for every $\mathbf{M}_{\mathbf{G}_{if}}$-legal $\mathcal{L}$-valuation v: if $\mathtt{X} \cap v(p_i) \neq \emptyset$ for every $\mathtt{X} \subseteq \pounds$ such that $\emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $1 \leq i \leq ar(\diamond)$, then $\mathtt{X} \cap v(\diamond(p_1, \dots, p_{ar(\diamond)})) \neq \emptyset$ for every $\mathtt{X} \subseteq \pounds$ such that $\emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$.

Proof. By definition, $\diamond$ admits axiom-expansion in $\mathbf{G}$ iff for every $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$, we have $\{(\mathtt{X}{:}p_i) \mid \emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}, 1 \leq i \leq ar(\diamond)\} \vdash_{\mathbf{G}_{if}} (\mathtt{Y}{:}\diamond(p_1, \dots, p_{ar(\diamond)}))$. The given condition is equivalent by Theorem 3.2.15. □

For ordinary pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculi in which the only identity axiom is (id) we obtain that a connective $\diamond \in \diamondsuit_{\mathcal{L}}$ admits axiom-expansion in $\mathbf{G}$ iff for every $\mathbf{M}_{\mathbf{G}_{if}}$-legal $\mathcal{L}$-valuation v: if $v(p_i) \neq \emptyset$ for every $1 \leq i \leq ar(\diamond)$ then $v(\diamond(p_1, \ldots, p_{ar(\diamond)})) \neq \emptyset$.

3.4 Soundness and Completeness Proofs

In this section we provide proofs of the three soundness and completeness theorems above (Theorems 3.2.15, 3.3.2 and 3.3.8). To avoid repetitions, we will prove one more general result, from which these theorems directly follow. We consider $\langle \mathcal{U}, \mathcal{C} \rangle$*-restricted* $\mathbf{M}_{\mathbf{G}}$*-legal partial* $\mathcal{L}$*-valuations*. These are defined as $\mathbf{M}_{\mathbf{G}}$-legal partial $\mathcal{L}$-valuations with the additional requirement that $v(\varphi) \in \mathcal{U}$ for every $\varphi \in \mathcal{C} \cap Dom_v$. In addition, we say that a $\langle \mathcal{U}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}}$-legal partial $\mathcal{L}$-valuation v is a model of an extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ if μ is an extended $\langle Dom_v, \pounds \rangle$-sequent (that is, all $\mathcal{L}$-formulas that occur in μ are in Dom_v), and $v \models \alpha$ for some $\alpha \in \mu$. The general soundness and completeness theorem is given by:

Theorem 3.4.1. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, $\mathcal{F}, \mathcal{C}$ sets of $\mathcal{L}$-formulas, $\mathcal{S}$ a set of $\langle \mathcal{L}, \pounds \rangle$-sequents, and μ an extended $\langle \mathcal{F}, \pounds \rangle$-sequent. Then the following are equivalent:

1. There exists a proof P in $\mathbf{G}$ of some $\langle \mathcal{L}, \pounds \rangle$-sequent $s \subseteq \mu$ from $\mathcal{S}$, that consists only of $\langle \mathcal{F}, \pounds \rangle$-sequents, and the cut-formula of every application of a cut rule in P is an element of $\mathcal{C}$.
2. For every $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuation v with $Dom_v = \mathcal{F}$, it holds that if v is a model of $\mathcal{S}$ then it is also a model of μ.

Note that the availability of the weakening rules ensures that when μ is finite (forming a usual $\langle \mathcal{F}, \pounds \rangle$-sequent), then 1 can be equivalently written as: there exists a proof P in $\mathbf{G}$ of μ from $\mathcal{S}$ that consists only of $\langle \mathcal{F}, \pounds \rangle$-sequents and the cut-formula of every application of a cut rule in P is an element of $\mathcal{C}$. Therefore:

- Theorem 3.3.2 follows by taking $\mathcal{C} = \mathcal{L}$ (since $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{L} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuations are exactly $\mathbf{M}_{\mathbf{G}}$-legal partial $\mathcal{L}$-valuations).
- Theorem 3.2.15 follows by taking $\mathcal{F} = \mathcal{C} = \mathcal{L}$ (since $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{L} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuations with $Dom_v = \mathcal{L}$ are exactly $\mathbf{M}_{\mathbf{G}}$-legal $\mathcal{L}$-valuations).

In addition, Theorem 3.3.8 follows by taking $\mathcal{F} = \mathcal{L}$ (since $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuations with $Dom_v = \mathcal{L}$ are exactly $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal $\mathcal{L}$-valuations).

The following simple proposition will be useful in the proof below:

Proposition 3.4.2. Let v be a partial $\mathcal{L}$-valuation with $Im_v \subseteq 2^{\pounds}$, s an $\langle \mathcal{L}, \pounds \rangle$-sequent, and σ an $\mathcal{L}$-substitution such that $\sigma(s)$ is a $\langle Dom_v, \pounds \rangle$-sequent. Then, $v \models \sigma(I(s))$ iff $v \models \sigma(s)$ (see Definition 3.2.10).

Proof. Directly follows from the definitions (note that $\sigma(I(s)) = I(\sigma(s))$). □

Next, we prove Theorem 3.4.1.

Soundness

Assume that 1 holds. Let v be an $\langle \mathcal{U}_{\mathbf{M_G}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuation with $Dom_v = \mathcal{F}$. Suppose that $v \models \mathcal{S}$. Using induction on the length of P, we show that $v \models s$ for every sequent s occurring in P. It follows that $v \models \mu$. This trivially holds for the sequents of $\mathcal{S}$ (note that only $\langle Dom_v, \pounds \rangle$-sequents in $\mathcal{S}$ can appear in P). We show that this property is also preserved by applications of the rules of $\mathbf{G}$. Suppose that $s = \sigma(s') \cup c_1 \cup ... \cup c_n$ is derived from $\sigma(s_1) \cup c_1, \dots, \sigma(s_n) \cup c_n$ using a pure $\langle \mathcal{L}, \pounds \rangle$-rule $r = s_1, \dots, s_n / s'$ of $\mathbf{G}$. Assume that $v \models \sigma(s_i) \cup c_i$ for every $1 \leq i \leq n$. We show that $v \models s$. Since s occurs in P, s is a $\langle Dom_v, \pounds \rangle$-sequent. Thus, by definition it suffices to show that $v \models \alpha$ for some $\alpha \in s$. If $v \models \alpha$ for some $\alpha \in c_1 \cup ... \cup c_n$, then we are done. Assume otherwise. Then our assumption entails that $v \models \sigma(s_i)$ for every $1 \leq i \leq n$. We show that $v \models \sigma(s')$. Distinguish between two cases:

- Suppose that r is a primitive $\pounds$-rule. Then, $\sigma(s_i) = (\mathtt{X}_i{:}\varphi)$ for $1 \leq i \leq n$ and $\sigma(s') = (\mathtt{X}{:}\varphi)$ for some $\mathtt{X}_1, \dots, \mathtt{X}_n, \mathtt{X} \subseteq \pounds$ and $\varphi \in \mathcal{L}$. For every $1 \leq i \leq n$, since $v \models \sigma(s_i)$, there exists some $\mathtt{x} \in \mathtt{X}_i$ such that $\mathtt{x} \in v(\varphi)$. In other words, for every $1 \leq i \leq n$, $\mathtt{X}_i \cap v(\varphi) \neq \emptyset$. Now, note that r cannot be a weakening rule. Indeed, if r were a weakening rule then $n = 1$ and $\mathtt{X}_1 = \emptyset$, and this contradicts the fact that $\mathtt{X}_1 \cap v(\varphi) \neq \emptyset$. In addition, r cannot be a cut rule (i.e. we must have $\mathtt{X} \neq \emptyset$). Indeed, if r were a cut rule, then since the cut-formula of every application of a cut rule in P is an element of $\mathcal{C}$, we would have that $\varphi \in \mathcal{C}$. In this case, since v is $\langle \mathcal{U}_{\mathbf{M_G}}, \mathcal{C} \rangle$-restricted, $v(\varphi) \in \mathcal{U}_{\mathbf{M_G}}$, and in particular $v(\varphi) \in \pounds(r)$. It would then follow that $\mathtt{X} \cap v(\varphi) \neq \emptyset$, but this is not possible when $\mathtt{X} = \emptyset$. Therefore, we have that $r \in \mathsf{P}_{\mathbf{G}_{cf}}$. Since v is $\mathbf{M}_{\mathbf{G}_{cf}}$-legal, $v(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{cf}}}$. In particular, $v(\varphi) \in \pounds(r)$, and we obtain that $\mathtt{X} \cap v(\varphi) \neq \emptyset$. Thus $\mathtt{x}{:}\varphi \in \sigma(s')$ for some $\mathtt{x} \in v(\varphi)$. It follows that $v \models \sigma(s')$.

- Suppose that r is not a primitive $\pounds$-rule. For every $1 \leq i \leq n$, since $v \models \sigma(s_i)$, we have that $v \models \sigma(I(s_i))$ (using Proposition 3.4.2). Since v is $\mathbf{M}_{\mathbf{G}_{cf}}$-legal, v satisfies $\mathbf{Sem}(r) = \{I(s_i) \mid 1 \leq i \leq n\} \Rightarrow I(s')$. Since $\sigma(\varphi) \in Dom_v$ for every φ that occurs in $\mathbf{Sem}(r)$ (since $\sigma(s_i)$ for $1 \leq i \leq n$ and $\sigma(s')$ occur in the proof P), and $v \models \sigma(I(s_i))$ for every $1 \leq i \leq n$, we have that $v \models \sigma(I(s'))$ as well. By Proposition 3.4.2, $v \models \sigma(s')$.

Completeness

We prove that 2 implies 1. For this proof, call an extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ' *provable* if there is a proof P in $\mathbf{G}$ of some $\langle \mathcal{L}, \pounds \rangle$-sequent $s' \subseteq \mu'$ from $\mathcal{S}$, that consists only of $\langle \mathcal{F}, \pounds \rangle$-sequents, and the cut-formula of every application of a cut rule in P is an element of $\mathcal{C}$. Otherwise, say that μ' is unprovable. Suppose that μ is unprovable. We construct an $\langle \mathcal{U}_{\mathbf{M_G}}, \mathcal{C} \rangle$-restricted $\mathbf{M}_{\mathbf{G}_{cf}}$-legal partial $\mathcal{L}$-valuation v with $Dom_v = \mathcal{F}$, that is a model of $\mathcal{S}$, but not of μ. Call an extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ^* *maximal* if it satisfies the following conditions:

1. $frm[\mu^*] \subseteq \mathcal{F}$.
2. μ^* is unprovable.
3. For every $\pounds$-labelled $\mathcal{F}$-formula $\alpha \notin \mu^*$, $\{\alpha\} \cup \mu^*$ is provable.

We first construct a maximal extended $\langle \mathcal{L}, \pounds \rangle$-sequent μ^*, such that $\mu \subseteq \mu^*$. Let $\alpha_1, \alpha_2, ...$ be an enumeration of all $\pounds$-labelled $\mathcal{F}$-formulas. Recursively define an (infinite) sequence of extended $\langle \mathcal{L}, \pounds \rangle$-sequents $\mu_0, \mu_1, ...$, as follows. Let $\mu_0 = \mu$. For $k \geq 1$, let $\mu_k = \mu_{k-1}$ iff $\{\alpha_k\} \cup \mu_{k-1}$ is provable. Otherwise, let $\mu_k = \{\alpha_k\} \cup \mu_{k-1}$. Finally, let $\mu^* = \bigcup_{k \geq 0} \mu_k$. It is easy to verify that μ^* has all required properties.

Now, define a partial $\mathcal{L}$-valuation v by $Dom_v = \mathcal{F}$ and $v(\varphi) = \{\mathtt{x} \in \pounds \mid \mathtt{x}{:}\varphi \notin \mu^*\}$ for every $\varphi \in \mathcal{F}$. Note that the following property holds:

(**a**) For every $\langle \mathcal{F}, \pounds \rangle$-sequent s, $v \models s$ iff $s \cup c$ is provable for some $\langle \mathcal{F}, \pounds \rangle$-sequent $c \subseteq \mu^*$.

Proof. Let s be an $\langle \mathcal{F}, \pounds \rangle$-sequent. Suppose that there exists an $\langle \mathcal{F}, \pounds \rangle$-sequent $c \subseteq \mu$ such that $s \cup c$ is provable. The maximality of μ^* entails that $s' \not\subseteq \mu^*$. Thus $\mathtt{x}{:}\varphi \notin \mu^*$ for some $\mathtt{x}{:}\varphi \in s'$. The definition of v entails that $\mathtt{x} \in v(\varphi)$, and so $v \models s'$. For the converse, assume that $v \models s'$. Hence there exists some $\mathtt{x}{:}\varphi \in s'$ such that $\mathtt{x} \in v(\varphi)$. By definition, $\mathtt{x}{:}\varphi \notin \mu^*$. The maximality of μ^* entails that there exists a sequent $c \subseteq \mu^*$ such that $\{\mathtt{x}{:}\varphi\} \cup c$ is provable. The availability of the weakening rules entails that $s' \cup c$ is provable as well. □

Next, we show that v is $\mathbf{M}_{\mathbf{G}_{cf}}$-legal. We first prove that $v(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{cf}}}$ for every $\varphi \in \mathcal{F}$. Thus we prove that for every $\varphi \in \mathcal{F}$, $v(\varphi) \in \pounds(r)$ for every $r \in \mathsf{P}_{\mathbf{G}_{cf}}$. Let $\varphi \in \mathcal{F}$, and let $r = (\mathtt{X}_1{:}p_1), \ldots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ be a primitive $\pounds$-rule in $\mathsf{P}_{\mathbf{G}_{cf}}$. To see that $v(\varphi) \in \pounds(r)$, we show that if $\mathtt{X}_i \cap v(\varphi) \neq \emptyset$ for every $1 \leq i \leq n$, then $\mathtt{X} \cap v(\varphi) \neq \emptyset$.

Suppose that $\mathtt{X}_i \cap v(\varphi) \neq \emptyset$ for every $1 \leq i \leq n$. Hence $v \models (\mathtt{X}_i{:}\varphi)$. (**a**) entails that for every $1 \leq i \leq n$, there exists some $\langle \mathcal{F}, \pounds \rangle$-sequent $c_i \subseteq \mu^*$ such that $(\mathtt{X}_i{:}\varphi) \cup c_i$ is provable. Using the rule r, we obtain that $(\mathtt{X}{:}\varphi) \cup c_1 \cup ... \cup c_n$ is provable. (**a**) again entails that $v \models (\mathtt{X}{:}\varphi)$, and so $\mathtt{X} \cap v(\varphi) \neq \emptyset$.

Now, we prove that $v \models \Lambda_{\mathbf{M}_{\mathbf{G}_{cf}}}$, that is: $v \models \mathbf{Sem}(r)$ for every $r \in \mathsf{R}_{\mathbf{G}}$ (by definition, $\mathsf{R}_{\mathbf{G}} = \mathsf{R}_{\mathbf{G}_{cf}}$). Let $r = s_1, ..., s_n/s$ be a pure $\langle \mathcal{L}, \pounds \rangle$-rule in $\mathsf{R}_{\mathbf{G}}$. By definition, $\mathbf{Sem}(r) = \{I(s_i) \mid 1 \leq i \leq n\} \Rightarrow I(s)$. Let σ be an $\mathcal{L}$-substitution, such that $\sigma(\varphi) \in Dom_v = \mathcal{F}$ for every φ that occurs in $\mathbf{Sem}(r)$. Suppose that $v \models \sigma(I(s_i))$ for every $1 \leq i \leq n$. Thus $v \models \sigma(s_i)$ (Proposition 3.4.2) for every $1 \leq i \leq n$. (**a**) entails that for every $1 \leq i \leq n$ there exists some $\langle \mathcal{F}, \pounds \rangle$-sequent $c_i \subseteq \mu^*$ such that $\sigma(s_i) \cup c_i$ is provable. Now, using these proofs for $1 \leq i \leq n$ and the rule r we obtain that $\sigma(s) \cup c_1 \cup ... \cup c_n$ is provable. (**a**) entails that $v \models \sigma(s)$. By Proposition 3.4.2, $v \models \sigma(I(s))$.

Next, we show that v is $\langle \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}, \mathcal{C} \rangle$-restricted. Let $\varphi \in \mathcal{C} \cap \mathcal{F}$. To see that $v(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$ it suffices to show that $v(\varphi) \in \pounds(r)$ for every cut rule $r \in \mathsf{P}_{\mathbf{G}}$ (we already have $v(\varphi) \in \pounds(r)$ for every other primitive rule r of $\mathbf{G}$). Let $(\mathtt{X}_1{:}p_1), ..., (\mathtt{X}_n{:}p_1)/\emptyset$ be a cut rule in $\mathsf{P}_{\mathbf{G}}$. To see that $v(\varphi) \in \pounds(r)$, we show that $\mathtt{X}_i \cap v(\varphi) = \emptyset$ for some $1 \leq i \leq n$. Suppose otherwise. Then for every $1 \leq i \leq n$ $v \models (\mathtt{X}_i{:}\varphi)$, and (**a**) entails that there exists some $\langle \mathcal{F}, \pounds \rangle$-sequent $c_i \subseteq \mu^*$ such that $(\mathtt{X}_i{:}\varphi) \cup c_i$ is provable. Applying the rule r (with $\varphi \in \mathcal{C}$ as the cut-formula), we obtain that $c_1 \cup ... \cup c_n$ is provable. But since $c_1 \cup ... \cup c_n \subseteq \mu^*$, this contradicts the properties of μ^*.

Finally, we show that $v \models \mathcal{S}$ but $v \not\models \mu$. Let $s \in \mathcal{S}$ be an $\langle \mathcal{F}, \pounds \rangle$-sequent. Obviously, s is provable, and by (**a**), $v \models s$. To see that $v \not\models \mu$, note that since $\mu \subseteq \mu^*$, $\mathtt{x} \notin v(\varphi)$ for every $\mathtt{x}{:}\varphi \in \mu$.

Chapter 4

Canonical Calculi

In the previous chapter we established a strong connection between pure calculi and many-valued systems. However, there is a price to pay for the high generality. Evidently, the set of semantic conditions of the many-valued pre-system $\mathbf{M_G}$ for some calculus $\mathbf{G}$ might be very complicated, which makes the search for a refuting valuation as complicated as proof-search in $\mathbf{G}$. In such cases the semantic criteria for cut-admissibility and analyticity might be useless. Thus it makes sense to study narrower families of sequent calculi, for which we can obtain *effective* semantics, as well as *simple and decidable* characterizations of their proof-theoretic properties.

A particular interesting family of this sort is the family of *canonical systems*, that was introduced in [17]. The idea behind canonical systems implicitly underlies a long tradition in the philosophy of logic, established by Gentzen in his seminal paper [56]. According to this tradition, the meaning of a connective $\diamond$ is determined by the derivation rules which are associated with it. For that matter, one should have rules of some "ideal" type, in each of which $\diamond$ is mentioned exactly once, and no other connective is involved. Formulating this idea, [17] introduced the notion of a "canonical (introduction) rule", and "canonical propositional Gentzen-type systems" were defined as two-sided sequent systems in which: (i) all logical rules are canonical rules; (ii) the usual cut rule, identity-axiom and all structural rules are included. The semantics of canonical systems was given using *non-deterministic matrices* (*Nmatrices*), a natural generalization of logical many-valued matrices. This revealed a remarkable triple correspondence in canonical systems between cut-admissibility, *sub*-analyticity, and the existence of a characteristic two-valued Nmatrix. Later the theory was generalized to many-sided sequent calculi that employ certain (fixed) cut rules and identity axioms [19, 21].

In this chapter we generalize previously studied canonical systems, and study a wide family of pure calculi employing canonical logical rules. Since all of them are pure sequent calculi, we utilize the general results of the previous chapter. We show that in this more restricted settings, the obtained many-valued systems are very simple, and they can be seen as a slight generalization the framework of Nmatrices from [17]. We call these many-valued systems *partial non-deterministic matrices* (*PNmatrices*). As Nmatrices, the main attractive property of (finite) PNmatrices is their effectiveness, as they always induce decision procedures for their underlying logics. In addition, we show that using the PNmatrix that characterizes a given canonical calculus **G**, it is easy to decide whether **G** admits strong *sub*-analyticity, strong cut-admissibility and axiom-expansion. In particular, our results show that strong *sub*-analyticity is equivalent to strong cut-admissibility in this family of calculi.

The structure of this chapter is as follows. First, we precisely define the family of canonical calculi that we study (which is a subfamily of pure calculi studied in the previous chapters). In Section 4.2 we present the framework of PNmatrices (which is a subfamily of many-valued systems introduced in the previous chapter) and show that it is effective. The connection between canonical calculi and PNmatrices is established and exemplified in Section 4.3. Finally, Section 4.4 provides simple decidable characterizations of proof-theoretic properties of canonical calculi based on their characteristic PNmatrices.

Publications Related to this Chapter

Most of the material in this chapter was included in [28, 29]. However, the notions and proofs in these papers consider canonical calculi and PNmatrices in their own right. Here we introduce them as particular cases of pure calculi and many-valued systems, and derive our results using the more general theorems from the previous chapters. In addition, specific cases of canonical two-sided calculi without cut and/or identity axiom were included in [68, 70].

4.1 Canonical Calculi

Canonical calculi are pure calculi that, in addition to the primitive rules, include only pure rules of a special well-behaved form, called *canonical rules*. Each canonical rule is associated with some connective $\diamond$, and $\diamond(p_1, \dots, p_{ar(\diamond)})$ is the only formula occurring in its conclusion. In turn, its premises are composed only from $p_1, \dots, p_{ar(\diamond)}$. Formally, this is defined as follows:

Definition 4.1.1. A pure $\langle \mathcal{L}, \pounds \rangle$-rule r is called *canonical* if its conclusion has the form $\{\mathtt{X}\colon \diamond (p_1, \dots, p_{ar(\diamond)})\}$ for some non-empty $\mathtt{X} \subseteq \pounds$ and $\diamond \in \Diamond_{\mathcal{L}}$, and its premises are all $\langle \{p_1, \dots, p_{ar(\diamond)}\}, \pounds \rangle$-sequents. Canonical pure $\langle \mathcal{L}, \pounds \rangle$-rules will be also called *canonical* $\langle \mathcal{L}, \pounds \rangle$*-rules*, and in the case above we will say that r is a canonical $\langle \mathcal{L}, \pounds \rangle$-rule *for* $\diamond$.

Note that by definition canonical $\langle\mathcal{L}, \pounds\rangle$-rules for a connective $\diamond \in \diamondsuit_{\mathcal{L}}$ have the form:

$$(\mathtt{X}_1^1{:}p_1) \cup ... \cup (\mathtt{X}_1^{ar(\diamond)}{:}p_{ar(\diamond)}), \dots, (\mathtt{X}_m^1{:}p_1) \cup ... \cup (\mathtt{X}_m^{ar(\diamond)}{:}p_{ar(\diamond)})/(\mathtt{X}{:}\diamond(p_1, \dots, p_{ar(\diamond)})).$$

for some $m \geq 0$ and $\mathtt{X}_1^1, \dots, \mathtt{X}_1^{ar(\diamond)}, \dots, \mathtt{X}_m^1, \dots, \mathtt{X}_m^{ar(\diamond)}, \mathtt{X} \subseteq \pounds$ (where $\mathtt{X}$ is not empty). An application of a rule of this form is any inference step of the following form:

$$\frac{(\mathtt{X}_1^1{:}\varphi_1) \cup ... \cup (\mathtt{X}_1^{ar(\diamond)}{:}\varphi_{ar(\diamond)}) \cup c_1 \quad \dots \quad (\mathtt{X}_m^1{:}\varphi_1) \cup ... \cup (\mathtt{X}_m^{ar(\diamond)}{:}\varphi_{ar(\diamond)}) \cup c_m}{(\mathtt{X}{:}\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})) \cup c_1 \cup ... \cup c_m}$$

where $\varphi_1, \dots, \varphi_{ar(\diamond)}$ are $\mathcal{L}$-formulas, and c_i is an $\langle\mathcal{L}, \pounds\rangle$-sequent for every $1 \leq i \leq m$.

Example 4.1.2. The pure $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rules from Example 2.2.6 $\{\mathtt{t}{:}p_1\}, \{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1 \supset p_2\}$, $\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \supset p_2\}$, and $\{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \supset p_2\}$, are all canonical. On the other hand, the rules $\{\mathtt{f}{:}\neg p_1\}, \{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1 \wedge p_2)\}$ and $\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}\neg p_1\}/\{\mathtt{f}{:}\neg(p_1 \wedge \neg p_1)\}$ from Example 2.2.22 are not canonical.

Example 4.1.3. Note that we allow the formula in the conclusion to appear with more than one label. For example, $\emptyset/\{\mathtt{f}{:}\star p_1, \mathtt{t}{:}\star p_1\}$ is a canonical $\langle\{\star^1\}, \{\mathtt{f}, \mathtt{t}, \mathtt{i}\}\rangle$-rule. In fact, this rule is also a canonical $\langle\{\star^1\}, \pounds_{\mathbf{2}}\rangle$-rule, that may be useful when (id) is not available in general.

In turn, canonical calculi are defined as follows.

Definition 4.1.4. A *canonical* $\langle\mathcal{L}, \pounds\rangle$*-calculus* $\mathbf{G}$ is a pure $\langle\mathcal{L}, \pounds\rangle$-calculus, such that $\mathsf{R}_{\mathbf{G}}$ (the set of non-primitive rules of $\mathbf{G}$) consists only of canonical $\langle\mathcal{L}, \pounds\rangle$-rules.

Note that we do impose any limitations on the primitive rules, in particular allowing any set of cuts and identity axioms in canonical calculi. Hence the canonical calculi studied here are substantially more general than previously studied canonical systems:

- The canonical systems of [17] correspond to canonical $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-calculi whose primitive rules consist of (cut) and (id) (except for the weakening rules), and the conclusion of each non-primitive rule takes the form $\{\mathtt{x}{:}\diamond(p_1, \dots, p_{ar(\diamond)})\}$ for some $\mathtt{x} \in \pounds_{\mathbf{2}}$ and $\diamond \in \diamondsuit_{\mathcal{L}}$. In particular, **LK** (see Example 2.2.20) is such a canonical $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-calculus.

- The signed canonical calculi of [19, 21] are similar to our canonical calculi, but they all employ $\{\mathtt{x}{:}p_1\}, \{\mathtt{y}{:}p_1\}/\emptyset$ for every $\mathtt{x} \neq \mathtt{y}$ as cut rules and $\emptyset/(\pounds : p_1)$ as the only identity axiom.

In addition, the family of canonical calculi studied here includes all labelled calculi from [26] for many-valued logics.

Due to the special form of the rules of canonical calculi (which, except for cuts, enjoy the subformula property), in this family of calculi cut-admissibility guarantees *sub*-analyticity:

Proposition 4.1.5. If a canonical calculus enjoys (strong) cut-admissibility, then it is (strongly) *sub*-analytic.

Using the semantics, we show below that these two properties are actually equivalent (Corollary 4.4.4).

4.2 Partial Non-deterministic Matrices

In this section we introduce the semantic framework of *partial non-deterministic matrices (PNmatrices)*, and show that the consequence relations induced by (finite) PNmatrices are always decidable. In the next section we show that this framework is indeed suitable for all canonical calculi.

PNmatrices are many-valued systems (see Definition 3.1.5) in which all semantic conditions have the following well-behaved form:

Definition 4.2.1. An $\mathcal{L}$-semantic condition $\mathcal{I} \Rightarrow I$ over $\mathcal{U}$ is called *canonical* if there is some $\diamond \in \Diamond_{\mathcal{L}}$, such that the formulas occurring in $\mathcal{I}$ are all in $\{p_1, \dots, p_{ar(\diamond)}\}$, and I has the form $\{\diamond(p_1, \dots, p_{ar(\diamond)}) \doteqdot u_i \mid u_i \in \mathcal{U}'\}$ for some non-empty (finite) $\mathcal{U}' \subseteq \mathcal{U}$. In this case we also say that $\mathcal{I} \Rightarrow I$ is a *canonical $\mathcal{L}$-semantic condition for $\diamond$*.

The semantic conditions in Example 3.2.12 are canonical $\mathcal{L}$-semantic conditions for $\supset$. On the other hand, the $\mathcal{L}$-semantic condition in Example 3.2.13 is not canonical.

Example 4.2.2. Suppose that $\supset \in \Diamond^2_{\mathcal{L}}$, and let $\mathcal{U} = \{f, t\}$. The following are canonical $\mathcal{L}$-semantic conditions for $\supset$ (over $\mathcal{U}$):

$$\{\{p_1 \doteqdot t\}, \{p_2 \doteqdot f\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot f\}$$

$$\{\{p_2 \doteqdot t\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot t\}$$

Definition 4.2.3. A *(pre-) partial non-deterministic matrix* ((pre-) *PNmatrix*, for short) for $\mathcal{L}$ is a many-valued (pre-) system $\mathbf{M}$ for $\mathcal{L}$, in which all $\mathcal{L}$-semantic conditions in $\Lambda_{\mathbf{M}}$ are canonical.

Now, the semantic conditions in (pre-) PNmatrices can be easily presented using *many-valued truth tables*. However, usual *deterministic* truth tables would not suffice. Thus, as done in [17], we use *non-deterministic* truth tables, where there might be more than one option to determine the value of a compound formula according to the values of its immediate subformulas. In fact, we slightly extend this notion, by introducing *partial* non-deterministic truth tables, in which we also allow empty sets of options to appear. This extension will enable us to have "truth tables presentation" for *all* (pre-) PNmatrices.

Definition 4.2.4. Let $\diamond \in \Diamond_{\mathcal{L}}$ and let $\mathcal{U}$ be a set of truth vales.

1. A *partial non-deterministic truth table* (*PNtable*, for short) for $\diamond$ over $\mathcal{U}$ is a function from $\mathcal{U}^{ar(\diamond)}$ to $2^{\mathcal{U}}$.

2. A partial $\mathcal{L}$-valuation v with $Im_v \subseteq \mathcal{U}$ *respects* a PNtable Ξ for $\diamond$ over $\mathcal{U}$ if
$$v(\sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))) \in \Xi(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)}))$$
for every $\mathcal{L}$-substitution σ such that $\sigma(\{p_1, \dots, p_{ar(\diamond)}, \diamond(p_1, \dots, p_{ar(\diamond)})\}) \subseteq Dom_v$.

Example 4.2.5. Let $\mathcal{L} = \{\star^1\}$. Consider the following PNtables for $\star$ over $\mathcal{U} = \{u_1, u_2\}$:

Ξ_1	
u_1	$\{u_2\}$
u_2	$\{u_1\}$

Ξ_2	
u_1	$\{u_2\}$
u_2	$\{u_1, u_2\}$

Ξ_3	
u_1	$\emptyset$
u_2	$\{u_2\}$

The $\mathcal{L}$-valuation assigning u_2 to all formulas respects Ξ_2 and Ξ_3, but not Ξ_1. In addition, this is the only $\mathcal{L}$-valuation (whose image is contained in $\mathcal{U}$) that respects Ξ_3.

Now, any set of canonical $\mathcal{L}$-semantic conditions for some connective naturally induces a PNtable.

Example 4.2.6. Suppose that $\supset \in \Diamond^2_{\mathcal{L}}$. Let $\mathbf{M}$ be a pre-PNmatrix for $\mathcal{L}$ such that $\mathcal{V}_{\mathbf{M}} = \mathcal{U}_{\mathbf{M}} = \{f, t\}$, and $\Lambda_{\mathbf{M}}$ consists of the canonical $\mathcal{L}$-semantic conditions for $\supset$ from Example 4.2.2. Consider the PNtable Xi for $\supset$ over $\{f, t\}$:[1]

Ξ	f	t
f	$\{f, t\}$	$\{t\}$
t	$\{f\}$	$\{t\}$

It is easy to verify that a partial $\mathcal{L}$-valuation v with Dom_v closed under subformulas and $Im_v \subseteq \{f, t\}$ is $\mathbf{M}$-legal iff it respects Ξ. Note that this PNtable corresponds to the semantics of "primal implication" from [62].

The general construction is given by:

Definition 4.2.7. Let $\mathbf{M}$ be a (pre-) PNmatrix. For every $\diamond \in \Diamond_{\mathcal{L}}$, $\diamond_{\mathbf{M}}$ is the PNtable for $\diamond$ over $\mathcal{V}_{\mathbf{M}}$ defined as follows. For every $u_1, \dots, u_{ar(\diamond)} \in \mathcal{V}_{\mathbf{M}}$, $\diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)})$ consists of all truth values $u \in \mathcal{V}_{\mathbf{M}}$ such that for every canonical $\mathcal{L}$-semantic condition $\mathcal{I} \Rightarrow I \in \Lambda_{\mathbf{M}}$ for $\diamond$, $\diamond(p_1, \dots, p_{ar(\diamond)}) \doteqdot u \in I$ whenever for every $I' \in \mathcal{I}$ we have $p_i \doteqdot u_i \in I'$ for some $1 \leq i \leq ar(\diamond)$.

Example 4.2.8. Let $\mathcal{L} = \{\star^1\}$. Consider the three pre-PNmatrices $\mathbf{M}_1, \mathbf{M}_2$ and $\mathbf{M}_3$ for $\mathcal{L}$ defined by: $\mathcal{V}_{\mathbf{M}_i} = \mathcal{U}_{\mathbf{M}_i} = \{u_1, u_2\}$ and:

[1] We represent PNtables for binary connectives by two-dimensional tables. The lines range over the first argument, and the columns over the second one.

1. $\Lambda_{\mathbf{M}_1} = \{\{\{p_1 \doteqdot u_1\}\} \Rightarrow I_2, \{\{p_1 \doteqdot u_2\}\} \Rightarrow I_1\}$
2. $\Lambda_{\mathbf{M}_2} = \{\{\{p_1 \doteqdot u_1\}\} \Rightarrow I_2\}$
3. $\Lambda_{\mathbf{M}_3} = \{\{\{p_1 \doteqdot u_1\}\} \Rightarrow I_1, \{\{p_1 \doteqdot u_1\}\} \Rightarrow I_2, \{\{p_1 \doteqdot u_2\}\} \Rightarrow I_2\}$

where: $I_1 = \{\star p_1 \doteqdot u_1\}$ and $I_2 = \{\star p_1 \doteqdot u_2\}$. Then for every $1 \leq i \leq 3$, $\star_{\mathbf{M}_i} = \Xi_i$ given in Example 4.2.5.

Proposition 4.2.9. Let $\mathbf{M}$ be a (pre-) PNmatrix. A partial $\mathcal{L}$-valuation v whose domain is closed under subformulas is $\mathbf{M}$-legal iff $Im_v \subseteq \mathcal{U}_{\mathbf{M}}$ and v respects $\diamond_{\mathbf{M}}$ for every $\diamond \in \diamondsuit_{\mathcal{L}}$.

Proof. The proof is completely straightforward using the definitions above. Let v be a partial $\mathcal{L}$-valuation whose domain is closed under subformulas.

1. Suppose that v is $\mathbf{M}$-legal. By definition, $Im_v \subseteq \mathcal{U}_{\mathbf{M}}$. Let $\diamond \in \diamondsuit_{\mathcal{L}}$ and let $\varphi = \diamond(p_1, \dots, p_{ar(\diamond)})$. We show that v respects $\diamond_{\mathbf{M}}$. Let σ be an $\mathcal{L}$-substitution with $\sigma(\{p_1, \dots, p_{ar(\diamond)}, \varphi\}) \subseteq Dom_v$. We show that $v(\sigma(\varphi)) \in \diamond_{\mathbf{M}}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)}))$. By definition, it suffices to show that for every canonical $\mathcal{L}$-semantic condition $\mathcal{I} \Rightarrow I \in \Lambda_{\mathbf{M}}$ for $\diamond$, $\diamond(p_1, \dots, p_{ar(\diamond)}) \doteqdot v(\sigma(\varphi)) \in I$ whenever for every $I' \in \mathcal{I}$ we have $p_i \doteqdot v(\sigma(p_i)) \in I'$ for some $1 \leq i \leq ar(\diamond)$. Suppose that for every $I' \in \mathcal{I}$ we have $p_i \doteqdot v(\sigma(p_i)) \in I'$ for some $1 \leq i \leq ar(\diamond)$. Then for every $I' \in \mathcal{I}$, v satisfies $\sigma(I')$. Since v is $\mathbf{M}$-legal it satisfies $\mathcal{I} \Rightarrow I$. Hence, v satisfies $\sigma(I)$ (note that $\sigma(\psi) \in Dom_v$ for each formula ψ that occurs in $\mathcal{I} \Rightarrow I$). Since $\mathcal{I} \Rightarrow I$ is a canonical $\mathcal{L}$-semantic condition for $\diamond$, it follows that $\sigma(\varphi) \doteqdot v(\sigma(\varphi)) \in \sigma(I)$. Thus $\varphi \doteqdot v(\sigma(\varphi)) \in I$.

2. Suppose that $Im_v \subseteq \mathcal{U}_{\mathbf{M}}$ and v respects $\diamond_{\mathbf{M}}$ for every $\diamond \in \diamondsuit_{\mathcal{L}}$. We show that v is $\mathbf{M}$-legal. Thus we have to show that v satisfies every $\mathcal{L}$-semantic condition $\mathcal{I} \Rightarrow I$ in $\Lambda_{\mathbf{M}}$. Let $\mathcal{I} \Rightarrow I \in \Lambda_{\mathbf{M}}$. Since $\mathbf{M}$ is a (pre-) PNmatrix, $\mathcal{I} \Rightarrow I$ is a canonical $\mathcal{L}$-semantic condition for some connective $\diamond$. Let σ be an $\mathcal{L}$-substitution such that $\sigma(\varphi) \in Dom_v$ for every φ that occurs in $\mathcal{I} \Rightarrow I$. Let $\varphi = \diamond(p_1, \dots, p_{ar(\diamond)})$. In particular, φ occurs in I, and thus $\sigma(\varphi) \in Dom_v$. Since Dom_v is closed under subformulas we also have $\sigma(\{p_1, \dots, p_{ar(\diamond)}\}) \subseteq Dom_v$. Suppose that v satisfies $\sigma(I')$ for every $I' \in \mathcal{I}$. Then for every $I' \in \mathcal{I}$, there is some $1 \leq i \leq ar(\diamond)$ such that $p_i \doteqdot v(\sigma(p_i)) \in I$. The definition of $\diamond_{\mathbf{M}}$ ensures that $\varphi \doteqdot u \in I$ for every $u \in \diamond_{\mathbf{M}}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)})))$. Now, since v respects $\diamond_{\mathbf{M}}$, we have that $v(\sigma(\varphi)) \in \diamond_{\mathbf{M}}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)})))$. Therefore, $\varphi \doteqdot v(\sigma(\varphi)) \in I$. It follows that v satisfies $\sigma(I)$. □

4.2.1 Effectiveness

The major attractive property of PNmatrices is their *effectiveness*: the logics induced by finite PNmatrices are all decidable. This can be easily shown for the subclass of *proper* PNmatrices, that corresponds to the Nmatrices of [17] (where no empty sets of truth values are allowed to appear in the truth tables of the logical connectives):

Definition 4.2.10. A (pre-) PNmatrix $\mathbf{M}$ for $\mathcal{L}$, is called *proper* if $\mathcal{U}_{\mathbf{M}}$ is non-empty, and for every connective $\diamond \in \diamondsuit_{\mathcal{L}}$ and $u_1, \dots, u_{ar(\diamond)} \in \mathcal{U}_{\mathbf{M}}$, $\diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M}} \neq \emptyset$.

Example 4.2.11. Let $\mathcal{L} = \{\star^1\}$. Consider the three pre-PNmatrices $\mathbf{M}_1, \mathbf{M}_2$ and $\mathbf{M}_3$ for $\mathcal{L}$ given in Example 4.2.8. Then $\mathbf{M}_1$ and $\mathbf{M}_2$ are proper, but $\mathbf{M}_3$ is not proper since $\star_{\mathbf{M}_3}(u_1) \cap \{u_1, u_2\} = \emptyset$.

Proper PNmatrices are exactly these in which every partial valuation (whose domain is closed under subformulas) can be extended to a full one, and thus they are *sub*-analytic many-valued systems (see Definition 3.1.16). Hence the effectiveness of proper PNmatrices directly follows from Theorem 3.1.18. In addition, the characterization of strong *sub*-analyticity in canonical sequent calculi (Corollary 4.4.1 below) will immediately follow from this observation as well.

Proposition 4.2.12. A (pre-) PNmatrix $\mathbf{M}$ (for $\mathcal{L}$) is proper iff it is *sub*-analytic.

Proof (Outline). Suppose that $\mathbf{M}$ is proper. The extension of an $\mathbf{M}$-legal partial $\mathcal{L}$-valuation whose domain is closed under subformulas is recursively defined by induction on the complexity of formulas. For atomic formulas that are not in the original domain, we arbitrarily choose a value in $\mathcal{U}_{\mathbf{M}}$. For non-atomic formulas that are not in the original domain, we arbitrarily choose a value from $\mathcal{U}_{\mathbf{M}}$ that occurs in the (non-empty) set of options allowed by the corresponding PNtable.

For the converse, note first that if $\mathcal{U}_{\mathbf{M}}$ is empty, then the empty valuation (whose domain is the empty set) cannot be extended. Otherwise, there is some $\diamond \in \diamondsuit_{\mathcal{L}}$ and $u_1, \dots, u_{ar(\diamond)} \in \mathcal{U}_{\mathbf{M}}$, such that $\diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M}} = \emptyset$. Define $v : \{p_1, \dots, p_{ar(\diamond)}\} \to \mathcal{U}_{\mathbf{M}}$ by $v(p_i) = u_i$. Obviously, v is an $\mathbf{M}$-legal partial $\mathcal{L}$-valuation, whose domain is finite and closed under subformulas. However, v cannot be extended to an $\mathbf{M}$-legal $\mathcal{L}$-valuation, since there is no truth value to assign to $\diamond(p_1, \dots, p_{ar(\diamond)})$. □

Corollary 4.2.13. Let $\mathbf{M}$ be a finite proper PNmatrix for $\mathcal{L}$. Given a finite set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash_{\mathbf{M}} \varphi$ or not.

Proof. Directly follows from Theorem 3.1.18 and Proposition 4.2.12. □

Now, we show that also *non-proper* finite PNmatrices are effective. This is an immediate corollary of Theorem 3.1.19 using the following theorem:

Theorem 4.2.14. Let $\mathbf{M}$ be a finite PNmatrix for $\mathcal{L}$. Given an $\mathbf{M}$-legal partial $\mathcal{L}$-valuation v, whose domain is finite and closed under subformulas, it is decidable whether v can be extended to an $\mathbf{M}$-legal (full) $\mathcal{L}$-valuation or not.

In other words, the framework of finite PNmatrices allows to effectively check whether a certain partial valuation (whose domain is closed under subformulas) is a restriction of a full one. To prove this theorem we use the following notation and proposition:

Notation 4.2.15. For a (pre-) PNmatrix $\mathbf{M}$ and $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$, we denote by $\mathbf{M} \cap \mathcal{U}$ the (pre-) PNmatrix which is identical to $\mathbf{M}$ except for $\mathcal{U}_{\mathbf{M}\cap\mathcal{U}} = \mathcal{U}$.

Obviously, if an $\mathcal{L}$-valuation is $\mathbf{M} \cap \mathcal{U}$-legal for some $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$, then it is $\mathbf{M}$-legal. It follows that for every PNmatrix $\mathbf{M}$ and $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$, $\vdash_{\mathbf{M}} \subseteq \Vdash_{\mathbf{M}\cap\mathcal{U}}$.

Proposition 4.2.16. Let $\mathbf{M}$ be a (pre-) PNmatrix for $\mathcal{L}$, and let $\mathcal{F} \subseteq \mathcal{L}$ be closed under subformulas. An $\mathbf{M}$-legal partial $\mathcal{L}$-valuation v, whose domain is closed under subformulas, can be extended to a (full) $\mathbf{M}$-legal $\mathcal{L}$-valuation iff v is $\mathbf{M}\cap\mathcal{U}$-legal for some $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$ such that $\mathbf{M} \cap \mathcal{U}$ is proper.

Proof. Suppose that there is some $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$ such that $\mathbf{M} \cap \mathcal{U}$ is proper and v is $\mathbf{M} \cap \mathcal{U}$-legal. By Proposition 4.2.12, there exists an $\mathbf{M} \cap \mathcal{U}$-legal $\mathcal{L}$-valuation v' that extends v. Clearly, v' is also $\mathbf{M}$-legal.

For the converse, let v' be an $\mathbf{M}$-legal $\mathcal{L}$-valuation that extends v. Choose $\mathcal{U} = Im_{v'}$. Obviously, $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$ and v is $\mathbf{M} \cap \mathcal{U}$-legal. We show that $\mathbf{M} \cap \mathcal{U}$ is proper. Obviously, $\mathcal{U}$ is non-empty. Let $\diamond \in \Diamond_{\mathcal{L}}$ and $u_1, \dots, u_{ar(\diamond)} \in \mathcal{U}$. Since $\mathcal{U} = Im_{v'}$, there are some $\varphi_1, \dots, \varphi_{ar(\diamond)} \in \mathcal{L}$, such that $v'(\varphi_i) = u_i$ for every $1 \leq i \leq ar(\diamond)$. Since v' is $\mathbf{M}$-legal, $v'(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})) \in \diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)})$. By definition $v'(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})) \in \mathcal{U}$. Hence $\diamond_{\mathbf{M}\cap\mathcal{U}}(u_1, \dots, u_{ar(\diamond)}) \cap \mathcal{U} = \diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)}) \cap \mathcal{U} \neq \emptyset$. □

Proof of Theorem 4.2.14. To verify that v can be extended to an $\mathbf{M}$-legal $\mathcal{L}$-valuation, go over all finite sets $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{M}}$ such that $\mathbf{M}\cap\mathcal{U}$ is proper, and check whether v is $\mathbf{M}\cap\mathcal{U}$-legal for each of them. Return a positive answer iff such $\mathcal{U}$ have been found. The correctness is guaranteed by Proposition 4.2.16. □

As a corollary we have the following:

Corollary 4.2.17. Let $\mathbf{M}$ be a finite PNmatrix for $\mathcal{L}$. Given a finite set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash_{\mathbf{M}} \varphi$ or not.

Proof. Directly follows from Theorem 3.1.19 and Theorem 4.2.14. □

Remark 4.2.18. As done for ordinary matrices (see, e.g., [93]) it is also possible to define $\Vdash_{\wp}$, the consequence relation induced by a *family* $\wp$ *of PNmatrices* to be $\bigcap_{\mathbf{M}\in\wp} \Vdash_{\mathbf{M}}$. A PNmatrix can then be thought of as a succinct presentation of a family of proper PNmatrices: following the proof of Proposition 4.2.16, the consequence relation induced by a PNmatrix $\mathbf{M}$ can be shown to be equivalent to the relation induced by the family of all the proper PNmatrices $\mathbf{M}\cap\mathcal{U}$ for $\mathcal{U}\subseteq\mathcal{U}_{\mathbf{M}}$. Conversely, for every family of proper PNmatrices it is possible to construct an equivalent PNmatrix.

4.3 PNmatrices for Canonical Calculi

To show that the semantics of canonical calculi can be characterized by PNmatrices, it suffices to note that the semantic conditions induced by *canonical* rules are all canonical semantic conditions.

Proposition 4.3.1. Let r be a canonical $\langle\mathcal{L},\pounds\rangle$-rule for a connective $\diamond$. Then $\mathbf{Sem}(r)$ is a canonical $\mathcal{L}$-semantic condition for $\diamond$.

Proof. Directly follows from the definitions. □

It follows that for every canonical $\langle\mathcal{L},\pounds\rangle$-calculus $\mathbf{G}$, the many-valued pre-system $\mathbf{M_G}$ is actually a pre-PNmatrix:

Corollary 4.3.2. Let $\mathbf{G}$ be a canonical $\langle\mathcal{L},\pounds\rangle$-calculus. Then, the many-valued pre-system $\mathbf{M_G}$ is a pre-PNmatrix for which $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M}_G}$. Furthermore, for every set $\mathtt{X}\subseteq\pounds$, the logic induced by $\mathbf{G}$ and $\mathtt{X}$ is identical to the logic induced by the PNmatrix $\mathbf{M}_{\mathbf{G}}^{\mathtt{X}}$.

Proof. Directly follow from Proposition 4.3.1, Theorem 3.2.15, and Corollary 3.2.18 □

Example 4.3.3. Let $\mathcal{L}=\{\supset^2,\wedge^2,T^2\}$, and $\mathbf{G}$ be a canonical $\langle\mathcal{L},\pounds_{\mathbf{2}}\rangle$-calculus that consists of the following rules:

$(\mathtt{f}{:}weak)$	$\{\emptyset\}/\{\mathtt{f}{:}p_1\}$	$(\mathtt{t}{:}weak)$	$\{\emptyset\}/\{\mathtt{t}{:}p_1\}$
(cut)	$\{\mathtt{f}{:}p_1\},\{\mathtt{t}{:}p_1\}/\emptyset$	(id)	$\emptyset/\{\mathtt{f}{:}p_1,\mathtt{t}{:}p_1\}$
$(\mathtt{f}{:}\supset)$	$\{\mathtt{t}{:}p_1\},\{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1\supset p_2\}$	$(\mathtt{t}{:}\supset)$	$\{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1\supset p_2\}$
$(\mathtt{f}{:}\wedge)$	$\{\mathtt{f}{:}p_1,\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1\wedge p_2\}$	$(\mathtt{t}{:}\wedge)$	$\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1\wedge p_2\}$
$(\mathtt{f}{:}T)$	$\{\mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1Tp_2\}$	$(\mathtt{t}{:}T)$	$\{\mathtt{t}{:}p_1\}/\{\mathtt{t}{:}p_1Tp_2\}$

The rules for $\wedge$ are the usual ones (those of **LK**), while the rules for $\supset$ are those employed in primal logic [62] (see Example 2.2.6). T is standing for the "Tonk" connective, and the two rules above are equivalent to its original introduction rule from [83]. We sketch the modular construction of the pre-PNmatrix $\mathbf{M_G}$.

1. Begin with $\mathbf{G}_0$, a canonical $\langle \mathcal{L}, \mathcal{E}_2 \rangle$-calculus, that does not have any rules besides the two weakening rules above. Then, $\mathcal{V}_{\mathbf{M}_{\mathbf{G}_0}} = 2^{\mathcal{E}_2} = \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$. Now since $\mathsf{P}_{\mathbf{G}_0}$ is empty, $\mathcal{U}_{\mathbf{M}_{\mathbf{G}_0}} = \mathcal{V}_{\mathbf{M}_{\mathbf{G}_0}}$. Since $\mathsf{R}_{\mathbf{G}_0}$ is empty as well, there are no semantic $\mathcal{L}$-conditions in $\Lambda_{\mathbf{M}_{\mathbf{G}_0}}$, and thus it is represented by "completely non-deterministic" PNtables, that is $\diamond_{\mathbf{M}_{\mathbf{G}_0}}(\mathtt{X}_1, \mathtt{X}_2) = \mathcal{V}_{\mathbf{M}_{\mathbf{G}_0}}$ for every $\diamond \in \{\supset, \wedge, T\}$ and $\mathtt{X}_1, \mathtt{X}_2 \in \mathcal{V}_{\mathbf{M}_{\mathbf{G}_0}}$.

2. Add $(\mathtt{f}\colon \supset)$ to $\mathbf{G}_0$ to obtain $\mathbf{G}_1$. This introduces the following canonical $\mathcal{L}$-semantic condition for $\supset$ in $\Lambda_{\mathbf{M}_{\mathbf{G}_1}}$:

$$\mathbf{Sem}((\mathtt{f}\colon \supset)) = \{\{p_1 \doteqdot \{\mathtt{t}\}, p_1 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}, \{p_2 \doteqdot \{\mathtt{f}\}, p_2 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}\} \Rightarrow \{p_1 \supset p_2 \doteqdot \{\mathtt{f}\}, p_1 \supset p_2 \doteqdot \{\mathtt{f}, \mathtt{t}\}\}.$$

Consequently the PNtable of $\supset$ in $\mathbf{M}_{\mathbf{G}_1}$ is:

$\supset_{\mathbf{M}_{\mathbf{G}_1}}$	$\emptyset$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$	$\{\mathtt{f},\mathtt{t}\}$
$\emptyset$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$
$\{\mathtt{f}\}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$
$\{\mathtt{t}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f},\mathtt{t}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$

3. Add all other canonical rules of $\mathbf{G}$ to $\mathbf{G}_1$ to obtain $\mathbf{G}_2$. This introduces one more canonical $\mathcal{L}$-semantic condition for $\supset$, two canonical $\mathcal{L}$-semantic conditions for $\wedge$, and two for T. The PNtables in $\mathbf{M}_{\mathbf{G}_2}$ are given by:

$\supset_{\mathbf{M}_{\mathbf{G}_2}}$	$\emptyset$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$	$\{\mathtt{f},\mathtt{t}\}$
$\emptyset$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f}\}$	$2^{\mathcal{E}_2}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f},\mathtt{t}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$

$\wedge_{\mathbf{M}_{\mathbf{G}_2}}$	$\emptyset$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$	$\{\mathtt{f},\mathtt{t}\}$
$\emptyset$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f},\mathtt{t}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$

$T_{\mathbf{M}_{\mathbf{G}_2}}$	$\emptyset$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$	$\{\mathtt{f},\mathtt{t}\}$
$\emptyset$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$2^{\mathcal{E}_2}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f},\mathtt{t}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$

4. Add (*id*) to $\mathbf{G}_2$ to obtain $\mathbf{G}_3$. This does not effect $\wedge_{\mathbf{M}_{\mathbf{G}_2}}$, but it does change the legal truth values, and we now have $\mathcal{U}_{\mathbf{M}_{\mathbf{G}_3}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$ (see Example 3.2.9).

5. Finally add (*cut*) to $\mathbf{G}_3$ to obtain $\mathbf{G}$. This overrules another truth value, and we finally have $\mathcal{U}_{\mathbf{M}_{\mathbf{G}}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$. Note that the PNtables of $\mathbf{M}_G$ are the same as in $\mathbf{M}_{\mathbf{G}_2}$. Nevertheless, since $\mathcal{U}_{\mathbf{M}_{\mathbf{G}}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$, we can reduce them to the following tables:

$\supset_{\mathbf{M}_{\mathbf{G}}}$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$
$\{\mathtt{f}\}$	$\{\{\mathtt{f}\}, \{\mathtt{t}\}\}$	$\{\{\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$\{\{\mathtt{f}\}\}$	$\{\{\mathtt{t}\}\}$

$\wedge_{\mathbf{M}_{\mathbf{G}}}$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$
$\{\mathtt{f}\}$	$\{\{\mathtt{f}\}\}$	$\{\{\mathtt{f}\}\}$
$\{\mathtt{t}\}$	$\{\{\mathtt{f}\}\}$	$\{\{\mathtt{t}\}\}$

$T_{\mathbf{M}_{\mathbf{G}}}$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$
$\{\mathtt{f}\}$	$\{\{\mathtt{f}\}\}$	$\{\{\mathtt{f}\}, \{\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$\emptyset$	$\{\{\mathtt{t}\}\}$

Note that $\mathbf{M}_{\mathbf{G}}$ is not proper since $T_{\mathbf{M}_{\mathbf{G}}}(\{\mathtt{t}\}, \{\mathtt{f}\}) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}}} = \emptyset$.

Example 4.3.4. Suppose that $\mathcal{L}$ contains a unary connective denoted by $\neg$, and let $\mathbf{G}$ be a canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus. Let $(\mathtt{f}{:}\neg)$ and $(\mathtt{t}{:}\neg)$ be the two usual rules for $\neg$, that is the rules $\{\mathtt{t}{:}p_1\}/\{\mathtt{f}{:}\neg p_1\}$ and $\{\mathtt{f}{:}p_1\}/\{\mathtt{t}{:}\neg p_1\}$ (respectively). The following tables present $\neg_{\mathbf{M}_{\mathbf{G}}}$ for four different options: (*i*) there are no canonical rules for $\neg$ in $\mathbf{G}$; (*ii*) $(\mathtt{f}{:}\neg)$ is the only canonical rules for $\neg$ in $\mathbf{G}$; (*iii*) $(\mathtt{t}{:}\neg)$ is the only canonical rules for $\neg$ in $\mathbf{G}$; and (*iv*) $(\mathtt{f}{:}\neg)$ and $(\mathtt{t}{:}\neg)$ are the only canonical rules for $\neg$ in $\mathbf{G}$.

(*i*)	
$\emptyset$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}\}$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{t}\}$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}, \mathtt{t}\}$	$2^{\pounds_{\mathbf{2}}}$

(*ii*)	
$\emptyset$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}\}$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{t}\}$	$\{\{\mathtt{f}\}, \{\mathtt{f}, \mathtt{t}\}\}$
$\{\mathtt{f}, \mathtt{t}\}$	$\{\{\mathtt{f}\}, \{\mathtt{f}, \mathtt{t}\}\}$

(*iii*)	
$\emptyset$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}\}$	$\{\{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$
$\{\mathtt{t}\}$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}, \mathtt{t}\}$	$\{\{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$

(*iv*)	
$\emptyset$	$2^{\pounds_{\mathbf{2}}}$
$\{\mathtt{f}\}$	$\{\{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$
$\{\mathtt{t}\}$	$\{\{\mathtt{f}\}, \{\mathtt{f}, \mathtt{t}\}\}$
$\{\mathtt{f}, \mathtt{t}\}$	$\{\{\mathtt{f}, \mathtt{t}\}\}$

Remark 4.3.5. The semantics of PNmatrices obtained for canonical calculi according to the definitions above coincide with the Nmatrices semantics suggested in [17, 19] (see also [21]) for the (narrower) families of canonical systems studied there. The transition from Nmatrices to PNmatrices makes it possible to provide semantics for *every* canonical calculi, while [17, 19] handle only a subset of them, called *coherent* canonical calculi. Indeed, the calculus $\mathbf{G}$ from Example 4.3.3 is not coherent (because of the rules for T), and it is captured by the (non-proper) PNmatrix given above.

It can be easily verified that for **LK** (see Example 2.2.20) we obtain the usual two-valued truth tables semantics of classical logic. In addition, the PNmatrix obtained for

LK's (*cut*)-free fragment is practically equivalent to the semantics of three-valued Schütte valuations for this calculus (see [58]). Obviously, since **LK** has cut-elimination, this semantics coincides with the usual classical semantics when it comes to the provability of sequents from the *empty set of assumptions*. Similarly, the PNmatrix for **LK**'s (*id*)-free fragment coincides with the three-valued semantics introduced in [64] for this fragment.[2] Next, we demonstrate how this semantics can be used to prove that certain applications of cuts or identity axioms are unavoidable.

Example 4.3.6. Using the PNmatrix semantics of canonical calculi, one can easily verify that using (*cut*) is unavoidable in a given derivation. For example, we obviously have $\{\mathtt{t}{:}p_1 \supset p_2\} \vdash_{\mathbf{LK}} \{\mathtt{t}{:}p_1 \supset (p_3 \supset p_2)\}$ (see Example 2.2.20 for a precise definition of **LK**). We show that the $\mathcal{L}_{cl}$-sequent $\{\mathtt{t}{:}p_1 \supset (p_3 \supset p_2)\}$ has no cut-free derivation in **LK** from $\{\mathtt{t}{:}p_1 \supset p_2\}$. Consider a partial $\mathcal{L}_{cl}$-valuation v with:

1. $Dom_v = \{p_1, p_2, p_3, p_1 \supset p_2, p_3 \supset p_2, p_1 \supset (p_3 \supset p_2)\}$.
2. $v(p_1) = v(p_3) = \{\mathtt{t}\}$, $v(p_2) = v(p_3 \supset p_2) = \{\mathtt{f}\}$, $v(p_1 \supset (p_3 \supset p_2)) = \{\mathtt{f}\}$, $v(p_1 \supset p_2) = \{\mathtt{f}, \mathtt{t}\}$.

Then v is $\mathbf{M}_{\mathbf{LK}_{cf}}$-legal. To see this note that $\mathcal{U}_{\mathbf{M}_{\mathbf{LK}_{cf}}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}, \{\mathtt{f}, \mathtt{t}\}\}$, and the PNtable $\supset_{\mathbf{M}_{\mathbf{LK}_{cf}}}$ is the following one (omitting the occurrences of the non-legal truth-value $\emptyset$):

$\supset_{\mathbf{M}_{\mathbf{LK}_{cf}}}$	$\{\mathtt{f}\}$	$\{\mathtt{t}\}$	$\{\mathtt{f},\mathtt{t}\}$
$\{\mathtt{f}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{t}\}$	$\{\{\mathtt{f}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$
$\{\mathtt{f},\mathtt{t}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{t}\},\{\mathtt{f},\mathtt{t}\}\}$	$\{\{\mathtt{f},\mathtt{t}\}\}$

Since $\mathbf{M}_{\mathbf{LK}_{cf}}$ is proper, the partial $\mathcal{L}_{cl}$-valuation v can be extended to an $\mathbf{M}_{\mathbf{LK}_{cf}}$-legal $\mathcal{L}_{cl}$-valuation v'. v' is a model of $\{\mathtt{t}{:}p_1 \supset p_2\}$ but not of $\{\mathtt{t}{:}p_1 \supset (p_3 \supset p_2)\}$, and so we have $\{\mathtt{t}{:}p_1 \supset p_2\} \nvdash_{\mathbf{M}_{\mathbf{LK}_{cf}}} \{\mathtt{t}{:}p_1 \supset (p_3 \supset p_2)\}$. Corollary 4.3.2 entails that $\{\mathtt{t}{:}p_1 \supset p_2\} \nvdash_{\mathbf{LK}_{cf}} \{\mathtt{t}{:}p_1 \supset (p_3 \supset p_2)\}$.

Example 4.3.7. Using the PNmatrix semantics, one can easily verify that using (*id*) is unavoidable in a given derivation. For example, clearly $\{\mathtt{t}{:}\neg p_1\} \vdash_{\mathbf{LK}} \{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$. Consider a partial $\mathcal{L}_{cl}$-valuation v with $Dom_v = \{p_1, p_2, \neg p_1, p_1 \wedge p_2, \neg(p_1 \wedge p_2)\}$, defined by $v(p_1) = \emptyset$, $v(\neg p_1) = v(p_2) = v(p_1 \wedge p_2) = \{\mathtt{t}\}$, and $v(\neg(p_1 \wedge p_2)) = \{\mathtt{f}\}$. It is easy to verify that v is $\mathbf{M}_{\mathbf{LK}_{if}}$-legal: $\mathcal{U}_{\mathbf{M}_{\mathbf{LK}_{if}}} = \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}\}$, the PNtable $\wedge_{\mathbf{M}_{\mathbf{LK}_{if}}}$ is the same as $\wedge_{\mathbf{M}_{\mathbf{G}_2}}$ from Example 4.3.3, and the PNtable $\neg_{\mathbf{M}_{\mathbf{LK}_{if}}}$ is the one given in case (*iv*) in Example 4.3.4. Since $\mathbf{M}_{\mathbf{LK}_{if}}$ is proper, v can be extended to an $\mathbf{M}_{\mathbf{LK}_{if}}$-legal $\mathcal{L}_{cl}$-valuation v'.

[2]Note that [58] and [64] concern also the usual quantifiers of **LK**, while here we only investigate its *propositional* fragment.

v' is a model of $\{\mathtt{t}{:}\neg p_1\}$ but not of $\{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$. Hence, $\{\mathtt{t}{:}\neg p_1\} \nvdash_{\mathbf{M}_{\mathbf{LK}_{if}}} \{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$. Corollary 4.3.2 entails that $\{\mathtt{t}{:}\neg p_1\} \nvdash_{\mathbf{LK}_{if}} \{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$. In other words, $\{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$ has no identity-axiom-free derivation in **LK** from $\{\mathtt{t}{:}\neg p_1\}$.

Now, the decidability of canonical calculi and logics induced by canonical calculi are immediate corollaries.

Corollary 4.3.8. Given a canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, a finite set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents and an $\langle \mathcal{L}, \pounds \rangle$-sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathbf{G}} s$ or not. In addition, given a canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, a set of labels $\mathtt{X} \subseteq \pounds$, a finite set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash^{\mathtt{X}}_{\mathbf{G}} \varphi$ or not.

Proof. Construct $\mathbf{M}_{\mathbf{G}}$ according to the definitions above and use Corollary 3.2.16, Theorem 4.2.14, and Corollary 4.3.2. □

We note that the use of *non-deterministic* truth tables is essential for charactering the logics induced by arbitrary canonical calculi. Indeed, the use of non-deterministic semantics is unavoidable in cases of "syntactic under-specification" in the canonical rules for the connectives (see [21]). But, even when the calculus employs the usual connectives with their *ordinary* introduction rules, non-deterministic truth tables are required to characterize the cut-free and the identity-axiom-free fragments of the calculus. The next proposition demonstrates this claim.

Definition 4.3.9. Let $\mathbf{M}$ be a (pre-) PNmatrix for $\mathcal{L}$. A connective $\diamond \in \diamondsuit_{\mathcal{L}}$ is called *deterministic in* $\mathbf{M}$ if $\diamond_{\mathbf{M}}(u_1, \dots, u_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M}}$ is a singleton for every $u_1, \dots, u_{ar(\diamond)} \in \mathcal{U}_{\mathbf{M}}$.

Proposition 4.3.10. Suppose that $\mathcal{L}$ contains a unary connective denoted by $\neg$, and let $\mathbf{G}$ be a canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus, whose rules for $\neg$ are the usual rules: $\{\mathtt{t}{:}p_1\}/\{\mathtt{f}{:}\neg p_1\}$ and $\{\mathtt{f}{:}p_1\}/\{\mathtt{t}{:}\neg p_1\}$. Let $\mathbf{M}$ be any finite PNmatrix for $\mathcal{L}$. If $\neg$ is deterministic in $\mathbf{M}$ then: $\Vdash_{\mathbf{M}} \neq \Vdash^{\{\mathtt{t}\}}_{\mathbf{G}_{cf}}$ and $\Vdash_{\mathbf{M}} \neq \Vdash^{\{\mathtt{t}\}}_{\mathbf{G}_{if}}$.

Proof. Assume that $\neg$ is deterministic in $\mathbf{M}$.

1. Suppose that $\Vdash_{\mathbf{M}} = \Vdash^{\{\mathtt{t}\}}_{\mathbf{G}_{cf}}$. Note that for every $n \geq 0$, $\{\neg^i p_1 \mid i > n\} \nVdash^{\{\mathtt{t}\}}_{\mathbf{G}_{cf}} \neg^n p_1$ (it is easy to verify this using Corollary 4.3.2). Consequently, $\{\neg^i p_1 \mid i > n\} \nVdash_{\mathbf{M}} \neg^n p_1$ for every $n \geq 0$. For every $n \geq 0$, let v_n be an $\mathbf{M}$-legal $\mathcal{L}$-valuation, which is a model (with respect to $\mathbf{M}$) of $\{\neg^i p_1 \mid i > n\}$, and not of $\neg^n p_1$. We show that $v_m(p_1) \neq v_n(p_1)$ for every $n > m \geq 0$ (and so, $\mathbf{M}$ is infinite). Let $n > m \geq 0$. Since v_m is a model of $\neg^n p_1$, and v_n is not a model of $\neg^n p_1$, we have $v_m(\neg^n p_1) \neq v_n(\neg^n p_1)$. This implies (using the fact that $\neg_{\mathbf{M}}$ is deterministic) that $v_m(p_1) \neq v_n(p_1)$.

2. Suppose that $\Vdash_{\mathbf{M}} = \Vdash^{\{\mathtt{t}\}}_{\mathbf{G}_{if}}$. Note that (i) $\neg^n p_1 \Vdash^{\{\mathtt{t}\}}_{\mathbf{G}_{if}} \neg^m p_1$ for every $n, m \in \mathbb{N}_{even}$ provided that $n \leq m$; and (ii) $\neg^{n+2} p_1 \nVdash^{\{\mathtt{t}\}}_{\mathbf{G}_{if}} \neg^n p_1$ for every $n \in \mathbb{N}$ (it is easy to verify this using Corollary 4.3.2). For every $n \geq 0$, let v_n be an $\mathbf{M}$-legal $\mathcal{L}$-valuation, which is a model (with respect to $\mathbf{M}$) of $\neg^{n+2} p_1$, but not of $\neg^n p_1$. We show that $v_n(p_1) \neq v_m(p_1)$ for every $n, m \in \mathbb{N}_{even}$ such that $n < m$ (and so, $\mathbf{M}$ is infinite). Let $n, m \in \mathbb{N}_{even}$ such that $n < m$. Then, since v_n is a model of $\neg^{n+2} p_1$, (i) implies that v_n is a model of $\neg^m p_1$. On the other hand, v_m is not a model of $\neg^m p_1$. This implies (using the fact that $\neg_{\mathbf{M}}$ is deterministic) that $v_n(p_1) \neq v_m(p_1)$. □

4.4 Characterization of Proof-Theoretic Properties

In addition to decision procedures, the semantics of PNmatrices is useful for checking proof-theoretic properties of canonical calculi. In this section we use the semantic characterizations of strong *sub*-analyticity, strong cut-admissibility, and axiom-expansion from the previous chapter to obtain simple *decidable* criteria for these properties in canonical calculi.

Using Theorem 3.3.3 and Proposition 4.2.12, the criterion for strong *sub*-analyticity is immediate:

Corollary 4.4.1. A canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ is strongly *sub*-analytic iff $\mathbf{M}_{\mathbf{G}}$ is proper.

Proof. By Theorem 3.3.3, $\mathbf{G}$ is strongly *sub*-analytic iff $\mathbf{M}_{\mathbf{G}}$ is *sub*-analytic. By Proposition 4.2.12, this holds iff $\mathbf{M}_{\mathbf{G}}$ is proper. □

To characterize strong cut-admissibility, we use the following lemma:

Lemma 4.4.2. Let $\mathbf{G}$ be a canonical $\langle \mathcal{L}, \pounds \rangle$-calculus. For every $\diamond \in \Diamond_{\mathcal{L}}$, and every $\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)}, \mathtt{X}'_1, \dots, \mathtt{X}'_{ar(\diamond)} \subseteq \pounds$ such that $\mathtt{X}_i \subseteq \mathtt{X}'_i$ for every $1 \leq i \leq ar(\diamond)$, we have $\diamond_{\mathbf{M}_{\mathbf{G}}}(\mathtt{X}'_1, \dots, \mathtt{X}'_{ar(\diamond)}) \subseteq \diamond_{\mathbf{M}_{\mathbf{G}}}(\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)})$.

Proof. Directly follows from the definitions. □

Theorem 4.4.3. A canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ enjoys strong cut-admissibility iff $\mathbf{M}_{\mathbf{G}}$ is proper.

Proof. First, if $\mathbf{M}_{\mathbf{G}}$ is not proper, then (by Corollary 4.4.1) $\mathbf{G}$ is not *sub*-analytic. By Proposition 4.1.5, $\mathbf{G}$ does not enjoy strong cut-admissibility. Now suppose that $\mathbf{M}_{\mathbf{G}}$ is proper. If $\mathbf{G}$ has no cut rules then we are obviously done. Assume otherwise. We use Theorem 3.3.9 to show that $\mathbf{G}$ enjoys strong cut-admissibility. Thus, we have to show that for every $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v, there exists an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation v' such that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$. Note that we have:

- $\mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}} = \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$ since the additional primitive $\pounds$-rules in $s(\mathbf{G})$ do not have any effect on the legal truth values.
- $\mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}} = \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})}} \cup \{\pounds\}$ by Lemma 3.3.11.
- $\pounds \notin \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$ since $\mathbf{G}$ has at least one cut rule.
- $\diamond_{\mathbf{M}_{s(\mathbf{G})_{cf}}} = \diamond_{\mathbf{M}_{\mathbf{G}}}$ for every $\diamond \in \diamondsuit_{\mathcal{L}}$ since $s(\mathbf{G})_{cf}$ and $\mathbf{G}$ differ only in their primitive rules.

Now, let v be an $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal $\mathcal{L}$-valuation. The construction of v' is done by recursion on the complexity of formulas. First, for atomic formulas, if $v(p) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$, we choose $v'(p) = v(p)$. Otherwise, we (arbitrarily) choose $v'(p)$ to be an element of $\mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$ (which is non-empty since $\mathbf{M}_{\mathbf{G}}$ is proper). Now, let $\diamond \in \diamondsuit_{\mathcal{L}}$, $\varphi = \diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})$, and suppose $v'(\varphi_i)$ was defined for every $1 \leq i \leq ar(\diamond)$. We choose $v'(\varphi)$ to be equal to $v(\varphi)$ if the latter is in $\mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$. Otherwise, $v(\varphi) = \pounds$, and we choose $v'(\varphi)$ to be some element of $\diamond_{\mathbf{M}_{\mathbf{G}}}(v'(\varphi_1), \dots, v'(\varphi_{ar(\diamond)})) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$ (such an element exists since $\mathbf{M}_{\mathbf{G}}$ is proper). We show that for every $\varphi \in \mathcal{L}$: $v'(\varphi) = v(\varphi)$ iff $v(\varphi) \neq \pounds$. First, if $v(\varphi) \neq \pounds$ then $v(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$. In this case we chose $v'(\varphi) = v(\varphi)$. Now, if $v(\varphi) = \pounds$ then $v(\varphi) \notin \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$, and $v'(\varphi) \neq v(\varphi)$ since we chose $v'(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$. It remains to show that v' is $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal. By definition, $v'(\varphi) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}}} \subseteq \mathcal{U}_{\mathbf{M}_{s(\mathbf{G})_{cf}}}$ for every $\varphi \in \mathcal{L}$. Suppose (for contradiction) that $v'(\varphi) \notin \diamond_{\mathbf{M}_{s(\mathbf{G})_{cf}}}(v'(\varphi_1), \dots, v'(\varphi_{ar(\diamond)}))$ for some formula $\varphi = \diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})$. Thus $v'(\varphi) \notin \diamond_{\mathbf{M}_{\mathbf{G}}}(v'(\varphi_1), \dots, v'(\varphi_{ar(\diamond)}))$. If $v(\varphi) = \pounds$, then our construction ensures that $v'(\varphi) \in \diamond_{\mathbf{M}_{\mathbf{G}}}(v'(\varphi_1), \dots, v'(\varphi_{ar(\diamond)})) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}}}$. Hence, $v(\varphi) \neq \pounds$, and so $v'(\varphi) = v(\varphi)$. Now, for every $1 \leq i \leq ar(\diamond)$, $v'(\varphi_i) = v(\varphi_i)$ iff $v(\varphi_i) \neq \pounds$, and thus $v'(\varphi_i) \subseteq v(\varphi_i)$. Lemma 4.4.2 entails that $v(\varphi) \notin \diamond_{\mathbf{M}_{\mathbf{G}}}(v(\varphi_1), \dots, v(\varphi_{ar(\diamond)}))$. This contradicts the fact that v is $\mathbf{M}_{s(\mathbf{G})_{cf}}$-legal. □

As a corollary we obtain the following correspondence in canonical $\langle \mathcal{L}, \pounds \rangle$-calculi:

Corollary 4.4.4. The following are equivalent for every canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$:

- $\mathbf{M}_{\mathbf{G}}$ is proper.
- $\mathbf{G}$ is strongly *sub*-analytic.
- $\mathbf{G}$ enjoys strong cut-admissibility.

Proof. The equivalence follows by Corollary 4.4.1 and Theorem 4.4.3. □

Corollary 4.4.5. The question whether a given canonical $\langle \mathcal{L}, \pounds \rangle$-calculus enjoys strong *sub*-analyticity and strong cut-admissibility is decidable.

Proof. Construct $\mathbf{M}_{\mathbf{G}}$ and check whether it is proper or not. □

Remark 4.4.6. A similar equivalence was proved in [17] for the (narrower) family of canonical calculi studied there. Note that [17] included also a necessary and sufficient condition for strong cut-admissibility using a simple syntactic condition of *coherence*. It can be shown that the coherence of a calculus $\mathbf{G}$ from the family studied in [17] is equivalent to the fact that $\mathbf{M_G}$ is proper.

Since canonical calculi are decidable, the property of *axiom-expansion* in them is decidable as well (see Definition 2.3.21). Indeed, one has to verify that

$$\{(\mathtt{X}{:}p_i) \mid \emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}, 1 \leq i \leq ar(\diamond)\} \vdash_{\mathbf{G}_{if}} (\mathtt{Y}{:} \diamond (p_1, \dots, p_{ar(\diamond)}))$$

for every $\mathtt{Y} \subseteq \pounds$ such that $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and connective $\diamond \in \Diamond_{\mathcal{L}}$ (recall that $\mathbf{G}_{if}$ denotes the calculus obtained from $\mathbf{G}$ by discarding all identity axioms). Alternatively, the following semantic criterion can be used:

Corollary 4.4.7. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds \rangle$-calculus. A connective $\diamond \in \Diamond_{\mathcal{L}}$ of arity n admits axiom-expansion in $\mathbf{G}$ if $\diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}} \subseteq \mathcal{U}_{\mathbf{M_G}}$ for every $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$. If $\mathbf{M}_{\mathbf{G}_{if}}$ is proper, then the converse holds as well.

Proof. Suppose that $\diamond$ does not admit axiom-expansion in $\mathbf{G}$. By Corollary 3.3.12, there exists an $\mathbf{M}_{\mathbf{G}_{if}}$-legal $\mathcal{L}$-valuation v, such that $\mathtt{X} \cap v(p_i) \neq \emptyset$ for every $\mathtt{X} \subseteq \pounds$ such that $\emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $1 \leq i \leq n$, but $\mathtt{X}_0 \cap v(\diamond(p_1, \dots, p_n)) = \emptyset$ for some $\mathtt{X}_0 \subseteq \pounds$ such that $\emptyset/(\mathtt{X}_0{:}p_1) \in \mathsf{P}_{\mathbf{G}}$. Since v is $\mathbf{M}_{\mathbf{G}_{if}}$-legal, we have $v(\diamond(p_1, \dots, p_n)) \in \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}}$ and $v(\diamond(p_1, \dots, p_n)) \in \diamond_{\mathbf{M}_{\mathbf{G}_{if}}}(v(p_1), \dots, v(p_n))$. Since $\mathtt{X}_0 \cap v(\diamond(p_1, \dots, p_n)) = \emptyset$, we have that $v(\diamond(p_1, \dots, p_n)) \notin \pounds(\emptyset/(\mathtt{X}_0{:}p_1))$, and so $v(\diamond(p_1, \dots, p_n)) \notin \mathcal{U}_{\mathbf{M_G}}$. By definition, we have $\diamond_{\mathbf{M}_{\mathbf{G}_{if}}} = \diamond_{\mathbf{M_G}}$, and hence, $\diamond_{\mathbf{M_G}}(v(p_1), \dots, v(p_n)) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}} \not\subseteq \mathcal{U}_{\mathbf{M_G}}$. Finally, note that $v(p_1), \dots, v(p_n) \in \mathcal{U}_{\mathbf{M_G}}$, since v is $\mathbf{M}_{\mathbf{G}_{if}}$-legal and $\mathtt{X} \cap v(p_i) \neq \emptyset$ for every $\mathtt{X} \subseteq \pounds$ such that $\emptyset/(\mathtt{X}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $1 \leq i \leq n$.

For the converse, suppose that $\mathbf{M}_{\mathbf{G}_{if}}$ is proper. Assume that there are $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$ and $\mathtt{X} \in \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}}$ such that $\mathtt{X} \notin \mathcal{U}_{\mathbf{M_G}}$. Consider the partial $\mathcal{L}$-valuation v, defined by: $Dom_v(v) = \{p_1, \dots, p_n, \diamond(p_1, \dots, p_n)\}$, $v(p_i) = \mathtt{X}_i$ for $1 \leq i \leq n$, and $v(\diamond(p_1, \dots, p_n)) = \mathtt{X}$. v is $\mathbf{M}_{\mathbf{G}_{if}}$-legal, and since $\mathbf{M}_{\mathbf{G}_{if}}$ is proper, there is an $\mathbf{M}_{\mathbf{G}_{if}}$-legal (full) $\mathcal{L}$-valuation v' that extends v (Proposition 4.2.12). Now, $\mathtt{Y} \cap v'(p_i) \neq \emptyset$ for every $\mathtt{Y} \subseteq \pounds$ such that $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ and $1 \leq i \leq n$ (since $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$), but $\mathtt{Y} \cap v'(\diamond(p_1, \dots, p_n)) \neq \emptyset$ for some $\mathtt{Y} \subseteq \pounds$ such that $\emptyset/(\mathtt{Y}{:}p_1) \in \mathsf{P}_{\mathbf{G}}$ (since $\mathtt{X} \notin \mathcal{U}_{\mathbf{M_G}}$). By Corollary 3.3.12, $\diamond$ does not admit axiom-expansion in $\mathbf{G}$. □

In the case of ordinary canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculi (as those studied in [17]) it is possible to obtain a simpler characterization of axiom-expansion, showing that the connectives that admit axiom-expansion are exactly the *deterministic* ones. This correspondence was originally proved in [11]. Here we provide a new semantic proof of it, using the more general characterization given in Corollary 4.4.7.

Corollary 4.4.8. Let $\mathbf{G}$ be a pure $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus with $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$, in which the conclusion of each non-primitive rule takes the form $\{\mathtt{x}{:}\diamond(p_1, \dots, p_{ar(\diamond)})\}$ for some $\mathtt{x} \in \pounds_{\mathbf{2}}$ and $\diamond \in \Diamond_{\mathcal{L}}$. Suppose that $\mathbf{M_G}$ is proper. A connective $\diamond \in \Diamond^n_{\mathcal{L}}$ admits axiom-expansion in $\mathbf{G}$ iff $\diamond_{\mathbf{M_G}}$ is deterministic (see Definition 4.3.9).

Proof. Following Example 3.2.9, $\mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}} = \{\emptyset, \{\mathtt{f}\}, \{\mathtt{t}\}\}$, and $\mathcal{U}_{\mathbf{M_G}} = \{\{\mathtt{f}\}, \{\mathtt{t}\}\}$. In addition, for the given $\mathbf{G}$, the following properties of $\mathbf{M_G}$ are easily obtained from the definitions:

- $\{\{\mathtt{f}\}, \{\mathtt{t}\}\} \subseteq \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n)$ iff $\emptyset \in \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n)$ for every $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$.
- $\mathbf{M}_{\mathbf{G}_{if}}$ is proper. Otherwise, $\diamond_{\mathbf{M}_{\mathbf{G}_{if}}}(\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}} = \emptyset$ for some $\diamond \in \Diamond_{\mathcal{L}}$ and $\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)} \in \mathcal{U}_{M_{\mathbf{G}_{if}}}$. Let $\mathtt{X}'_i = \mathtt{X}_i \cup \{\mathtt{f}\}$ for every $1 \leq i \leq ar(\diamond)$. By Lemma 4.4.2, $\diamond_{\mathbf{M}_{\mathbf{G}_{if}}}(\mathtt{X}'_1, \dots, \mathtt{X}'_{ar(\diamond)}) \subseteq \diamond_{\mathbf{M}_{\mathbf{G}_{if}}}(\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)})$, and so $\diamond_{\mathbf{M_G}}(\mathtt{X}'_1, \dots, \mathtt{X}'_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M_G}} = \emptyset$. This contradicts the fact that $\mathbf{M_G}$ is proper.

Suppose that $\diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_{ar(\diamond)}) \cap \mathcal{U}_{\mathbf{M_G}}$ is not a singleton for some $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$. Thus $\{\{\mathtt{f}\}, \{\mathtt{t}\}\} \subseteq \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n)$. Hence, $\emptyset \in \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n)$. Therefore, we have $\emptyset \in \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}}$ but $\emptyset \notin \mathcal{U}_{\mathbf{M_G}}$. By Corollary 4.4.7, since $\mathbf{M}_{\mathbf{G}_{if}}$ is proper, $\diamond$ does not admit axiom-expansion in $\mathbf{G}$.

For the converse, suppose that $\diamond$ does not admit axiom-expansion in $\mathbf{G}$. By Corollary 4.4.7, there are $\mathtt{X}_1, \dots, \mathtt{X}_n \in \mathcal{U}_{\mathbf{M_G}}$ and $\mathtt{X} \in \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n) \cap \mathcal{U}_{\mathbf{M}_{\mathbf{G}_{if}}}$ such that $\mathtt{X} \notin \mathcal{U}_{\mathbf{M_G}}$. We must have $\mathtt{X} = \emptyset$, and hence $\{\{\mathtt{f}\}, \{\mathtt{t}\}\} \subseteq \diamond_{\mathbf{M_G}}(\mathtt{X}_1, \dots, \mathtt{X}_n)$. Hence $\diamond_{\mathbf{M_G}}$ is non-deterministic. ∎

Chapter 5

Quasi-canonical Calculi

In the previous chapter we studied the family of *canonical calculi*, and showed how to semantically characterize each logic induced by a canonical calculus using a PNmatrix, and use this PNmatrix to give a decision procedure for the logic. However, a variety of important logics have pure sequent calculi that are *not* canonical, but still have somewhat similar nature. In particular, this is true for the family of *quasi-canonical systems*, introduced in [13]. These are propositional fully-structural two-sided systems, which in addition to the usual weakening rules, cut and identity axiom, include also logical rules with the following properties (the language is assumed to have a unary connective $\neg$):

1. Exactly one formula is introduced in the conclusion of the rule, on exactly one of its two sides.
2. The formula being introduced is either of the form $\diamond(p_1, \dots, p_{ar(\diamond)})$ or of the form $\neg \diamond (p_1, \dots, p_{ar(\diamond)})$.
3. All formulas in the premises of a rule introducing a connective $\diamond$ belong to the set $\{p_1, \dots, p_{ar(\diamond)}, \neg p_1, \dots, \neg p_{ar(\diamond)}\}$.
4. There are no restrictions on the side formulas in the rule application (i.e. the rules are *pure*).

Of course, rules of this kind are not canonical in the sense of Definition 4.1.1, due to the following two "violations": (i) the introduced formula can be not only $\diamond(p_1, \dots, p_{ar(\diamond)})$, but also $\neg \diamond (p_1, \dots, p_{ar(\diamond)})$, and (ii) the premises may contain not only atomic formulas p_i, but also $\neg p_i$. Hence the results obtained in Chapter 4 for canonical calculi do not directly apply.

In this chapter we show that the theory of canonical calculi can still be exploited for quasi-canonical systems by translating them into equivalent (in the sense defined below) *canonical* calculi. In fact, this is possible for a substantially larger family of many-sided calculi of which the quasi-canonical systems of [13] are particular examples. In particular, in the calculi studied in this chapter $\neg$ does not have any special status, and all unary connectives of a given language may occur in the premises or precede the main connective in the conclusions of the logical rules. Note that various calculi for important many-valued logics (e.g. for the propositional part of the relevance logic of first degree entailment [1]), as well as the (cut-free) calculi proposed in [13] and [8] for many paraconsistent logics (particularly, for *C-systems* [40]), fall in the family of quasi-canonical calculi studied in this chapter.

Publications Related to this Chapter

The material in this chapter was included in [29]. However, [29] concerned only two-sided quasi-canonical calculi with the usual cut and identity axiom, while here we naturally consider quasi-canonical calculi employing any finite set of labels and primitive rules.

5.1 Quasi-canonical Calculi

As noted above, the language of quasi-canonical systems is assumed in [13] to include a unary connective $\neg$. This restriction can be lifted by allowing any unary connective (possibly in addition to $\neg$). Similarly, we shall not restrict ourselves to two-sided sequents, and continue working in the full framework of pure $\langle\mathcal{L}, \pounds\rangle$-calculi. The notion of a quasi-canonical rule can then be formalized in our terms as follows:

Definition 5.1.1. A pure $\langle\mathcal{L}, \pounds\rangle$-rule r is called *quasi-canonical* if its conclusion has the form $\{\mathtt{X}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ or $\{\mathtt{X}{:} \star \diamond(p_1, \dots, p_{ar(\diamond)})\}$ for some non-empty $\mathtt{X} \subseteq \pounds$, $\diamond \in \Diamond_{\mathcal{L}}$, and $\star \in \Diamond^1_{\mathcal{L}}$, and its premises are all $\langle\{p_1, \dots, p_{ar(\diamond)}\} \cup \{\star p_i \mid \star \in \Diamond^1_{\mathcal{L}}, 1 \leq i \leq ar(\diamond)\}, \pounds\rangle$-sequents. Quasi-canonical pure $\langle\mathcal{L}, \pounds\rangle$-rules will be also called *quasi-canonical* $\langle\mathcal{L}, \pounds\rangle$-*rules.*

Note that this definition is more liberal then Definition 4.1.1, and every canonical $\langle\mathcal{L}, \pounds\rangle$-rule is also a quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-rule.

Example 5.1.2. Various sequent calculi for paraconsistent logics and relevance logics employ (some of) the following quasi-canonical $\langle\mathcal{L}_{cl}, \pounds_{\mathbf{2}}\rangle$-rules (see [8]):

$(\mathtt{f}{:}\neg\neg)$	$\{\mathtt{f}{:}p_1\}/\{\mathtt{f}{:}\neg\neg p_1\}$	$(\mathtt{t}{:}\neg\neg)$	$\{\mathtt{t}{:}p_1\}/\{\mathtt{t}{:}\neg\neg p_1\}$
$(\mathtt{f}{:}\neg\wedge)$	$\{\mathtt{f}{:}\neg p_1\}, \{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1 \wedge p_2)\}$	$(\mathtt{t}{:}\neg\wedge)$	$\{\mathtt{f}{:}\neg p_1, \mathtt{f}{:}\neg p_2\}/\{\mathtt{t}{:}\neg(p_1 \wedge p_2)\}$
$(\mathtt{f}{:}\neg\vee)$	$\{\mathtt{f}{:}\neg p_1, \mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1 \vee p_2)\}$	$(\mathtt{t}{:}\neg\vee)$	$\{\mathtt{t}{:}\neg p_1\}, \{\mathtt{t}{:}\neg p_2\}/\{\mathtt{t}{:}\neg(p_1 \vee p_2)\}$
$(\mathtt{f}{:}\neg\supset)$	$\{\mathtt{f}{:}p_1, \mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1 \supset p_2)\}$	$(\mathtt{t}{:}\neg\supset)$	$\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}\neg p_2\}/\{\mathtt{t}{:}\neg(p_1 \supset p_2)\}$

Note that non of these rules is a canonical $\langle\mathcal{L}_{cl}, \pounds_{\mathbf{2}}\rangle$-rule.

Example 5.1.3. All (non-primitive) rules of $\mathbf{G_{C_1}}$ (see Example 2.2.22) except for the rule $\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}\neg p_1\} / \{\mathtt{f}{:}\neg(p_1 \wedge \neg p_1)\}$ are quasi-canonical.

In turn, quasi-canonical calculi are defined as follows:

Definition 5.1.4. A *quasi-canonical* $\langle \mathcal{L}, \pounds \rangle$*-calculus* $\mathbf{G}$ is a pure $\langle \mathcal{L}, \pounds \rangle$-calculus, such that $\mathsf{R}_{\mathbf{G}}$ (the set of non-primitive rules of $\mathbf{G}$) consists only of quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-rules.

Various well-known logics are induced by quasi-canonical calculi. This includes many important three and four-valued logics (e.g. for the relevance logic of first degree entailment [1]), a large family of paraconsistent logics known as C-systems ([40]), for which cut-free quasi-canonical systems were proposed in [13], and various other paraconsistent extensions of positive classical logic studied in [8]. Note that the quasi-canonical systems of [13] correspond to quasi-canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculi with $\neg \in \diamondsuit^1_{\mathcal{L}}$, $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$, $\neg$ is the only connective that may appear in the premises of non-primitive rules, and the conclusions of non-primitive rules have the form $\{\mathtt{x}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ or $\{\mathtt{x}{:}\neg \diamond (p_1, \dots, p_{ar(\diamond)})\}$ for $\mathtt{x} \in \pounds_{\mathbf{2}}$ and $\diamond \in \diamondsuit_{\mathcal{L}}$.

5.2 From Quasi-canonical to Canonical Calculi

We provide a translation of a given quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$ into a canonical one $T(\mathbf{G})$, which is equivalent to $\mathbf{G}$ in a sense defined below. The idea is to "encode" the information related to the connectives from $\diamondsuit^1_{\mathcal{L}}$ in the labels employed in $\mathbf{G}$, so that connectives from $\diamondsuit^1_{\mathcal{L}}$ "violating" canonicity are removed. To this end, we use $\pounds \cup \{\mathtt{x}^\star \mid \mathtt{x} \in \pounds, \star \in \diamondsuit^1_{\mathcal{L}}\}$ as the set of labels for $T(\mathbf{G})$. We denote this set by $\pounds^{\diamondsuit^1_{\mathcal{L}}}$. Note that this may be seen as a generalization of the original idea behind two-sided sequents in classical logic. Indeed, often one-sided sequents are translated to two-sided ones by differentiating the negated formulas from the non-negated ones using two different labels (sides).

Definition 5.2.1. For an $\pounds$-labelled $\mathcal{L}$-formula α, $T(\alpha)$ is the $\pounds^{\diamondsuit^1_{\mathcal{L}}}$-labelled $\mathcal{L}$-formula, defined as follows: $T(\alpha) = \mathtt{x}^\star{:}\varphi$ if $\alpha = \mathtt{x}{:} \star \varphi$ for some $\mathtt{x} \in \pounds$, $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\varphi \in \mathcal{L}$, and otherwise $T(\alpha) = \alpha$. T is extended to $\langle \mathcal{L}, \pounds \rangle$-sequents and sets of $\langle \mathcal{L}, \pounds \rangle$-sequents in the obvious way (e.g. $T(s) = \{T(\alpha) \mid \alpha \in s\}$).

Example 5.2.2. Suppose that $\neg \in \diamondsuit^1_{\mathcal{L}}$ and $\pounds = \pounds_{\mathbf{2}}$. Then:

$$T(\{\mathtt{f}{:}\neg\neg p_1, \mathtt{t}{:}\neg p_1, \mathtt{t}{:}p_2\}) = \{\mathtt{f}^\neg{:}\neg p_1, \mathtt{t}^\neg{:}p_1, \mathtt{t}{:}p_2\}.$$

The translation of a quasi-canonical calculus into a canonical one is given by:

Notation 5.2.3. Given $\mathtt{X} \subseteq \pounds$ and $\star \in \diamondsuit^1_{\mathcal{L}}$, $\mathtt{X}^\star$ denotes the subset $\{\mathtt{x}^\star \mid \mathtt{x} \in \mathtt{X}\}$ of $\pounds^{\diamondsuit^1_{\mathcal{L}}}$.

Definition 5.2.4. Let $\mathbf{G}$ be a quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-calculus. $T(\mathbf{G})$ is the canonical $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-calculus that consists of the following rules (except for the weakening rules):

- $T(\mathbf{G})$ includes all primitive $\pounds$-rule in $\mathsf{P}_{\mathbf{G}}$, and, in addition, for every primitive $\pounds$-rule in $\mathsf{P}_{\mathbf{G}}$, $r = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$, and $\star \in \Diamond^1_{\mathcal{L}}$, $T(\mathbf{G})$ has the primitive $\pounds^{\Diamond^1_{\mathcal{L}}}$-rule $(\mathtt{X}_1^\star{:}p_1), \dots, (\mathtt{X}_n^\star{:}p_1)/(\mathtt{X}^\star{:}p_1)$, denoted below by $r^\star$.

- For every quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-rule $r = \mathcal{S}/s$ of $\mathbf{G}$, $T(\mathbf{G})$ includes the $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-rule $T(\mathcal{S})/T(s)$, denoted by $T(r)$.

- For every $\star \in \Diamond^1_{\mathcal{L}}$, $T(\mathbf{G})$ includes the canonical $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-rule $\{\mathtt{x}^\star{:}p_1\}/\{\mathtt{x}{:}\star p_1\}$, denoted below by $(\mathtt{x}^\star \to \mathtt{x})$.

Example 5.2.5. Let $\mathcal{L} = \{\neg^1, \wedge^2\}$. The system $PLK[\{(\neg\wedge \Rightarrow)\}]$ from [8] is practically a quasi-canonical $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-calculus. Besides the weakening rules, (cut), and (id) this calculus includes the following quasi-canonical rules:

$(\mathtt{f}{:}\wedge)$	$\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$	$(\mathtt{t}{:}\wedge)$	$\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \wedge p_2\}$
$(\mathtt{t}{:}\neg)$	$\{\mathtt{f}{:}p_1\}/\{\mathtt{t}{:}\neg p_1\}$	$(\mathtt{f}{:}\neg\wedge)$	$\{\mathtt{f}{:}\neg p_1\}, \{\mathtt{f}{:}\neg p_2\}/\{\mathtt{f}{:}\neg(p_1 \wedge p_2)\}$

We denote this system by $\mathbf{G}_0$. Now, ${\pounds_{\mathbf{2}}}^{\Diamond^1_{\mathcal{L}}} = \{\mathtt{f}, \mathtt{t}, \mathtt{f}^\neg, \mathtt{t}^\neg\}$. $T(\mathbf{G}_0)$ is the canonical $\langle\mathcal{L}, {\pounds_{\mathbf{2}}}^{\Diamond^1_{\mathcal{L}}}\rangle$-calculus, with the following rules:

$(\mathtt{f}{:}weak)$	$\{\emptyset\}/\{\mathtt{f}{:}p_1\}$	$(\mathtt{t}{:}weak)$	$\{\emptyset\}/\{\mathtt{t}{:}p_1\}$
$(\mathtt{f}^\neg{:}weak)$	$\{\emptyset\}/\{\mathtt{f}^\neg{:}p_1\}$	$(\mathtt{t}^\neg{:}weak)$	$\{\emptyset\}/\{\mathtt{t}^\neg{:}p_1\}$
(cut)	$\{\mathtt{f}{:}p_1\}, \{\mathtt{t}{:}p_1\}/\emptyset$	(id)	$\emptyset/\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_1\}$
$(cut)^\neg$	$\{\mathtt{f}^\neg{:}p_1\}, \{\mathtt{t}^\neg{:}p_1\}/\emptyset$	$(id)^\neg$	$\emptyset/\{\mathtt{f}^\neg{:}p_1, \mathtt{t}^\neg{:}p_1\}$
$T((\mathtt{f}{:}\wedge))$	$\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$	$T((\mathtt{t}{:}\wedge))$	$\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}/\{\mathtt{t}{:}p_1 \wedge p_2\}$
$T((\mathtt{t}{:}\neg))$	$\{\mathtt{f}{:}p_1\}/\{\mathtt{t}^\neg{:}p_1\}$	$T((\mathtt{f}{:}\neg\wedge))$	$\{\mathtt{f}^\neg{:}p_1\}, \{\mathtt{f}^\neg{:}p_2\}/\{\mathtt{f}^\neg{:}p_1 \wedge p_2\}$
$(\mathtt{f}^\neg \to \mathtt{f})$	$\{\mathtt{f}^\neg{:}p_1\}/\{\mathtt{f}{:}\neg p_1\}$	$(\mathtt{t}^\neg \to \mathtt{t})$	$\{\mathtt{t}^\neg{:}p_1\}/\{\mathtt{t}{:}\neg p_1\}$

It is easy to see that all rules in $T(\mathbf{G})$ are primitive $\pounds^{\Diamond^1_{\mathcal{L}}}$-rules or canonical $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-rules, and thus $T(\mathbf{G})$ is a canonical $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-calculus. In particular, note that for a quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-rule r, $T(r)$ as defined above, can either be a canonical $\langle\mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}}\rangle$-rule (as $T((\mathtt{f}{:}\neg\wedge))$ above) or a primitive $\pounds^{\Diamond^1_{\mathcal{L}}}$-rule (as $T((\mathtt{t}{:}\neg))$ above) .

To prove the equivalence between $\mathbf{G}$ and $T(\mathbf{G})$ we use the following lemmas:

Lemma 5.2.6. Let $\mathtt{x}{:}\varphi$ be an $\pounds$-labelled $\mathcal{L}$-formula, and σ an $\mathcal{L}$-substitution, such that $\varphi \notin at_{\mathcal{L}}$ or $\sigma(\varphi)$ is not of the form $\star\psi$ for $\star \in \Diamond^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$. Then, $T(\sigma(\mathtt{x}{:}\varphi)) = \sigma(T(\mathtt{x}{:}\varphi))$.

Proof. If $\varphi = \star\psi$ for some $\star \in \Diamond^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$, then:

$$T(\sigma(\mathtt{x}{:}\varphi)) = T(\mathtt{x}{:}\star\sigma(\psi)) = \mathtt{x}^\star{:}\sigma(\psi) = \sigma(\mathtt{x}^\star{:}\psi) = \sigma(T(\mathtt{x}{:}\varphi)).$$

Otherwise, φ is either atomic or $\varphi = \diamond(p_1, \dots, p_{ar(\diamond)})$ for $ar(\diamond) \neq 1$, and in both cases $\sigma(\varphi)$ is not of the form $\star\psi$ for $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$. Then:

$$T(\sigma(\mathtt{x}{:}\varphi)) = T(\mathtt{x}{:}\sigma(\varphi)) = \mathtt{x}{:}\sigma(\varphi) = \sigma(\mathtt{x}{:}\varphi) = \sigma(T(\mathtt{x}{:}\varphi)). \qquad \square$$

Lemma 5.2.7. Let **G** be a quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-calculus, and σ an $\mathcal{L}$-substitution. For every $\pounds$-labelled $\mathcal{L}$-formula α: $\{T(\sigma(\alpha))\} \vdash_{T(\mathbf{G})} \{\sigma(T(\alpha))\}$.

Proof. Let $\alpha = \mathtt{x}{:}\varphi$ (for $\mathtt{x} \in \pounds$ and $\varphi \in \mathcal{L}$). If $\varphi \notin at_{\mathcal{L}}$ or $\sigma(\varphi)$ is not of the form $\star\psi$ for $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$, then by Lemma 5.2.6, $T(\sigma(\alpha)) = \sigma(T(\alpha))$, and obviously, $\{T(\sigma(\alpha))\} \vdash_{T(\mathbf{G})} \{\sigma(T(\alpha))\}$. Otherwise, $\varphi \in at_{\mathcal{L}}$ and $\sigma(\varphi) = \star\psi$ for some $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$. In this case, $T(\sigma(\alpha)) = T(\mathtt{x}{:}\star\psi) = \mathtt{x}^{\star}{:}\psi$, and $\sigma(T(\alpha)) = \sigma(\alpha) = \mathtt{x}{:}\star\psi$. By applying the rule $(\mathtt{x}^{\star} \to \mathtt{x})$, we obtain that $\{T(\sigma(\alpha))\} \vdash_{T(\mathbf{G})} \{\sigma(T(\alpha))\}$. $\square$

Proposition 5.2.8. For every quasi-canonical $\langle\mathcal{L}, \pounds\rangle$-calculus **G**, set $\mathcal{S}$ of $\langle\mathcal{L}, \pounds\rangle$-sequents and $\langle\mathcal{L}, \pounds\rangle$-sequent s: if $\mathcal{S} \vdash_{\mathbf{G}} s$ then $T(\mathcal{S}) \vdash_{T(\mathbf{G})} T(s)$.

Proof. It suffices to show that for every application of a rule of **G** deriving the $\langle\mathcal{L}, \pounds\rangle$-sequent s from the $\langle\mathcal{L}, \pounds\rangle$-sequents $s_1, \dots, s_n$, we can derive $T(s)$ from $T(s_1), \dots, T(s_n)$ in $T(\mathbf{G})$. Consider the possible cases:

- $s = \{\mathtt{x}{:}\varphi\} \cup c$ is derived from c by applying a weakening rule. In this case, we can derive $T(s) = T(\{\mathtt{x}{:}\varphi\}) \cup T(c)$ from $T(c)$ by applying weakening as well.

- $s = (\mathtt{X}{:}\varphi) \cup c_1 \cup \dots \cup c_n$ is derived from $s_1 = (\mathtt{X}_1{:}\varphi) \cup c_1, \dots, s_n = (\mathtt{X}_n{:}\varphi) \cup c_n$ by applying of a primitive $\pounds$-rule $r = (\mathtt{X}_1{:}p_1), \dots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ of **G**. If φ does not have the form $\star\psi$ (for $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$), then $T(s_i) = \{\mathtt{X}_i{:}\varphi\} \cup T(c_i)$ for every $1 \leq i \leq n$, and $T(s) = \{\mathtt{X}{:}\varphi\} \cup T(c_1) \cup \dots \cup T(c_n)$. By applying the same primitive $\pounds$-rule r, we can derive $T(s)$ from $T(s_1), \dots, T(s_n)$ in $T(\mathbf{G})$. Otherwise, $\varphi = \star\psi$ (for $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$). Here, $T(s_i) = (\mathtt{X}^{\star}_i{:}\psi) \cup T(c_i)$ for every $1 \leq i \leq n$, and $T(s) = (\mathtt{X}^{\star}{:}\psi) \cup T(c_1) \cup \dots \cup T(c_n)$. By applying $r^{\star}$, we can derive $T(s)$ from $T(s_1), \dots, T(s_n)$ in $T(\mathbf{G})$.

- $s = \sigma(s') \cup c_1 \cup \dots \cup c_n$ is derived from $s_1 = \sigma(s'_1) \cup c_1, \dots, s_n = \sigma(s'_n) \cup c_n$ by applying a quasi-canonical rule $r = s'_1, \dots, s'_n/s'$ of **G**. For every $1 \leq i \leq n$, $T(s_i) = T(\sigma(s'_i)) \cup T(c_i)$, and thus by Lemma 5.2.7 (using also Proposition 2.2.17), $T(s_i) \vdash_{T(\mathbf{G})} \sigma(T(s'_i)) \cup T(c_i)$. By applying the rule $T(r)$ of $T(\mathbf{G})$ we can derive $\sigma(T(s')) \cup T(c_1) \cup \dots \cup T(c_n)$. Since r is quasi-canonical, s' consists solely of non-atomic formulas, and by Lemma 5.2.6, $\sigma(T(s')) = T(\sigma(s'))$. Thus, we derived $T(s)$ in $T(\mathbf{G})$. $\square$

For the other direction, we define a translation T^{-1}, mapping $\langle\mathcal{L}, \pounds^{\diamondsuit^1_{\mathcal{L}}}\rangle$-sequents to $\langle\mathcal{L}, \pounds\rangle$-sequents:

Definition 5.2.9. For an $\pounds^{\Diamond^1_{\mathcal{L}}}$-labelled $\mathcal{L}$-formula α, $T^{-1}(\alpha)$ is the $\pounds$-labelled $\mathcal{L}$-formula, defined as follows: $T^{-1}(\alpha) = \mathtt{x}{:} \star \varphi$ if $\alpha = \mathtt{x}^\star{:}\varphi$ for some $\mathtt{x} \in \pounds$, $\star \in \Diamond^1_{\mathcal{L}}$ and $\varphi \in \mathcal{L}$, and otherwise $T^{-1}(\alpha) = \alpha$. T^{-1} is extended to $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequents and sets of $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequents in the obvious way (e.g. $T^{-1}(s) = \{T^{-1}(\alpha) \mid \alpha \in s\}$).

Lemma 5.2.10. $T^{-1}(\sigma(T(s))) = \sigma(s)$ for every $\langle \mathcal{L}, \pounds \rangle$-sequent s, and $\mathcal{L}$-substitution σ.

Proposition 5.2.11. For every quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequents and $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequent s: if $\mathcal{S} \vdash_{T(\mathbf{G})} s$ then $T^{-1}(\mathcal{S}) \vdash_{\mathbf{G}} T^{-1}(s)$.

Proof. It suffices to show that for every application of a rule of $T(\mathbf{G})$ deriving the $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequent s from the $\langle \mathcal{L}, \pounds^{\Diamond^1_{\mathcal{L}}} \rangle$-sequents $s_1, \ldots, s_n$, we can derive $T^{-1}(s)$ from $T^{-1}(s_1), \ldots, T^{-1}(s_n)$ in $\mathbf{G}$. Consider the possible cases:

- $s = \{\mathtt{x}{:}\varphi\} \cup c$ is derived from c by applying a weakening rule. In this case, we can derive $T^{-1}(s) = T^{-1}(\{\mathtt{x}{:}\varphi\}) \cup T^{-1}(c)$ from $T^{-1}(c)$ by applying weakening as well.

- $s = (\mathtt{X}{:}\varphi) \cup c_1 \cup \ldots \cup c_n$ is derived from $s_1 = (\mathtt{X}_1{:}\varphi) \cup c_1, \ldots, s_n = (\mathtt{X}_n{:}\varphi) \cup c_n$ by applying the primitive $\pounds$-rule $(\mathtt{X}_1{:}p_1), \ldots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ of $T(\mathbf{G})$, that occurs also in $\mathbf{G}$ itself. For every $1 \leq i \leq n$, $T^{-1}(s_i) = (\mathtt{X}_i{:}\varphi) \cup T^{-1}(c_i)$. By applying the same rule we can derive $T^{-1}(s) = (\mathtt{X}{:}\varphi) \cup T^{-1}(c_1) \cup \ldots \cup T^{-1}(c_n)$ in $\mathbf{G}$.

- $s = (\mathtt{X}^\star{:}\varphi) \cup c_1 \cup \ldots \cup c_n$ is derived from $s_1 = (\mathtt{X}^\star_1{:}\varphi) \cup c_1, \ldots, s_n = (\mathtt{X}^\star_n{:}\varphi) \cup c_n$ by applying the primitive $\pounds^{\Diamond^1_{\mathcal{L}}}$-rule $r^\star = (\mathtt{X}^\star_1{:}p_1), \ldots, (\mathtt{X}^\star_n{:}p_1)/(\mathtt{X}^\star{:}p_1)$ of $T(\mathbf{G})$. In this case $r = (\mathtt{X}_1{:}p_1), \ldots, (\mathtt{X}_n{:}p_1)/(\mathtt{X}{:}p_1)$ is a primitive rule of $\mathbf{G}$. For every $1 \leq i \leq n$, $T^{-1}(s_i) = (\mathtt{X}_i{:} \star \varphi) \cup T^{-1}(c_i)$. By applying the rule r we can derive the $\langle \mathcal{L}, \pounds \rangle$-sequent $T^{-1}(s) = (\mathtt{X}{:} \star \varphi) \cup T^{-1}(c_1) \cup \ldots \cup T^{-1}(c_n)$ in $\mathbf{G}$.

- $s = \sigma(T(s')) \cup c_1 \cup \ldots \cup c_n$ is derived from $s_1 = \sigma(T(s'_1)) \cup c_1, \ldots, s_n = \sigma(T(s'_n)) \cup c_n$ by applying a rule $T(r)$ of $T(\mathbf{G})$, where $r = s'_1, \ldots, s'_n/s'$ is a rule of $\mathbf{G}$. By Lemma 5.2.10, for every $1 \leq i \leq n$, $T^{-1}(s_i) = \sigma(s'_1) \cup T^{-1}(c_i)$. By applying r we can derive $T^{-1}(s) = \sigma(s') \cup T^{-1}(c_1) \cup \ldots \cup T^{-1}(c_n)$ in $\mathbf{G}$.

- $s = \{\mathtt{x}{:} \star \varphi\} \cup c$ is derived from $s_1 = \{\mathtt{x}^\star{:}\varphi) \cup c$ using the rule $(\mathtt{x}^\star \to \mathtt{x})$. In this case we have $T^{-1}(s) = T^{-1}(s_1)$, and obviously we are done. □

It follows that $T(\mathbf{G})$ is equivalent to the original quasi-canonical calculus $\mathbf{G}$ in the following sense:

Theorem 5.2.12. For every quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents and $\langle \mathcal{L}, \pounds \rangle$-sequent s: $\mathcal{S} \vdash_{\mathbf{G}} s$ iff $T(\mathcal{S}) \vdash_{T(\mathbf{G})} T(s)$.

Proof. By Lemma 5.2.10 $T^{-1}(T(s)) = s$ for every $\langle \mathcal{L}, \pounds \rangle$-sequent s (take the "identity substitution"). Thus the claim directly follows from Propositions 5.2.8 and 5.2.11. □

A general decidability result for all quasi-canonical calculi immediately follows:

Corollary 5.2.13.

1. Given a quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, a finite set $\mathcal{S}$ of $\langle \mathcal{L}, \pounds \rangle$-sequents, and an $\langle \mathcal{L}, \pounds \rangle$-sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathbf{G}} s$ or not.
2. Given a quasi-canonical $\langle \mathcal{L}, \pounds \rangle$-calculus $\mathbf{G}$, a set $\mathtt{X} \subseteq \pounds$, a finite set Γ of $\mathcal{L}$-formulas, and an $\mathcal{L}$-formula φ, it is decidable whether $\Gamma \Vdash^{\mathtt{X}}_{\mathbf{G}} \varphi$ or not.

Proof. Directly follows from Corollary 4.3.8 and Theorem 5.2.12. Note that the construction of $T(\mathbf{G})$ from $\mathbf{G}$ is obviously computable. □

In addition to decidability, the results of the previous chapters provide a method to obtain a (pre-) PNmatrix semantics for $T(\mathbf{G})$ and the logics it induces. Next we show that in the most common and interesting case, dealing with a quasi-canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus $\mathbf{G}$ with $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$, we are also able to use the semantic framework of PNmatrices to characterize $\mathbf{G}$ *itself* and the logics it induces.

Proposition 5.2.14. Let $\mathbf{G}$ be a quasi-canonical $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-calculus, and v an $\mathbf{M}_{T(\mathbf{G})}$-legal $\mathcal{L}$-valuation. Suppose that $\mathsf{P}_{\mathbf{G}} = \{(cut), (id)\}$. Then, for every $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequent s: $v \models s$ iff $v \models T(s)$.

Proof. Suppose that $v \models T(s)$. By definition, there exists some $\pounds_{\mathbf{2}}^{\diamondsuit^1_{\mathcal{L}}}$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi \in T(s)$ such that $\mathtt{x} \in v(\varphi)$. If $\mathtt{x} \in \pounds_{\mathbf{2}}$ then $\mathtt{x}{:}\varphi \in s$ as well, and clearly $v \models s$. Otherwise, $\mathtt{x} = \mathtt{y}^\star$ for some $\mathtt{y} \in \pounds_{\mathbf{2}}$ and $\star \in \diamondsuit^1_{\mathcal{L}}$, and we have $\mathtt{y}{:} \star \varphi \in s$. Now, since v is $\mathbf{M}_{T(\mathbf{G})}$-legal and $T(\mathbf{G})$ includes the rule $(\mathtt{y}^\star \to \mathtt{y})$, we should have $\mathtt{y} \in v(\star\varphi)$, and consequently, $v \models s$. To see this, note that since $T(\mathbf{G})$ includes the rule $(\mathtt{y}^\star \to \mathtt{y})$, we have that $\mathbf{Sem}((\mathtt{y}^\star \to \mathtt{y})) \in \Lambda_{\mathbf{M}_{T(\mathbf{G})}}$. Since v is $\mathbf{M}_{T(\mathbf{G})}$-legal, it satisfies $\mathbf{Sem}((\mathtt{y}^\star \to \mathtt{y})$. Now, $\mathbf{Sem}((\mathtt{y}^\star \to \mathtt{y}) = \{\{p_1 \doteqdot \mathtt{X} \mid \{\mathtt{y}^\star\} \subseteq \mathtt{X} \subseteq \pounds\}\} \Rightarrow \{\star p_1 \doteqdot \mathtt{X} \mid \{\mathtt{y}\} \subseteq \mathtt{X} \subseteq \pounds\}$. Consequently, since $\mathtt{y}^\star \in v(\varphi)$, we have $\mathtt{y} \in v(\star\varphi)$.

For the converse, suppose that $v \models s$. By definition, there exists some $\pounds_{\mathbf{2}}$-labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi \in s$ such that $\mathtt{x} \in v(\varphi)$. If φ does not have a form $\star\psi$ for $\star \in \diamondsuit^1_{\mathcal{L}}$, then $\mathtt{x}{:}\varphi \in T(s)$, and clearly $v \models T(s)$. Otherwise, $\varphi = \star\psi$ for some $\star \in \diamondsuit^1_{\mathcal{L}}$ and $\psi \in \mathcal{L}$, and $\mathtt{x}^\star{:}\psi \in T(s)$. We show that we have $\mathtt{x}^\star \in v(\psi)$ in this case (and so, $v \models T(s)$). First, since v is $\mathbf{M}_{T(\mathbf{G})}$-legal and $(cut) \in \mathsf{P}_{T(\mathbf{G})}$, we have $\overline{\mathtt{x}} \notin v(\varphi)$ (where $\overline{\mathtt{f}} = \mathtt{t}$ and $\overline{\mathtt{t}} = \mathtt{f}$, see Example 3.2.9). Since $T(\mathbf{G})$ includes the rule $(\overline{\mathtt{x}}^\star \to \overline{\mathtt{x}})$, this entails that $\overline{\mathtt{x}}^\star \notin v(\psi)$ (this is proved as in the first direction above). Finally, since $(id)^\star \in \mathsf{P}_{T(\mathbf{G})}$, it follows that $\mathtt{x}^\star \in v(\psi)$. □

Corollary 5.2.15. Let $\mathbf{G}$ be a quasi-canonical $\langle\mathcal{L},\mathscr{L}_{\mathbf{2}}\rangle$-calculus with $\mathsf{P}_{\mathbf{G}}=\{(cut),(id)\}$. Then, $\vdash_{\mathbf{G}}=\vdash_{\mathbf{M}_{T(\mathbf{G})}}$ and $\Vdash_{\mathbf{G}}^{\mathtt{X}}=\Vdash_{\mathbf{M}_{T(\mathbf{G})}^{\mathtt{X}}}$ for every $\mathtt{X}\subseteq\mathscr{L}_{\mathbf{2}}$.

Proof. Easily follows from Proposition 5.2.14, Theorem 5.2.12, and Theorem 3.2.15. □

Example 5.2.16. Consider the logic induced by the quasi-canonical calculus $\mathbf{G}_0$ from Example 5.2.5 and the set of labels $\{\mathtt{t}\}$. The corresponding PNmatrix $\mathbf{M}=\mathbf{M}_{T(\mathbf{G}_0)}^{\{\mathtt{t}\}}$, which is obtained from the canonical calculus $T(\mathbf{G}_0)$ constructed in that example, consists of:[1]

1. $\mathcal{U}_{\mathbf{M}}=\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\},\{\mathtt{f},\mathtt{t}^{\neg}\}\}$.
2. $\mathcal{D}_{\mathbf{M}}=\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$.
3. $\wedge$ and $\neg$ have the following PNtables:

$\wedge_{\mathbf{M}}$	$\{\mathtt{t},\mathtt{f}^{\neg}\}$	$\{\mathtt{t},\mathtt{t}^{\neg}\}$	$\{\mathtt{f},\mathtt{t}^{\neg}\}$
$\{\mathtt{t},\mathtt{f}^{\neg}\}$	$\{\{\mathtt{t},\mathtt{f}^{\neg}\}\}$	$\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$
$\{\mathtt{t},\mathtt{t}^{\neg}\}$	$\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$	$\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$
$\{\mathtt{f},\mathtt{t}^{\neg}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$

$\neg_{\mathbf{M}}$	
$\{\mathtt{t},\mathtt{f}^{\neg}\}$	$\{\{\mathtt{f},\mathtt{t}^{\neg}\}\}$
$\{\mathtt{t},\mathtt{t}^{\neg}\}$	$\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$
$\{\mathtt{f},\mathtt{t}^{\neg}\}$	$\{\{\mathtt{t},\mathtt{t}^{\neg}\},\{\mathtt{t},\mathtt{f}^{\neg}\}\}$

Corollary 5.2.15 entails that $\mathbf{M}$ characterizes the tcr $\Vdash_{\mathbf{G}_0}^{\{\mathtt{t}\}}$ (that is: $\mathcal{T}\Vdash_{\mathbf{M}}\varphi$ iff $\mathcal{T}\Vdash_{\mathbf{G}_0}^{\{\mathtt{t}\}}\varphi$). Note that $\mathbf{M}$ is isomorphic to the three-valued Nmatrix for this logic given in [8]. Furthermore, it is easy to check that this is the case for *all* the PNmatrices obtained by this general procedure for the quasi-canonical calculi considered in [8, 13].

[1]For simplicity, we use a reduced presentation of $\mathbf{M}$ as explained in Remark 3.1.12.

Chapter 6

Non-pure Sequent Calculi

For various important non-classical logics, such as modal logics and intuitionistic logic, there is no known (cut-free) pure calculus. Indeed, a major restriction in pure calculi is that unlimited context sequents may be freely used in all inference steps. Well-known sequent calculi for modal logics and intuitionistic logic do not meet this requirement, and thus they do not belong to the family of pure calculi studied in the previous chapters. For example, consider the following schemes of applications written in the usual notation of two-sided sequents:

$$(1)\ \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2} \qquad (2)\ \frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad (3)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

These schemes demonstrate different possibilities regarding context sequents, and non of them can be presented as a pure rule:

1. Scheme (1) allows only left context formulas, that is: all context sequents should be subsets of $\{\mathtt{f}{:}\varphi \mid \varphi \in \mathcal{L}\}$. This scheme is employed in the multiple-conclusion sequent calculus for intuitionistic logic [90].

2. Scheme (2) again allows only left context formulas, but all of them should begin with $\Box$ ($\Box\Gamma$ is an abbreviation for $\{\Box\varphi \mid \varphi \in \Gamma\}$). In other words, all context sequents should be subsets of $\{\mathtt{f}{:}\Box\varphi \mid \varphi \in \mathcal{L}\}$. This scheme is employed in the usual sequent calculus for the modal logic $S4$ [96].

3. Scheme (3) exhibits more complicated treatment of the context formulas: each labelled formula $\mathtt{f}{:}\varphi$ in the premise "becomes" $\mathtt{f}{:}\Box\varphi$ in the conclusion of the application. In other words, any sequent $c \subseteq \{\mathtt{f}{:}\varphi \mid \varphi \in \mathcal{L}\}$ can serve as a context sequent in the premise of the application, provided that $\{\mathtt{f}{:}\Box\varphi \mid \mathtt{f}{:}\varphi \in c\}$ is the context sequent of the conclusion. This scheme is employed in the usual sequent

calculus for the modal logic K.[1]

In this chapter we introduce a general framework of sequent calculi, called *basic calculi*, that allow context restrictions of certain kinds (including those demonstrated above). Unlike in the previous chapters, we restrict our attention only to *two-sided sequent calculi*. Various sequent calculi that seem to have completely different natures can be directly presented as basic calculi. This includes standard sequent calculi for modal logics, as well as the usual multiple-conclusion systems for intuitionistic logic, its dual, and bi-intuitionistic logic. Our goal is to carry out a general and uniform semantic study of these systems, that will provide useful semantics for them, as well as semantic criteria for their proof-theoretic properties.

Publications Related to this Chapter

The material in this chapter was included in [15, 73].

6.1 Basic Calculi

In this section we precisely define the general structure of derivation rules that are allowed to appear in basic calculi. Rules of this structure will be called *basic rules*. As in Section 2.2, we explicitly differentiate between a rule and its applications. Derivations in a certain basic calculus consist of *applications* of rules, and the rules themselves are just succinct formulations of their sets of applications. In addition, for the formulation of the rules, we differentiate between two parts of their applications, namely the *context* part and the *non-context* part (see [92]). The non-context part is obtained by instantiating a rigid structure that is given in the rule. In turn, the structure of the context part is determined using *context-relations*. This structure is less restrictive, as the number of context formulas is completely free. Next we turn to the formal definitions.

Notation 6.1.1. Throughout this chapter, dealing only with two-sided sequents, we will not mention the set of labels in the aforementioned notions. For example, we refer to $\pounds_{\mathbf{2}}$-labelled $\mathcal{L}$-formulas and $\langle \mathcal{L}, \pounds_{\mathbf{2}} \rangle$-sequents simply as *labelled $\mathcal{L}$-formulas* and *$\mathcal{L}$-sequents*. We may use the usual sequent notation $\Gamma \Rightarrow \Delta$, where Γ and Δ are (possibly empty) finite sets of formulas, interpreted as $\{\mathtt{f}{:}\varphi \mid \varphi \in \Gamma\} \cup \{\mathtt{t}{:}\varphi \mid \varphi \in \Delta\}$, and employ the standard abbreviations, e.g., $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $\Gamma \Rightarrow$ instead of $\Gamma \Rightarrow \emptyset$.

[1]Our intuitive distinction between context formulas and non-context formulas in rule schemes is based on the following principle: the exact number of non-context formulas is explicitly specified in the scheme, while any (finite) number of context formulas may be employed. Taking this into account, note that context formulas may "change" (as in (3)).

Name	Context-Relation	Instances (all pair of the form:)
π_0	$\{\langle \mathtt{f}{:}p_1, \mathtt{f}{:}p_1\rangle, \langle \mathtt{t}{:}p_1, \mathtt{t}{:}p_1\rangle\}$	$\langle \Gamma \Rightarrow \Delta \;,\; \Gamma \Rightarrow \Delta\rangle$
π_{int}	$\{\langle \mathtt{f}{:}p_1, \mathtt{f}{:}p_1\rangle\}$	$\langle \Gamma \Rightarrow \;,\; \Gamma \Rightarrow \rangle$
π_{d}	$\{\langle \mathtt{t}{:}p_1, \mathtt{t}{:}p_1\rangle\}$	$\langle \Rightarrow \Delta \;,\; \Rightarrow \Delta\rangle$
$\emptyset$	$\emptyset$	$\langle \Rightarrow \;,\; \Rightarrow \rangle$
π_{K}	$\{\langle \mathtt{f}{:}p_1, \mathtt{f}{:}\Box p_1\rangle\}$	$\langle \Gamma \Rightarrow \;,\; \Box\Gamma \Rightarrow \rangle$
π_{K4}	$\{\langle \mathtt{f}{:}p_1, \mathtt{f}{:}\Box p_1\rangle, \langle \mathtt{f}{:}\Box p_1, \mathtt{f}{:}\Box p_1\rangle\}$	$\langle \Gamma_1, \Box\Gamma_2 \Rightarrow \;,\; \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \rangle$
π_{S4}	$\{\langle \mathtt{f}{:}\Box p_1, \mathtt{f}{:}\Box p_1\rangle\}$	$\langle \Box\Gamma \Rightarrow \;,\; \Box\Gamma \Rightarrow \rangle$
π_{B}	$\{\langle \mathtt{f}{:}p_1, \mathtt{f}{:}\Box p_1\rangle, \langle \mathtt{t}{:}\Box p_1, \mathtt{t}{:}p_1\rangle\}$	$\langle \Gamma \Rightarrow \Box\Delta \;,\; \Box\Gamma \Rightarrow \Delta\rangle$
π_{S5}	$\{\langle \mathtt{f}{:}\Box p_1, \mathtt{f}{:}\Box p_1\rangle, \langle \mathtt{t}{:}\Box p_1, \mathtt{t}{:}\Box p_1\rangle\}$	$\langle \Box\Gamma \Rightarrow \Box\Delta \;,\; \Box\Gamma \Rightarrow \Box\Delta\rangle$

Table 6.1: Important Context-Relations

Definition 6.1.2. An $\mathcal{L}$-*context-relation* is a finite binary relation on the set of labelled $\mathcal{L}$-formulas. Given an $\mathcal{L}$-context-relation π, $\bar{\pi}$ is the binary relation between labelled $\mathcal{L}$-formulas consisting of all substitution instances of π, that is:

$$\bar{\pi} = \{\langle \sigma(\alpha), \sigma(\beta)\rangle \mid \sigma \textit{ is an } \mathcal{L}\textit{-substitution and } \langle \alpha, \beta\rangle \in \pi\}.$$

A π-*instance* is an ordered pair of $\mathcal{L}$-sequents $\langle s_1, s_2\rangle$ for which there exist (not necessarily distinct) labelled $\mathcal{L}$-formulas $\alpha_1, \ldots, \alpha_n$, and $\beta_1, \ldots, \beta_n$ such that $s_1 = \{\alpha_1, \ldots, \alpha_n\}$, $s_2 = \{\beta_1, \ldots, \beta_n\}$, and $\alpha_i \bar{\pi} \beta_i$ for every $1 \leq i \leq n$.

Several context-relations, that are used in the definitions and examples below, are given in Table 6.1 (in some of them the language is assumed to have a unary connective denoted by $\Box$).

Definition 6.1.3. A *basic $\mathcal{L}$-premise* is an ordered pair of the form $\langle s, \pi\rangle$, where s is an $\mathcal{L}$-sequent and π is an $\mathcal{L}$-context-relation. A *basic $\mathcal{L}$-rule* is a pair of the form $\mathcal{P}/s$, where $\mathcal{P}$ is a finite set of basic $\mathcal{L}$-premise, and s is an $\mathcal{L}$-sequent. The elements of $\mathcal{P}$ are called the *premises* of the rule, and s is called the *conclusion* of the rule. To improve readability, we usually drop the set braces of the set $\mathcal{P}$ of premises. An *application* of a basic $\mathcal{L}$-rule $\langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle/s$ is any inference step of the following form:

$$\frac{\sigma(s_1) \cup c_1 \quad \ldots \quad \sigma(s_n) \cup c_n}{\sigma(s) \cup c'_1 \cup \ldots \cup c'_n}$$

where σ is an $\mathcal{L}$-substitution, and for every $1 \leq i \leq n$, c_i and c'_i are $\mathcal{L}$-sequents such that $\langle c_i, c'_i\rangle$ is a π_i-instance. The sequents $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n$ are called the *premises* of the application, while $\sigma(s) \cup c'_1 \cup \ldots \cup c'_n$ is called the *conclusion* of the application.

Table 6.2 provides some examples of basic rules and the forms of their applications. Note that pure $\langle \mathcal{L}, \mathcal{L}_2\rangle$-rules as defined in Chapter 2 are basic $\mathcal{L}$-rules in which all premises have the form $\langle s, \pi_0\rangle$.

Name	**Basic Rule**	**Application**
(T)	$\langle\{\mathtt{f}{:}p_1\}, \pi_0\rangle / \{\mathtt{f}{:}\Box p_1\}$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}$
$(S4)$	$\langle\{\mathtt{t}{:}p_1\}, \pi_{\mathsf{S4}}\rangle / \{\mathtt{t}{:}p_1\}$	$\dfrac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$
$(K4)$	$\langle\{\mathtt{t}{:}p_1\}, \pi_{\mathsf{K4}}\rangle / \{\mathtt{t}{:}\Box p_1\}$	$\dfrac{\Gamma_1, \Box\Gamma_2 \Rightarrow \varphi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi}$
(D_0)	$\langle\{\mathtt{f}{:}p_1\}, \emptyset\rangle / \{\mathtt{f}{:}\Box p_1\}$	$\dfrac{\varphi \Rightarrow}{\Box\varphi \Rightarrow}$
(D)	$\langle\emptyset, \pi_{\mathsf{K}}\rangle / \emptyset$	$\dfrac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}$
$(\mathtt{f}{:}\supset)$	$\langle\{\mathtt{t}{:}p_1\}, \pi_0\rangle, \langle\{\mathtt{f}{:}p_2\}, \pi_0\rangle / \{\mathtt{f}{:}p_1 \supset p_2\}$	$\dfrac{\Gamma_1 \Rightarrow \varphi_1, \Delta_1 \quad \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow \Delta_1, \Delta_2}$
$(\mathtt{f}{:}\supset')$	$\langle\{\mathtt{t}{:}p_1\}, \pi_0\rangle, \langle\{\mathtt{f}{:}p_2\}, \pi_{\mathsf{int}}\rangle / \{\mathtt{f}{:}p_1 \supset p_2\}$	$\dfrac{\Gamma_1 \Rightarrow \varphi_1, \Delta \quad \Gamma_2, \varphi_2 \Rightarrow}{\Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow \Delta}$
$(\mathtt{t}{:}\supset^*)$	$\langle\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}, \pi\rangle / \{\mathtt{t}{:}p_1 \supset p_2\}$ where $\pi = \{\langle \mathtt{f}{:}p_1 \supset p_2, \mathtt{f}{:}p_1 \supset p_2\rangle\}$	$\dfrac{\psi_1 \supset \psi_1', \dots, \psi_n \supset \psi_n', \varphi_1 \Rightarrow \varphi_2}{\psi_1 \supset \psi_1', \dots, \psi_n \supset \psi_n' \Rightarrow \varphi_1 \supset \varphi_2}$

Table 6.2: Basic Rules Examples

Convention 6.1.4. Henceforth, we identify pure $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rules (see Definition 2.2.5) with basic $\mathcal{L}$-rules that employ π_0 as the context-relation in all of their premises. Thus we refer to all pure $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-rules as basic rules.

Definition 6.1.5. A *basic $\mathcal{L}$-calculus* consists of a finite set of basic $\pounds$-rules, that includes the primitive $\pounds_{\mathbf{2}}$-rules: ($\mathtt{f}$:*weak*), ($\mathtt{t}$:*weak*), (*cut*) and (*id*) (see Page 14). A *proof* in a basic $\mathcal{L}$-calculus is defined exactly as in pure $\langle\mathcal{L}, \pounds\rangle$-calculus (see Definition 2.2.14), and we write $\mathcal{S} \vdash_{\mathbf{G}} s$ to denote the existence of a proof of an $\mathcal{L}$-sequent s from a set $\mathcal{S}$ of $\mathcal{L}$-sequents in a basic $\mathcal{L}$-calculus $\mathbf{G}$.

Notation 6.1.6. We denote by $\mathsf{R}_{\mathbf{G}}$ all basic $\mathcal{L}$-rules of a basic $\mathcal{L}$-calculus $\mathbf{G}$, except for ($\mathtt{f}$:*weak*), ($\mathtt{t}$:*weak*), (*cut*) and (*id*). $\Pi_{\mathbf{G}}$ denotes the set of $\mathcal{L}$-context-relations appearing in the basic rules of $\mathbf{G}$ (in particular, since (*cut*) is always included, $\pi_0 \in \Pi_{\mathbf{G}}$ for every basic calculus $\mathbf{G}$).

Note that pure $\langle\mathcal{L}, \pounds_{\mathbf{2}}\rangle$-calculi whose primitive rules include (*cut*) and (*id*) are obtained as a particular instance in which $\Pi_{\mathbf{G}} = \{\pi_0\}$. In addition, the above notion of a basic rule is sufficiently general, so that many known sequent systems for various propositional logics can be easily presented in this framework. Next, we list some of these sequent systems, and present their formulation as basic calculi. In the sequel, we will return to some of these calculi, provide a semantics for them, and use it to study their proof-theoretic properties.

Remark 6.1.7. When we say that a basic $\mathcal{L}$-calculus $\mathbf{G}$ is a calculus *for a logic* $\mathbf{L} = \langle \mathcal{L}, \Vdash \rangle$, we mean that $\{\mathtt{t}{:}\psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{G}} \{\mathtt{t}{:}\varphi\}$ iff $\mathcal{T} \Vdash \varphi$. In cases where there is difference between the local version of the logic and the global one, as happens in modal logics, we refer to the global version (see [35]). In addition, this chapter deals only with propositional logics. Throughout, when we mention a known Gentzen-type system, we refer only to its propositional fragment.

Example 6.1.8 (**LJ**)**.** The most famous sequent system for intuitionistic logic is of-course Gentzen's LJ [56]. This system manipulates single-conclusion sequents, and thus it does not fall in our framework.[2] However, there is an equivalent multiple-conclusion system, called LJ' in [90], that can be naturally presented as a basic calculus, that we call **LJ**. Its propositional language $\mathcal{L}_{\mathbf{LJ}}$ is $\{\bot^0, \wedge^2, \vee^2, \supset^2\}$ ($\neg\varphi$ can be defined by $\varphi \supset \bot$). The rules of **LJ** are the same rules of **LK** (see Example 2.2.20), except for $(\mathtt{t}{:} \supset)$, in which π_{int} is used instead of π_{0} (see Table 6.1). Thus this rule has now the form $\langle \{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}, \pi_{\mathsf{int}} \rangle / \{\mathtt{t}{:}p_1 \supset p_2\}$, and its applications allow to infer sequents of the form $\Gamma \Rightarrow \varphi_1 \supset \varphi_2$ from $\Gamma, \varphi_1 \Rightarrow \varphi_2$ (note that right context-formulas are forbidden). In addition, since we do not include $\neg$ in $\mathcal{L}_{\mathbf{LJ}}$, we discard its rules, and add the following basic $\mathcal{L}_{\mathbf{LJ}}$-rule for $\bot$: $\emptyset / \{\mathtt{f}{:}\ \bot\}$.

Example 6.1.9 (**BLJ**)**.** Bi-intuitionistic logic (see, e.g., [59]) is the extension of intuitionistic logic with a binary connective dual to implication (denoted here by $\prec$). Thus its language is $\{\bot^0, \wedge^2, \vee^2, \supset^2, \prec^2\}$, and we denote it by $\mathcal{L}_{\mathbf{BLJ}}$. A sequent system for this logic (see [80]) can be presented as a basic $\mathcal{L}_{\mathbf{BLJ}}$-calculus, which we call **BLJ**, obtained by augmenting **LJ** with the following rules:

$(\mathtt{f}{:} \prec)$ $\langle \{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}, \pi_{\mathsf{d}} \rangle / \{\mathtt{f}{:}p_1 \prec p_2\}$ $(\mathtt{t}{:} \prec)$ $\langle \{\mathtt{t}{:}p_1\}, \pi_{\mathsf{0}} \rangle, \langle \{\mathtt{f}{:}p_2\}, \pi_{\mathsf{0}} \rangle / \{\mathtt{t}{:}p_1 \prec p_2\}$

Applications of these rules have the forms:

$$(\mathtt{f}{:} \prec) \quad \frac{\varphi_1 \Rightarrow \varphi_2, \Delta}{\varphi_1 \prec \varphi_2 \Rightarrow \Delta} \qquad\qquad (\mathtt{t}{:} \prec) \quad \frac{\Gamma_1 \Rightarrow \varphi_1, \Delta_1 \quad \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \varphi_1 \prec \varphi_2, \Delta_1, \Delta_2}$$

Example 6.1.10 (**PLJ**)**.** Sequent systems for many paraconsistent logics that extend the positive fragment of intuitionistic logic are defined and studied in [8]. All of them belong to the family of basic calculi. For example, we present the system $PLJ(\{(\Rightarrow \neg \supset)\})$ from [8] as a basic calculus, that we call **PLJ**. Let $\mathcal{L}_{\mathbf{PLJ}} = \{\neg^1, \wedge^2, \vee^2, \supset^2\}$. The basic

[2] Note that canonical *single-conclusion* (two-sided) *sequent* calculi, of which LJ is the prototype example, were introduced and studied in the author's M.Sc. thesis (see also [14]).

$\mathcal{L}_{\mathbf{PLJ}}$-calculus **PLJ** is obtained from **LJ** by adding the following rules (and discarding the rule for $\bot$):

$(\mathtt{t}{:}\neg)\quad \langle\{\mathtt{f}{:}p_1\}, \pi_0\rangle/\{\mathtt{t}{:}\neg p_1\}\qquad (\mathtt{t}{:}\neg\supset)\quad \langle\{\mathtt{t}{:}p_1\}, \pi_0\rangle, \langle\{\mathtt{t}{:}\neg p_2\}, \pi_0\rangle/\{\mathtt{t}{:}\neg(p_1 \supset p_2)\}$

Applications of these rules have the forms:

$$(\mathtt{t}{:}\neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \qquad (\mathtt{t}{:}\neg\supset)\ \frac{\Gamma_1 \Rightarrow \varphi_1, \Delta_1 \quad \Gamma_2 \Rightarrow \neg\varphi_2, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \neg(\varphi_1 \supset \varphi_2), \Delta_1, \Delta_2}$$

Example 6.1.11 (Systems for Modal Logics)**.** Ordinary sequent systems for modal logics are surveyed in [96] and [81]. All of them belong to the family of basic calculi. As examples we present as basic calculi six of them (used later to demonstrate certain semantic phenomena). Let $\mathcal{L}_\Box = \{\Box^1, \neg^1, \wedge^2, \vee^2, \supset^2\}$. We use the basic rules (K), (B) and $(S5)$ (in addition to some of the rules presented in Table 6.2). (K), (B) and $(S5)$ all have the form $\langle\{\mathtt{t}{:}p_1\}, \pi\rangle/\{\mathtt{t}{:}\Box p_1\}$, where π is π_{K}, π_{B}, and π_{S5} respectively (see Table 6.1). Applications of these rules have the form:

$$(K)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad (B)\ \frac{\Gamma \Rightarrow \varphi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Delta} \qquad (S5)\ \frac{\Box\Gamma \Rightarrow \varphi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Box\Delta}$$

Based on **LK**, six basic $\mathcal{L}_\Box$-calculi are defined as follows:

$$\mathbf{K} = \mathbf{LK} + (K) \qquad \mathbf{K4} = \mathbf{LK} + (K4) \qquad \mathbf{KD} = \mathbf{K} + (D)$$
$$\mathbf{KB} = \mathbf{LK} + (B) \qquad \mathbf{S4} = \mathbf{LK} + (S4) + (T) \qquad \mathbf{S5} = \mathbf{LK} + (S5) + (T)$$

Note that $\Box$ is the only primitive modality in the language $\mathcal{L}_\Box$, and $\Diamond\varphi$ can be defined as $\neg\Box\neg\varphi$. For an extended language with two dual primitive modal operators, one should modify some of the context-relations in the rules for $\Box$, and add dual rules for $\Diamond$. For example, for the logic $S4$ the following four schemes are used:

$$\frac{\Box\Gamma \Rightarrow \varphi, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box\varphi, \Diamond\Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta} \qquad \frac{\Box\Gamma, \varphi \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond\varphi \Rightarrow \Diamond\Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Diamond\varphi, \Delta}$$

Example 6.1.12 (**GL**)**.** The logic GL (the modal logic of provability, see [94]) is obtained by adding the axiom $\Box(\Box\varphi \supset \varphi) \supset \Box\varphi$ to the usual Hilbert system for the modal logic K. In addition, GL has a well-known sequent system (see, e.g., [75, 84, 3]), that can be presented as a basic $\mathcal{L}_\Box$-calculus, that we call **GL**. **GL** is obtained from **LK** by adding (GL) – the basic $\mathcal{L}_\Box$-rule $\langle\{\mathtt{f}{:}\Box p_1, \mathtt{t}{:}p_1\}, \pi_{\mathsf{K4}}\rangle/\{\mathtt{t}{:}\Box p_1\}$. Applications of (GL) allow to infer sequents of the form $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi$ from $\Gamma_1, \Box\Gamma_2, \Box\varphi \Rightarrow \varphi$.

Example 6.1.13 (**IS5**)**.** Sequent systems for intuitionistic modal logics provide an interesting source of examples to be studied in the framework of basic calculi, as they naturally employ more than one (non-trivial) context-relation. For example, the system G_3 from [78] can be presented as the basic $\mathcal{L}^\Box_{\mathbf{LJ}}$-calculus obtained from **LJ** by adding the rules $(S5)$ and (T) ($\mathcal{L}^\Box_{\mathbf{LJ}}$ denotes the language obtained by augmenting $\mathcal{L}_{\mathbf{LJ}}$ with a unary connective $\Box$). In the sequel, we refer to this basic calculus as **IS5**.

Example 6.1.14. In [74] several sequent calculi for weak modal logics are introduced. All of them belong to the family of basic calculi. For example, the first system from [74] (called $Mseq$ there), is the basic $\mathcal{L}_{\Box}$-calculus obtained from **LK** by adding (M) – the basic rule $\langle\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}, \emptyset\rangle/\{\mathtt{f}{:}\Box p_1, \mathtt{t}{:}\Box p_2\}$. Its applications allow to infer sequents of the form $\Box\varphi_1 \Rightarrow \Box\varphi_2$ from $\varphi_1 \Rightarrow \varphi_2$.

Example 6.1.15. Several sequent systems for logics of strict implication are provided in [65], and can be presented as basic calculi. For example, $\mathbf{GS4}^I$ (from [65]) is equivalent to the basic $\mathcal{L}_{cl}$-calculus, obtained from **LK** by replacing the rule $(\mathtt{t}{:}\supset)$ with the rule $(\mathtt{t}{:}\supset^*)$ (see Table 6.2). Note that $\mathbf{GS4}^I$ includes also a less standard rule (denoted by $(\to \mathsf{K}^I)$ in [65]) that cannot be presented as one basic rule. However, one can show that it is redundant in $\mathbf{GS4}^I$.

Example 6.1.16 (GP). Primal logic was defined and studied in [31]. As explained there, this logic is used in the context of the access control language DKAL. We consider here the sequent system GP from [31] for primal logic with disjunction and quotations. Given a finite set Q of constants denoting "principals", let

$$\mathcal{L}^Q_{\mathbf{GP}} = \{\bot^0, \top^0, \wedge^2, \vee^2, \supset^2\} \cup \{q\ \mathtt{said}^1, q\ \mathtt{implied}^1 \mid q \in Q\}.$$

The sequent system GP (over Q) can be presented as a basic $\mathcal{L}^Q_{\mathbf{GP}}$-calculus, that we call $\mathbf{GP}^Q$. The rules of $\mathbf{GP}^Q$ are the rules of **LK** for $\wedge, \vee$, and the following rules for the other connectives (for every $q \in Q$):

$(\mathtt{f}{:}\bot)\quad \emptyset/\{\mathtt{f}{:}\bot\}$ $\qquad (\mathtt{t}{:}\top)\quad \emptyset/\{\mathtt{t}{:}\top\}$

$(\mathtt{f}{:}\supset)\quad \langle\{\mathtt{t}{:}p_1\}, \pi_0\rangle, \langle\{\mathtt{f}{:}p_2\}, \pi_0\rangle/\{\mathtt{f}{:}p_1 \supset p_2\}$ $\qquad (\mathtt{t}{:}\supset)\quad \langle\{\mathtt{t}{:}p_2\}, \pi_0\rangle/\{\mathtt{t}{:}p_1 \supset p_2\}$

$(\mathtt{Said}_q)\quad \langle\{\mathtt{t}{:}p_1\}, \pi^q_s\rangle/\{\mathtt{t}{:}q\ \mathtt{said}\ p_1\}$

$(\mathtt{Implied}_q)\quad \langle\{\mathtt{t}{:}p_1\}, \pi^q_i\rangle/\{\mathtt{t}{:}q\ \mathtt{implied}\ p_1\}$

where $\pi^q_s = \{\langle\mathtt{f}{:}p_1, \mathtt{f}{:}q\ \mathtt{said}\ p_1\rangle\}$, and $\pi^q_i = \pi^q_s \cup \{\langle\mathtt{f}{:}p_1, \mathtt{f}{:}q\ \mathtt{implied}\ p_1\rangle\}$ for every $q \in Q$. Applications of these rules have the form:

$$(\mathtt{f}{:}\bot)\ \frac{}{\bot\Rightarrow} \qquad (\mathtt{t}{:}\top)\ \frac{}{\Rightarrow \top}$$

$$(\mathtt{f}{:}\supset)\ \frac{\Gamma_1 \Rightarrow \varphi_1, \Delta_1 \quad \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\mathtt{t}{:}\supset)\ \frac{\Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}$$

$$(\mathtt{Said}_q)\ \frac{\Gamma \Rightarrow \varphi}{q\ \mathtt{said}\ \Gamma \Rightarrow q\ \mathtt{said}\ \varphi} \qquad (\mathtt{Implied}_q)\ \frac{\Gamma, \Delta \Rightarrow \varphi}{q\ \mathtt{said}\ \Gamma, q\ \mathtt{implied}\ \Delta \Rightarrow q\ \mathtt{implied}\ \varphi}$$

6.1.1 Proof-Theoretic Properties

Generally speaking, the definition of the proof-theoretic properties from Chapter 2 are adapted in the obvious way to basic calculi (see Section 2.3). Formally, since we explicitly required that each basic calculus includes (*cut*) and (*id*), we can not consider the calculi obtained from a basic calculus by discarding (*cut*) or (*id*) as basic calculi. Thus we find it convenient to restrict the proofs in basic calculi via *proof specifications*:

Definition 6.1.17. An $\mathcal{L}$*-proof-specification* is a triple of sets of $\mathcal{L}$-formulas $\langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$. Given an $\mathcal{L}$-proof-specification $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$, a proof P in a basic $\mathcal{L}$-calculus $\mathbf{G}$ is called a ρ*-proof* if the following conditions hold:

1. P contains only $\mathcal{F}$-sequents (that is $\mathcal{L}$-sequents consisting only of formulas from $\mathcal{F}$, see Definition 2.3.3).
2. The cut-formula of every application of (*cut*) in P is in $\mathcal{C}$.
3. The id-formula of every application of (*id*) in P is in $\mathcal{A}$.

We write $\mathcal{S} \vdash_{\mathbf{G} \upharpoonright \rho} s$ if there exists a ρ-proof in a basic $\mathcal{L}$-calculus $\mathbf{G}$ of an $\mathcal{L}$-sequent s from a set $\mathcal{S}$ of $\mathcal{L}$-sequents.

Note that $\vdash_{\mathbf{G}}$ is a special case of $\vdash_{\mathbf{G} \upharpoonright \rho}$, obtained by choosing $\rho = \langle \mathcal{L}, \mathcal{L}, \mathcal{L} \rangle$. In addition, the following are equivalent:

1. $\mathcal{S} \vdash_{\mathbf{G} \upharpoonright \rho} s$ for $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$.
2. $\{s' \in \mathcal{S} \mid \mathit{frm}[s'] \subseteq \mathcal{F}\} \vdash_{\mathbf{G} \upharpoonright \rho'} s$ for $\rho' = \langle \mathcal{F}, \mathcal{C} \cap \mathcal{F}, \mathcal{A} \cap \mathcal{F} \rangle$.
3. $\mathcal{S} \cup \{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\} \mid \varphi \in \mathcal{A}\} \vdash_{\mathbf{G} \upharpoonright \rho'} s$ for $\rho' = \langle \mathcal{F}, \mathcal{C}, \emptyset \rangle$.

Now, we can uniformly define *(strong)* $\leq$*-analyticity*, *(strong) cut-admissibility*, and *axiom-expansion* using proof-specifications.

Definition 6.1.18. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus.

1. Let $\leq$ be a safe partial order on $\mathcal{L}$. $\mathbf{G}$ is $\leq$*-analytic* if for every $\mathcal{L}$-sequent s, $\vdash_{\mathbf{G}} s$ implies $\vdash_{\mathbf{G} \upharpoonright \rho} s$ for $\rho = \langle \downarrow^{\leq}[s], \mathcal{L}, \mathcal{L} \rangle$. $\mathbf{G}$ is *strongly* $\leq$*-analytic* if for every set $\mathcal{S}$ of $\mathcal{L}$-sequents and $\mathcal{L}$-sequent s, $\mathcal{S} \vdash_{\mathbf{G}} s$ implies $\mathcal{S} \vdash_{\mathbf{G} \upharpoonright \rho} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], \mathcal{L}, \mathcal{L} \rangle$.

2. $\mathbf{G}$ enjoys *cut-admissibility* if for every $\mathcal{L}$-sequent s, $\vdash_{\mathbf{G} \upharpoonright \rho} s$ for $\rho = \langle \mathcal{L}, \emptyset, \mathcal{L} \rangle$ whenever $\vdash_{\mathbf{G}} s$. $\mathbf{G}$ enjoys *strong cut-admissibility* if for every set $\mathcal{S}$ of $\mathcal{L}$-sequents and $\mathcal{L}$-sequent s, $\mathcal{S} \vdash_{\mathbf{G} \upharpoonright \rho} s$ for $\rho = \langle \mathcal{L}, \mathit{frm}[\mathcal{S}], \mathcal{L} \rangle$ whenever $\mathcal{S} \vdash_{\mathbf{G}} s$.

3. A connective $\diamond \in \diamondsuit_{\mathcal{L}}$ admits axiom-expansion in $\mathbf{G}$ if $\vdash_{\mathbf{G} \upharpoonright \rho} \{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$ for the formula $\varphi = \diamond(p_1, \dots, p_{ar(\diamond)})$ and $\rho = \langle \mathcal{L}, \mathcal{L}, \{p_1, \dots, p_{ar(\diamond)}\} \rangle$.

Note also that the consequences of (strong) $\leq$-analyticity discussed in Section 2.3.1 (consistency, conservativity, and decidability) hold for (strong) $\leq$-analytic basic calculi as well, with minor modification in their proofs.

6.2 Kripke-style Semantics for Basic Calculi

In this section we introduce a method for providing semantics for any given basic calculus and proof-specification. Thus, given a basic calculus **G** and a proof-specification ρ, we show how to uniformly recognize a class of semantic structures $\mathcal{K}_{\mathbf{G}\restriction\rho}$, that naturally induce a semantic consequence relation $\vdash_{\mathcal{K}_{\mathbf{G}\restriction\rho}}$ between sequents, for which we have soundness and completeness, i.e. $\vdash_{\mathbf{G}\restriction\rho}=\vdash_{\mathcal{K}_{\mathbf{G}\restriction\rho}}$. The semantic framework employed for this purpose is a generalization of Kripke-style semantics for modal and intuitionistic logic, where instead of usual Kripke frames and models, we will have (partial) *Kripke valuations*. These are defined as follows:

Definition 6.2.1. A *partial Kripke $\mathcal{L}$-valuation* (partial *$\mathcal{L}$-Kvaluation*, for short) is a function v from the Cartesian product of some set W_v (whose elements are called *worlds*) and some set $Dom_v \subseteq \mathcal{L}$ to $2^{\{\mathtt{f},\mathtt{t}\}}$. A partial $\mathcal{L}$-Kvaluation v with $Dom_v = \mathcal{L}$ is also called an $\mathcal{L}$-Kvaluation.

Note that as in Chapter 3, the truth values are subsets of labels. Next, we introduce the semantic consequence relation associated with a given set of such valuations.

Definition 6.2.2. Let v be a partial $\mathcal{L}$-Kvaluation.

1. A labelled $\mathcal{L}$-formula $\mathtt{x}{:}\varphi$ is *true* in some $w \in W_v$ with respect to v (denoted by: $v, w \models \mathtt{x}{:}\varphi$) if $\varphi \in Dom_v$ and $\mathtt{x} \in v(w, \varphi)$.

2. An $\mathcal{L}$-sequent s is *true* with respect to v:

   (a) in some $w \in W_v$ (denoted by: $v, w \models s$) if $v, w \models \alpha$ for some $\alpha \in s$.

   (b) in some set $W \subseteq W_v$ (denoted by: $v, W \models s$) if $v, w \models s$ for every $w \in W$.

3. v is a *model* of:

   (a) an *$\mathcal{L}$-sequent s* (denoted by: $v \models s$) if s is a Dom_v-sequent and $v, W_v \models s$.

   (b) a *set $\mathcal{S}$ of $\mathcal{L}$-sequents* (denoted by: $v \models \mathcal{S}$) if $v \models s$ for every Dom_v-sequent $s \in \mathcal{S}$.

Now, the given proof-specification $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ enforces some simple conditions on Kvaluations: (1) As in Chapter 3, Dom_v should consist exactly of the formulas in $\mathcal{F}$, those that are allowed to appear in ρ-proofs; (2) If some formula φ may serve as a cut-formula (i.e., if $\varphi \in \mathcal{C} \cap \mathcal{F}$), the value $\{\mathtt{f}, \mathtt{t}\}$ should be never assigned to φ; and (3) Similarly, $\emptyset$ should not be assigned to φ, if φ may serve as an id-formula (i.e., if $\varphi \in \mathcal{A} \cap \mathcal{F}$). These restrictions are formulated in the next definition.

Definition 6.2.3. Let $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ be an $\mathcal{L}$-proof-specification. A partial $\mathcal{L}$-Kvaluation v is called *ρ-legal* if the following hold:

1. $Dom_v = \mathcal{F}$.
2. $v(w, \varphi) \neq \{\mathtt{f}, \mathtt{t}\}$ for every $w \in W_v$ and $\varphi \in \mathcal{C} \cap \mathcal{F}$.
3. $v(w, \varphi) \neq \emptyset$ for every $w \in W_v$ and $\varphi \in \mathcal{A} \cap \mathcal{F}$.

A special kind of Kvaluations will be particularly needed below:

Definition 6.2.4. A partial $\mathcal{L}$-Kvaluation is called *normal* if it is $\langle \mathcal{F}, \mathcal{L}, \mathcal{L} \rangle$-legal for some $\mathcal{F} \subseteq \mathcal{L}$.

Note that a normal partial $\mathcal{L}$-Kvaluation v assigns either $\{\mathtt{f}\}$ or $\{\mathtt{t}\}$ to any pair $\langle w, \varphi \rangle \in W_v \times Dom_v$.

Next we turn to restrictions on Kvaluations imposed by the basic system itself. As in Chapter 3, the intuitive idea is that each syntactic ingredient of **G** imposes a certain constraint on Kvaluations. Taking all of these constraints together, we get a set of Kvaluations for which **G** is sound and complete. The exact constraints are formulated below.

First, we associate with each context-relation π of **G** a binary ("accessibility") relation on W_v, and enforce certain conditions on the associated accessibility relations.

Definition 6.2.5. Given a set W, a $\langle \mathbf{G}, W \rangle$-coupling is a function assigning a binary relation on W to every $\pi \in \Pi_{\mathbf{G}}$.

Definition 6.2.6. Let **G** be a basic $\mathcal{L}$-calculus, v a partial $\mathcal{L}$-Kvaluation, and $\Re$ a $\langle \mathbf{G}, W_v \rangle$-coupling.

1. Given an $\mathcal{L}$-context-relation π, R^v_π denotes the binary relation on W_v defined as follows: $w_1 R^v_\pi w_2$ iff for every two labelled Dom_v-formulas α_1, α_2, if $\alpha_2 \bar{\pi} \alpha_1$ and $v, w_2 \models \alpha_2$ then $v, w_1 \models \alpha_1$.

2. $\langle v, \Re \rangle$ is called:

   (a) *π-legal* for some $\pi \in \Pi_{\mathbf{G}}$ if $\Re(\pi) \subseteq R^v_\pi$.
   (b) *Π-legal* for some $\Pi \subseteq \Pi_{\mathbf{G}}$ if it is π-legal for every $\pi \in \Pi$.

Example 6.2.7. Let **G** be a basic $\mathcal{L}$-calculus, v a partial $\mathcal{L}$-Kvaluation, and $\Re$ a $\langle \mathbf{G}, W_v \rangle$-coupling.

1. Consider the $\mathcal{L}$-context-relation π_0. By definition, $\alpha_2 \bar{\pi}_0 \alpha_1$ iff $\alpha_2 = \alpha_1$. Thus $w_1 R^v_{\pi_0} w_2$ iff for every labelled Dom_v-formula α such that $v, w_2 \models \alpha$, we have

$v, w_1 \models \alpha$. Equivalently, $w_1 R^v_{\pi_0} w_2$ iff $v(w_2, \varphi) \subseteq v(w_1, \varphi)$ for every $\varphi \in Dom_v$. Therefore, $\langle v, \Re \rangle$ is π_0-legal iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_0) w_2$, we have that $v(w_2, \varphi) \subseteq v(w_1, \varphi)$ for every $\varphi \in Dom_v$. In particular, if $\Re(\pi_0)$ is identity (on W_v) then $\langle v, \Re \rangle$ is π_0-legal. Note that if v is normal, then $w_1 R^v_{\pi_0} w_2$ iff $v(w_1, \varphi) = v(w_2, \varphi)$ for every $\varphi \in Dom_v$. In this case, $\langle v, \Re \rangle$ is π_0-legal iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_0) w_2$, we have that $v(w_2, \varphi) = v(w_1, \varphi)$ for every $\varphi \in Dom_v$.

2. Suppose that π_{int} appears in $\Pi_{\mathbf{G}}$. Here, $\alpha_2 \bar{\pi}_{\mathsf{int}} \alpha_1$ iff $\alpha_2 = \alpha_1 = \mathtt{f}{:}\varphi$ for some $\varphi \in \mathcal{L}$. Thus $w_1 R^v_{\pi_{\mathsf{int}}} w_2$ iff for every labelled Dom_v-formula α of the form $\mathtt{f}{:}\varphi$, if $v, w_2 \models \alpha$, then $v, w_1 \models \alpha$. Equivalently, $w_1 R^v_{\pi_{\mathsf{int}}} w_2$ iff $\mathtt{f} \in v(w_2, \varphi)$ implies that $\mathtt{f} \in v(w_1, \varphi)$ for every $\varphi \in Dom_v$. Therefore, $\langle v, \Re \rangle$ is π_{int}-legal iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_{\mathsf{int}}) w_2$, we have that $\mathtt{f} \in v(w_2, \varphi)$ implies $\mathtt{f} \in v(w_1, \varphi)$ for every $\varphi \in Dom_v$. Note that if v is normal, then $\langle v, \Re \rangle$ is π_{int}-legal iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_{\mathsf{int}}) w_2$, we have that $v(w_1, \varphi) = \{\mathtt{t}\}$ implies $v(w_2, \varphi) = \{\mathtt{t}\}$ for every $\varphi \in Dom_v$. This restriction corresponds to the persistence (or "monotonicity") condition that is employed in intuitionistic Kripke semantics, where $\Re(\pi_{\mathsf{int}})$ serves as the accessibility relation.

3. Suppose that π_{K} appears in $\Pi_{\mathbf{G}}$. Here, $w_1 R^v_{\pi_{\mathsf{K}}} w_2$ iff $v, w_2 \models \mathtt{f}{:}\varphi$ implies $v, w_1 \models \mathtt{f}{:}\Box\varphi$ whenever $\varphi \in Dom_v$ and $\Box\varphi \in Dom_v$. Equivalently, $w_1 R^v_{\pi_{\mathsf{K}}} w_2$ iff $\mathtt{f} \in v(w_2, \varphi)$ implies $\mathtt{f} \in v(w_1, \Box\varphi)$ whenever $\varphi \in Dom_v$ and $\Box\varphi \in Dom_v$. Therefore, $\langle v, \Re \rangle$ is π_{K}-legal iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_{\mathsf{K}}) w_2$, we have that $\mathtt{f} \in v(w_2, \varphi)$ implies $\mathtt{f} \in v(w_1, \Box\varphi)$ whenever $\varphi \in Dom_v$ and $\Box\varphi \in Dom_v$. Roughly speaking, this provides "one half" of the usual semantics of $\Box$.

4. Suppose that $\emptyset$ appears in $\Pi_{\mathbf{G}}$ (this context-relation is used in the rule (M), see Example 6.1.14). Since there do not exist labelled $\mathcal{L}$-formulas α_1, α_2 such that $\alpha_2 \bar{\emptyset} \alpha_1$, $w_1 R^v_{\emptyset} w_2$ trivially holds for every $w_1, w_2 \in W_v$. Thus $R^v_{\emptyset} = W_v \times W_v$, and every pair $\langle v, \Re \rangle$ is trivially $\emptyset$-legal.

Next we formulate the effect of the basic rules appearing in a basic calculus.

Notation 6.2.8. Given a set W, a binary relation $R \subseteq W \times W$, and an element $w \in W$, we denote the set $\{w' \in W \mid wRw'\}$ by $R[w]$.

Definition 6.2.9. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, v a partial $\mathcal{L}$-Kvaluation, and $\Re$ a $\langle \mathbf{G}, W_v \rangle$-coupling. $\langle v, \Re \rangle$ is called:

1. *r-legal* for some $r = \langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / s$ in $\mathsf{R}_{\mathbf{G}}$ if the following condition holds for every $w \in W_v$ and $\mathcal{L}$-substitution σ such that $frm[\sigma(\{s_1, \ldots, s_n, s\})] \subseteq Dom_v$: if $v, \Re(\pi_i)[w] \models \sigma(s_i)$ for every $1 \leq i \leq n$, then $v, w \models \sigma(s)$.

2. *R-legal* for some $R \subseteq \mathsf{R}_{\mathbf{G}}$ if it is r-legal for every $r \in R$.

Example 6.2.10. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, v a partial $\mathcal{L}$-Kvaluation, and $\mathfrak{R}$ a $\langle \mathbf{G}, W_v \rangle$-coupling.

1. Suppose that $\mathsf{R}_{\mathbf{G}}$ includes a rule r of the form $\langle \{\mathtt{t}{:}p_1\}, \pi \rangle / \{\mathtt{t}{:}\Box p_1\}$ (such a rule appears in various basic calculi for modal logics presented above). $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and $\mathcal{L}$-substitution σ: if $\mathit{frm}[\sigma(\{\mathtt{t}{:}p_1, \mathtt{t}{:}\Box p_1\})] \subseteq \mathit{Dom}_v$, and $v, \mathfrak{R}(\pi)[w] \models \sigma(\{\mathtt{t}{:}p_1\})$, then $v, w \models \sigma(\{\mathtt{t}{:}\Box p_1\})$. Equivalently, $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and formula φ: if $\{\varphi, \Box\varphi\} \subseteq \mathit{Dom}_v$, and $\mathtt{t} \in v(w', \varphi)$ for every $w' \in \mathfrak{R}(\pi)[w]$, then $\mathtt{t} \in v(w, \Box\varphi)$. Roughly speaking, this provides the "other half" of the usual semantics of $\Box$ (see Example 6.2.7, Item 3).

2. Suppose that $\mathsf{R}_{\mathbf{G}}$ includes a rule r of the form $\langle \{\mathtt{t}{:}p_1\}, \pi \rangle, \langle \{\mathtt{f}{:}p_2\}, \pi \rangle / \{\mathtt{f}{:}p_1 \supset p_2\}$ (a rule of this form appears in **LK** and **LJ** with $\pi = \pi_0$). $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and $\mathcal{L}$-substitution σ: if $\mathit{frm}[\sigma(\{\mathtt{t}{:}p_1, \mathtt{f}{:}p_2, \mathtt{f}{:}p_1 \supset p_2\})] \subseteq \mathit{Dom}_v$, $v, \mathfrak{R}(\pi)[w] \models \sigma(\{\mathtt{t}{:}p_1\})$ and $v, \mathfrak{R}(\pi)[w] \models \sigma(\{\mathtt{f}{:}p_2\})$, then $v, w \models \sigma(\{\mathtt{f}{:}p_1 \supset p_2\})$. Equivalently, $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and two formulas φ_1, φ_2: if $\{\varphi_1, \varphi_2, \varphi_1 \supset \varphi_2\} \subseteq \mathit{Dom}_v$, and for every $w' \in \mathfrak{R}(\pi)[w]$ it holds that $\mathtt{t} \in v(w', \varphi_1)$ and $\mathtt{f} \in v(w', \varphi_2)$, then $\mathtt{f} \in v(w, \varphi_1 \supset \varphi_2)$.

3. Suppose that $\mathsf{R}_{\mathbf{G}}$ includes a rule r of the form $\langle \{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}, \pi \rangle / \{\mathtt{t}{:}p_1 \supset p_2\}$ (a rule of this form appears in **LK** with $\pi = \pi_0$, and in **LJ** with $\pi = \pi_{\mathsf{int}}$). $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and $\mathcal{L}$-substitution σ: if $\mathit{frm}[\sigma(\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2, \mathtt{t}{:}p_1 \supset p_2\})] \subseteq \mathit{Dom}_v$, and $v, \mathfrak{R}(\pi)[w] \models \sigma(\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\})$, then $v, w \models \sigma(\{\mathtt{t}{:}p_1 \supset p_2\})$. Equivalently, $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$ and two formulas φ_1, φ_2: if $\{\varphi_1, \varphi_2, \varphi_1 \supset \varphi_2\} \subseteq \mathit{Dom}_v$, and $\mathtt{f} \in v(w', \varphi_1)$ or $\mathtt{t} \in v(w', \varphi_2)$ for every $w' \in \mathfrak{R}(\pi)[w]$, then $\mathtt{t} \in v(w, \varphi_1 \supset \varphi_2)$.

4. Suppose that $\mathsf{R}_{\mathbf{G}}$ includes a rule r of the form $\langle \emptyset, \pi \rangle / \emptyset$ (for example, this is the form of the rule (D), see Table 6.2). Applications of this rule allow to infer an $\mathcal{L}$-sequent s' from an $\mathcal{L}$-sequent s whenever $\langle s, s' \rangle$ is a π-instance. $\langle v, \mathfrak{R} \rangle$ is r-legal iff for every $w \in W_v$: if $v, \mathfrak{R}(\pi)[w] \models \emptyset$, then $v, w \models \emptyset$ (note that $\sigma(\emptyset) = \emptyset$ for every $\mathcal{L}$-substitution σ). Since the empty sequent is not true in any world, this condition would hold iff for every world w there exists some $w' \in \mathfrak{R}(\pi)[w]$. In other words, $\langle v, \mathfrak{R} \rangle$ is r-legal iff $\mathfrak{R}(\pi)$ is a serial relation.

Now, by collecting the semantic restrictions introduced by the context-relations and the basic rules, as well as those of the proof-specification, we obtain the set $\mathcal{K}_{\mathbf{G}\restriction\rho}$ of partial $\mathcal{L}$-Kvaluations for which a given basic calculus $\mathbf{G}$ and a proof-specification ρ are sound and complete.

Definition 6.2.11. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and v a partial $\mathcal{L}$-Kvaluation.

1. Given a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$, the pair $\langle v, \Re\rangle$ is called $\mathbf{G}$-*legal* if it is both $\Pi_{\mathbf{G}}$-legal, and $\mathsf{R}_{\mathbf{G}}$-legal.
2. v is called $\mathbf{G}$-*legal* if $\langle v, \Re\rangle$ is $\mathbf{G}$-legal for *some* $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$.
3. Given an $\mathcal{L}$-proof-specification ρ, v is called $\mathbf{G}{\restriction}_{\rho}$-*legal* if it is both ρ-legal and $\mathbf{G}$-legal. $\mathcal{K}_{\mathbf{G}\restriction_{\rho}}$ denotes the set of all $\mathbf{G}{\restriction}_{\rho}$-legal partial $\mathcal{L}$-Kvaluations.

Theorem 6.2.12. $\vdash_{\mathbf{G}\restriction_{\rho}} = \vdash_{\mathcal{K}_{\mathbf{G}\restriction_{\rho}}}$ for every basic $\mathcal{L}$-calculus $\mathbf{G}$ and $\mathcal{L}$-proof-specification ρ.

The proof is given in Section 6.4. In order to obtain a very general *soundness* result, we chose above the set $\mathcal{K}_{\mathbf{G}\restriction_{\rho}}$ to be as large as possible. On the other hand, a stronger *completeness* result can be obtained by considering a smaller set of Kvaluations:

Notation 6.2.13. Given a partial $\mathcal{L}$-Kvaluation v, we denote by Id_v the identity relation on W_v.

Definition 6.2.14. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and v a partial $\mathcal{L}$-Kvaluation.

1. $\Re^v_{\mathbf{G}}$ denotes the $\langle \mathbf{G}, W_v\rangle$-coupling defined by $\Re^v_{\mathbf{G}}(\pi_0) = Id_v$, and $\Re^v_{\mathbf{G}}(\pi) = R^v_{\pi}$ for every other $\pi \in \Pi_{\mathbf{G}}$ (see Definition 6.2.6).
2. v is called *strongly* $\mathbf{G}$-*legal* if $\langle v, \Re^v_{\mathbf{G}}\rangle$ is $\mathsf{R}_{\mathbf{G}}$-legal.
3. v is called *differentiated* if $R^v_{\pi_0} = Id_v$.
4. Given an $\mathcal{L}$-proof-specification ρ, v is called *strongly* $\mathbf{G}{\restriction}_{\rho}$-*legal* if it is both ρ-legal and strongly $\mathbf{G}$-legal. $\mathcal{K}^*_{\mathbf{G}\restriction_{\rho}}$ denotes the set of all strongly $\mathbf{G}{\restriction}_{\rho}$-legal and differentiated partial $\mathcal{L}$-Kvaluations.

By definition (and following Example 6.2.7, Item 1), for every partial $\mathcal{L}$-Kvaluation v, the pair $\langle v, \Re^v_{\mathbf{G}}\rangle$ is π-legal for every $\pi \in \Pi_{\mathbf{G}}$. Thus, a strongly $\mathbf{G}$-legal partial $\mathcal{L}$-Kvaluation is $\mathbf{G}$-legal.

Remark 6.2.15. Following Example 6.2.7 (Item 1), a normal partial $\mathcal{L}$-Kvaluation v is differentiated iff $w_1 = w_2$ whenever $v(w_1, \varphi) = v(w_2, \varphi)$ for every $\varphi \in Dom_v$. The name of this property is taken from [41].

Theorem 6.2.16. $\vdash_{\mathbf{G}\restriction_{\rho}} = \vdash_{\mathcal{K}^*_{\mathbf{G}\restriction_{\rho}}}$ for every basic $\mathcal{L}$-calculus $\mathbf{G}$ and $\mathcal{L}$-proof-specification ρ.

The proof is given in Section 6.4. The two last theorems are combined in the following theorem, that provides an "interval" of possible semantics for a given basic calculus.

Theorem 6.2.17. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and ρ an $\mathcal{L}$-proof-specification. Then, $\vdash_{\mathbf{G}\upharpoonright\rho} = \vdash_{\mathcal{K}}$ for every set $\mathcal{K}$ of partial $\mathcal{L}$-Kvaluations satisfying $\mathcal{K}^*_{\mathbf{G}\upharpoonright\rho} \subseteq \mathcal{K} \subseteq \mathcal{K}_{\mathbf{G}\upharpoonright\rho}$.

The proof is given in Section 6.4. The following is a useful instance that does not consider proof-specifications at all:

Corollary 6.2.18. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus. Then, $\vdash_{\mathbf{G}} = \vdash_{\mathcal{K}}$ for every set $\mathcal{K}$ of normal $\mathbf{G}$-legal $\mathcal{L}$-Kvaluations that contains all normal strongly $\mathbf{G}$-legal differentiated $\mathcal{L}$-Kvaluations.

Proof. Since $\langle \mathcal{L}, \mathcal{L}, \mathcal{L} \rangle$-legal partial $\mathcal{L}$-Kvaluations are exactly normal $\mathcal{L}$-Kvaluations, the claim directly follows from Theorem 6.2.17. □

Theorem 6.2.17 provides a general soundness and completeness result applicable to every basic $\mathcal{L}$-calculus $\mathbf{G}$ and $\mathcal{L}$-proof-specification ρ. Its exact content depends on the choice of set $\mathcal{K}$ of partial $\mathcal{L}$-Kvaluations. $\mathcal{K}$ should meet two conditions: first, it should contain only $\mathbf{G}\upharpoonright\rho$-legal partial $\mathcal{L}$-Kvaluations; and second, it should contain *all* strongly $\mathbf{G}\upharpoonright\rho$-legal differentiated partial $\mathcal{L}$-Kvaluations. In many cases, using the structure of the context-relations in $\Pi_{\mathbf{G}}$, it is possible to recognize some properties common to all strongly $\mathbf{G}\upharpoonright\rho$-legal differentiated partial $\mathcal{L}$-Kvaluations, and derive specific soundness and completeness results with respect to the set of all $\mathbf{G}\upharpoonright\rho$-legal partial $\mathcal{L}$-Kvaluations satisfying these properties. The following proposition is particularly useful for this purpose.

Notation 6.2.19. Given a labelled formula of the form $\texttt{f}{:}\varphi$, we denote by $\overline{\texttt{f}{:}\varphi}$ the labelled formula $\texttt{t}{:}\varphi$. Similarly, $\overline{\texttt{t}{:}\varphi}$ denotes the labelled formula $\texttt{f}{:}\varphi$.

Proposition 6.2.20. Let v be a partial $\mathcal{L}$-Kvaluation, and π_1, π_2, π_3 context-relations.

1. Suppose that $\bar{\pi}_3 = \bar{\pi}_1 \cup \bar{\pi}_2$. Then $R^v_{\pi_3} = R^v_{\pi_2} \cap R^v_{\pi_1}$. In particular, if $\bar{\pi}_1 \subseteq \bar{\pi}_2$ then $R^v_{\pi_2} \subseteq R^v_{\pi_1}$.

2. Suppose that for every labelled Dom_v-formulas α_1, α_2, if $\alpha_2 \bar{\pi}_3 \alpha_1$ then there exists $\alpha' \in Dom_v$ such that $\alpha_2 \bar{\pi}_1 \alpha'$ and $\alpha' \bar{\pi}_2 \alpha_1$. Then $R^v_{\pi_2} \circ R^v_{\pi_1} \subseteq R^v_{\pi_3}$.[3] In particular, if for every labelled Dom_v-formulas α_1 and α_2, $\alpha_2 \bar{\pi}_1 \alpha_1$ implies that there exists a labelled Dom_v-formula α' such that $\alpha_2 \bar{\pi}_1 \alpha'$ and $\alpha' \bar{\pi}_1 \alpha_1$, then $R^v_{\pi_1}$ is a transitive relation.

[3] Given two relations $R_1, R_2 \subseteq A^2$, $aR_1 \circ R_2 b$ if there exists some $c \in A$ such that aR_1c and cR_2b.

3. Assume that v is normal. If $\overline{\alpha_1}\bar{\pi}_1\overline{\alpha_2}$ whenever $\alpha_2\bar{\pi}_2\alpha_1$, then $R^v_{\pi_1} \subseteq (R^v_{\pi_2})^{-1}$. In particular, (i) if $\alpha_1\bar{\pi}_1\alpha_2$ implies $\overline{\alpha_2}\bar{\pi}_2\overline{\alpha_1}$ and vice-versa, then $R^v_{\pi_2} = (R^v_{\pi_1})^{-1}$, and (ii) if $\alpha_1\bar{\pi}_1\alpha_2$ implies $\overline{\alpha_2}\bar{\pi}_1\overline{\alpha_1}$ and vice-versa, then $R^v_{\pi_1}$ is a symmetric relation.

Proof.
1. Suppose first that $w_1 R^v_{\pi_3} w_2$. By definition, this means that for every labelled Dom_v-formulas α_1, α_2, if $\alpha_2\bar{\pi}_3\alpha_1$ and $v, w_2 \models \alpha_2$ then $v, w_1 \models \alpha_1$. Since $\bar{\pi}_1 \subseteq \bar{\pi}_3$, this implies that for every labelled Dom_v-formulas α_1, α_2, if $\alpha_2\bar{\pi}_1\alpha_1$ and $v, w_2 \models \alpha_2$ then $v, w_1 \models \alpha_1$. Hence, $w_1 R^v_{\pi_1} w_2$. Similarly, $w_1 R^v_{\pi_2} w_2$. For the converse, suppose that $w_1 R^v_{\pi_1} w_2$ and $w_1 R^v_{\pi_2} w_2$. By definition, this means that for every labelled Dom_v-formulas α_1, α_2, if $\alpha_2\bar{\pi}_1\alpha_1$ or $\alpha_2\bar{\pi}_2\alpha_1$, and $v, w_2 \models \alpha_2$ then $v, w_1 \models \alpha_1$. Since $\bar{\pi}_3 \subseteq \bar{\pi}_1 \cup \bar{\pi}_2$, this implies that for every labelled Dom_v-formulas α_1, α_2, if $\alpha_2\bar{\pi}_3\alpha_1$ and $v, w_2 \models \alpha_2$ then $v, w_1 \models \alpha_1$. Hence, $w_1 R^v_{\pi_3} w_2$.

2. Let $w_1, w_2 \in W_v$ such that $w_1 R^v_{\pi_2} \circ R^v_{\pi_1} w_2$. Then there exists $w' \in W_v$, such that $w_1 R^v_{\pi_2} w'$ and $w' R^v_{\pi_1} w_2$. We show that $w_1 R^v_{\pi_3} w_2$. Let α_1, α_2 be labelled Dom_v-formulas, such that $\alpha_2\bar{\pi}_3\alpha_1$, and $v, w_2 \models \alpha_2$. Therefore, there exists a labelled Dom_v-formula α' such that $\alpha_2\bar{\pi}_1\alpha'$ and $\alpha'\bar{\pi}_2\alpha_1$. Since $w' R^v_{\pi_1} w_2$, we have $v, w' \models \alpha'$. Since $w_1 R^v_{\pi_2} w'$, we have $v, w_1 \models \alpha_1$.

3. Let $w R^v_{\pi_1} w'$. We show that $w' R^v_{\pi_2} w$. Let α_1, α_2 be labelled Dom_v-formulas, such that $\alpha_2\bar{\pi}_2\alpha_1$ and $v, w \models \alpha_2$. This implies that $\overline{\alpha_1}\bar{\pi}_1\overline{\alpha_2}$. Now, since v is normal and $v, w \models \alpha_2$, we have that $v, w \not\models \overline{\alpha_2}$. Since $w R^v_{\pi_1} w'$, we have $v, w' \not\models \overline{\alpha_1}$. Since v is normal, this entails that $v, w' \models \alpha_1$. □

The following soundness and completeness results are easily obtained using Proposition 6.2.20:

Corollary 6.2.21. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and ρ an $\mathcal{L}$-proof-specification. Suppose that $\bar{\pi} = \bar{\pi} \circ \bar{\pi}$ for some $\pi \in \Pi_{\mathbf{G}}$. Let $\mathcal{K}$ be the set of all ρ-legal partial $\mathcal{L}$-Kvaluations v for which there exists a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$, such that $\Re(\pi)$ is a transitive relation, and $\langle v, \Re \rangle$ is $\mathbf{G}$-legal. Then, $\vdash_{\mathbf{G}} = \vdash_{\mathcal{K}}$.

Proof. Clearly, $\mathcal{K}$ is a set of $\mathbf{G}\lceil_\rho$-legal partial $\mathcal{L}$-Kvaluations. By Theorem 6.2.17 it suffices to show that $\mathcal{K}$ contains all strongly $\mathbf{G}\lceil_\rho$-legal partial $\mathcal{L}$-Kvaluations. Let v be a strongly $\mathbf{G}\lceil_\rho$-legal partial $\mathcal{L}$-Kvaluation. Then $\langle v, \Re^v_{\mathbf{G}} \rangle$ is $\mathbf{G}$-legal. By Proposition 6.2.20 (Item 2), $\Re^v_{\mathbf{G}}(\pi) = R^v_\pi$ is transitive. It follows that $v \in \mathcal{K}$. □

Corollary 6.2.22. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and ρ an $\mathcal{L}$-proof-specification. Suppose that for some $\pi \in \Pi_{\mathbf{G}}$, $\bar{\pi}$ includes only pairs of the form $\langle \alpha, \alpha \rangle$. Let $\mathcal{K}$ be the set of all ρ-legal partial $\mathcal{L}$-Kvaluations v for which there exists a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$, such that $\Re(\pi)$ is a reflexive relation, and $\langle v, \Re \rangle$ is $\mathbf{G}$-legal. Then, $\vdash_{\mathbf{G}} = \vdash_{\mathcal{K}}$.

Proof. As in the proof of Corollary 6.2.21, it suffices to show that $\mathcal{K}$ contains all strongly $\mathbf{G}{\restriction}_{\rho}$-legal partial $\mathcal{L}$-Kvaluations. Let v be a strongly $\mathbf{G}{\restriction}_{\rho}$-legal partial $\mathcal{L}$-Kvaluation. Then $\langle v, \Re^v_{\mathbf{G}}\rangle$ is $\mathbf{G}$-legal. Proposition 6.2.20 (Item 1) entails that $R^v_{\pi_0} \subseteq R^v_{\pi}$. Since $Id_v \subseteq R^v_{\pi_0}$, $\Re^v_{\mathbf{G}}(\pi) = R^v_{\pi}$ is reflexive. It follows that $v \in \mathcal{K}$. □

Semantic characterizations of $\leq$-analyticity, cut-admissibility, and axiom-expansion in basic calculi follow from Theorem 6.2.17. This is the topic of Section 6.3. To end this section, we prove a useful property of differentiated ρ-legal partial $\mathcal{L}$-Kvaluations:

Proposition 6.2.23. Let v be a differentiated $\langle\mathcal{F},\mathcal{C},\mathcal{A}\rangle$-legal partial $\mathcal{L}$-Kvaluation.

1. If $v(w,\varphi) = v(w',\varphi)$ for every $\varphi \in \mathcal{F}$ then $w = w'$.
2. $|W_v| \leq 2^{|\mathcal{F}\cap\mathcal{C}\cap\mathcal{A}|} \cdot 3^{|\mathcal{F}\cap\overline{\mathcal{C}}\cap\mathcal{A}|+|\mathcal{F}\cap\mathcal{C}\cap\overline{\mathcal{A}}|} \cdot 4^{|\mathcal{F}\cap\overline{\mathcal{C}}\cap\overline{\mathcal{A}}|}$.

Proof. 2 directly follows from 1 by counting the number of possible functions from $\mathcal{F}$ to $2^{\{\mathtt{f},\mathtt{t}\}}$, that can be used in an $\langle\mathcal{F},\mathcal{C},\mathcal{A}\rangle$-legal partial $\mathcal{L}$-Kvaluation. For 1, suppose that $v(w,\varphi) = v(w',\varphi)$ for every $\varphi \in \mathcal{F}$. It follows that $wR^v_{\pi_0}w'$. Since v is differentiated, $R^v_{\pi_0} = Id_v$, and so $w = w'$. □

Together with Theorem 6.2.17, the last proposition makes it possible to have a *semantic decision procedure* for deciding whether $\mathcal{S} \vdash_{\mathbf{G}{\restriction}_{\rho}} s$ given a basic $\mathcal{L}$-calculus $\mathbf{G}$, an $\mathcal{L}$-proof-specification $\rho = \langle\mathcal{F},\mathcal{C},\mathcal{A}\rangle$ with finite $\mathcal{F},\mathcal{C}$ and $\mathcal{A}$, finite set $\mathcal{S}$ of $\mathcal{L}$-sequents, and a single $\mathcal{L}$-sequent s. Indeed, it is possible to check all functions of the form $v : W \times \mathcal{F} \to 2^{\{\mathtt{f},\mathtt{t}\}}$, where $|W|$ is bounded according to the last proposition. Theorem 6.2.17 and the last proposition entail that $\mathcal{S} \not\vdash_{\mathbf{G}{\restriction}_{\rho}} s$ iff one of these functions is a strongly $\mathbf{G}{\restriction}_{\rho}$-legal partial $\mathcal{L}$-Kvaluation, which is a model of $\mathcal{S}$ but not of s. In this case the semantics is effective, leading to a counter-model search procedure. (Note that a *syntactic* decision procedure for this problem is trivial, as one can simply construct and check one-by-one all possible proof candidates.) It follows that we have a semantic decision procedure to decide whether $\mathcal{S} \vdash_{\mathbf{G}{\restriction}_{\rho}} s$ for strongly $\leq$-analytic basic calculus $\mathbf{G}$ (where $\leq$ is safe, see Definition 2.3.1). Indeed, in this case $\mathcal{S} \vdash_{\mathbf{G}} s$ iff $\mathcal{S} \vdash_{\mathbf{G}{\restriction}_{\rho}} s$ for $\rho = \langle{\downarrow^{\leq}}[\mathcal{S}\cup\{s\}], {\downarrow^{\leq}}[\mathcal{S}\cup\{s\}], {\downarrow^{\leq}}[\mathcal{S}\cup\{s\}]\rangle$.

6.2.1 Examples

In this section we provide various examples of applications of Corollary 6.2.18, by applying it to some of the basic calculi presented above. In particular, many fundamental soundness and completeness theorems for known logics and calculi are easily obtained as special cases.

Example 6.2.24 (LJ). Using Corollary 6.2.18, we are able to obtain a sound and complete semantics for **LJ**, which is practically identical to the usual Kripke semantics for intuitionistic logic. For this purpose, let $\mathcal{K}_{\mathbf{LJ}}$ be the set of normal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluations v that respect the usual truth tables of $\wedge, \vee, \perp$ in each world (where obviously, $\{\mathtt{f}\}$ and $\{\mathtt{t}\}$ are identified with $false$ and $true$), and in addition there exists a partial order $\leq$ on W_v satisfying the following conditions:

(persistence) If $v(w, \varphi) = \{\mathtt{t}\}$ then $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \geq w$.

(implication) $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{t}\}$ iff $v(w', \varphi_1) = \{\mathtt{f}\}$ or $v(w', \varphi_2) = \{\mathtt{t}\}$ for every $w' \geq w$.

We show that (1) $\mathcal{K}_{\mathbf{LJ}}$ is a set of normal **LJ**-legal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluations, and (2) $\mathcal{K}_{\mathbf{LJ}}$ contains all normal strongly **LJ**-legal differentiated $\mathcal{L}_{\mathbf{LJ}}$-Kvaluations. Corollary 6.2.18 implies then that $\vdash_{\mathbf{LJ}} = \vdash_{\mathcal{K}_{\mathbf{LJ}}}$.

1. Let $v \in \mathcal{K}_{\mathbf{LJ}}$, and let $\leq$ be a partial order on W_v satisfying *(persistence)* and *(implication)*. Recall that $\Pi_{\mathbf{LJ}} = \{\pi_0, \pi_{\mathsf{int}}\}$. Choose $\mathfrak{R}$ to be the $\langle \mathbf{LJ}, W_v \rangle$-coupling assigning Id_v to π_0, and $\leq$ to π_{int}. Clearly, $\langle v, \mathfrak{R} \rangle$ is π_0-legal. By Example 6.2.7 (Item 2), condition *(persistence)* ensures that $\langle v, \mathfrak{R} \rangle$ is π_{int}-legal. It is straightforward to show that $\langle v, \mathfrak{R} \rangle$ is $\mathsf{R}_{\mathbf{LJ}}$-legal. For example, following Example 6.2.10 (Item 3), *(implication)* above immediately implies that $\langle v, \mathfrak{R} \rangle$ is $(\mathtt{t}\colon \supset)$-legal.

2. Let v be a normal strongly **LJ**-legal differentiated $\mathcal{L}_{\mathbf{LJ}}$-Kvaluation. It is easy to show that v respects the usual truth tables of $\wedge, \vee, \perp$ in each world. We show that $R^v_{\pi_{\mathsf{int}}}$ is a partial order satisfying *(persistence)* and *(implication)*. Since $\bar{\pi_{\mathsf{int}}} \subseteq \bar{\pi_{\mathsf{int}}} \circ \bar{\pi_{\mathsf{int}}}$, Proposition 6.2.20 (Item 2) entails that $R^v_{\pi_{\mathsf{int}}}$ is transitive. Next, note that $\pi_{\mathsf{int}} \subseteq \pi_0$, hence, by Proposition 6.2.20 (Item 1), $R^v_{\pi_0} \subseteq R^v_{\pi_{\mathsf{int}}}$; since $Id_v \subseteq R^v_{\pi_0}$, $R^v_{\pi_{\mathsf{int}}}$ is reflexive. To see that $R^v_{\pi_{\mathsf{int}}}$ is anti-symmetric, suppose that $w R^v_{\pi_{\mathsf{int}}} w'$ and $w' R^v_{\pi_{\mathsf{int}}} w$. This implies that $v(w, \varphi) = v(w', \varphi)$ for every $\varphi \in \mathcal{L}_{\mathbf{LJ}}$. Since v is differentiated, $w = w'$. It remains to show that *(persistence)* and *(implication)* hold for $R^v_{\pi_{\mathsf{int}}}$. Following Example 6.2.7 (Item 2), since $\langle v, \mathfrak{R}^v_{\mathbf{LJ}} \rangle$ is π_{int}-legal, condition *(persistence)* holds. By Example 6.2.10 (Item 2), since $\langle v, \mathfrak{R}^v_{\mathbf{LJ}} \rangle$ is $(\mathtt{f}\colon \supset)$-legal, we have that for every $w \in W_v$, if $v(w, \varphi_1) = \{\mathtt{t}\}$ and $v(w, \varphi_2) = \{\mathtt{f}\}$ then $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{f}\}$. By Example 6.2.10 (Item 3), since $\langle v, \mathfrak{R}^v_{\mathbf{LJ}} \rangle$ is $(\mathtt{t}\colon \supset)$-legal, we have that for every $w \in W_v$, if $v(w', \varphi_1) = \{\mathtt{f}\}$ or $v(w', \varphi_2) = \{\mathtt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{int}}}[w]$, then $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{t}\}$. These two facts together with *(persistence)* establish *(implication)*.

Now, what happens if we simply apply Theorem 6.2.12 for **LJ** (perhaps without knowing about the usual Kripke semantics for intuitionistic logic)? In this case, we obtain that

LJ is sound and complete for the set of normal **LJ**-legal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluations. This set can be defined exactly like $\mathcal{K}_{\mathbf{LJ}}$ without restricting $\leq$ to be a partial order. Thus, we obtain a semantics which is less restrictive than the usual one. On the other hand, we can apply Theorem 6.2.16, and obtain that **LJ** is sound and complete for the set of normal strongly **LJ**-legal differentiated $\mathcal{L}_{\mathbf{LJ}}$-Kvaluations. This set is a subset of $\mathcal{K}_{\mathbf{LJ}}$ obtained by imposing also the converse of *(persistence)* (if $v(w, \varphi) = \{\mathtt{t}\}$ implies $v(w', \varphi) = \{\mathtt{t}\}$ for every φ, then $w \leq w'$). Here we obtain a more restrictive semantics than the usual one.

Example 6.2.25 (**BLJ**)**.** Using Corollary 6.2.18, we obtain a sound and complete semantics for **BLJ**, which is practically the same as the usual Kripke semantics for bi-intuitionistic logic (see, e.g., [59]). For this purpose, let $\mathcal{K}_{\mathbf{BLJ}}$ be the set of all normal $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluations v satisfying the conditions from Example 6.2.24, and the following additional condition:

(exclusion) $v(w, \varphi_1 \prec \varphi_2) = \{\mathtt{f}\}$ iff $v(w', \varphi_1) = \{\mathtt{f}\}$ or $v(w', \varphi_2) = \{\mathtt{t}\}$ for every $w' \leq w$.

Now, $\mathcal{K}_{\mathbf{BLJ}}$ is a set of **BLJ**-legal $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluations, that contains all normal strongly **BLJ**-legal differentiated $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluations, and so $\vdash_{\mathbf{BLJ}} = \vdash_{\mathcal{K}_{\mathbf{BLJ}}}$ by Corollary 6.2.18. This is shown similarly as for **LJ**. In particular, the rules of $\prec$ correspond to *(exclusion)*, and Proposition 6.2.20 (Item 3) entails that in strongly **BLJ**-legal $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluations $R^v_{\pi_{\mathsf{d}}} = (R^v_{\pi_{\mathsf{int}}})^{-1}$ (since $\alpha_1 \bar{\pi}_{\mathsf{d}} \alpha_2$ iff $\overline{\alpha_2} \bar{\pi}_{\mathsf{int}} \overline{\alpha_1}$).

Example 6.2.26 (**PLJ**)**.** Using Corollary 6.2.18, **PLJ** is sound and complete with respect to the set $\mathcal{K}$ of normal $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluations v satisfying the conditions from Example 6.2.24 (ignoring the condition involving $\perp$), and the following two conditions:

- If $v(w, \varphi) = \{\mathtt{f}\}$ then $v(w, \neg\varphi) = \{\mathtt{t}\}$.
- If $v(w, \varphi_1) = \{\mathtt{t}\}$ and $v(w, \neg\varphi_2) = \{\mathtt{t}\}$ then $v(w, \neg(\varphi_1 \supset \varphi_2)) = \{\mathtt{t}\}$.

To see this it suffices to show that $\mathcal{K}$ is a set of **PLJ**-legal $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluations containing all normal strongly **PLJ**-legal differentiated $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluations. This is done straightforwardly. Clearly, this semantics is non-deterministic, as the truth values of φ in every world may not determine the truth values of $\neg\varphi$. For example, in an $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluation with a single world w, if $v(w, p_1) = \{\mathtt{t}\}$, then $v(w, \neg p_1)$ can be either $\{\mathtt{t}\}$ or $\{\mathtt{f}\}$. Note that this semantics is different from the (three-valued) semantics given in [8] for this system.

Example 6.2.27 (**K**)**.** The usual Kripke semantics of the modal logic K can be described using the set $\mathcal{K}_{\mathbf{K}}$ of normal $\mathcal{L}_{\Box}$-Kvaluations defined as follows: $v \in \mathcal{K}_{\mathbf{K}}$ iff v respects the usual truth tables of the classical connectives in each world, and there exists a binary relation R on W_v such that the following condition holds:

(necessity) $v(w, \Box\varphi) = \{\mathtt{t}\}$ iff $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R[w]$.

Now Corollary 6.2.18 implies that $\vdash_{\mathbf{K}} = \vdash_{\mathcal{K}_{\mathbf{K}}}$. To see this, we prove that $\mathcal{K}_{\mathbf{K}}$ is a set of normal **K**-legal $\mathcal{L}_\Box$-Kvaluations that contains all normal strongly **K**-legal $\mathcal{L}_\Box$-Kvaluations:

1. Let $v \in \mathcal{K}_{\mathbf{K}}$, and let R be a relation on W_v satisfying *(necessity)*. Choose $\Re$ to be the $\langle \mathbf{K}, W_v \rangle$-coupling assigning Id_v to π_0, and R to π_{K}. Following Example 6.2.7 (Items 1 and 3), $\langle v, \Re \rangle$ is $\Pi_{\mathbf{K}}$-legal. It remains to show that $\langle v, \Re \rangle$ is $\mathsf{R}_{\mathbf{K}}$-legal. We show it here only for the rule (K). Following Example 6.2.10 (Item 1), it suffices to see that for every $w \in W_v$ and formula φ: if $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R[w]$, then $v(w, \Box\varphi) = \{\mathtt{t}\}$. This follows from the definition of $\mathcal{K}_{\mathbf{K}}$.

2. Let v be a normal strongly **K**-legal Kvaluation. It is easy to show that v respects the usual truth tables of the classical connectives in each world. We claim that $R^v_{\pi_{\mathsf{K4}}}$ is a relation satisfying *(necessity)*. To see this, note that since $\langle v, \Re^v_{\mathbf{K}} \rangle$ is (K)-legal, we have that if $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{K4}}}[w]$, then $v(w, \Box\varphi) = \{\mathtt{t}\}$. The converse is obtained from the fact that (by definition) $w_1 R^v_{\pi_{\mathsf{K4}}} w_2$ iff $v(w_2, \varphi) = \{\mathtt{f}\}$ implies $v(w_1, \Box\varphi) = \{\mathtt{f}\}$ for every $\varphi \in \mathcal{L}_\Box$.

Example 6.2.28 (Systems for modal logics)**.** The usual Kripke semantics of the modal logics $K4$, KD, KB, $S4$ and $S5$ can be described as variations on the set $\mathcal{K}_{\mathbf{K}}$ (from Example 6.2.27), obtained by imposing an additional requirement on R:

- $\mathcal{K}_{\mathbf{K4}}$ – R is transitive.
- $\mathcal{K}_{\mathbf{KD}}$ – R is serial.
- $\mathcal{K}_{\mathbf{KB}}$ – R is symmetric.
- $\mathcal{K}_{\mathbf{S4}}$ – R is reflexive and transitive.
- $\mathcal{K}_{\mathbf{S5}}$ – R is an equivalence relation.

For every $\mathbf{G} \in \{\mathbf{K4}, \mathbf{KD}, \mathbf{KB}, \mathbf{S4}, \mathbf{S5}\}$, Corollary 6.2.18 implies that $\vdash_{\mathbf{G}} = \vdash_{\mathcal{K}_{\mathbf{G}}}$. Indeed, we prove that in each of these cases $\mathcal{K}_{\mathbf{G}}$ is a set of **G**-legal $\mathcal{L}_\Box$-Kvaluations that contains all strongly **G**-legal $\mathcal{L}_\Box$-Kvaluations. Let $\mathbf{G} \in \{\mathbf{K4}, \mathbf{KD}, \mathbf{KB}\}$ (the proofs for **S4** and **S5** are similar and left for the reader).

1. Let $v \in \mathcal{K}_{\mathbf{G}}$, and R be a relation on W_v satisfying *(necessity)* and the additional condition of $\mathcal{K}_{\mathbf{G}}$ (transitivity, seriality, or symmetry). Choose $\Re$ to be the $\langle \mathbf{G}, W_v \rangle$-coupling assigning Id_v to π_0, and R to the other context relation in $\Pi_{\mathbf{G}}$ (π_{K4},π_{K}, or π_{B}). We show that v is **G**-legal:

   K4 As in Example 6.2.27, $\langle v, \Re \rangle$ is π_0-legal and $\mathsf{R}_{\mathbf{K4}}$-legal. It remains to show that $R \subseteq R^v_{\pi_{\mathsf{K4}}}$ (and so $\langle v, \Re \rangle$ is π_{K4}-legal). Suppose that $w_1 R w_2$. Let α_1 and α_2 be labelled $\mathcal{L}_\Box$-formulas such that $\alpha_2 \bar{\pi}_{\mathsf{K4}} \alpha_1$ and $v, w_2 \models \alpha_2$. Then, $\alpha_2 = \mathtt{f}{:}\varphi$ and $\alpha_1 = \mathtt{f}{:}\Box\varphi$, or $\alpha_2 = \alpha_1 = \mathtt{f}{:}\Box\varphi$ (for some formula φ). In the first case, *(necessity)* directly implies that $v, w_1 \models \alpha_1$. Suppose now that $\alpha_2 = \alpha_1 = \mathtt{f}{:}\Box\varphi$ for some formula φ. Since $v, w_2 \models \alpha_2$ (i.e., $v(w_2, \Box\varphi) = \{\mathtt{f}\}$), *(necessity)* entails that $v, w \models \mathtt{f}{:}\varphi$ (i.e., $v(w, \varphi) = \{\mathtt{f}\}$) for some $w \in R[w_2]$. The transitivity of R then ensures that $w_1 R w$. Again *(necessity)* implies that $v, w_1 \models \alpha_1$. It follows that $w_1 R^v_{\pi_{\mathsf{K4}}} w_2$.

KD As in Example 6.2.27, $\langle v, \Re\rangle$ is $\Pi_{\mathbf{KD}}$-legal, and $\mathsf{R}_{\mathbf{KD}} \setminus \{(D)\}$-legal. In addition, following Example 6.2.10 (Item 4), the seriality of R ensures that $\langle v, \Re\rangle$ is (D)-legal. Therefore v is **KD**-legal.

KB As in Example 6.2.27, $\langle v, \Re\rangle$ is π_0-legal and $\mathsf{R}_{\mathbf{KB}}$-legal. It remains to show that $R \subseteq R^v_{\pi_{\mathsf{B}}}$ (and so $\langle v, \Re\rangle$ is also π_{B}-legal). Suppose that $w_1 R w_2$. Let α_1 and α_2 be labelled $\mathcal{L}_\Box$-formulas such that $\alpha_2 \bar{\pi} \alpha_1$ and $v, w_2 \models \alpha_2$. Then, $\alpha_2 = \mathtt{f}{:}\varphi$ and $\alpha_1 = \mathtt{f}{:}\Box\varphi$, or $\alpha_2 = \mathtt{t}{:}\Box\varphi$ and $\alpha_1 = \mathtt{t}{:}\varphi$ (for some formula φ). In the first case, *(necessity)* directly implies that $v, w_1 \models \alpha_1$. Suppose now that $\alpha_2 = \mathtt{t}{:}\Box\varphi$ and $\alpha_1 = \mathtt{t}{:}\varphi$. Since $v, w_2 \models \alpha_2$ (i.e., $v(w_2, \Box\varphi) = \{\mathtt{t}\}$), *(necessity)* entails $v, w \models \alpha_1$ for every $w \in R[w_2]$. The symmetry of R ensures that $w_2 R w_1$, and so $v, w_1 \models \alpha_1$. It follows that $w_1 R^v_{\pi_{\mathsf{B}}} w_2$.

2. Let v be a normal strongly **G**-legal $\mathcal{L}_\Box$-Kvaluation. Similarly to Example 6.2.27, one shows that $v \in \mathcal{K}_{\mathbf{K}}$. In addition:

   K4 Since $\bar{\pi}_{\mathsf{K4}} \subseteq \bar{\pi}_{\mathsf{K4}} \circ \bar{\pi}_{\mathsf{K4}}$, Proposition 6.2.20 (Item 2) entails that $R^v_{\pi_{\mathsf{K4}}}$ is transitive.

   KD Since $\langle v, \Re^v_{\mathbf{KD}}\rangle$ is (D)-legal, $R^v_{\pi_{\mathsf{D}}}$ is serial (see Example 6.2.10, Item 4).

   KB Since $\alpha_1 \bar{\pi}_{\mathsf{B}} \alpha_2$ iff $\overline{\alpha_2} \bar{\pi}_{\mathsf{B}} \overline{\alpha_1}$, Proposition 6.2.20 (Item 3) entails that $R^v_{\pi_{\mathsf{B}}}$ is symmetric.

Example 6.2.29 (**GL**)**.** Semantically, the modal logic GL is characterized by the set of Kripke frames whose accessibility relation is transitive and conversely well-founded. However, GL is not *strongly* complete with respect to models built on this set of frames (i.e. we have $\Vdash_{GL} \varphi$ iff every such frame is a model of φ, but we do *not* have $\mathcal{T} \Vdash_{GL} \varphi$ iff every such frame which is a model of $\mathcal{T}$ is a model of φ, see [94]). Using our method, starting from the basic calculus **GL**, we obtain a (different) *strongly* sound and complete semantics for GL. Indeed, by Corollary 6.2.18, **GL** is (strongly) sound and complete with respect to the set $\mathcal{K}_{\mathbf{GL}}$ of normal $\mathcal{L}_\Box$-Kvaluations, defined similarly to $\mathcal{K}_{\mathbf{K}}$ (see Example 6.2.27), with two additional requirements: (1) R is transitive; and (2) If $v(w', \varphi) = \{\mathtt{f}\}$ for some $w' \in R[w]$, then there is some $w'' \in R[w]$ such that $v(w'', \varphi) = \{\mathtt{f}\}$ and $v(w'', \Box\varphi) = \{\mathtt{t}\}$. To see this we prove that $\mathcal{K}_{\mathbf{GL}}$ is a set of **GL**-legal $\mathcal{L}_\Box$-Kvaluations that contains all normal strongly **GL**-legal $\mathcal{L}_\Box$-Kvaluations:

1. Let $v \in \mathcal{K}_{\mathbf{GL}}$, and let R be a transitive relation on W_v, satisfying *(necessity)* and condition (2) above. Choose $\mathfrak{R}$ to be the $\langle \mathbf{GL}, W_v \rangle$-coupling assigning Id_v to π_{0}, and R to π_{K4}. Similarly to Example 6.2.28, one proves that $R \subseteq R^v_{\pi_{\mathsf{K4}}}$ (using the transitivity of R), and so $\langle v, \mathfrak{R} \rangle$ is π_{K4}-legal. It remains to show that $\langle v, \mathfrak{R} \rangle$ is $\mathsf{R}_{\mathbf{GL}}$-legal. We show it here for (GL). Let $w \in W_v$, and let σ be an $\mathcal{L}_\Box$-substitution. Suppose that $v, \mathfrak{R}(\pi_{\mathsf{K4}})[w] \models \sigma(\{\mathtt{f}{:}\Box p_1, \mathtt{t}{:}p_1\})$. We show that $v, w \models \sigma(\{\mathtt{t}{:}\Box p_1\})$. Assume otherwise. Then $v(w, \Box\sigma(p_1)) = \{\mathtt{f}\}$. Thus *(necessity)* implies that there exists some $w' \in R[w]$, such that $v(w', \sigma(p_1)) = \{\mathtt{f}\}$. By condition (2), there is some $w'' \in R[w]$ such that $v(w'', \sigma(p_1)) = \{\mathtt{f}\}$ and $v(w'', \Box\sigma(p_1)) = \{\mathtt{t}\}$. Clearly, $v, w'' \not\models \sigma(\{\mathtt{f}{:}\Box p_1, \mathtt{t}{:}p_1\})$. But, since $\mathfrak{R}(\pi_{\mathsf{K4}}) = R$, this contradicts the fact that $v, \mathfrak{R}(\pi_{\mathsf{K4}})[w] \models \sigma(\{\mathtt{f}{:}\Box p_1, \mathtt{t}{:}p_1\})$.

2. Let v be a normal strongly **GL**-legal $\mathcal{L}_\Box$-Kvaluation. It is straightforward to show that v respects the usual truth tables of the classical connectives in each world. We show that there exists a transitive relation R on W_v satisfying *(necessity)* and condition (2) above. We show that $R^v_{\pi_{\mathsf{K4}}}$ has this property (its transitivity is proved exactly as in Example 6.2.28):

   (a) Since $\langle v, \mathfrak{R}^v_{\mathbf{GL}} \rangle$ is (GL)-legal, if $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{K4}}}[w]$, then $v(w, \Box\varphi) = \{\mathtt{t}\}$. The converse holds since $\langle v, \mathfrak{R}^v_{\mathbf{GL}} \rangle$ is π_{K4}-legal.

   (b) We prove that $R^v_{\pi_{\mathsf{K4}}}$ satisfies condition (2) above. Suppose (for contradiction) that there exist some $\varphi \in \mathcal{L}_\Box$ and $w \in W_v$, such that $v(w', \varphi) = \{\mathtt{f}\}$ for some $w' \in R^v_{\pi_{\mathsf{K4}}}[w]$, and there does not exist $w'' \in R^v_{\pi_{\mathsf{K4}}}[w]$, such that $v(w'', \varphi) = \{\mathtt{f}\}$ and $v(w'', \Box\varphi) = \{\mathtt{t}\}$. It follows that $v, \mathfrak{R}(\pi_{\mathsf{K4}})[w] \models \{\mathtt{f}{:}\Box\varphi, \mathtt{t}{:}\varphi\}$. Since $\langle v, \mathfrak{R} \rangle$ is (GL)-legal, $v(w, \Box\varphi) = \{\mathtt{t}\}$. But, this contradicts *(necessity)*.

We note it is not clear whether this semantics for GL is useful (in particular, whether it leads to a decision procedure). This question is left open for a future work.

Example 6.2.30 ($\mathbf{GP}^Q$)**.** Using Corollary 6.2.18, we obtain a sound and complete semantics for $\mathbf{GP}^Q$, which is practically identical to the semantics presented in [31]. For this purpose, let $\mathcal{K}$ be the set of $\mathcal{L}^Q_{\mathbf{GP}}$-Kvaluations v that respect the usual truth tables of $\wedge, \vee, \bot, \top$ in each world, and satisfy the following conditions:

1. If $v(w, \varphi_1) = \{\mathtt{t}\}$ and $v(w, \varphi_2) = \{\mathtt{f}\}$ then $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{f}\}$.
2. If $v(w, \varphi_2) = \{\mathtt{t}\}$ then $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{t}\}$.
3. For every $q \in Q$, there exist binary relations, S^q and I^q, on W_v, satisfying:

   (a) $I^q \subseteq S^q$.

   (b) $v(w, q \;\mathtt{said}\; \varphi) = \{\mathtt{t}\}$ iff $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in S^q[w]$.

(c) $v(w, q\ \texttt{implied}\ \varphi) = \{\texttt{t}\}$ iff $v(w', \varphi) = \{\texttt{t}\}$ for every $w' \in I^q[w]$.

Clearly, this semantics is *non-deterministic*, as the truth values of φ_1 and φ_2 in every world may not determine the value of $\varphi_1 \supset \varphi_2$. As in previous examples, it is straightforward to show that $\mathcal{K}$ is a set of normal $\mathbf{GP}^Q$-legal $\mathcal{L}^Q_{\mathbf{GP}}$-Kvaluations, that contains all normal strongly $\mathbf{GP}^Q$-legal $\mathcal{L}^Q_{\mathbf{GP}}$-Kvaluations (the fact that in all strongly $\mathbf{GP}^Q$-legal $\mathcal{L}^Q_{\mathbf{GP}}$-Kvaluations $\Re(\pi^q_i) \subseteq \Re(\pi^q_s)$ for every $q \in Q$ follows from Proposition 6.2.20, Item 1).

Example 6.2.31 (IS5). Using Corollary 6.2.18, we obtain a sound and complete Kripke semantics for **IS5**. For this purpose, let $\mathcal{K}$ be the set of normal $\mathcal{L}^{\Box}_{\mathbf{LJ}}$-Kvaluations v satisfying the conditions from Example 6.2.24, and in addition, there exists an equivalence relation $\sim$, such that $v(w, \Box\varphi) = \{\texttt{t}\}$ iff $v(w', \varphi) = \{\texttt{t}\}$ for every $w' \in \sim[w]$. (Note that if $v \in \mathcal{K}$, then for every $w, w' \in W_v$, we have that if $w \leq w'$ and $v(w'', \varphi) = \{\texttt{t}\}$ for every $w'' \in \sim[w]$, then $v(w'', \varphi) = \{\texttt{t}\}$ for every $w'' \in \sim[w']$.) As in previous examples, it is straightforward to show that $\mathcal{K}$ is a set of normal **IS5**-legal $\mathcal{L}^{\Box}_{\mathbf{LJ}}$-Kvaluations, that contains all normal strongly **IS5**-legal $\mathcal{L}^{\Box}_{\mathbf{LJ}}$-Kvaluations. Interestingly, the Kripke semantics presented in [78] is not identical to this one. In particular, in our semantics $\sim$ should be an equivalence relation, and no direct conditions bind $\leq$ and $\sim$.

6.3 Characterization of Proof-Theoretic Properties

In this section we use Theorem 6.2.17 to derive characterizations of strong $\leq$-analyticity, strong cut-admissibility, and axiom-expansion in basic calculi. First, note that the soundness part of Theorem 6.2.17 can be utilized for providing relatively simple semantic arguments for the failure of certain proof-theoretic properties. For example, by exhibiting an $\langle \mathcal{L}, \emptyset, \mathcal{L} \rangle$-legal **G**-legal $\mathcal{L}$-Kvaluation which is not a model of some sequent s, we show that every proof of s requires to use (*cut*). If we also have $\vdash_{\mathbf{G}} s$, then it follows that **G** does not enjoy cut-admissibility. Similarly, by using the other two components of the proof-specification, we can show that a certain calculus **G** is not $\leq$-analytic (for some $\leq$) or that some connective does not admit axiom-expansion in **G**. Note that proving facts of this kind using proof-theoretic methods is sometimes very challenging! Next we provide some concrete examples of such applications.

Example 6.3.1. Let s be the $\mathcal{L}_{\mathbf{BLJ}}$-sequent $\{\texttt{f}{:}p_1, \texttt{t}{:}p_2, \texttt{t}{:}p_1 \supset (p_1 \prec p_2)\}$. We show that $\not\vdash_{\mathbf{BLJ}\restriction_\rho} s$ for $\rho = \langle \mathcal{L}_{\mathbf{BLJ}}, \emptyset, \mathcal{L}_{\mathbf{BLJ}} \rangle$ (i.e., there does not exist a proof of s in **BLJ** without cuts). By Theorem 6.2.17, it suffices to find a $\mathbf{BLJ}\restriction_\rho$-legal $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluation which is not a model of s. Let v be an $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluation defined by $W_v = \{w_1, w_2\}$, $v(w, \varphi) = \{\texttt{f}, \texttt{t}\}$ for every $w \in W_v$ and $\mathcal{L}_{\mathbf{BLJ}}$-formula φ except for: $v(w_1, p_1) = v(w_2, p_1) = v(w_2, p_2) = \{\texttt{t}\}$, and $v(w_1, p_2) = v(w_1, p_1 \supset (p_1 \prec p_2)) = v(w_2, p_1 \prec p_2) = \{\mathtt{f}\}$. Let $\Re$ be the $\langle \mathbf{BLJ}, W_v \rangle$-coupling defined by $\Re(\pi_0) = \{\langle w_2, w_2 \rangle\}$, $\Re(\pi_{\mathsf{int}}) = \{\langle w_1, w_2 \rangle\}$, and $\Re(\pi_{\mathsf{d}}) = \emptyset$. One can straightforwardly verify that $\langle v, \Re \rangle$ is **BLJ**-legal, and clearly, $v \not\models s$. However, it is easy to find a proof for s in **BLJ**, and thus $\vdash_{\mathbf{BLJ}} s$. This provides a semantic demonstration of the fact that **BLJ** does not enjoy cut-admissibility (the sequent s is a simplified version of the one used in [80] to syntactically prove this fact).

Example 6.3.2. It is well-known that **S5** does not enjoy cut-admissibility. We provide a semantic demonstration of this fact. Let s be the $\mathcal{L}_\Box$-sequent $\{\mathtt{t}{:}p_1, \mathtt{t}{:}\Box\neg\Box p_1\}$. It is easy to see that s is provable in **S5** (using a cut on $\Box p_1$). Now, let $\rho = \langle \mathcal{L}_\Box, \{p_1, \Box\neg\Box p_1\}, \mathcal{L}_\Box \rangle$. We show that $\not\vdash_{\mathbf{S5}\restriction\rho} s$ (and so, in particular, there does exist a cut-free proof of s). Let v be an $\mathcal{L}_\Box$-Kvaluation defined by $W_v = \{w_1, w_2\}$, $v(w, \varphi) = \{\mathtt{f}, \mathtt{t}\}$ for every $w \in W_v$ and $\mathcal{L}_\Box$-formula φ except for:

$$v(w_2, p_1) = v(w_2, \Box p_1) = \{\mathtt{t}\},$$
$$v(w_1, p_1) = v(w_1, \Box\neg\Box p_1) = v(w_2, \neg\Box p_1) = v(w_2, \Box\neg\Box p_1) = \{\mathtt{f}\}.$$

Clearly, v is ρ-legal and $v \not\models s$. Let $\Re$ be the $\langle \mathbf{S5}, W_v \rangle$-coupling with $\Re(\pi_0) = \{\langle w_2, w_2 \rangle\}$ and $\Re(\pi_{\mathsf{S5}}) = \{\langle w_1, w_2 \rangle, \langle w_2, w_2 \rangle\}$. One can straightforwardly verify that $\langle v, \Re \rangle$ is **S5**-legal, and thus v is **S5**-legal. For example:

- $\langle v, \Re \rangle$ is π_{S5}-legal since the following conditions are met: (1) if $w \Re(\pi_{\mathsf{S5}}) w'$ and $\mathtt{t} \in v(w', \Box\varphi)$ then $\mathtt{t} \in v(w, \Box\varphi)$; (2) if $w \Re(\pi_{\mathsf{S5}}) w'$ and $\mathtt{f} \in v(w', \Box\varphi)$ then $\mathtt{f} \in v(w, \Box\varphi)$.
- $\langle v, \Re \rangle$ is $(S5)$-legal since the following condition is met: if $\mathtt{t} \in v(w', \varphi)$ for every $w' \in \Re(\pi_{\mathsf{S5}})[w]$, then $\mathtt{t} \in v(w, \Box\varphi)$.
- $\langle v, \Re \rangle$ is (T)-legal since the following condition is met: if $\mathtt{f} \in v(w', \varphi)$ for every $u \in \Re(\pi_0)[w]$, then $\mathtt{f} \in v(w, \Box\varphi)$.

Example 6.3.3. PLJ is not *sub*-analytic. This is shown in [8], by proving that the $\mathcal{L}_{\mathbf{PLJ}}$-sequent $s = \{\mathtt{t}{:}p_1, \mathtt{t}{:}p_2 \supset \neg(p_2 \supset p_1)\}$ is provable, but every proof of it must include a formula that does not occur in $sub[s]$. Using Theorem 6.2.17, we can provide a semantic demonstration of this fact. Let $\rho = \langle sub[s], \mathcal{L}_{\mathbf{PLJ}}, \mathcal{L}_{\mathbf{PLJ}} \rangle$. Consider the ρ-legal partial $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluation v, defined by $W_v = \{w_1, w_2\}$, and:

$$v(w_1, p_1) = v(w_1, p_2) = v(w_1, \neg(p_2 \supset p_1)) = v(w_1, p_2 \supset \neg(p_2 \supset p_1)) = \{\mathtt{f}\},$$
$$v(w_1, p_2 \supset p_1) = \{\mathtt{t}\},$$
$$v(w_2, \neg(p_2 \supset p_1)) = v(w_2, p_2 \supset \neg(p_2 \supset p_1)) = \{\mathtt{f}\},$$
$$v(w_2, p_1) = v(w_2, p_2) = v(w_2, p_2 \supset p_1) = \{\mathtt{t}\}.$$

Let $\Re$ be the $\langle \mathbf{PLJ}, W_v \rangle$-coupling defined by $\Re(\pi_{\mathsf{int}}) = \{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle, \langle w_1, w_2 \rangle\}$ and $\Re(\pi_0) = Id_v$. It is straightforward to show that $\langle v, \Re \rangle$ is **PLJ**-legal, and so v is $\mathbf{PLJ}{\restriction}_\rho$-legal. Clearly, $v \not\models s$. By Theorem 6.2.17, $\not\vdash_{\mathbf{PLJ}{\restriction}_\rho} s$. In other words, there does not exist a proof of s in **PLJ** consisting solely of $sub[s]$-sequents.

Next, we present characterizations of strong $\leq$-analyticity, strong cut-admissibility, and axiom-expansion in basic calculi, that may be used to prove these properties. For simplicity of the presentation, we shall not discuss the weak versions of $\leq$-analyticity and cut-admissibility, but note that they are always implied by the strong property (by takings $\mathcal{S} = \emptyset$). We use the following additional notion:

Definition 6.3.4. An *instance* of a partial $\mathcal{L}$-Kvaluation v is a normal (full) $\mathcal{L}$-Kvaluation v' such that $W_{v'} = W_v$, and $v'(w, \varphi) \subseteq v(w, \varphi)$ for every $w \in W_v$ and $\varphi \in Dom_v$.

The following proposition immediately follows from the definitions.

Proposition 6.3.5. Let v be a partial $\mathcal{L}$-Kvaluation, and let v' be an instance of v. Then, for every Dom_v-sequent s: if $v' \models s$ then $v \models s$. If v is $\langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$-legal and $frm[s] \subseteq \mathcal{C}$, the converse holds as well.

The following characterization of strong $\leq$-analyticity follows from the previous results.

Corollary 6.3.6. A basic $\mathcal{L}$-calculus **G** is strongly $\leq$-analytic iff for every set $\mathcal{S}$ of $\mathcal{L}$-sequents and $\mathcal{L}$-sequent s, $\mathcal{S} \vdash_{\mathcal{K}_1} s$ implies $\mathcal{S} \vdash_{\mathcal{K}_2} s$, where $\mathcal{K}_1$ is the set of all normal **G**-legal $\mathcal{L}$-Kvaluations, and $\mathcal{K}_2$ is the set of all normal strongly **G**-legal partial $\mathcal{L}$-Kvaluations whose domain is $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$.

Proof. Suppose that **G** is strongly $\leq$-analytic. Assume that $\mathcal{S} \vdash_{\mathcal{K}_1} s$ for some set $\mathcal{S}$ of $\mathcal{L}$-sequents and $\mathcal{L}$-sequent s. By Corollary 6.2.18, $\mathcal{S} \vdash_{\mathbf{G}} s$, and so $\mathcal{S} \vdash_{\mathbf{G}{\restriction}_\rho} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], \mathcal{L}, \mathcal{L} \rangle$. Note that $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], \mathcal{L}, \mathcal{L} \rangle$-legal partial $\mathcal{L}$-Kvaluations are exactly normal partial $\mathcal{L}$-Kvaluations whose domain is $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$. Therefore, by Theorem 6.2.17, $\mathcal{S} \vdash_{\mathcal{K}_2} s$.

For the converse, suppose that $\mathcal{S} \vdash_{\mathbf{G}} s$. By Corollary 6.2.18, $\mathcal{S} \vdash_{\mathcal{K}_1} s$, and so our assumption entails that $\mathcal{S} \vdash_{\mathcal{K}_2} s$. Theorem 6.2.17 again entails that $\mathcal{S} \vdash_{\mathbf{G}{\restriction}_\rho} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], \mathcal{L}, \mathcal{L} \rangle$. ☐

The above characterization might be quite complicated to be used in practice. Therefore, we now present a simpler semantic criterion, that turns out to be useful for many basic calculi.

Corollary 6.3.7. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus. Suppose that every normal strongly $\mathbf{G}$-legal partial $\mathcal{L}$-Kvaluation whose domain is closed under $\leq$ has a $\mathbf{G}$-legal instance. Then $\mathbf{G}$ is strongly $\leq$-analytic.

Proof. We use Corollary 6.3.6. Let $\mathcal{S}$ be a set of $\mathcal{L}$-sequents, and s a single $\mathcal{L}$-sequent. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be defined as in Corollary 6.3.6. Assume that $\mathcal{S} \vdash_{\mathcal{K}_1} s$. We prove that $\mathcal{S} \vdash_{\mathcal{K}_2} s$. Let $v \in \mathcal{K}_2$, and suppose that $v \models \mathcal{S}$. Since $Dom_v = \downarrow^{\leq}[\mathcal{S} \cup \{s\}]$ is closed under $\leq$, our assumption entails that there exists a $\mathbf{G}$-legal instance v' of v. Thus $v' \in \mathcal{K}_1$. By Proposition 6.3.5, we have $v' \models \mathcal{S}$. Since $\mathcal{S} \vdash_{\mathcal{K}_1} s$, we have $v' \models s$. Proposition 6.3.5 entails that $v \models s$. □

Before turning to some examples of applying the criterion above, we present a characterization of strong cut-admissibility. Its proof is similar to the proof of Corollary 6.3.6.

Definition 6.3.8. Given a set $\mathcal{C} \subseteq \mathcal{L}$, an $\mathcal{L}$-Kvaluation is called $\mathcal{C}$*-cut-restricted* if it is $\langle \mathcal{L}, \mathcal{C}, \mathcal{L} \rangle$-legal. An $\mathcal{L}$-Kvaluation is called *cut-restricted* if it is $\emptyset$-cut-restricted.

Corollary 6.3.9. A basic $\mathcal{L}$-calculus $\mathbf{G}$ enjoys strong cut-admissibility iff for every set $\mathcal{S}$ of $\mathcal{L}$-sequents and $\mathcal{L}$-sequent s, $\mathcal{S} \vdash_{\mathcal{K}_1} s$ implies $\mathcal{S} \vdash_{\mathcal{K}_2} s$, where $\mathcal{K}_1$ is the set of all normal $\mathbf{G}$-legal $\mathcal{L}$-Kvaluations, and $\mathcal{K}_2$ is the set of all $frm[\mathcal{S}]$-cut-restricted strongly $\mathbf{G}$-legal $\mathcal{L}$-Kvaluations.

Again, the following provides a simpler *sufficient* criterion:

Corollary 6.3.10. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus. Suppose that every cut-restricted strongly $\mathbf{G}$-legal $\mathcal{L}$-Kvaluation has a $\mathbf{G}$-legal instance. Then $\mathbf{G}$ enjoys strong cut-admissibility.

Proof. We use Corollary 6.3.9. Let $\mathcal{S}$ be a set of $\mathcal{L}$-sequents, and s a single $\mathcal{L}$-sequent. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be defined as in Corollary 6.3.9. Assume that $\mathcal{S} \vdash_{\mathcal{K}_1} s$. We prove that $\mathcal{S} \vdash_{\mathcal{K}_2} s$. Let $v \in \mathcal{K}_2$, and suppose that $v \models \mathcal{S}$. Since v is cut-restricted, our assumption entails that there exists a $\mathbf{G}$-legal instance v' of v. Thus $v' \in \mathcal{K}_1$. By Proposition 6.3.5, we have $v' \models \mathcal{S}$ (since v' is $\langle \mathcal{L}, frm[\mathcal{S}], \mathcal{L} \rangle$-legal). Since $\mathcal{S} \vdash_{\mathcal{K}_1} s$, we have $v' \models s$. Proposition 6.3.5 entails that $v \models s$. □

Next we apply the previous criteria to prove strong $\leq$-analyticity and/or strong cut-admissibility for some of the basic calculi presented in the examples above.

Example 6.3.11. We use Corollary 6.3.10 to show that $\mathbf{LJ}$ enjoys strong cut-admissibility. Let v be a cut-restricted strongly $\mathbf{LJ}$-legal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluation. We recursively construct an instance v' of v. For every $w \in W_v$ and for every atomic formula p, $v'(w, p) = \{\mathtt{x}\}$ if $v(w, p) = \{\mathtt{x}\}$, and otherwise $v'(w, p) = \{\mathtt{t}\}$ (say). Now suppose that $v'(w, \varphi_1)$ and $v'(w, \varphi_2)$ were defined for every $w \in W_v$:

- $v'(w, \varphi_1 \supset \varphi_2)$ is defined by: if $v(w, \varphi_1 \supset \varphi_2) = \{\texttt{x}\}$ then $v'(w, \varphi_1 \supset \varphi_2) = \{\texttt{x}\}$. Otherwise $v'(w, \varphi_1 \supset \varphi_2) = \{\texttt{t}\}$ iff for every $w' \in R^v_{\pi_{\mathsf{int}}}[w]$, either $v'(w', \varphi_1) = \{\texttt{f}\}$ or $v'(w', \varphi_2) = \{\texttt{t}\}$.
- $v'(w, \varphi_1 \wedge \varphi_2)$ is defined by: if $v(w, \varphi_1 \wedge \varphi_2) = \{\texttt{x}\}$ then $v'(w, \varphi_1 \wedge \varphi_2) = \{\texttt{x}\}$. Otherwise $v'(w, \varphi_1 \wedge \varphi_2) = \{\texttt{t}\}$ iff $v'(w, \varphi_1) = \{\texttt{t}\}$ and $v'(w, \varphi_2) = \{\texttt{t}\}$. Similar definitions are used for the other connectives of **LJ**.

Clearly, v' is an instance of v. Based on the fact that v is a cut-restricted strongly **LJ**-legal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluation, it is easy to prove that $\langle v', \Re^v_{\mathbf{LJ}} \rangle$ is **LJ**-legal (and so v' is **LJ**-legal).

Example 6.3.12. While **BLJ** does not enjoy cut-admissibility (Example 6.3.1), we use Corollary 6.3.7 to show that it is still strongly *sub*-analytic. This answers a question raised in [80].[4] Let v be a normal strongly **BLJ**-legal partial $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluation, whose domain is closed under subformulas. A construction of an instance v' of v, is done as in Example 6.3.11 with the following addition:

- If $v(w, \varphi_1 \prec \varphi_2) = \{\texttt{x}\}$ then $v'(w, \varphi_1 \prec \varphi_2) = \{\texttt{x}\}$. Otherwise $v'(w, \varphi_1 \prec \varphi_2) = \{\texttt{f}\}$ iff $v'(w', \varphi_1) = \{\texttt{f}\}$ or $v'(w', \varphi_2) = \{\texttt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{d}}}[w]$.

Clearly, v' is an instance of v. Based on the facts that v is a normal strongly **BLJ**-legal partial $\mathcal{L}_{\mathbf{BLJ}}$-Kvaluation, and that Dom_v is closed under subformulas, it is straightforward to prove that $\langle v', \Re^v_{\mathbf{BLJ}} \rangle$ is **BLJ**-legal.

Example 6.3.13. Following Example 6.3.3, **PLJ** is not *sub*-analytic. As a substitute, a weaker property is proved for this system in [8] (called the n-subformula property). Roughly speaking, this property means that whenever a sequent s is provable, there also exists a proof of s that includes only formulas from $sub[s]$ and some of their negations. To be more precise, it is equivalent to strong *nsub*-analyticity, where *nsub* is the transitive closure of the union of the relation *sub* and

$$\{\langle \neg\varphi_i, \neg(\varphi_1 \diamond \varphi_2)\rangle \mid \varphi_1, \varphi_2 \in \mathcal{L}_{\mathbf{PLJ}}, \diamond \in \{\wedge, \vee, \supset\}, i = 1, 2\}.$$

Note that *nsub* is safe, and so strong *nsub*-analyticity suffices to establish decidability. Next, we prove strong *nsub*-analyticity for **PLJ** using Corollary 6.3.7. Let v be a normal strongly **PLJ**-legal partial $\mathcal{L}_{\mathbf{PLJ}}$-Kvaluation, whose domain is closed under subformulas. A construction of an instance v' of v is done as in Example 6.3.11 with the following addition: $v'(w, \neg\varphi) = \{\texttt{x}\}$ if $v(w, \neg\varphi) = \{\texttt{x}\}$, and $v'(w, \neg\varphi) = \{\texttt{t}\}$ otherwise. Clearly, v' is an instance of v. Based on the fact that v is a normal strongly **PLJ**-legal partial

[4] Note that other systems for this logic, that enjoy cut-admissibility, were devised in [59] and [80]. However, these systems do not employ the standard notion of a sequent used in Gentzen-type systems, but more complicated data-structures.

$\mathcal{L}_{\mathbf{PLJ}}$-Kvaluation, we show that v' is **PLJ**-legal, since $\langle v', \Re^v_{\mathbf{PLJ}}\rangle$ is **PLJ**-legal. To see that $\langle v', \Re^v_{\mathbf{PLJ}}\rangle$ is π_{int}-legal, it suffices to note that for every $\varphi \in \mathcal{L}_{\mathbf{PLJ}}$, if $v'(w, \varphi) = \{\mathtt{t}\}$ then $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in \Re(\pi_{\mathsf{int}})[w]$. We claim that $\langle v', \Re^v_{\mathbf{PLJ}}\rangle$ is $\mathsf{R}_{\mathbf{PLJ}}$-legal. We demonstrate it here only for the rule ($\mathtt{t}$:$\neg\supset$). Thus we show that if $v'(w, \varphi_1) = \{\mathtt{t}\}$ and $v'(w, \neg\varphi_2) = \{\mathtt{t}\}$ then $v'(w, \neg(\varphi_1 \supset \varphi_2)) = \{\mathtt{t}\}$. Assume that $v'(w, \neg(\varphi_1 \supset \varphi_2)) = \{\mathtt{f}\}$. Our construction then ensures that $\neg(\varphi_1 \supset \varphi_2) \in Dom_v$, and $v(w, \neg(\varphi_1 \supset \varphi_2)) = \{\mathtt{f}\}$ as well. Since $\{\varphi_1, \neg\varphi_2\} \subseteq \downarrow^{nsub}[\neg(\varphi_1 \supset \varphi_2)]$ and Dom_v is closed under $nsub$, we have that $\{\varphi_1, \neg\varphi_2\} \subseteq Dom_v$. Since $\langle v', \Re^v_{\mathbf{PLJ}}\rangle$ is ($\mathtt{t}$:$\neg\supset$)-legal, either $v(w, \varphi_1) = \{\mathtt{f}\}$ or $v(w, \neg\varphi_2) = \{\mathtt{f}\}$. By our construction, $v'(w, \varphi_1) = \{\mathtt{f}\}$ or $v'(w, \neg\varphi_2) = \{\mathtt{f}\}$.

Example 6.3.14. Each of the four basic calculi **K**, **K4**, **KD**, and **S4** admits the semantic criterion given in Corollary 6.3.10 (and so they all enjoy strong cut-admissibility). To see this, let **G** be any one of these calculi, and v a cut-restricted strongly **G**-legal $\mathcal{L}_\Box$-Kvaluation. We recursively construct an instance v' of v. For every $w \in W_v$ and for every atomic formula p, $v'(w, p) = \{\mathtt{x}\}$ if $v(w, p) = \{\mathtt{x}\}$, and otherwise $v'(w, p) = \{\mathtt{t}\}$ (say). Now suppose that $v'(w, \varphi_1)$ and $v'(w, \varphi_2)$ were defined, define $v'(w, \varphi_1 \supset \varphi_2)$ as follows (similar definitions for the other classical connectives): if $v(w, \varphi_1 \supset \varphi_2) = \{\mathtt{x}\}$ then $v'(w, \varphi_1 \supset \varphi_2) = \{\mathtt{x}\}$, and otherwise $v'(w, \varphi_1 \supset \varphi_2) = \{\mathtt{t}\}$ iff either $v'(w, \varphi_1) = \{\mathtt{f}\}$ or $v'(w, \varphi_2) = \{\mathtt{t}\}$. In addition, $v'(w, \Box\varphi) = \{\mathtt{x}\}$ if $v(w, \Box\varphi) = \{\mathtt{x}\}$, and otherwise we set $v'(w, \Box\varphi) = \{\mathtt{t}\}$ iff $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^v_\pi[w]$ (where π is the context-relation in $\Pi_{\mathbf{G}}$, that is not π_0). Clearly, v' is an instance of v. Using the fact that v is a cut-restricted strongly **G**-legal $\mathcal{L}_\Box$-Kvaluation, it is easy to show that $\langle v', \Re^v_{\mathbf{G}}\rangle$ is **G**-legal.

Example 6.3.15. While **KB** and **S5** do not enjoy cut-admissibility (for **S5**, see Example 6.3.2), Corollary 6.3.7 can be used to show that they are still strongly *sub*-analytic. We demonstrate it here for **KB**. Let v be a normal strongly **KB**-legal partial $\mathcal{L}_\Box$-Kvaluation, whose domain is closed under subformulas. A construction of an instance v' of v is done exactly as in Example 6.3.14. We show that v' is indeed a **KB**-legal Kvaluation, as $\langle v', \Re^v_{\mathbf{KB}}\rangle$ is **KB**-legal. To see that $\langle v', \Re^v_{\mathbf{KB}}\rangle$ is π_{B}-legal, we show that $R^v_{\pi_{\mathsf{B}}} \subseteq R^{v'}_{\pi_{\mathsf{B}}}$. Suppose that $w_1 R^v_{\pi_{\mathsf{B}}} w_2$. Note that by Proposition 6.2.20 (Item 3), we have that $w_2 R^v_{\pi_{\mathsf{B}}} w_1$ (because of the structure of π_{B}). We prove that $w_1 R^{v'}_{\pi_{\mathsf{B}}} w_2$. Let α_1 and α_2 be labelled $\mathcal{L}_\Box$-formulas such that $\alpha_2 \bar{\pi}_{\mathsf{B}} \alpha_1$ and $v, w_2 \models \alpha_2$. The structure of π_{B} ensures that there exists some $\varphi \in \mathcal{L}_\Box$ such that either $\alpha_2 = \mathtt{f}{:}\varphi$ and $\alpha_1 = \mathtt{f}{:}\Box\varphi$, or $\alpha_2 = \mathtt{t}{:}\Box\varphi$ and $\alpha_1 = \mathtt{t}{:}\varphi$. If $\Box\varphi \in Dom_v$ then α_1 and α_2 are labelled Dom_v-formulas (since Dom_v is closed under subformulas). In this case, since $w_1 R^v_{\pi_{\mathsf{B}}} w_2$, we have that $v, w_1 \models \alpha_1$, and we are done. Otherwise, for every $w \in W_v$, $v'(w, \Box\varphi) = \{\mathtt{t}\}$ iff $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{B}}}[w]$. Now, if $\alpha_2 = \mathtt{f}{:}\varphi$ and $\alpha_1 = \mathtt{f}{:}\Box\varphi$, then $v, w_2 \models \alpha_2$ directly entails that $v, w_1 \models \alpha_1$. Otherwise, $\alpha_2 = \mathtt{t}{:}\Box\varphi$ and $\alpha_1 = \mathtt{t}{:}\varphi$. It follows that $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^v_{\pi_{\mathsf{B}}}[w_2]$. Since $w_2 R^v_{\pi_{\mathsf{B}}} w_1$, $v, w_1 \models \alpha_1$ in this case as well. Finally, we claim that

$\langle v', \Re^{v}_{\mathbf{KB}}\rangle$ is $\mathsf{R}_{\mathbf{KB}}$-legal. We show it here only for the rule (B). Following Example 6.2.10 (Item 1), we should prove that for every $w \in W_v$ and $\mathcal{L}_\Box$-formula φ: if $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^{v}_{\pi_\mathsf{B}}[w]$, then $v'(w, \Box\varphi) = \{\mathtt{t}\}$. Let $w \in W_v$, and $\varphi \in \mathcal{L}_\Box$. Suppose that $v'(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^{v}_{\pi_\mathsf{B}}[w]$. If $\Box\varphi \notin Dom_v$, then the construction of v' directly entails that $v'(w, \Box\varphi) = \{\mathtt{t}\}$. Otherwise, $\varphi \in Dom_v$ as well, and the construction of v' entails that $v(w', \varphi) = \{\mathtt{t}\}$ for every $w' \in R^{v}_{\pi_\mathsf{B}}[w]$. Since v is strongly **KB**-legal, $\langle v, \Re^{v}_{\mathbf{KB}}\rangle$ is (B)-legal. Thus we have that $\mathtt{t} \in v(w, \Box\varphi)$ (see Definition 6.2.9, Item 1). It then follows that $v'(w, \Box\varphi) = \{\mathtt{t}\}$.

Example 6.3.16. Using the semantic criterion of Corollary 6.3.10, it is easy to see that $\mathbf{GP}^Q$ enjoys strong cut-admissibility. The construction of a $\mathbf{GP}^Q$-legal instance for every cut-restricted strongly $\mathbf{GP}^Q$-legal $\mathcal{L}^{Q}_{\mathbf{GP}}$-Kvaluation is done as for **K** (see Example 6.3.14), with straightforward modifications for q `said` and q `implied`. In addition we replace the $\{\mathtt{f}, \mathtt{t}\}$ values assigned to formulas of the form $\varphi_1 \supset \varphi_2$ by the value assigned to φ_2 in each world.

Example 6.3.17. IS5 does not enjoy cut-admissibility, since the $\mathcal{L}^{\Box}_{\mathbf{LJ}}$-sequent

$$s = \{\mathtt{f}{:}\Box(\Box p_1 \vee p_2), \mathtt{t}{:}\Box p_1, \mathtt{t}{:}(\Box p_2 \supset \bot) \supset \bot\}$$

is provable, but not cut-free provable (see [78]). Using Theorem 6.2.12, one can semantically verify that there is no cut-free proof of s, by constructing an **IS5**-legal $\langle \mathcal{L}^{\Box}_{\mathbf{LJ}}, \emptyset, \mathcal{L}^{\Box}_{\mathbf{LJ}}\rangle$-legal $\mathcal{L}^{\Box}_{\mathbf{LJ}}$-Kvaluation which is not a model of it. In addition, the condition for strong *sub*-analyticity given in Corollary 6.3.7 does not hold for **IS5**. Since this condition is only proven to be sufficient, it does not mean that **IS5** is not strongly *sub*-analytic, and this question remains open.

Finally, Theorem 6.2.17 also naturally leads to the following semantic characterization of axiom-expansion.

Corollary 6.3.18. Let $\diamond \in \diamondsuit_{\mathcal{L}}$, and let $\varphi = \diamond(p_1, \dots, p_{ar(\diamond)})$. $\diamond$ admits axiom-expansion in a basic $\mathcal{L}$-calculus **G** iff every $\langle \mathcal{L}, \mathcal{L}, \{p_1, \dots, p_{ar(\diamond)}\}\rangle$-legal strongly **G**-legal $\mathcal{L}$-Kvaluation is also $\langle \mathcal{L}, \mathcal{L}, \{\varphi\}\rangle$-legal.

Proof. We prove one direction. The converse is similar. Assume that $\diamond$ admits axiom-expansion in **G**. By definition, $\vdash_{\mathbf{G}\restriction_\rho} \{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$ for $\rho = \langle \mathcal{L}, \mathcal{L}, \{p_1, \dots, p_{ar(\diamond)}\}\rangle$. Theorem 6.2.17 entails that every strongly $\mathbf{G}\restriction_\rho$-legal $\mathcal{L}$-Kvaluation is a model of $\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$. It follows that in every strongly $\mathbf{G}\restriction_\rho$-legal $\mathcal{L}$-Kvaluation v, $v(w, \varphi) \neq \emptyset$ for every $w \in W_v$. Thus, every strongly $\mathbf{G}\restriction_\rho$-legal $\mathcal{L}$-Kvaluation is $\langle \mathcal{L}, \mathcal{L}, \{\varphi\}\rangle$-legal. □

Example 6.3.19 (LJ). Using the criterion given in Corollary 6.3.18, it is straightforward to prove that every connective of $\mathcal{L}_{\mathbf{LJ}}$ admits axiom-expansion in **LJ**. We do it here for

$\supset$. Let v be an $\langle\mathcal{L},\mathcal{L},\{p_1,p_2\}\rangle$-legal strongly **LJ**-legal $\mathcal{L}_{\mathbf{LJ}}$-Kvaluation. We show that $v(w,p_1\supset p_2)\neq\emptyset$ for every $w\in W_v$, and so v is $\langle\mathcal{L},\mathcal{L},\{p_1\supset p_2\}\rangle$-legal. Suppose that $\mathtt{t}\notin v(w,p_1\supset p_2)$ for some $w\in W_v$. Since $\langle v,\Re^v_{\mathbf{LJ}}\rangle$ is ($\mathtt{f}$: $\supset$)-legal, $\mathtt{f}\notin v(w',p_1)$ and $\mathtt{t}\notin v(w',p_2)$ for some $w'\in\Re^v_{\mathbf{LJ}}(\pi_{\mathsf{int}})[w]$. Since v is $\langle\mathcal{L},\mathcal{L},\{p_1,p_2\}\rangle$-legal, $v(w',p_1)\neq\emptyset$ and $v(w',p_2)\neq\emptyset$. This entails that $v(w',p_1)=\{\mathtt{t}\}$ and $v(w',p_2)=\{\mathtt{f}\}$. Since $\langle v,\Re^v_{\mathbf{LJ}}\rangle$ is ($\mathtt{f}$: $\supset$)-legal and $\Re^v_{\mathbf{LJ}}(\pi_0)=Id_v$, we have that $\mathtt{f}\in v(w',p_1\supset p_2)$. Since $\langle v,\Re^v_{\mathbf{LJ}}\rangle$ is π_{int}-legal, $\mathtt{f}\in v(w,p_1\supset p_2)$ as well.

6.4 Soundness and Completeness Proofs

This section is devoted to prove Theorem 6.2.17. Theorems 6.2.12 and 6.2.16 are immediately obtained as special cases. Let $\mathbf{G}$ be a basic $\mathcal{L}$-calculus, and $\rho=\langle\mathcal{F},\mathcal{C},\mathcal{A}\rangle$ an $\mathcal{L}$-proof-specification. Clearly, to show that $\vdash_{\mathbf{G}\restriction\rho}=\vdash_{\mathcal{K}}$ for every set $\mathcal{K}$ of partial $\mathcal{L}$-Kvaluations satisfying $\mathcal{K}^*_{\mathbf{G}\restriction\rho}\subseteq\mathcal{K}\subseteq\mathcal{K}_{\mathbf{G}\restriction\rho}$, it suffices to prove the following:

Soundness $\vdash_{\mathbf{G}\restriction\rho}\subseteq\vdash_{\mathcal{K}_{\mathbf{G}\restriction\rho}}$. **Completeness** $\vdash_{\mathcal{K}^*_{\mathbf{G}\restriction\rho}}\subseteq\vdash_{\mathbf{G}\restriction\rho}$.

Soundness

For the soundness proof we use the following simple lemmas.

Lemma 6.4.1. Let v be a partial $\mathcal{L}$-Kvaluation, let $w\in W_v$, and let s_1 and s_2 be two $\mathcal{L}$-sequents. Then, $v,w\models s_1\cup s_2$ iff either $v,w\models s_1$ or $v,w\models s_2$.

Lemma 6.4.2. Let $\langle s,s'\rangle$ be a π-instance for some $\mathcal{L}$-context-relation π. Let v be partial $\mathcal{L}$-Kvaluation, and let $w\in W_v$. Suppose that $v,w'\models s$ for some $w'\in R^v_\pi[w]$. Then either $frm[s']\not\subseteq Dom_v$ or $v,w\models s'$.

Proof. Suppose that $frm[s']\subseteq Dom_v$, we show that $v,w\models s'$. Since $v,w'\models s$, we have $v,w'\models\alpha_2$ for some $\alpha_2\in s$. Since $\langle s,s'\rangle$ is a π-instance, there exists $\alpha_1\in s'$ such that $\alpha_2\bar{\pi}\alpha_1$. Note that $frm[\alpha_2]\in Dom_v$ (because $v,w'\models\alpha_2$) and $frm[\alpha_1]\in Dom_v$ (because $frm[s']\subseteq Dom_v$). Then since $wR^v_\pi w'$, $v,w\models\alpha_1$. It follows that $v,w\models s'$. □

Now, assume that $\mathcal{S}\vdash_{\mathbf{G}\restriction\rho} s_0$. Thus there exists a ρ-proof P in $\mathbf{G}$ of s_0 from $\mathcal{S}$. Let $\mathcal{K}=\mathcal{K}_{\mathbf{G}\restriction\rho}$. We prove that $\mathcal{S}\vdash_{\mathcal{K}} s_0$. Let $v\in\mathcal{K}$. Then, $\langle v,\Re\rangle$ is $\mathbf{G}$-legal for some $\langle\mathbf{G},W_v\rangle$-coupling $\Re$. Suppose that $v\models\mathcal{S}$. Using induction on the length of P, we show that $v\models s$ for every sequent s appearing in P. It then follows that $v\models s_0$. Note first that since v is ρ-legal, $Dom_v=\mathcal{F}$, and so every sequent in P is a Dom_v-sequent. Thus it suffices to prove that for every sequent s' appearing in P, we have $v,W_v\models s'$. This trivially holds for the sequents of $\mathcal{S}$ that appear in P. We show that the property of being true in W_v is preserved by applications of the rules of **G**. Consider such an application in P, and assume that $v, W_v \models s$ for every premise s of this application. We show that its conclusion is also true in W_v. Let $w \in W_v$.

1. Suppose that $c \cup \{\mathtt{x}{:}\varphi\}$ is derived using from c, using $(\mathtt{x}{:}weak)$ $(x \in \mathcal{L}_{\mathbf{2}})$. Since $v, w \models c$, Lemma 6.4.1 entails that $v, w \models c \cup \{\mathtt{x}{:}\varphi\}$.

2. Suppose that $\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$ is derived using (id). In this case, $\varphi \in \mathcal{A}$. Since v is ρ-legal, $v(w, \varphi) \neq \emptyset$. This easily implies that $v, w \models \{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$.

3. Suppose that $c_1 \cup c_2$ is derived from $c_1 \cup \{\mathtt{f}{:}\varphi\}$ and $c_2 \cup \{\mathtt{t}{:}\varphi\}$ using (cut). In this case, $\varphi \in \mathcal{C}$. Since v is ρ-legal, $v(w, \varphi) \neq \{\mathtt{f}, \mathtt{t}\}$. This easily implies that either $v, w \not\models \{\mathtt{f}{:}\varphi\}$ or $v, w \not\models \{\mathtt{t}{:}\varphi\}$. Since $v, w \models c_1 \cup \{\mathtt{f}{:}\varphi\}$, Lemma 6.4.1 entails that either $v, w \models c_1$ or $v, w \models \{\mathtt{f}{:}\varphi\}$. Similarly, either $v, w \models c_2$ or $v, w \models \{\mathtt{t}{:}\varphi\}$. This entails that either $v, w \models c_1$ or $v, w \models c_2$. Therefore Lemma 6.4.1 entails that $v, w \models c_1 \cup c_2$.

4. Suppose that $\sigma(s) \cup c'_1 \cup \ldots \cup c'_n$ is derived from $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n$ using a basic $\mathcal{L}$-rule $r = \langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / s'$ in $\mathsf{R}_{\mathbf{G}}$. Thus $\langle c_i, c'_i \rangle$ is a π_i-instance for every $1 \leq i \leq n$. Now, if $v, w \models c'_i$ for some $1 \leq i \leq n$, then by Lemma 6.4.1, $v, w \models \sigma(s) \cup c'_1 \cup \ldots \cup c'_n$, and we are done. Assume otherwise. We show that $v, \Re(\pi_i)[w] \models \sigma(s_i)$ for every $1 \leq i \leq n$. Let $1 \leq i \leq n$ and $w' \in \Re(\pi_i)[w]$. Since $\langle v, \Re \rangle$ is π_i-legal, we have that $w R^v_{\pi_i} w'$. Now, since $\langle c_i, c'_i \rangle$ is a π_i-instance, Lemma 6.4.2 entails that $v, w' \not\models c_i$. By Lemma 6.4.1 (since we assumed that $v, w' \models \sigma(s_i) \cup c_i$), $v, w' \models \sigma(s_i)$. Finally, we have $v, w \models \sigma(s)$ since $\langle v, \Re \rangle$ is r-legal and $frm[\sigma(\{s_1, \ldots, s_n, s\})] \subseteq Dom_v$. By Lemma 6.4.1, $v, w \models \sigma(s) \cup c'_1 \cup \ldots \cup c'_n$.

Completeness

Recall that by extended $\mathcal{L}$-sequent we mean a (possibly infinite) set of labelled $\mathcal{L}$-formulas (see Definition 3.3.6). For this completeness proof, we call an extended $\mathcal{L}$-sequent μ *provable* if $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s$ for some $\mathcal{L}$-sequent $s \subseteq \mu$. Otherwise, we say that μ is *unprovable*. In addition, we call an extended $\mathcal{L}$-sequent μ^* *maximal* if it satisfies the following conditions: (1) $frm[\mu^*] \subseteq \mathcal{F}$; (2) μ^* is unprovable; and (3) for every labelled $\mathcal{F}$-formula $\alpha \notin \mu^*$, $\{\alpha\} \cup \mu^*$ is provable. As in Section 3.4, it is straightforward to show that:

(**a**) For every unprovable extended $\mathcal{F}$-sequent μ, there is a maximal extended $\mathcal{L}$-sequent μ^* such that $\mu \subseteq \mu^*$.

Now, suppose that $\mathcal{S} \nvdash_{\mathbf{G}\restriction\rho} s_0$. We show that $\mathcal{S} \nvdash_{\mathcal{K}^*_{\mathbf{G}\restriction\rho}} s_0$. Let v be the partial $\mathcal{L}$-Kvaluation defined by: W_v is the set of all maximal extended $\mathcal{L}$-sequents; $Dom_v = \mathcal{F}$; and $v(\mu, \varphi) = \{\mathtt{x} \in \{\mathtt{f}, \mathtt{t}\} \mid \mathtt{x}{:}\varphi \notin \mu\}$ for every $\mu \in W_v$ and $\varphi \in \mathcal{F}$. We show that $v \in \mathcal{K}^*_{\mathbf{G}\restriction\rho}$, and that v is a model of $\mathcal{S}$ but not of s_0.

Note first that v is ρ-legal. By definition $Dom_v = \mathcal{F}$. To see that $v(\mu, \varphi) \neq \{\mathtt{f}, \mathtt{t}\}$ for every $\varphi \in \mathcal{C}$, it suffices to prove that if $\varphi \in \mathcal{C} \cap \mathcal{F}$ then $\mathtt{f}{:}\varphi \in \mu$ or $\mathtt{t}{:}\varphi \in \mu$. Assume by way of contradiction that $\mathtt{f}{:}\varphi \notin \mu$ and $\mathtt{t}{:}\varphi \notin \mu$ for some $\varphi \in \mathcal{C} \cap \mathcal{F}$. It follows that there exist $\mathcal{F}$-sequents $s_1, s_2 \subseteq \mu$ such that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s_1 \cup \{\mathtt{f}{:}\varphi\}$ and $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s_2 \cup \{\mathtt{t}{:}\varphi\}$. Since $\varphi \in \mathcal{C}$, a (legal) application of (cut) ensured that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s_1 \cup s_2$. But this contradicts the properties of μ. To see that $v(\mu, \varphi) \neq \emptyset$ for every $\varphi \in \mathcal{A}$, it suffices to prove that $\varphi \in \mathcal{A} \cap \mathcal{F}$ implies that $\mathtt{f}{:}\varphi \notin \mu$ or $\mathtt{t}{:}\varphi \notin \mu$. Note that if $\varphi \in \mathcal{A} \cap \mathcal{F}$ then $\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$ is a (legal) application of (id), and so $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} \{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$. Since μ is maximal, either $\mathtt{f}{:}\varphi \notin \mu$ or $\mathtt{t}{:}\varphi \notin \mu$.

Now, for every $\pi \in \Pi_{\mathbf{G}}$, let $R^{\mathcal{F}}_{\pi}$ denote the binary relation on W_v defined by: $\mu_1 R^{\mathcal{F}}_{\pi} \mu_2$ iff for every labelled $\mathcal{F}$-formulas α_1, α_2, if $\alpha_2 \bar{\pi} \alpha_1$ and $\alpha_2 \in \mu_2$ then $\alpha_1 \in \mu_1$. We claim that the following hold:

(**b**) For every labelled $\mathcal{F}$-formula α and $\mu \in W_v$: $v, \mu \models \alpha$ iff $\alpha \notin \mu$.

> *Proof.* Suppose that $\alpha = \mathtt{x}{:}\varphi$ where $\mathtt{x} \in \{\mathtt{f}, \mathtt{t}\}$. Then, $\mathtt{x} \in v(\mu, \varphi)$ iff $\alpha \notin \mu$. Equivalently, $v, \mu \models \alpha$ iff $\alpha \notin \mu$. □

(**c**) $R^{\mathcal{F}}_{\pi} = R^{v}_{\pi}$ for every $\pi \in \Pi_{\mathbf{G}}$, and $R^{\mathcal{F}}_{\pi_0} = Id_v$.

> *Proof.* For every context-relation π, $R^{\mathcal{F}}_{\pi} = R^{v}_{\pi}$ follows from (**b**). To see that $R^{\mathcal{F}}_{\pi_0} = Id_v$, note that $\alpha_2 \bar{\pi}_0 \alpha_1$ iff $\alpha_2 = \alpha_1$. Thus, $\mu_1 R^{\mathcal{F}}_{\pi_0} \mu_2$ iff for every labelled $\mathcal{F}$-formula α, $v, \mu_2 \models \alpha$ implies that $v, \mu_1 \models \alpha$. By (**b**), we obtain that $\mu_1 R^{\mathcal{F}}_{\pi_0} \mu_2$ iff $\mu_1 \subseteq \mu_2$. Therefore, obviously, $\mu R^{\mathcal{F}}_{\pi_0} \mu$ for every $\mu \in W_v$. For the converse, we show that if $\mu_1, \mu_2 \in W_v$ and $\mu_1 \subseteq \mu_2$, then $\mu_1 = \mu_2$. Assume (by way of contradiction) that $\mu_1 \subseteq \mu_2$ and there exists $\alpha \in \mu_2 \setminus \mu_1$. Since μ_1 is maximal, there exists an $\mathcal{F}$-sequent $s \subseteq \mu_1$ such that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s \cup \{\alpha\}$. But, $s \cup \{\alpha\} \subseteq \mu_2$, and this contradicts the fact that μ_2 is unprovable. □

(**d**) For every $\mathcal{F}$-sequent s and $\mu \in W_v$: $s \not\subseteq \mu$ iff $v, \mu \models s$.

> *Proof.* Easily follows from (**b**). □

(**e**) For every $\mathcal{F}$-sequent s and $\mu \in W_v$: if there exists an $\mathcal{F}$-sequent $s' \subseteq \mu$ such that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s \cup s'$, then $v, \mu \models s$.

Proof. Assume that there exists a sequent $s' \subseteq \mu$ such that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s \cup s'$. Since μ is unprovable, $s \not\subseteq \mu$. Therefore, (**d**) entails that $v, \mu \models s$. □

(**f**) For every $\mathcal{F}$-sequent s, $\mu \in W_v$, and $\pi \in \Pi_{\mathbf{G}}$: if $v, R^{\mathcal{F}}_{\pi}[\mu] \models s$, then there exists a π-instance $\langle c, c' \rangle$ such that $c' \subseteq \mu$ and $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s \cup c$.

Proof. Assume that there is no π-instance $\langle c, c' \rangle$ such that $c' \subseteq \mu$ and $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s \cup c$. We show that $v, R^{\mathcal{F}}_{\pi}[\mu] \not\models s$. Let $\mu^* = \{\alpha \mid \mathit{frm}[\alpha] \in \mathcal{F} \text{ and } \exists \beta \in \mu. \alpha\bar{\pi}\beta\}$. Since ($\mathtt{f}$:*weak*) and ($\mathtt{t}$:*weak*) are available, our assumption implies that $s \cup \mu^*$ is unprovable. Since $\mathit{frm}[s \cup \mu^*] \subseteq \mathcal{F}$, (**a**) entails that there exists maximal extended $\mathcal{L}$-sequent μ', such that $s \cup \mu^* \subseteq \mu'$. (**d**) entails that $v, \mu' \not\models s$. By definition, $\mu R^{\mathcal{F}}_{\pi} \mu'$. Hence, $v, R^{\mathcal{F}}_{\pi}[\mu] \not\models s$. □

(**g**) For every $\mathcal{L}$-sequent s: $v \models s$ iff $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s$.

Proof. Note first that if $\mathit{frm}[s] \not\subseteq \mathcal{F}$, then by definition, $v \not\models s$ and $\mathcal{S} \not\vdash_{\mathbf{G}\restriction\rho} s$. Assume now that $\mathit{frm}[s] \subseteq \mathcal{F}$. One direction easily follows from (**e**). For the converse, assume that $\mathcal{S} \not\vdash_{\mathbf{G}\restriction\rho} s$. We show that $v \not\models s$. Because of the presence of ($\mathtt{f}$:*weak*) and ($\mathtt{t}$:*weak*), there does not exist $s' \subseteq s$ such that $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s'$. By (**a**), there exists $\mu \in W_v$, such that $s \subseteq \mu$. (**d**) entails that $v, \mu \not\models s$. Hence, $v \not\models s$. □

Next, we show that v is strongly $\mathbf{G}$-legal. Thus we prove that $\langle v, \Re^v_{\mathbf{G}} \rangle$ is $\mathsf{R}_{\mathbf{G}}$-legal. Let $r = \langle s_1, \pi_1 \rangle, \dots, \langle s_n, \pi_n \rangle / s$ be a rule in $\mathsf{R}_{\mathbf{G}}$. Let $\mu \in W_v$, and let σ be an $\mathcal{L}$-substitution. Suppose that $\mathit{frm}[\sigma(\{s_1, \dots, s_n, s\})] \subseteq \mathcal{F}$, and that $v, \Re^v_{\mathbf{G}}(\pi_i)[\mu] \models \sigma(s_i)$ for every $1 \leq i \leq n$. We prove that $v, \mu \models \sigma(s)$. By (**c**), $\Re^v_{\mathbf{G}}(\pi_i) = R^{\mathcal{F}}_{\pi_i}$ for every $1 \leq i \leq n$. Thus (**f**) entails that for every $1 \leq i \leq n$, there exists a π-instance $\langle c_i, c'_i \rangle$ such that $c'_i \subseteq \mu$ and $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} \sigma(s_i) \cup c_i$. Now we can use these proofs, and the rule r to obtain $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} \sigma(s) \cup c'_1 \cup \dots \cup c'_n$, where $c'_1 \cup \dots \cup c'_n \subseteq \mu$. (**e**) entails that $v, \mu \models \sigma(s)$.

It follows that $v \in \mathcal{K}^*_{\mathbf{G}\restriction\rho}$ (note that by (**c**), v is differentiated). Finally, we show that $v \models \mathcal{S}$ but $v \not\models s_0$. Since obviously $\mathcal{S} \vdash_{\mathbf{G}\restriction\rho} s'$ for every $\mathcal{F}$-sequent $s' \in \mathcal{S}$, (**g**) implies that $v \models s'$ for every such s'. Since $\mathcal{S} \not\vdash_{\mathbf{G}\restriction\rho} s_0$, (**g**) also implies that $v \not\models s_0$.

Chapter 7

Canonical Gödel Hypersequent Calculi

Gödel logic, known also as Gödel-Dummett logic, is perhaps the most prominent intermediate logic, and one of the three fundamental fuzzy logics [63]. It was introduced in [48] both semantically, by an infinite-valued matrix, and syntactically, with a simple axiomatization, namely the extension of (an axiomatization of) intuitionistic logic with the axiom scheme $(\varphi_1 \supset \varphi_2) \vee (\varphi_2 \supset \varphi_1)$ of linearity. The quest for a (cut-free) Gentzen-type formulation for (propositional) Gödel logic began later, and several calculi were proposed (see, e.g., [86, 46, 2, 50, 25, 51]). One of the most important cut-free calculi for Gödel logic is the calculus **HG**, introduced in [4] (see also [22] and [76]). **HG** is relatively simple, especially due to the fact that its logical rules are practically the same rules as in **LJ**, the well-known single-conclusion sequent calculus for intuitionistic logic. This is obtained by working in the slightly richer framework of (single-conclusion) *hypersequents*, that provides a natural generalization of Gentzen's original sequents framework.[1] The structural part of **HG** consists of all the usual structural rules, both on the sequent level (internal) and on the hypersequent level (external). In addition, it includes the *communication rule* that allows "exchange of information" between two hypersequents [7], and, needless to say, the identity axiom and the (admissible) cut rule.

In this chapter we introduce and study the family of *canonical Gödel hypersequent calculi* of which **HG** is the prototype example. The idea, just like in canonical sequent calculi (following [17], see Chapter 4), is to allow any "ideal" logical rules for introducing the logical connectives. Thus we define *canonical single-conclusion hypersequent rules*, and in turn, canonical hypersequent Gödel calculi are (two-sided, single-conclusion) hypersequent calculi that include all standard structural rules, the cut rule, the identity

[1] Note that hypersequents are currently the main proof-theoretic framework for fuzzy logics [76].

axiom, the communication rule, and an arbitrary finite set of canonical single-conclusion hypersequent rules.[2] Then, as in the previous chapters, we study canonical Gödel calculi from a *semantic* point of view. Note that we could similarly study *multiple-conclusion* hypersequent calculi in which some of the rules do not allow right context formulas (as the right rules for implication in the multiple-conclusion calculus for intuitionistic logic). We choose to work in the single-conclusion framework both in order to demonstrate the applicability of our methods in this framework, and because the single-conclusion framework is more common when it comes to Gödel logic.

First and foremost, our study includes a general method to obtain a sound and complete semantics for *every* canonical Gödel calculus. The semantics is based on totally ordered algebraic structures with (possibly) non-deterministic interpretations of the different connectives. Here we also consider the semantic effect of the cut rule and the identity axiom, and obtain semantics for canonical Gödel calculi in which these rules are restricted to apply only on some given set of formulas. This semantics is then used to characterize proof-theoretic properties of canonical Gödel calculi, and particularly to identify the "good" ones, namely those that enjoy (strong) cut-admissibility. In fact, we show that the simple *coherence* criterion of [17, 14] characterizes strong cut-admissibility in canonical Gödel calculi as well.

Publications Related to this Chapter

The material in this chapter was included in [71, 69]. Note that canonical *single-conclusion* (two-sided) *sequent* calculi were introduced and studied in the author's M.Sc. thesis (see also [14]).

7.1 Preliminaries

As in the previous chapter, in this chapter we only consider two-sided sequents, refer to them as $\mathcal{L}$-sequents, and employ the standard notation and abbreviations (see Notation 6.1.1). In turn, hypersequents are defined as follows:

Definition 7.1.1. An $\mathcal{L}$-*hypersequent* is a finite set of $\mathcal{L}$-sequents. Given a set $\mathcal{F} \subseteq \mathcal{L}$, an $\mathcal{F}$-hypersequent is a hypersequent consisting solely of $\mathcal{F}$-sequents.

[2] In fully-structural single-conclusion sequent calculi weakening on the right side can only be applied on a sequent whose right side is empty. Similarly, in fully-structural single-conclusion hypersequent calculi right internal weakening can only be applied on a hypersequent including a component with an empty right-hand side.

Notation 7.1.2. We usually use H as a metavariable for $\mathcal{L}$-hypersequents. We usually denote an $\mathcal{L}$-hypersequent $\{s_1, \dots, s_n\}$ by $s_1 \mid \dots \mid s_n$, and employ the standard abbreviations, e.g. $H_1 \mid H_2$ instead of $H_1 \cup H_2$, and $H \mid s$ instead of $H \cup \{s\}$.

Remark 7.1.3. As we did for sequents, we defined hypersequents using *sets*. This immediately entails that the external exchange rule, the external contraction rule and the external expansion rule (the converse of contraction) are built-in in all hypersequent calculi that we study.

Unlike previous chapters, we study here *single-conclusion* calculi:

Definition 7.1.4. An $\mathcal{L}$-sequent s satisfying $|\{\varphi \in \mathcal{L} \mid \mathtt{t}{:}\varphi \in s\}| \leq 1$ is called a *single-conclusion $\mathcal{L}$-sequent*. An $\mathcal{L}$-sequent s is called *negative* if $\{\varphi \in \mathcal{L} \mid \mathtt{t}{:}\varphi \in s\} = \emptyset$. A *single-conclusion $\mathcal{L}$-hypersequent* is an $\mathcal{L}$-hypersequent that consists solely of single-conclusion $\mathcal{L}$-sequents.

We usually use the metavariable E and F for singleton or empty sets of formulas (to represent the "right-side" of a single-conclusion sequent). Since in this chapter we discuss only the single-conclusion framework, we shall omit the prefix "single-conclusion" and refer to single-conclusion $\mathcal{L}$-(hyper)sequents simply as $\mathcal{L}$-(hyper)sequents.

7.2 Canonical Gödel Calculi

As defined below, all canonical Gödel calculi include the external and internal weakening rules (($\mathtt{f}$:*weak*) and ($\mathtt{t}$:*weak*)), and the rules (*com*), (*cut*) and (*id*):[3]

External Weakening This rule allows to infer $H \mid s$ from H for every $\mathcal{L}$-hypersequent H and $\mathcal{L}$-sequent s.

($\mathtt{f}$:*weak*) This rule allows to infer $H \mid s \cup \{\mathtt{f}{:}\varphi\}$ from $H \mid s$ for every $\mathcal{L}$-hypersequent H, $\mathcal{L}$-sequent s, and $\mathcal{L}$-formula φ.

($\mathtt{t}$:*weak*) This rule allows to infer $H \mid s \cup \{\mathtt{t}{:}\varphi\}$ from $H \mid s$ for every $\mathcal{L}$-hypersequent H, negative $\mathcal{L}$-sequent s, and $\mathcal{L}$-formula φ.

(*com*) This rule allows to infer $H \mid s_1 \cup c_2 \mid s_2 \cup c_1$ from $H \mid s_1 \cup c_1$ and $H \mid s_2 \cup c_2$ for every $\mathcal{L}$-hypersequent H, $\mathcal{L}$-sequents s_1 and s_2, and negative $\mathcal{L}$-sequents c_1 and c_2. In the more usual notation, applications of (*com*) have the form:

$$\frac{H \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \qquad H \mid \Gamma_2, \Gamma_1' \Rightarrow E_2}{H \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}$$

[3]We use the names ($\mathtt{f}$:*weak*), ($\mathtt{t}$:*weak*), (*cut*) and (*id*) in this context as well, but strictly speaking these are not the same rules that were defined in Chapter 2, but their hypersequential versions.

(*cut*) This rule allows to infer $H \mid s_1 \cup s_2$ from $H \mid s_1 \cup \{\mathtt{f}{:}\varphi\}$ and $H \mid s_2 \cup \{\mathtt{t}{:}\varphi\}$ for every $\mathcal{L}$-hypersequent H, $\mathcal{L}$-sequent s_1, and negative $\mathcal{L}$-sequent s_2.

(*id*) This rule provides all axioms of the form $\{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}\}$ for every $\mathcal{L}$-formula φ.

Unlike the structural rules, the logical rules of canonical Gödel calculi are not predefined, and they vary according to the concrete language of the calculus. Next we define the general form of the allowed logical rules. We first define right-introduction rules and their applications, and then deal with left-introduction rules.

Definition 7.2.1. A *canonical right single-conclusion hypersequent $\mathcal{L}$-rule* is a pair of the form $\mathcal{S}/\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$, where $\diamond \in \Diamond_{\mathcal{L}}$ and $\mathcal{S}$ is a finite set of $\{p_1, \dots, p_{ar(\diamond)}\}$-sequents. The elements of $\mathcal{S}$ are called the *premises* of the rule, and $\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ is called the *conclusion* of the rule. An *application* of $\{s_1, \dots, s_n\}/\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ is any inference step of the following form:

$$\frac{H \mid \sigma(s_1) \cup c \quad \dots \quad H \mid \sigma(s_n) \cup c}{H \mid \{\mathtt{t}{:}\sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))\} \cup c}$$

where σ is an $\mathcal{L}$-substitution, c is a negative $\mathcal{L}$-sequent (called *context sequent*), and H is an $\mathcal{L}$-hypersequent (called *context hypersequent*). $H \mid \sigma(s_1) \cup c, \dots, H \mid \sigma(s_n) \cup c$ are called the *premises* of the application, while $H \mid \{\mathtt{t}{:}\sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))\} \cup c$ is called the *conclusion* of the application.

Example 7.2.2. Suppose that $\wedge \in \Diamond^2_{\mathcal{L}}$, and consider the following canonical right single-conclusion hypersequent $\mathcal{L}$-rule:

$$\{\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \wedge p_2\}$$

Applications of this rule have the form:

$$\frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \wedge \varphi_2}$$

Obviously, in applications of canonical right single-conclusion hypersequent rules, we can not allow non-negative context sequents (otherwise there will not be "enough space" for the conclusion). The following definition of left rules is slightly more complicated, since a non-negative context sequent may be added to the negative premises. To have a general notion here, we allow also cases in which some negative premises disallow non-negative context (just like the negative premises of the right rules). Hence, the set of premises of canonical left rules is divided into two different sets: premises that disallow non-negative context (including all non-negative premises), and premises that allow non-negative context (of-course all of them are negative).[4]

[4] A similar division of the premises was used in the definition of canonical left single-conclusion *sequent* rules in [14].

Definition 7.2.3. A *canonical left single-conclusion hypersequent $\mathcal{L}$-rule* is a triple of the form $\mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:}\diamond(p_1, \dots, p_{ar(\diamond)})\}$, where $\diamond \in \Diamond_{\mathcal{L}}$, $\mathcal{S}_1$ is a finite set of $\{p_1, \dots, p_{ar(\diamond)}\}$-sequents, and $\mathcal{S}_2$ is a finite set of negative $\{p_1, \dots, p_{ar(\diamond)}\}$-sequents. The elements of $\mathcal{S}_1 \cup \mathcal{S}_2$ are called the *premises* of the rule, and $\{\mathtt{f}{:}\diamond(p_1, \dots, p_{ar(\diamond)})\}$ is called the *conclusion* of the rule. An *application* of $\{s_1, \dots, s_n\}, \{s'_1, \dots, s'_m\}/\{\mathtt{f}{:}\diamond(p_1, \dots, p_{ar(\diamond)})\}$ is any inference step of the following form:

$$\frac{H \mid \sigma(s_1) \cup c \quad \dots \quad H \mid \sigma(s_n) \cup c \qquad H \mid \sigma(s'_1) \cup c' \quad \dots \quad H \mid \sigma(s'_m) \cup c'}{H \mid \{\mathtt{f}{:}\sigma \diamond (p_1, \dots, p_{ar(\diamond)}))\} \cup c \cup c'}$$

where σ is an $\mathcal{L}$-substitution, c is a negative $\mathcal{L}$-sequent, and c' is an $\mathcal{L}$-sequent. The *premises*, *conclusion*, *context sequents*, and *context hypersequent* are defined exactly as in Definition 7.2.1.

Example 7.2.4. Suppose that $\star \in \Diamond^1_{\mathcal{L}}$, and consider the following canonical left single-conclusion hypersequent $\mathcal{L}$-rules:

$$\emptyset, \{\{\mathtt{f}{:}p_1\}\}/\{\mathtt{f}{:}\star p_1\} \qquad\qquad \{\{\mathtt{f}{:}p_1\}\}, \emptyset/\{\mathtt{f}{:}\star p_1\}$$

Applications of these rules have respectively the forms:

$$\frac{H \mid \Gamma, \varphi \Rightarrow}{H \mid \Gamma, \star\varphi \Rightarrow} \qquad\qquad \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, \star\varphi \Rightarrow E}$$

In this chapter we shall refer to "canonical right (left) single-conclusion hypersequent $\mathcal{L}$-rules" simply as "canonical right (left) $\mathcal{L}$-rules". By "canonical $\mathcal{L}$-rules" we mean either canonical right $\mathcal{L}$-rules or canonical left $\mathcal{L}$-rules. In addition, we say that a canonical $\mathcal{L}$-rule r is a rule *for* $\diamond$ if $\diamond$ is the connective that occurs in the conclusion of r.

Remark 7.2.5. Since both internal and external weakening rules are present in every hypersequent calculus we study, it is always possible to incorporate weakenings in the applications of the rules. Thus for example, we could have defined an application of a canonical right $\mathcal{L}$-rule as an inference step deriving $H \mid \sigma(s) \cup c$ from the $\mathcal{L}$-hypersequents $H_i \mid \sigma(s_i) \cup c_i$ for every $1 \leq i \leq n$, where $H, H_1, \dots, H_n$ are $\mathcal{L}$-hypersequents such that $H_1 \cup \dots \cup H_n \subseteq H$, $c, c_1, \dots, c_n$ are negative $\mathcal{L}$-sequents such that $c_1 \cup \dots \cup c_n \subseteq c$, and σ is an $\mathcal{L}$-substitution. A similar definition is possible for the canonical left rules. Henceforth, we may use freely this kind of applications (which formally might involve additional applications of the weakening rules).

In Table 7.1, we present all logical rules of the hypersequent system **HG** for the standard propositional Gödel logic (see [22]) as canonical rules.[5] It is also possible to introduce new connectives using canonical rules:

[5]By saying that **HG** is a system for propositional Gödel logic, we mean that $\{\{\{\mathtt{t}{:}\psi\}\} \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{HG}} \{\{\mathtt{t}{:}\varphi\}\}$ iff $\mathcal{T} \Vdash \varphi$ in propositional Gödel logic.

Canonical Rule	Application scheme
$\emptyset, \emptyset / \{\mathtt{f}{:}\ \bot\}$	$\dfrac{}{H \mid \Gamma, \bot \Rightarrow E}$
$\{\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}\} / \{\mathtt{t}{:}p_1 \wedge p_2\}$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \wedge \varphi_2}$
$\emptyset, \{\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}\} / \{\mathtt{f}{:}p_1 \wedge p_2\}$	$\dfrac{H \mid \Gamma, \varphi_1, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E}$
$\{\{\mathtt{t}{:}p_1\}\} / \{\mathtt{t}{:}p_1 \vee p_2\}$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_1}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}$
$\{\{\mathtt{t}{:}p_2\}\} / \{\mathtt{t}{:}p_1 \vee p_2\}$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}$
$\emptyset, \{\{\mathtt{f}{:}p_1\}, \{\mathtt{f}{:}p_2\}\} / \{\mathtt{f}{:}p_1 \vee p_2\}$	$\dfrac{H \mid \Gamma, \varphi_1 \Rightarrow E \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \vee \varphi_2 \Rightarrow E}$
$\{\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}\} / \{\mathtt{t}{:}p_1 \supset p_2\}$	$\dfrac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2}$
$\{\{\mathtt{t}{:}p_1\}\}, \{\{\mathtt{f}{:}p_2\}\} / \{\mathtt{f}{:}p_1 \supset p_2\}$	$\dfrac{H \mid \Gamma_1 \Rightarrow \varphi_1 \quad H \mid \Gamma_2, \varphi_2 \Rightarrow E}{H \mid \Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow E}$

Table 7.1: The Logical Rules of **HG**

Example 7.2.6. A primal-implication connective (see [62]) can be introduced with the following two rules:

$$\{\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \rightsquigarrow p_2\} \qquad \{\{\mathtt{t}{:}p_1\}\},\{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \rightsquigarrow p_2\}$$

Applications of the left rule are like those of the left rule of implication in **HG**, while applications of the right rule allow us to infer a hypersequent of the form $H \mid \Gamma \Rightarrow \varphi_1 \rightsquigarrow \varphi_2$ from a hypersequent of the form $H \mid \Gamma \Rightarrow \varphi_2$.

Example 7.2.7. It is possible to combine the usual right rule for conjunction with the usual left rule for disjunction, and introduce a new binary connective $\between$ with the following rules:

$$\{\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \between p_2\} \qquad \emptyset,\{\{\mathtt{f}{:}p_1\},\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \between p_2\}$$

Applications of the right rule are like those of the right rule of conjunction in **HG**, while applications of the left rule are like those of the left rule of disjunction in **HG** (see Table 7.1).

We can now define *canonical Gödel calculi*:

Definition 7.2.8. A *canonical Gödel $\mathcal{L}$-calculus* consists of all structural rules listed above (see Page 106), and any finite set of canonical $\mathcal{L}$-rules. The notion of a *proof* in a canonical Gödel $\mathcal{L}$-calculus $\mathbf{G}$ of an $\mathcal{L}$-hypersequent H from a set $\mathcal{H}$ of $\mathcal{L}$-hypersequents is defined as usual (see Definition 2.2.14). We write $\mathcal{H} \vdash_{\mathbf{G}} H$ to denote the existence of such a proof.

To speak about restricted proofs, as needed in order to define and characterize proof-theoretic, we will consider *proof-specifications*. These are defined exactly as in Chapter 6 (Definition 6.1.17), with the obvious modifications in the definition of a ρ-proof (reflecting the transition from sequents to hypersequents). We shall also employ the same notation and write $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$ if there exists a ρ-proof in $\mathbf{G}$ of H from $\mathcal{H}$, for a given $\mathcal{L}$-proof-specification ρ. For canonical Gödel calculi, we have the following:

Proposition 7.2.9. Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, and let $\rho = \langle \mathcal{L}, \mathcal{C}, \mathcal{A} \rangle$ be an $\mathcal{L}$-proof-specification. If $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$, then $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho'} H$ for $\rho' = \langle sub[\mathcal{C}] \cup sub[H], \mathcal{C}, \mathcal{A} \rangle$.

Proof. The claim is proved by usual induction on the length of the proof in $\mathbf{G}$. Note that all rules in canonical Gödel $\mathcal{L}$-calculi except for cut have the "local subformula property" (i.e., for each rule, the premises of its applications consist only of formulas occurring as subformulas of the corresponding conclusions). □

The proof-theoretic properties of (strong) *sub*-analyticity, (strong) cut-admissibility and axiom-expansion in canonical Gödel calculi are also defined as for basic calculi (Definition 6.1.18) with obvious modifications. From the last proposition, it easily follows that if a canonical Gödel $\mathcal{L}$-calculus $\mathbf{G}$ enjoys strong cut-admissibility (i.e. $\mathcal{H} \vdash_{\mathbf{G}} H$ implies that $\mathcal{H} \vdash_{\mathbf{G}\restriction_{\rho}} H$ for $\rho = \langle \mathcal{L}, frm[\mathcal{H}], \mathcal{L} \rangle$), then it is strongly *sub*-analytic (i.e. $\mathcal{H} \vdash_{\mathbf{G}} H$ implies that $\mathcal{H} \vdash_{\mathbf{G}\restriction_{\rho}} H$ for $\rho = \langle sub[\mathcal{H} \cup \{H\}], \mathcal{L}, \mathcal{L} \rangle$). In the sequel, we show that these two properties are actually equivalent for canonical Gödel calculi.

We end this section with two propositions that will turn out to be useful in connection with the proof of Theorem 7.3.17 below.

Proposition 7.2.10 (Generalized Communication)**.** Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, and let $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ be an $\mathcal{L}$-proof-specification. For all $\mathcal{F}$-hypersequents H_1, H_2, integers $n, m \geq 0$, $n+m$ $\mathcal{F}$-sequents $s_1, \dots, s_n, s'_1, \dots, s'_m$, and two negative $\mathcal{F}$-sequents c, c', the $\mathcal{L}$-hypersequent $H_1 \mid H_2 \mid s_1 \cup c' \mid \dots \mid s_n \cup c' \mid s'_1 \cup c \mid \dots \mid s'_m \cup c$ has a ρ-proof in $\mathbf{G}$ from the $\mathcal{L}$-hypersequents $H_1 \mid s_1 \cup c \mid \dots \mid s_n \cup c$ and $H_2 \mid s'_1 \cup c' \mid \dots \mid s'_m \cup c'$.

Proof. We prove this by induction on $n+m$. First, when $n = 0$ or $m = 0$, the claim follows by applying external weakening. Assume that $n, m > 0$, $n + m = l$ and that the claim holds for every n, m such that $n + m < l$. By the induction hypothesis, the following two hypersequents have a ρ-proof in **HIF** from $H_1 \mid s_1 \cup c \mid \dots \mid s_n \cup c$ and $H_2 \mid s'_1 \cup c' \mid \dots \mid s'_m \cup c'$:

$$H_1 \mid s_n \cup c \mid H_2 \mid s_1 \cup c' \mid \dots \mid s_{n-1} \cup c' \mid s'_1 \cup c \mid \dots \mid s'_m \cup c$$
$$H_1 \mid H_2 \mid s'_m \cup c' \mid s_1 \cup c' \mid \dots \mid s_{n-1} \cup c' \mid s'_1 \cup c \mid \dots \mid s'_{m-1} \cup c$$

An application of (*com*) on these two hypersequents provides the desired result. □

Proposition 7.2.11. Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ an $\mathcal{L}$-proof-specification, and $r = \mathcal{S}_1, \mathcal{S}_2 / \{\mathtt{f}{:}\varphi\}$ a canonical left $\mathcal{L}$-rule of $\mathbf{G}$, with $|\mathcal{S}_2| > 0$. Let σ be an $\mathcal{L}$-substitution such that $frm[\sigma(\mathcal{S}_1 \cup \mathcal{S}_2)] \cup \{\sigma(\varphi)\} \subseteq \mathcal{F}$. Let $\mathcal{S}_2 = \{s_1, \dots, s_m\}$ and $s_0 = \{\mathtt{f}{:}\sigma(\varphi)\}$. Let c be a negative $\mathcal{F}$-sequent and H an $\mathcal{F}$-hypersequent. Denote by $\mathcal{H}_1$ the set $\{H \mid c \cup \sigma(s)\}_{s \in \mathcal{S}_1}$. Then, for every $n_1, \dots, n_m \geq 0$, $\mathcal{F}$-hypersequent H' such that $H \subseteq H'$, and $n_1 + \dots + n_m$ $\mathcal{F}$-sequents $s^1_1, \dots, s^1_{n_1}, \dots, s^m_1, \dots, s^m_{n_m}$,

$$H' \mid c \cup s^1_1 \cup s_0 \mid \dots \mid c \cup s^1_{n_1} \cup s_0 \mid \dots \mid c \cup s^m_1 \cup s_0 \mid \dots \mid c \cup s^m_{n_m} \cup s_0$$

has a ρ-proof in $\mathbf{G}$ from

$$\mathcal{H}_1 \cup \{H' \mid s^i_1 \cup \sigma(s_i) \mid \dots \mid s^i_{n_i} \cup \sigma(s_i)\}_{1 \leq i \leq m}.$$

Proof. First, if $n_i = 0$ for some $1 \leq i \leq m$, the claim follows by applying external weakening on H'. Next, we prove the claim for the case that $n_1 = n_2 = \dots = n_m = 1$, by induction on the size S of the set $\{s^1_1, \dots, s^m_1\}$. If $S = 1$, then one application of r suffices. Now let $S \geq 2$, and assume that the claim holds for sets of size $S - 1$. Let $s^1_1, \dots, s^m_1$ be $\mathcal{F}$-sequents sets such that $|\{s^1_1, \dots, s^m_1\}| = S$, and let H' be some $\mathcal{F}$-hypersequent such that $H \subseteq H'$. Let $G_0 = H' \mid c \cup s^1_1 \cup s_0 \mid \dots \mid c \cup s^m_1 \cup s_0$, and for every $1 \leq i \leq m$, $G_i = H' \mid s^i_1 \cup \sigma(s_i)$. Let i_1, i_2 be two indices such that $s^{i_1}_1 \neq s^{i_2}_1$, and let $I_1 = \{1 \leq i \leq m \mid s^i_1 = s^{i_1}_1\}$ and

$I_2 = \{1 \leq i \leq m \mid s_1^i = s_1^{i_2}\}$. For every $j_1 \in I_1$ and $j_2 \in I_2$, we have that (using one application of (com)):

$$\{G_{j_1}, G_{j_2}\} \vdash_{\mathbf{G}\restriction_\rho} H' \mid s_1^{i_1} \cup \sigma(s_{j_2}) \mid s_1^{i_2} \cup \sigma(s_{j_1}).$$

For every $j_1 \in I_1$, the induction hypothesis and the availability of external weakening entail that $G_0 \mid s_1^{i_2} \cup \sigma(s_{j_1})$ has a ρ-proof in $\mathbf{G}$ from

$$\mathcal{H}_1 \cup \{G_j\}_{j \notin I_2} \cup \{H' \mid s_1^{i_1} \cup \sigma(s_{j_2}) \mid s_1^{i_2} \cup \sigma(s_{j_1})\}_{j_2 \in I_2}.$$

The induction hypothesis and the availability of external weakening again imply that G_0 has a ρ-proof in $\mathbf{G}$ from $\mathcal{H}_1 \cup \{G_j\}_{j \notin I_1} \cup \{G_0 \mid s_1^{i_2} \cup \sigma(s_{j_1})\}_{j_1 \in I_1}$. Together, we have $\mathcal{H}_1 \cup \{G_i\}_{1 \leq i \leq m} \vdash_{\mathbf{G}\restriction_\rho} G_0$.

Next we prove the claim for any $n_1, \dots, n_m \geq 1$ by induction on $n_1 + \dots + n_m$. Assume that $n_1 + \dots + n_m = l$ and that the claim holds for every $n_1, \dots, n_m$ such that $n_1 + \dots + n_m < l$. Let H' be an $\mathcal{F}$-hypersequent that extends H, and let $s_1^1, \dots, s_{n_1}^1, \dots, s_1^m, \dots, s_{n_m}^m$ be $\mathcal{F}$-sequents. Let G_0 denote the hypersequent

$$H' \mid c \cup s_1^1 \cup s_0 \mid \dots \mid c \cup s_{n_1}^1 \cup s_0 \mid \dots \mid c \cup s_1^m \cup s_0 \mid \dots \mid c \cup s_{n_m}^m \cup s_0,$$

and $\mathcal{H} = \{H' \mid s_1^i \cup \sigma(s_i) \mid \dots \mid s_{n_i}^i \cup \sigma(s_i)\}_{1 \leq i \leq m}$. For every $1 \leq i \leq m$, the induction hypothesis and the availability of external weakening entail that $\mathcal{H}_1 \cup \mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} G_0 \mid s_1^i \cup \sigma(s_i)$. By the proof for the case $n_1 = n_2 = \dots = n_m = 1$, we have that

$$\mathcal{H}_1 \cup \{G_0 \mid s_1^i \cup \sigma(s_i)\}_{1 \leq i \leq m} \vdash_{\mathbf{G}\restriction_\rho} G_0. \qquad \square$$

Example 7.2.12. Suppose that $\mathbf{G}$ includes the left rule $\emptyset, \{\{\mathtt{f}{:}p_1\}, \{\mathtt{f}{:}p_2\}\} / \{\mathtt{f}{:}p_1 \between p_2\}$ (see Example 7.2.7). By Proposition 7.2.11, the following rule (given by a scheme) is cut-free derivable in $\mathbf{G}$:

$$\frac{H \mid \Gamma_1^1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_{n_1}^1, \varphi_1 \Rightarrow E_{n_1} \qquad H \mid \Gamma_1^2, \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_{n_2}^2, \varphi_2 \Rightarrow F_{n_2}}{H \mid \Gamma_1^1, \varphi_1 \between \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_{n_1}^1, \varphi_1 \between \varphi_2 \Rightarrow E_{n_1} \mid \Gamma_1^2, \varphi_1 \between \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_{n_2}^2, \varphi_1 \between \varphi_2 \Rightarrow F_{n_2}}$$

7.3 Many-Valued Semantics

In this section we provide a method to obtain sound and complete many-valued semantics for any given canonical Gödel calculus and proof-specification. The semantic structures introduced for this task are called *Gödel valuations*. The truth values in these structures should form a *propositional Gödel set*, defined as follows:

Definition 7.3.1. A *(propositional) Gödel set* is a bounded linearly ordered set $\mathcal{V} = \langle V, \leq \rangle$. We denote by $0_\mathcal{V}$ and $1_\mathcal{V}$ the maximal and minimal elements (respectively) of V with respect to $\leq$. The operations $\min_\mathcal{V}, \max_\mathcal{V}$ are defined as usual (where $\min_\mathcal{V} \emptyset = 1_\mathcal{V}$ and $\max_\mathcal{V} \emptyset = 0_\mathcal{V}$). For every two elements $u_1, u_2 \in V$, $u_1 \to_\mathcal{V} u_2$ is defined to be $1_\mathcal{V}$ if $u_1 \leq u_2$, and u_2 otherwise. The relations $\geq$, $<$, and $>$ are also defined in the obvious way. We omit the subscript $\mathcal{V}$ when it is clear from the context, and sometimes identify $\mathcal{V}$ with the set V (e.g., when referring to the elements of V as elements of $\mathcal{V}$).

Now, (partial) Gödel $\mathcal{L}$-valuations are defined as follows:

Definition 7.3.2. A *partial Gödel $\mathcal{L}$-valuation* (partial *$\mathcal{L}$-Gvaluation*, for short) is a triple $\langle \mathcal{V}, Dom, v\rangle$, where $\mathcal{V}$ is a Gödel set, $Dom \subseteq \mathcal{L}$, and v is a function from Dom to $\mathcal{V}\times\mathcal{V}$. A partial $\mathcal{L}$-Gvaluation $\langle \mathcal{V}, Dom, v\rangle$ with $Dom = \mathcal{L}$ is also called an *$\mathcal{L}$-Gvaluation*.

Notation 7.3.3. Throughout, we identify a partial $\mathcal{L}$-Gvaluation $\langle \mathcal{V}, Dom, v\rangle$ with its underlying function v, and denote the Gödel set $\mathcal{V}$ by $\mathcal{V}_v$, and the set Dom by Dom_v. In addition, given a partial $\mathcal{L}$-Gvaluation v, we denote by $v^{\mathtt{f}}$ and $v^{\mathtt{t}}$ "the left and right projections of v", that is $v^{\mathtt{f}}(\varphi) = u_1$ and $v^{\mathtt{t}}(\varphi) = u_2$ iff $v(\varphi) = \langle u_1, u_2\rangle$.

There are two main ideas behind the definition of a partial $\mathcal{L}$-Gvaluation above. First, as in the previous chapters, these semantic structures may not assign truth values to *all* formulas. Their exact domain Dom should be determined according to the formulas that are allowed to appear in ρ-proofs for a given proof-specification ρ. Second, note that a partial $\mathcal{L}$-Gvaluation v assigns a *pair* $\langle v^{\mathtt{f}}(\varphi), v^{\mathtt{t}}(\varphi)\rangle$ of truth values to each formula $\varphi \in Dom_v$. Intuitively, $v^{\mathtt{f}}(\varphi)$ is the value of φ when it is "f-labelled" (occurs on the "left side" of a sequent), and $v^{\mathtt{t}}(\varphi)$ is its value when it is "t-labelled" (occurs on the "right side"). Roughly speaking, to have a complete semantics for proof-specifications in which (*cut*) and/or (*id*) may not be applied on some formula φ, we have to "disconnect" $v^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi)$. Obviously, certain restrictions on the relation between $v^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi)$ should be put when (*cut*) and/or (*id*) are allowed to apply on φ (see Definition 7.3.6 below). Before turning to these restrictions, we define the semantic consequence relation between hypersequents induced by a set of partial $\mathcal{L}$-Gvaluations.

Definition 7.3.4. Let v be a partial $\mathcal{L}$-Gvaluation.

1. Given a Dom_v-sequent s, $v^{\mathtt{f}}(s)$, $v^{\mathtt{t}}(s)$ and $v(s)$ denote the element of $\mathcal{V}_v$ defined by:

   (a) $v^{\mathtt{f}}(s) = \min\{v^{\mathtt{f}}(\varphi) \mid \mathtt{f}{:}\varphi \in s\}$.
   (b) $v^{\mathtt{t}}(s) = \max\{v^{\mathtt{t}}(\varphi) \mid \mathtt{t}{:}\varphi \in s\}$.
   (c) $v(s) = v^{\mathtt{f}}(s) \to v^{\mathtt{t}}(s)$.

2. v is a *model* of:

   (a) an *$\mathcal{L}$-sequent* s if s is a Dom_v-sequent and $v(s) = 1$.
   (b) an *$\mathcal{L}$-hypersequent* H if H is a Dom_v-hypersequent and v is a model of some component $s \in H$.

(c) a *set $\mathcal{H}$ of $\mathcal{L}$-hypersequents* if it is a model of every Dom_v-hypersequent $H \in \mathcal{H}$.

We write $v \models X$ to denote that v is a model of X (here X is either a sequent, a hypersequent, or a set of hypersequents).

Note that in order to check whether $v \models s$, one only needs $v^{\mathtt{f}}(\varphi)$ for every formula φ such that $\mathtt{f}{:}\varphi \in s$, and $v^{\mathtt{t}}(\varphi)$ for every formula φ such that $\mathtt{t}{:}\varphi \in s$. In turn, hypersequents are interpreted as "meta-disjunctions" of sequents. The consequence relation between hypersequents induced by a set $\mathcal{G}$ of partial $\mathcal{L}$-Gvaluations is defined as follows:

Definition 7.3.5. An $\mathcal{L}$-hypersequent H *follows* from a set $\mathcal{H}$ of $\mathcal{L}$-hypersequents with respect to a set $\mathcal{G}$ of partial $\mathcal{L}$-Gvaluations (denoted by: $\mathcal{H} \vdash_{\mathcal{G}} H$) if for every $v \in \mathcal{G}$: $v \models H$ whenever $v \models \mathcal{H}$.

As noted above, when (cut) and/or (id) are allowed to apply on some formula φ (according to the proof-specification), we have to "connect" $v^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi)$. Intuitively, (cut) and (id) have opposite semantic roles – while (cut) forces the "$\mathtt{f}$-value" to be greater than or equal to the "$\mathtt{t}$-value", (id) forces the "$\mathtt{f}$-value" to be lower than or equal to the $\mathtt{t}$-value. This is formulated in the next definition.

Definition 7.3.6. Let $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ be an $\mathcal{L}$-proof-specification. A partial $\mathcal{L}$-Gvaluation v is called *ρ-legal* if the following hold:

1. $Dom_v = \mathcal{F}$.
2. $v^{\mathtt{t}}(\varphi) \leq v^{\mathtt{f}}(\varphi)$ for every $\varphi \in \mathcal{C} \cap \mathcal{F}$.
3. $v^{\mathtt{f}}(\varphi) \leq v^{\mathtt{t}}(\varphi)$ for every $\varphi \in \mathcal{A} \cap \mathcal{F}$.

Note that when (cut) and (id) can be used for all formulas (that is: $\mathcal{C} = \mathcal{A} = \mathcal{L}$), we require that $v^{\mathtt{f}}(\varphi) = v^{\mathtt{t}}(\varphi)$ for every $\varphi \in Dom_v$. Such Gvaluations will be called *normal*:

Definition 7.3.7. A partial $\mathcal{L}$-Gvaluation v is called *normal* if $v^{\mathtt{f}}(\varphi) = v^{\mathtt{t}}(\varphi)$ for every $\varphi \in \mathcal{L}$. For normal partial $\mathcal{L}$-Gvaluation v, we may write $v(\varphi)$ instead of $v^{\mathtt{f}}(\varphi)$ (equivalently, $v^{\mathtt{t}}(\varphi)$).

Next, we turn to the semantic effect of the canonical rules included in the given canonical Gödel calculus. Roughly speaking, the idea here is that each rule for a connective $\diamond$ induces a function from $(\mathcal{V} \times \mathcal{V})^{ar(\diamond)}$ to $\mathcal{V}$. By applying these functions on the truth values of $\varphi_1, \ldots, \varphi_{ar(\diamond)}$, one obtain bounds on the truth value of $\diamond(\varphi_1, \ldots, \varphi_{ar(\diamond)})$. The functions induced by the right canonical rules for $\diamond$ provide lower bounds on $v^{\mathtt{t}}(\diamond(\varphi_1, \ldots, \varphi_{ar(\diamond)}))$, while those induced by the left rules for $\diamond$ are used as upper bounds on $v^{\mathtt{f}}(\diamond(\varphi_1, \ldots, \varphi_{ar(\diamond)}))$. Below we precisely formulate this idea.

Notation 7.3.8. Let v be a partial $\mathcal{L}$-Gvaluation. Given a finite set $\mathcal{S}$ of Dom_v-sequents, $v(\mathcal{S}) = \min\{v(s) \mid s \in \mathcal{S}\}$ and $v^{\mathtt{f}}(\mathcal{S}) = \max\{v^{\mathtt{f}}(s) \mid s \in \mathcal{S}\}$.

Definition 7.3.9. Let $\diamond$ be an n-ary connective of $\mathcal{L}$, $\mathcal{V}$ a Gödel set, and r a canonical $\mathcal{L}$-rule for $\diamond$. $\diamond^r_{\mathcal{V}}$ is a function from $(\mathcal{V} \times \mathcal{V})^n$ to $\mathcal{V}$ defined by:

$$\diamond^r_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \dots, \langle u^{\mathtt{f}}_n, u^{\mathtt{t}}_n\rangle) = \begin{cases} v(\mathcal{S}) & r = \mathcal{S}/\{\mathtt{t}{:}\varphi\} \\ v(\mathcal{S}_1) \to v^{\mathtt{f}}(\mathcal{S}_2) & r = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:}\varphi\} \end{cases}$$

where v is the partial $\mathcal{L}$-Gvaluation defined by $\mathcal{V}_v = \mathcal{V}$, $Dom_v = \{p_1, \dots, p_n\}$, and $v(p_i) = \langle u^{\mathtt{f}}_i, u^{\mathtt{t}}_i\rangle$ for every $1 \le i \le n$.

Example 7.3.10. Suppose that $\wedge \in \diamondsuit^2_{\mathcal{L}}$, and consider the usual rules:

$$(\mathtt{t}{:}\wedge) \quad \{\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \wedge p_2\} \qquad (\mathtt{f}{:}\wedge) \quad \emptyset, \{\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$$

Let $\mathcal{V}$ be a Gödel set, and $\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle \in \mathcal{V} \times \mathcal{V}$. We calculate $\wedge^{(\mathtt{t}:\wedge)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle)$, and $\wedge^{(\mathtt{f}:\wedge)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle)$. Let v be the partial $\mathcal{L}$-Gvaluation defined by $\mathcal{V}_v = \mathcal{V}$, $Dom_v = \{p_1, p_2\}$, and $v(p_i) = \langle u^{\mathtt{f}}_i, u^{\mathtt{t}}_i\rangle$ for $1 \le i \le 2$. We have

$$v(\{\mathtt{t}{:}p_1\}) = v^{\mathtt{f}}(\{\mathtt{t}{:}p_1\}) \to v^{\mathtt{t}}(\{\mathtt{t}{:}p_1\}) = 1 \to v^{\mathtt{t}}(p_1) = v^{\mathtt{t}}(p_1) = u^{\mathtt{t}}_1.$$

Similarly, $v(\{\mathtt{t}{:}p_2\}) = u^{\mathtt{t}}_2$. Thus

$$\wedge^{(\mathtt{t}:\wedge)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) = v(\{\{\mathtt{t}{:}p_1\}, \{\mathtt{t}{:}p_2\}\}) = \min\{u^{\mathtt{t}}_1, u^{\mathtt{t}}_2\}.$$

In addition, $v(\emptyset) = 1$, and

$$v^{\mathtt{f}}(\{\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}\}) = v^{\mathtt{f}}(\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}) = \min\{v^{\mathtt{f}}(p_1), v^{\mathtt{f}}(p_2)\} = \min\{u^{\mathtt{f}}_1, u^{\mathtt{f}}_2\}.$$

Therefore,

$$\wedge^{(\mathtt{f}:\wedge)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) = v(\emptyset) \to v^{\mathtt{f}}(\{\{\mathtt{f}{:}p_1, \mathtt{f}{:}p_2\}\}) = \min\{u^{\mathtt{f}}_1, u^{\mathtt{f}}_2\}.$$

Now, if we replace $(\mathtt{f}{:}\wedge)$ by $\emptyset, \{\{\mathtt{f}{:}p_1\}, \{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$ (see Example 7.2.7) we obtain

$$\wedge^{(\mathtt{f}:\wedge)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) = v(\emptyset) \to v^{\mathtt{f}}(\{\{\mathtt{f}{:}p_1\}, \{\mathtt{f}{:}p_2\}\}) = \max\{u^{\mathtt{f}}_1, u^{\mathtt{f}}_2\}.$$

Example 7.3.11. Suppose that $\supset \in \diamondsuit^2_{\mathcal{L}}$, and consider the usual rules:

$$(\mathtt{t}{:} \supset) \quad \{\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \supset p_2\} \qquad (\mathtt{f}{:} \supset) \quad \{\{\mathtt{t}{:}p_1\}\}, \{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \supset p_2\}$$

Let $\mathcal{V}$ be a Gödel set, and $\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle \in \mathcal{V} \times \mathcal{V}$. We calculate $\supset^{(\mathtt{t}:\supset)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle)$, and $\supset^{(\mathtt{f}:\supset)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle)$. Let v be the partial $\mathcal{L}$-Gvaluation defined by $\mathcal{V}_v = \mathcal{V}$, $Dom_v = \{p_1, p_2\}$, and $v(p_i) = \langle u^{\mathtt{f}}_i, u^{\mathtt{t}}_i\rangle$ for $1 \le i \le 2$. Then:

$$\begin{aligned} \supset^{(\mathtt{t}:\supset)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) &= v(\{\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}\}) = v^{\mathtt{f}}(\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}) \to v^{\mathtt{t}}(\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_2\}) = \\ &= v^{\mathtt{f}}(p_1) \to v^{\mathtt{t}}(p_2) = u^{\mathtt{f}}_1 \to u^{\mathtt{t}}_2 \\ \supset^{(\mathtt{f}:\supset)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) &= v(\{\{\mathtt{t}{:}p_1\}\}) \to v^{\mathtt{f}}(\{\{\mathtt{f}{:}p_2\}\}) = v^{\mathtt{t}}(p_1) \to v^{\mathtt{f}}(p_2) = u^{\mathtt{t}}_1 \to u^{\mathtt{f}}_2 \end{aligned}$$

Now, if we replace $(\mathtt{t}{:} \supset)$ by $\{\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \supset p_2\}$ (see Example 7.2.6) we obtain

$$\supset^{(\mathtt{t}:\supset)}_{\mathcal{V}}(\langle u^{\mathtt{f}}_1, u^{\mathtt{t}}_1\rangle, \langle u^{\mathtt{f}}_2, u^{\mathtt{t}}_2\rangle) = v(\{\{\mathtt{t}{:}p_2\}\}) = 1 \to v^{\mathtt{t}}(p_2) = u^{\mathtt{t}}_2$$

Note that the calculation of $\diamond^r_{\mathcal{V}}$ requires only applying "atomic Gödel functions", namely min, max, 0, 1, and Gödel implication. These functions enforce restrictions on Gvaluations as follows:

Definition 7.3.12. A partial $\mathcal{L}$-Gvaluation v is called *r-legal*:

- for a canonical *right* rule $r = \mathcal{S}/\{\mathtt{t}{:}\diamond(p_1,\dots,p_n)\}$ if for every $\mathcal{L}$-substitution σ such that $\sigma(\{p_1,\dots,p_n,\diamond(p_1,\dots,p_n)\}) \subseteq Dom_v$:
$$\diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)),\dots,v(\sigma(p_n))) \leq v^{\mathtt{t}}(\sigma(\diamond(p_1,\dots,p_n))).$$
- for a canonical *left* rule $r = \mathcal{S}_1,\mathcal{S}_2/\{\mathtt{f}{:}\diamond(p_1,\dots,p_n)\}$ if for every $\mathcal{L}$-substitution σ such that $\sigma(\{p_1,\dots,p_n,\diamond(p_1,\dots,p_n)\}) \subseteq Dom_v$:
$$v^{\mathtt{f}}(\sigma(\diamond(p_1,\dots,p_n))) \leq \diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)),\dots,v(\sigma(p_n))).$$

Note that the right rules for $\diamond$ impose restrictions on $v^{\mathtt{t}}(\sigma(\diamond(p_1,\dots,p_n)))$, while the left rules for $\diamond$ restrict $v^{\mathtt{f}}(\sigma(\diamond(p_1,\dots,p_n)))$.

Example 7.3.13. Let v be a ρ-legal partial $\mathcal{L}$-Gvaluation for $\rho = \langle \mathcal{F},\mathcal{C},\mathcal{A}\rangle$. Below we present the condition for v to be r-legal for the canonical rules presented in the examples above. Note that the condition on the right should hold for every $\mathcal{L}$-substitution σ such that $\{\sigma(p_1 \diamond p_2),\sigma(p_1),\sigma(p_2)\} \subseteq \mathcal{F}$ where $\diamond$ is the connective in the corresponding rule.

	Canonical Rule	*Semantic Condition*
$(\mathtt{t}{:}\wedge)$	$\{\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \wedge p_2\}$	$\min\{v^{\mathtt{t}}(\sigma(p_1)),v^{\mathtt{t}}(\sigma(p_2))\} \leq v^{\mathtt{t}}(\sigma(p_1 \wedge p_2))$
$(\mathtt{f}{:}\wedge)$	$\emptyset,\{\{\mathtt{f}{:}p_1,\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$	$v^{\mathtt{f}}(\sigma(p_1 \wedge p_2)) \leq \min\{v^{\mathtt{f}}(\sigma(p_1)),v^{\mathtt{f}}(\sigma(p_2))\}$
$(\mathtt{t}{:}\supset)$	$\{\{\mathtt{f}{:}p_1,\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \supset p_2\}$	$v^{\mathtt{f}}(\sigma(p_1)) \to v^{\mathtt{t}}(\sigma(p_2)) \leq v^{\mathtt{t}}(\sigma(p_1 \supset p_2))$
$(\mathtt{f}{:}\supset)$	$\{\{\mathtt{t}{:}p_1\}\},\{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \supset p_2\}$	$v^{\mathtt{f}}(\sigma(p_1 \supset p_2)) \leq v^{\mathtt{t}}(\sigma(p_1)) \to v^{\mathtt{f}}(\sigma(p_2))$
$(\mathtt{t}{:}\rightsquigarrow)$	$\{\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \rightsquigarrow p_2\}$	$v^{\mathtt{t}}(\sigma(p_2)) \leq v^{\mathtt{t}}(\sigma(p_1 \rightsquigarrow p_2))$
$(\mathtt{f}{:}\rightsquigarrow)$	$\{\{\mathtt{t}{:}p_1\}\},\{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \rightsquigarrow p_2\}$	$v^{\mathtt{f}}(\sigma(p_1 \rightsquigarrow p_2)) \leq v^{\mathtt{t}}(\sigma(p_1)) \to v^{\mathtt{f}}(\sigma(p_2))$
$(\mathtt{t}{:}\between)$	$\{\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \between p_2\}$	$\min\{v^{\mathtt{t}}(\sigma(p_1)),v^{\mathtt{t}}(\sigma(p_2))\} \leq v^{\mathtt{t}}(\sigma(p_1 \between p_2))$
$(\mathtt{f}{:}\between)$	$\emptyset,\{\{\mathtt{f}{:}p_1\},\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \between p_2\}$	$v^{\mathtt{f}}(\sigma(p_1 \between p_2)) \leq \max\{v^{\mathtt{f}}(\sigma(p_1)),v^{\mathtt{f}}(\sigma(p_2))\}$

Now, if we only consider *normal* (full) $\mathcal{L}$-Gvaluations, these conditions can be simplified as follows.

	Canonical Rule	*Semantic Condition*
$(\mathtt{t}{:}\wedge)$	$\{\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \wedge p_2\}$	$\min\{v(\varphi_1),v(\varphi_2)\} \leq v(\varphi_1 \wedge \varphi_2)$
$(\mathtt{f}{:}\wedge)$	$\emptyset,\{\{\mathtt{f}{:}p_1,\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \wedge p_2\}$	$v(\varphi_1 \wedge \varphi_2) \leq \min\{v(\varphi_1),v(\varphi_2)\}$
$(\mathtt{t}{:}\supset)$	$\{\{\mathtt{f}{:}p_1,\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \supset p_2\}$	$v(\varphi_1) \to v(\varphi_2) \leq v(\varphi_1 \supset \varphi_2)$
$(\mathtt{f}{:}\supset)$	$\{\{\mathtt{t}{:}p_1\}\},\{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \supset p_2\}$	$v(\varphi_1 \supset \varphi_2) \leq v(\varphi_1) \to v(\varphi_2)$
$(\mathtt{t}{:}\rightsquigarrow)$	$\{\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \rightsquigarrow p_2\}$	$v(\varphi_2) \leq v(\varphi_1 \rightsquigarrow \varphi_2)$

$(\mathtt{f}{:}\rightsquigarrow)$	$\{\{\mathtt{t}{:}p_1\}\},\{\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \rightsquigarrow p_2\}$	$v(\varphi_1 \rightsquigarrow \varphi_2) \leq v(\varphi_1) \to v(\varphi_2)$
$(\mathtt{t}{:}\between)$	$\{\{\mathtt{t}{:}p_1\},\{\mathtt{t}{:}p_2\}\}/\{\mathtt{t}{:}p_1 \between p_2\}$	$\min\{v(\varphi_1), v(\varphi_2)\} \leq v(\varphi_1 \between \varphi_2)$
$(\mathtt{f}{:}\between)$	$\emptyset, \{\{\mathtt{f}{:}p_1\},\{\mathtt{f}{:}p_2\}\}/\{\mathtt{f}{:}p_1 \between p_2\}$	$v(\varphi_1 \between \varphi_2) \leq \max\{v(\varphi_1), v(\varphi_2)\}$

In this case the condition in the right should hold for every $\varphi_1, \varphi_2 \in \mathcal{L}$. Recall that for normal Kvaluations we write $v(\varphi)$ instead of $v^{\mathtt{f}}(\varphi)$ (or equivalently $v^{\mathtt{t}}(\varphi)$). Note that the two rules for $\wedge$ together impose the usual Gödel logic semantics of $\wedge$:

$$v(\varphi_1 \wedge \varphi_2) = \min\{v(\varphi_1), v(\varphi_2)\}.$$

Similarly, the two rules for $\supset$ impose its usual semantics:

$$v(\varphi_1 \supset \varphi_2) = v(\varphi_1) \to v(\varphi_2).$$

Now, the rules for $\rightsquigarrow$ and $\between$ together impose the conditions:

$$v(\varphi_2) \leq v(\varphi_1 \rightsquigarrow \varphi_2) \leq v(\varphi_1) \to v(\varphi_2),$$
$$\min\{v(\varphi_1), v(\varphi_2)\} \leq v(\varphi_1 \between \varphi_2) \leq \max\{v(\varphi_1), v(\varphi_2)\}.$$

In these two cases we obtain *non-deterministic semantics* since the value assigned to $\varphi_1 \rightsquigarrow \varphi_2$ (similarly, to $\varphi_1 \between \varphi_2$) is not *uniquely* determined by the value assigned to φ_1 and φ_2. A deterministic semantics is obtained only when the lower bound determined by the right rules is equal to the upper bound determined by the left rules.

Remark 7.3.14. Note that in non-normal (partial) $\mathcal{L}$-Gvaluations, when we may have $v^{\mathtt{f}}(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})) \neq v^{\mathtt{t}}(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)}))$, then the semantics of $\diamond$ is non-deterministic by definition, since both $v^{\mathtt{f}}(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)}))$ and $v^{\mathtt{t}}(\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)}))$ are restricted only from one side. Therefore, even when $\diamond$ is a usual connective with ordinary introduction rules, non-deterministic semantics is employed to handle proof-specifications in which (*cut*) and/or (*id*) are not allowed on formulas of the form $\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})$.

The following technical lemma will be useful below.

Lemma 7.3.15. Let v be a partial $\mathcal{L}$-Gvaluation, r a canonical $\mathcal{L}$-rule for an n-ary connective $\diamond$, and σ an $\mathcal{L}$-substitution such that $\sigma(\{p_1, \dots, p_n\}) \subseteq Dom_v$. Then $\diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)), \dots, v(\sigma(p_n))$ is equal to:

- $v(\sigma(\mathcal{S}))$ when $r = \mathcal{S}/\{\mathtt{t}{:}\diamond(p_1, \dots, p_n)\}$ is a right rule.
- $v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$ when $r = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:}\diamond(p_1, \dots, p_n)\}$ is a left rule.

We are now ready to identify the set of Gvaluations for which a given canonical Gödel calculus and a proof-specification are sound and complete.

Definition 7.3.16. Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, and ρ an $\mathcal{L}$-proof-specification. A partial $\mathcal{L}$-Gvaluation v is called $\mathbf{G}$-*legal* if it is r-legal for every canonical $\mathcal{L}$-rule r of $\mathbf{G}$. v is called $\mathbf{G}{\restriction}\rho$-legal if it is both ρ-legal and $\mathbf{G}$-legal. $\mathcal{G}_{\mathbf{G}{\restriction}\rho}$ denotes the set of all $\mathbf{G}{\restriction}\rho$-legal partial $\mathcal{L}$-Gvaluations.

Next, we state the general soundness and completeness theorem connecting $\vdash_{\mathbf{G}\upharpoonright\rho}$ and $\vdash_{\mathcal{G}_{\mathbf{G}\upharpoonright\rho}}$. Its proof is given in Section 7.5.

Theorem 7.3.17. For every canonical Gödel $\mathcal{L}$-calculus $\mathbf{G}$, $\mathcal{L}$-proof-specification $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ where $\mathcal{F}$ is closed under subformulas, set $\mathcal{H}$ of $\mathcal{L}$-hypersequents, and $\mathcal{F}$-hypersequent H: $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H$ iff $\mathcal{H} \vdash_{\mathcal{G}_{\mathbf{G}\upharpoonright\rho}} H$.

For the simpler case without a proof-specification we obtain the following:

Corollary 7.3.18. For every canonical Gödel $\mathcal{L}$-calculus $\mathbf{G}$, $\vdash_{\mathbf{G}} = \vdash_{\mathcal{G}}$, where $\mathcal{G}$ is the set of all $\mathbf{G}$-legal normal $\mathcal{L}$-Gvaluations.

Proof. Since $\langle \mathcal{L}, \mathcal{L}, \mathcal{L} \rangle$-legal partial $\mathcal{L}$-Gvaluations are exactly normal $\mathcal{L}$-Gvaluations, the claim directly follows from Theorem 7.3.17. □

The following is an immediate corollary of the completeness proof.

Corollary 7.3.19. Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, and $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ an $\mathcal{L}$-proof-specification. If $\mathcal{H} \not\vdash_{\mathbf{G}\upharpoonright\rho} H$, then there exists a $\mathbf{G}\upharpoonright\rho$-legal partial $\mathcal{L}$-Gvaluation v, which is a model of $\mathcal{H}$ but not of H, satisfying $|\mathcal{V}_v| \leq 2|\mathcal{F}| + 2$.

Proof. Directly follows from the fact that the Gödel set $\mathcal{V}_v$ constructed in the completeness proof contains at most $2|\mathcal{F}| + 2$ elements (see Section 7.5). □

It follows that the semantics of $\mathbf{G}\upharpoonright\rho$-legal partial $\mathcal{L}$-Gvaluations is effective, in the sense that it naturally induces a procedure to decide whether $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H$ or not for a given canonical Gödel $\mathcal{L}$-calculus $\mathbf{G}$, $\mathcal{L}$-proof-specification $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A} \rangle$ with finite $\mathcal{F}, \mathcal{C}$ and $\mathcal{A}$, finite set $\mathcal{H}$ of $\mathcal{L}$-hypersequents and single $\mathcal{L}$-hypersequent H. Note that a syntactic decision procedure in this case is trivial, since the number of $\mathcal{F}$-hypersequents is bounded by $M = 2^{2^{2|\mathcal{F}|}}$. Obviously, one can enumerate all lists of $\mathcal{F}$-hypersequents of size at most M, and return "true" iff one of them is a ρ-proof in $\mathbf{G}$ of H from $\mathcal{H}$. Of-course, the problem is more interesting when ρ is *not* given, and one has to decide whether $\mathcal{H} \vdash_{\mathbf{G}} H$ or not. We consider this problem in the next section.

Remark 7.3.20. The linearly ordered set of truth values employed in the completeness proof is countable, and can be embedded into the unit interval $[0, 1]$. Thus we can fix $\mathcal{V} = [0, 1]$ in the definition of a Gvaluation, and obtain "standard" semantics.

7.4 Applications of the Semantics

In this section we use Theorem 7.3.17 to derive semantic characterizations of strong *suba*-analyticity, strong cut-admissibility, and axiom-expansion in canonical Gödel calculi, and use them to obtain a general decidability result for these calculi. First, note that the soundness part of Theorem 7.3.17 can be utilized to prove that certain applications of identity axioms or cuts are unavoidable, or perhaps to show that in all derivations of some hypersequent H, a certain formula φ appears.

Example 7.4.1. Let $H = \{\{\mathtt{t}{:}(p_1 \supset p_2) \vee (p_2 \supset p_1)\}\}$ and $\mathcal{F} = sub[H]$. We show that all proofs of H in **HG** that consist solely of formulas from $\mathcal{F}$ include the application $\{\{\mathtt{f}{:}p_1, \mathtt{t}{:}p_1\}\}$ of (id). Let $\rho = \langle \mathcal{F}, \mathcal{F}, \mathcal{F} \setminus \{p_1\}\rangle$. By Theorem 7.3.17, it suffices to provide a $\mathbf{HG}\lceil_\rho$-legal partial $\mathcal{L}$-Gvaluation which is not a model of H. For that we can choose $\mathcal{V}_v = [0, 1]$, $Dom_v = \mathcal{F}$ and:

- $v^{\mathtt{f}}(p_1) = 1, v^{\mathtt{t}}(p_1) = 0, v^{\mathtt{f}}(p_2) = v^{\mathtt{t}}(p_2) = 0.5$,
- $v^{\mathtt{f}}(p_1 \supset p_2) = v^{\mathtt{t}}(p_1 \supset p_2) = 0.5$,
- $v^{\mathtt{f}}(p_2 \supset p_1) = v^{\mathtt{t}}(p_2 \supset p_1) = 0$,
- $v^{\mathtt{f}}((p_1 \supset p_2) \vee (p_2 \supset p_1)) = v^{\mathtt{t}}((p_1 \supset p_2) \vee (p_2 \supset p_1)) = 0.5$.

It is straightforward to verify that v is a $\mathbf{HG}\lceil_\rho$-legal partial $\mathcal{L}$-Gvaluation which is not a model of H.

Next, we present a simple *coherence* criterion and prove that it is necessary and sufficient for strong *sub*-analyticity and strong cut-admissibility in canonical Gödel calculi.

Definition 7.4.2. A set R of canonical $\mathcal{L}$-rules for a connective $\diamond \in \diamondsuit_\mathcal{L}$ is called *coherent* if $\mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is classically inconsistent whenever R contains both $\mathcal{S}/\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ and $\mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$. A canonical Gödel $\mathcal{L}$-calculus **G** is called *coherent* if for for each $\diamond \in \diamondsuit_\mathcal{L}$, the set of rules in **G** for $\diamond$ is coherent.

Example 7.4.3. Clearly, the set of sequents $\{\{\mathtt{t}{:}p_2\}, \{\mathtt{t}{:}p_1\}, \{\mathtt{f}{:}p_2\}\}$ is classically inconsistent. Thus the two rules of $\rightsquigarrow$ from Example 7.2.6 form a coherent set of rules.

Note that this is exactly the same criterion used for single-conclusion canonical sequent systems in [14]. It is easy to verify that **HG** is coherent. Moreover, all sets of rules considered in previous examples are coherent. Coherence is a natural requirement for any canonical Gödel calculus. Indeed, in non-coherent calculi the existence of one provable hypersequent of the form $\{\{\mathtt{t}{:}\varphi\}\}$ and another provable hypersequent of the form $\{\{\mathtt{f}{:}\varphi\}\}$ implies that all (non-empty) hypersequents are provable:

Proposition 7.4.4. Let **G** be a canonical Gödel $\mathcal{L}$-calculus. If **G** is not coherent then $\{\{\mathtt{t}{:}p_1\}\}, \{\{\mathtt{f}{:}p_2\}\} \vdash_\mathbf{G} H$ for every non-empty $\mathcal{L}$-hypersequent H.

Proof. Similar to the proof of Theorem 4.10 in [14] for single conclusion canonical systems. The fact that **G** manipulates hypersequents is immaterial here. □

This easily entails that non-coherent calculi do not enjoy strong cut-admissibility. To see this, take $H = \{\emptyset\}$, i.e. the hypersequent consisting solely of the empty sequent. Obviously, in any canonical Gödel $\mathcal{L}$-calculus there does not exist an $\langle \mathcal{L}, \{p_1, p_2\}, \mathcal{L}\rangle$-proof of H from $\{\{\mathtt{t}{:}p_1\}\}$ and $\{\{\mathtt{f}{:}p_2\}\}$. Similarly, it is also clear that there does not exist a $\langle \{p_1, p_2\}, \mathcal{L}, \mathcal{L}\rangle$-proof of H from $\{\{\mathtt{t}{:}p_1\}\}$ and $\{\{\mathtt{f}{:}p_2\}\}$, and so non-coherent calculi are not strongly *sub*-analytic.

Next we show that coherence is sufficient for strong *sub*-analyticity and strong cut-admissibility.

Lemma 7.4.5. Let $\mathbf{G}$ be a coherent canonical Gödel $\mathcal{L}$-calculus. For every Gödel set $\mathcal{V}$, n-ary connective $\diamond \in \Diamond_{\mathcal{L}}$, canonical right and left $\mathcal{L}$-rules $r_{\mathtt{t}}, r_{\mathtt{f}}$ for $\diamond$ of $\mathbf{G}$ (respectively), and $u_1^{\mathtt{f}}, u_1^{\mathtt{t}}, \dots, u_n^{\mathtt{f}}, u_n^{\mathtt{t}} \in \mathcal{V}$ such that $u_i^{\mathtt{f}} \geq u_i^{\mathtt{t}}$ for every $1 \leq i \leq n$:

$$\diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle, \dots, \langle u_n^{\mathtt{f}}, u_n^{\mathtt{t}}\rangle) \leq \diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle, \dots, \langle u_n^{\mathtt{f}}, u_n^{\mathtt{t}}\rangle).$$

Proof. Suppose that $\diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle, \dots, \langle u_n^{\mathtt{f}}, u_n^{\mathtt{t}}\rangle) > \diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle, \dots, \langle u_n^{\mathtt{f}}, u_n^{\mathtt{t}}\rangle)$ for canonical $\mathcal{L}$-rules $r_{\mathtt{t}} = \mathcal{S}/\{\mathtt{t}{:}\diamond(p_1, \dots, p_n)\}$ and $r_{\mathtt{f}} = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:}\diamond(p_1, \dots, p_n)\}$. Let v be the partial $\mathcal{L}$-Gvaluation with $\mathcal{V}_v = \mathcal{V}$, $Dom_v = \{p_1, \dots, p_n\}$, and $v(p_i) = \langle u_i^{\mathtt{f}}, u_i^{\mathtt{t}}\rangle$ for every $1 \leq i \leq n$. Then, by definition, $v(\mathcal{S}) > v(\mathcal{S}_1) \to v^{\mathtt{f}}(\mathcal{S}_2)$. Hence $v(\mathcal{S}) > v^{\mathtt{f}}(\mathcal{S}_2)$ and $v(\mathcal{S}_1) > v^{\mathtt{f}}(\mathcal{S}_2)$. Consider the classical valuation c on $p_1, \dots, p_n$ defined by $c(p_i) = t$ iff $v^{\mathtt{t}}(p_i) > v^{\mathtt{f}}(\mathcal{S}_2)$. We prove that c satisfies every $\mathcal{L}$-sequent in $\mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$, and so $\mathbf{G}$ is not coherent.

Let $s \in \mathcal{S} \cup \mathcal{S}_1$. Since $v(\mathcal{S}) > v^{\mathtt{f}}(\mathcal{S}_2)$ and $v(\mathcal{S}_1) > v^{\mathtt{f}}(\mathcal{S}_2)$, $v(s) > v^{\mathtt{f}}(\mathcal{S}_2)$, and so $v^{\mathtt{f}}(s) \to v^{\mathtt{t}}(s) > v^{\mathtt{f}}(\mathcal{S}_2)$. If $v^{\mathtt{t}}(s) > v^{\mathtt{f}}(\mathcal{S}_2)$, it follows that $\mathtt{t}{:}p_i \in s$ for some $1 \leq i \leq n$ and $v^{\mathtt{t}}(p_i) > v^{\mathtt{f}}(\mathcal{S}_2)$, and so $c(p_i) = t$. Thus c classically satisfies s. Assume now that $v^{\mathtt{t}}(s) \leq v^{\mathtt{f}}(\mathcal{S}_2)$. This implies that $v^{\mathtt{f}}(s) \leq v^{\mathtt{t}}(s)$. It follows that $v^{\mathtt{f}}(s) \leq v^{\mathtt{f}}(\mathcal{S}_2)$, and so there exists some $\mathtt{f}{:}p_i \in s$ such that $v^{\mathtt{f}}(p_i) \leq v^{\mathtt{f}}(\mathcal{S}_2)$. Since $u_i^{\mathtt{f}} \geq u_i^{\mathtt{t}}$, $v^{\mathtt{t}}(p_i) \leq v^{\mathtt{f}}(p_i)$, and so $c(p_i) = f$. Thus c classically satisfies s.

Now, let $s \in \mathcal{S}_2$. Obviously, $v^{\mathtt{f}}(s) \leq v^{\mathtt{f}}(\mathcal{S}_2)$. This implies that there exists some $\mathtt{f}{:}p_i \in s$ such that $v^{\mathtt{f}}(p_i) \leq v^{\mathtt{f}}(\mathcal{S}_2)$. Since $u_i^{\mathtt{f}} \geq u_i^{\mathtt{t}}$, $v^{\mathtt{t}}(p_i) \leq v^{\mathtt{f}}(p_i)$, and so $c(p_i) = f$. Again c classically satisfies s. □

Theorem 7.4.6. All coherent canonical Gödel $\mathcal{L}$-calculi enjoy strong cut-admissibility.

Proof. Let $\mathbf{G}$ be a coherent canonical Gödel $\mathcal{L}$-calculus, $\mathcal{H}$ a set of $\mathcal{L}$-hypersequents and H an $\mathcal{L}$-hypersequent. Let $\rho = \langle \mathcal{L}, \mathcal{L}, \mathcal{A}\rangle$ and $\rho' = \langle \mathcal{L}, frm[\mathcal{H}], \mathcal{A}\rangle$. Suppose that $\mathcal{H} \nvdash_{\mathbf{G}\restriction_{\rho'}} H$. We show that $\mathcal{H} \nvdash_{\mathbf{G}\restriction_{\rho}} H$.[6] By Theorem 7.3.17, there is some $\mathbf{G}\restriction_{\rho'}$-legal

[6] In fact, this proves a stronger claim: $\mathcal{H} \vdash_{\mathbf{G}\restriction_{\rho}} H$ for $\rho = \langle \mathcal{L}, \mathcal{L}, \mathcal{A}\rangle$ implies that $\mathcal{H} \vdash_{\mathbf{G}\restriction_{\rho'}} H$ for $\rho' = \langle \mathcal{L}, frm[\mathcal{H}], \mathcal{A}\rangle$. Roughly speaking, this means that one never has to introduce new applications of the identity axioms for eliminating cuts. The usual notion of strong cut-admissibility is obtained by taking $\mathcal{A} = \mathcal{L}$.

$\mathcal{L}$-Gvaluation q, such that $q \models \mathcal{H}$ but $q \not\models H$. We construct a $\mathbf{G}{\restriction_\rho}$-legal $\mathcal{L}$-Gvaluation v, such that $v \models \mathcal{H}$ but $v \not\models H$. By Theorem 7.3.17, it then follows that $\mathcal{H} \not\vdash_{\mathbf{G}\restriction_\rho} H$. First, we define $\mathcal{V}_v = \mathcal{V}_q$. Next, for every $\varphi \in \mathcal{L}$, $v^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi)$ are determined as follows. If $q^{\mathtt{t}}(\varphi) \leq q^{\mathtt{f}}(\varphi)$ then $v^{\mathtt{f}}(\varphi) = q^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi) = q^{\mathtt{t}}(\varphi)$. Otherwise, if φ is an atomic formula, $v^{\mathtt{f}}(\varphi) = v^{\mathtt{t}}(\varphi) = q^{\mathtt{f}}(\varphi)$. Finally, if φ is a compound formula of the form $\diamond(\varphi_1, \ldots, \varphi_{ar(\diamond)})$ and $q^{\mathtt{f}}(\varphi) < q^{\mathtt{t}}(\varphi)$, we define:

$$v^{\mathtt{f}}(\varphi) = v^{\mathtt{t}}(\varphi) = \begin{cases} q^{\mathtt{f}}(\varphi) & v^{\mathbf{G}}(\varphi) < q^{\mathtt{f}}(\varphi) \\ v^{\mathbf{G}}(\varphi) & q^{\mathtt{f}}(\varphi) \leq v^{\mathbf{G}}(\varphi) \leq q^{\mathtt{t}}(\varphi) \\ q^{\mathtt{t}}(\varphi) & q^{\mathtt{t}}(\varphi) < v^{\mathbf{G}}(\varphi) \end{cases}$$

where $v^{\mathbf{G}}(\varphi) = \min\{\diamond^r_{\mathcal{V}_q}(v(\varphi_1), \ldots, v(\varphi_{ar(\diamond)})) \mid r \text{ is a canonical left } \mathcal{L}\text{-rule for } \diamond \text{ in } \mathbf{G}\}$. Note that $v^{\mathbf{G}}(\varphi)$ depends only on the values assigned to proper subformulas of φ, and hence this construction is well-defined. We first show that v is ρ-legal. Obviously, $v^{\mathtt{t}}(\varphi) \leq v^{\mathtt{f}}(\varphi)$ for every formula φ. It remains to prove that $v^{\mathtt{f}}(\varphi) \leq v^{\mathtt{t}}(\varphi)$ for every $\varphi \in \mathcal{A}$. To see this, note that the only case in which we have $v^{\mathtt{f}}(\varphi) \neq v^{\mathtt{t}}(\varphi)$ is when $q^{\mathtt{t}}(\varphi) < q^{\mathtt{f}}(\varphi)$. Since q is ρ'-legal, this can only happen for $\varphi \notin \mathcal{A}$. Next, we show that v is $\mathbf{G}$-legal. For this we use the following properties:

1. $q^{\mathtt{f}}(\varphi) \leq v^{\mathtt{f}}(\varphi)$ and $v^{\mathtt{t}}(\varphi) \leq q^{\mathtt{t}}(\varphi)$ for every formula φ.
2. $q^{\mathtt{f}}(s) \leq v^{\mathtt{f}}(s)$ and $v^{\mathtt{t}}(s) \leq q^{\mathtt{t}}(s)$ for every sequent s.
3. $v(s) \leq q(s)$ for every sequent s.
4. $v(\mathcal{S}) \leq q(\mathcal{S})$ and $q^{\mathtt{f}}(\mathcal{S}) \leq v^{\mathtt{f}}(\mathcal{S})$ for every finite set $\mathcal{S}$ of sequents.
5. $q(\mathcal{S}_1) \to q^{\mathtt{f}}(\mathcal{S}_2) \leq v(\mathcal{S}_1) \to v^{\mathtt{f}}(\mathcal{S}_2)$ for every finite sets $\mathcal{S}_1, \mathcal{S}_2$ of sequents.

The proofs of these properties easily follow from the definitions (note that if $u_1 \leq u_2$ and $u_3 \leq u_4$ then $u_2 \to u_3 \leq u_1 \to u_4$). Now, we show that v is $\mathbf{G}$-legal.

- Let $r = \mathcal{S}/\{\mathtt{t}\colon \diamond(p_1, \ldots, p_n)\}$ be a canonical right $\mathcal{L}$-rule of $\mathbf{G}$, and σ be an $\mathcal{L}$-substitution. We show that $\diamond^r_{\mathcal{V}_q}(v(\sigma(p_1)), \ldots, v(\sigma(p_n))) \leq v^{\mathtt{t}}(\sigma(\diamond(p_1, \ldots, p_n)))$. Let $\varphi = \sigma(\diamond(p_1, \ldots, p_n))$ and $u = \diamond^r_{\mathcal{V}_q}(v(\sigma(p_1)), \ldots, v(\sigma(p_n)))$. Since q is $\mathbf{G}$-legal, we have $\diamond^r_{\mathcal{V}_q}(q(\sigma(p_1)), \ldots, q(\sigma(p_n))) \leq q^{\mathtt{t}}(\varphi)$. By Lemma 7.3.15, $u = v(\sigma(\mathcal{S}))$, and $\diamond^r_{\mathcal{V}_q}(q(\sigma(p_1)), \ldots, q(\sigma(p_n))) = q(\sigma(\mathcal{S}))$. Thus if $v^{\mathtt{t}}(\varphi) = q^{\mathtt{t}}(\varphi)$, the claim follows by Item 4. Otherwise, the construction of v ensures that $v^{\mathtt{t}}(\varphi) = \max\{v^{\mathbf{G}}(\varphi), q^{\mathtt{f}}(\varphi)\}$. Thus $v^{\mathbf{G}}(\varphi) \leq v^{\mathtt{t}}(\varphi)$. Now, by Lemma 7.4.5, the coherence of $\mathbf{G}$ entails that $u \leq v^{\mathbf{G}}(\varphi)$.

- Let $r = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}\colon \diamond(p_1, \ldots, p_n)\}$ be a canonical left $\mathcal{L}$-rule of $\mathbf{G}$, and σ an $\mathcal{L}$-substitution. We show that $v^{\mathtt{f}}(\sigma(\diamond(p_1, \ldots, p_n))) \leq \diamond^r_{\mathcal{V}_q}(v(\sigma(p_1)), \ldots, v(\sigma(p_n)))$. Let $\varphi = \sigma(\diamond(p_1, \ldots, p_n))$ and $u = \diamond^r_{\mathcal{V}_q}(v(\sigma(p_1)), \ldots, v(\sigma(p_n)))$. Since q is $\mathbf{G}$-legal, we have

$q^{\mathtt{f}}(\varphi) \leq \diamond^{r}_{\mathcal{V}_q}(q(\sigma(p_1)), \dots, q(\sigma(p_n)))$. By Lemma 7.3.15, $u = v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$ and $\diamond^{r}_{\mathcal{V}_q}(q(\sigma(p_1)), \dots, q(\sigma(p_n))) = q(\sigma(\mathcal{S}_1)) \to q^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. Thus if $v^{\mathtt{t}}(\varphi) = q^{\mathtt{t}}(\varphi)$, then the claim follows by Item 5. Otherwise, the construction of v ensures that we have $v^{\mathtt{f}}(\varphi) = \min\{v^{\mathbf{G}}(\varphi), q^{\mathtt{t}}(\varphi)\}$, and so $v^{\mathtt{f}}(\varphi) \leq u$.

It remains to show that $v \models \mathcal{H}$ but $v \not\models H$. Let $H' \in \mathcal{H}$. Since $q \models \mathcal{H}$, there exists some $s \in H'$ such that $q^{\mathtt{f}}(s) \leq q^{\mathtt{t}}(s)$. Since q is ρ'-legal and $frm[s] \subseteq frm[\mathcal{H}]$, $q^{\mathtt{t}}(\varphi) \leq q^{\mathtt{f}}(\varphi)$ for every $\varphi \in frm[s]$. The construction of v ensures that $v(\varphi) = q(\varphi)$ whenever $\varphi \in frm[s]$. Hence, $v^{\mathtt{f}}(s) \leq v^{\mathtt{t}}(s)$, and consequently $v \models H'$. Finally, we show that $v \not\models H$. Let $s \in H$. Since $q \not\models H$, we have $q \not\models s$. Thus $q(s) < 1$. Item 3 above entails that $v(s) < 1$ as well, and so $v \not\models s$. □

Example 7.4.7. Since **HG** is coherent, it enjoys strong cut-admissibility. The extension of **HG** with the rules for $\rightsquigarrow$ and $\between$ from Examples 7.2.6 and 7.2.7 enjoys strong cut-admissibility as well.

This leads to the following triple equivalence in canonical Gödel calculi:

Corollary 7.4.8. The following are equivalent for a canonical Gödel $\mathcal{L}$-calculus **G**:

- **G** is coherent.
- **G** is strongly *sub*-analytic.
- **G** enjoys strong cut-admissibility.

Proof. Following Proposition 7.2.9, if **G** enjoys strong cut-admissibility then it is strongly *sub*-analytic. In addition, following Proposition 7.4.4, if **G** is not coherent then it is not strongly *sub*-analytic. The third link follows by Theorem 7.4.6. □

The decidability of coherent calculi is an easy corollary.

Corollary 7.4.9. Given a coherent canonical Gödel $\mathcal{L}$-calculus **G**, a finite set $\mathcal{H}$ of $\mathcal{L}$-hypersequents and an $\mathcal{L}$-hypersequent H, it is decidable whether $\mathcal{H} \vdash_{\mathbf{G}} H$ or not.

Proof. By Corollary 7.4.8, if **G** is coherent then it is strongly *sub*-analytic. Thus it suffices to check whether $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H$ for $\rho = \langle sub[\mathcal{H} \cup \{H\}], sub[\mathcal{H} \cup \{H\}], sub[\mathcal{H} \cup \{H\}]\rangle$. This can be as described in the discussion after Corollary 7.3.19. □

Note that the same equivalence of Corollary 7.4.8 holds for the family of canonical (two-sided) *sequent* systems [17] and their single-conclusion counterparts [14]. Another similarity between canonical Gödel calculi and canonical sequent calculi arises when studying the property of axiom-expansion. Indeed, as shown in Corollary 4.4.8, canonical sequent calculi exhibit a strong connection between determinism of the semantics of a certain connective, and the fact that this connective admits axiom-expansion (see also [11, 43]). Next, we show that a similar connection exists in canonical Gödel calculi.

Roughly speaking, a connective $\diamond$ is *deterministic* in a calculus $\mathbf{G}$ if for every formula $\diamond(\varphi_1, \dots, \varphi_{ar(\diamond)})$, the truth values assigned to $\varphi_1, \dots, \varphi_{ar(\diamond)}$ in $\mathbf{G}$-legal normal $\mathcal{L}$-Gvaluations uniquely determine the truth value assigned to $\diamond(\varphi_1, \dots, \varphi_n)$. Formally, this property is defined follows:

Definition 7.4.10. Let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus. An n-ary connective $\diamond \in \Diamond_{\mathcal{L}}$ is *deterministic* in $\mathbf{G}$ if for every Gödel set $\mathcal{V}$ and elements $u_1, \dots, u_n \in \mathcal{V}$, there exist canonical right and left $\mathcal{L}$-rule $r_{\mathtt{t}}, r_{\mathtt{f}}$ for $\diamond$ in $\mathbf{G}$ (respectively), such that

$$\diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(\langle u_1, u_1\rangle, \dots, \langle u_n, u_n\rangle) = \diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(\langle u_1, u_1\rangle, \dots, \langle u_n, u_n\rangle).$$

Indeed, if v is a $\mathbf{G}$-legal normal $\mathcal{L}$-Gvaluation, then for every compound formula $\varphi = \diamond(\varphi_1, \dots, \varphi_n)$, $v(\varphi)$ is forced to be greater than or equal to $\diamond_{\mathcal{V}}^{r}(v(\varphi_1), \dots, v(\varphi_n))$ for every right rule r for $\diamond$, and less than or equal to $\diamond_{\mathcal{V}}^{r}(v(\varphi_1), \dots, v(\varphi_n))$ for every left rule r for $\diamond$. If the condition above holds, then this leaves exactly one option for $v(\varphi)$ given $v(\varphi_1), \dots, v(\varphi_n)$.

Theorem 7.4.11. Let $\mathbf{G}$ be a coherent canonical Gödel $\mathcal{L}$-calculus. A connective admits axiom-expansion in $\mathbf{G}$ iff it is deterministic in $\mathbf{G}$.

Proof. Let $\diamond$ be an n-ary connective, and let $\varphi = \diamond(p_1, \dots, p_n)$. By definition, $\diamond$ admits axiom-expansion in $\mathbf{G}$ iff $\vdash_{\mathbf{G}\restriction_{\rho}} H$ for $H = \{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}\}$ and $\rho = \langle \mathcal{L}, \mathcal{L}, \{p_1, \dots, p_n\}\rangle$.

- Suppose that $\diamond$ is deterministic in $\mathbf{G}$. We show that $\vdash_{\mathbf{G}\restriction_{\rho}} H$. By Theorem 7.3.17, it suffices to show that every $\mathbf{G}\restriction_{\rho}$-legal $\mathcal{L}$-Gvaluation is a model of H. Let v be a $\mathbf{G}\restriction_{\rho}$-legal $\mathcal{L}$-Gvaluation. Since $\diamond$ is deterministic in $\mathbf{G}$, there are canonical right and left $\mathcal{L}$-rules $r_{\mathtt{t}}, r_{\mathtt{f}}$ for $\diamond$ in $\mathbf{G}$, such that $\diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(v(p_1), \dots, v(p_n)) = \diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(v(p_1), \dots, v(p_n))$. Since v is $\mathbf{G}$-legal, $\diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(v(p_1), \dots, v(p_n)) \leq v^{\mathtt{t}}(\varphi)$ and $v^{\mathtt{f}}(\varphi) \leq \diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(v(p_1), \dots, v(p_n))$. It follows that $v^{\mathtt{f}}(\varphi) \leq v^{\mathtt{t}}(\varphi)$, and so $v \models H$.

- Suppose that $\diamond$ is not deterministic in $\mathbf{G}$. Hence, there is a Gödel set $\mathcal{V}$ and $u_1, \dots, u_n \in \mathcal{V}$, such that $\diamond_{\mathcal{V}}^{r_{\mathtt{t}}}(\langle u_1, u_1\rangle, \dots, \langle u_n, u_n\rangle) \neq \diamond_{\mathcal{V}}^{r_{\mathtt{f}}}(\langle u_1, u_1\rangle, \dots, \langle u_n, u_n\rangle)$ for every canonical right and left $\mathcal{L}$-rules $r_{\mathtt{t}}, r_{\mathtt{f}}$ for $\diamond$ in $\mathbf{G}$. Let $\mathcal{F} = \{p_1, \dots, p_n, \varphi\}$, and define a partial $\mathcal{L}$-Gvaluation v by: $\mathcal{V}_v = \mathcal{V}$, $Dom_v = \mathcal{F}$, $v(p_i) = \langle u_i, u_i\rangle$ for every $1 \leq i \leq n$, and

$$v(\varphi) = \langle \min\{\diamond_{\mathcal{V}}^{r}(v(p_1), \dots, v(p_n)) \mid r \text{ is a canonical left } \mathcal{L}\text{-rule for } \diamond \text{ in } \mathbf{G}\},$$
$$\max\{\diamond_{\mathcal{V}}^{r}(v(p_1), \dots, v(p_n)) \mid r \text{ is a canonical right } \mathcal{L}\text{-rule for } \diamond \text{ in } \mathbf{G}\}\rangle.$$

  Then our assumption entails that $v^{\mathtt{t}}(\varphi) \neq v^{\mathtt{f}}(\varphi)$. Let $\rho' = \langle \mathcal{F}, \mathcal{F}, \{p_1, \dots, p_n\}\rangle$. It is easy to see that v is a $\mathbf{G}\restriction_{\rho'}$-legal partial $\mathcal{L}$-Gvaluation. In particular, $v^{\mathtt{t}}(\varphi) \leq v^{\mathtt{f}}(\varphi)$

since $\mathbf{G}$ is coherent (see Lemma 7.4.5). Hence $v^{\mathtt{t}}(\varphi) < v^{\mathtt{f}}(\varphi)$, and so v is not a model of H. By Theorem 7.3.17, $\nvdash_{\mathbf{G}\restriction\rho'} H$. Now, suppose for contradiction that $\vdash_{\mathbf{G}\restriction\rho} H$. By the proof of Theorem 7.4.6 (again using the fact that $\mathbf{G}$ is coherent), it follows that $\vdash_{\mathbf{G}\restriction\rho_1} H$ for $\rho_1 = \langle \mathcal{L}, \emptyset, \{p_1, \dots, p_n\}\rangle$. By Proposition 7.2.9, $\vdash_{\mathbf{G}\restriction\rho_2} H$ for $\rho_2 = \langle sub[H], \emptyset, \{p_1, \dots, p_n\}\rangle$. This contradicts the fact that $\nvdash_{\mathbf{G}\restriction\rho'} H$. □

7.5 Soundness and Completeness Proofs

In this section we prove Theorem 7.3.17. For the rest of this section, let $\mathbf{G}$ be a canonical Gödel $\mathcal{L}$-calculus, $\rho = \langle \mathcal{F}, \mathcal{C}, \mathcal{A}\rangle$ an $\mathcal{L}$-proof-specification where $\mathcal{F}$ is closed under subformulas, $\mathcal{H}$ a set of $\mathcal{L}$-hypersequents, and H_0 an $\mathcal{F}$-hypersequent.

Soundness

Suppose that $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_0$, and so there exists a ρ-proof P in $\mathbf{G}$ of H_0 from $\mathcal{H}$. Let v be a $\mathbf{G}\restriction\rho$-legal partial $\mathcal{L}$-Gvaluation, which is a model of $\mathcal{H}$. Using induction on the length of P, we show that $v \models H$ for every $\mathcal{L}$-hypersequent H appearing in P. It then follows that $v \models H_0$. Note that since all $\mathcal{L}$-hypersequents in P are $\mathcal{F}$-hypersequents, it suffices to prove that v is a model of some component $s \in H$ for every $\mathcal{L}$-hypersequent H in P. This trivially holds for the $\mathcal{L}$-hypersequents of $\mathcal{H}$ that appear in P. We show that this property is also preserved by applications in P of the rules of $\mathbf{G}$. Consider such an application, and assume that v is a model of some component of every premise of this application. We show that v is also a model of some component of the conclusion:

Weakenings For applications of the weakening rules, this is obvious.

(cut) Suppose that $H \mid s_1 \cup s_2$ is derived from $H \mid s_1 \cup \{\mathtt{f}{:}\varphi\}$ and $H \mid s_2 \cup \{\mathtt{t}{:}\varphi\}$ using (*cut*) (here s_2 must be a negative sequent). Thus $\varphi \in \mathcal{C}$. If $v \models s$ for some component $s \in H$, then we are done. Otherwise, $v \models s_1 \cup \{\mathtt{f}{:}\varphi\}$ and $v \models s_2 \cup \{\mathtt{t}{:}\varphi\}$. We show that $v \models s_1 \cup s_2$. By definition, we have $v^{\mathtt{f}}(s_1 \cup \{\mathtt{f}{:}\varphi\}) \leq v^{\mathtt{t}}(s_1)$ and $v^{\mathtt{f}}(s_2) \leq v^{\mathtt{t}}(\varphi)$. Since $\varphi \in \mathcal{C}$, $v^{\mathtt{t}}(\varphi) \leq v^{\mathtt{f}}(\varphi)$, and so $v^{\mathtt{f}}(s_2) \leq v^{\mathtt{f}}(\varphi)$. It follows that $v^{\mathtt{f}}(s_1 \cup s_2) \leq v^{\mathtt{f}}(s_1 \cup \{\mathtt{f}{:}\varphi\})$. Therefore, $v^{\mathtt{f}}(s_1 \cup s_2) \leq v^{\mathtt{t}}(s_1)$, and so $v \models s_1 \cup s_2$.

(id) Suppose that $\{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}\}$ is derived using (*id*). Thus $\varphi \in \mathcal{A}$, and so $v^{\mathtt{f}}(\varphi) \leq v^{\mathtt{t}}(\varphi)$. Consequently, $v \models \{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}$.

(com) Suppose that $H \mid s_1 \cup c_2 \mid s_2 \cup c_1$ is derived from $H \mid s_1 \cup c_1$ and $H \mid s_2 \cup c_2$ using (*com*), where c_1, c_2 are negative sequents. If $v \models s$ for some $s \in H$, then we are done. Otherwise, $v \models s_1 \cup c_1$ and $v \models s_2 \cup c_2$. We show that $v \models s_1 \cup c_2$ or $v \models s_2 \cup c_1$.

Our assumption entails that $v^{\mathtt{f}}(s_1 \cup c_1) \leq v^{\mathtt{t}}(s_1 \cup c_1)$ and $v^{\mathtt{f}}(s_2 \cup c_2) \leq v^{\mathtt{t}}(s_2 \cup c_2)$. By definition, $v^{\mathtt{t}}(s_1 \cup c_1) = v^{\mathtt{t}}(s_1)$ and $v^{\mathtt{t}}(s_2 \cup c_2) = v^{\mathtt{t}}(s_2)$. Hence, we have both $v^{\mathtt{f}}(s_1 \cup c_1) \leq v^{\mathtt{t}}(s_1)$ and $v^{\mathtt{f}}(s_2 \cup c_2) \leq v^{\mathtt{t}}(s_2)$. If $v^{\mathtt{f}}(s_1) \leq v^{\mathtt{t}}(s_1)$ then $v \models s_1 \cup c_2$ and we are done. Similarly, if $v^{\mathtt{f}}(s_2) \leq v^{\mathtt{t}}(s_2)$, then $v \models s_2 \cup c_1$ and we are done. Otherwise, we obtain that $v^{\mathtt{f}}(c_1) \leq v^{\mathtt{t}}(s_1)$ and $v^{\mathtt{f}}(c_2) \leq v^{\mathtt{t}}(s_2)$. Now, if $v^{\mathtt{t}}(s_1) \leq v^{\mathtt{f}}(c_2)$, we obtain that $v^{\mathtt{f}}(c_1) \leq v^{\mathtt{t}}(s_2)$ and so $v \models s_2 \cup c_1$, and otherwise (using the fact that $\leq$ is linear) we have $v^{\mathtt{f}}(c_2) < v^{\mathtt{t}}(s_1)$ and so $v \models s_1 \cup c_2$.

Right rules Suppose $H \mid \{\mathtt{t}{:}\sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))\} \cup c$ is derived from $\{H \mid \sigma(s) \cup c\}_{s \in \mathcal{S}}$, using the right rule $r = \mathcal{S}/\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ (here c must be a negative sequent). If $v \models s$ for some $s \in H$, then we are done. Otherwise, $v \models \sigma(s) \cup c$ for every $s \in \mathcal{S}$. Let $\varphi = \sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))$, and suppose for a contradiction that $v \not\models \{\mathtt{t}{:}\varphi\} \cup c$. Since c is a negative sequent, we have $v^{\mathtt{f}}(c) > v^{\mathtt{t}}(\varphi)$. Now, since $\varphi \in \mathcal{F}$ and $\mathcal{F}$ is closed under subformulas, we have also $\sigma(\{p_1, \dots, p_n\}) \subseteq \mathcal{F}$. Since v is r-legal we have $\diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)}))) \leq v^{\mathtt{t}}(\varphi)$. By Lemma 7.3.15, $\diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)}))) = v(\sigma(\mathcal{S}))$, and so $v^{\mathtt{f}}(c) > v(\sigma(\mathcal{S}))$. By definition, we have $v^{\mathtt{f}}(c) > v(\sigma(s)) = v^{\mathtt{f}}(\sigma(s)) \to v^{\mathtt{t}}(\sigma(s))$ for some $s \in \mathcal{S}$. It follows that $v^{\mathtt{f}}(\sigma(s)) > v^{\mathtt{t}}(\sigma(s))$ and $v^{\mathtt{f}}(c) > v^{\mathtt{t}}(\sigma(s))$. But then $v^{\mathtt{f}}(\sigma(s) \cup c) > v^{\mathtt{t}}(\sigma(s) \cup c)$, in contradiction to the fact that $v \models \sigma(s) \cup c$.

Left rules Suppose $H \mid \{\mathtt{f}{:}\sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))\} \cup c \cup c'$ is derived from $\{H \mid \sigma(s) \cup c\}_{s \in \mathcal{S}_1}$ and $\{H \mid \sigma(s'_i) \cup c'\}_{s \in \mathcal{S}_2}$, using the left rule $r = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ (here c must be a negative sequent). If $v \models s$ for some $s \in H$, then we are done. Otherwise, $v \models \sigma(s) \cup c$ for every $s \in \mathcal{S}_1$, and $v \models \sigma(s) \cup c'$ for every $s \in \mathcal{S}_2$. Hence: (1) for every $s \in \mathcal{S}_1$, either $v^{\mathtt{f}}(c) \leq v^{\mathtt{t}}(\sigma(s))$ or $v^{\mathtt{f}}(\sigma(s)) \leq v^{\mathtt{t}}(\sigma(s))$; and (2) either $v^{\mathtt{f}}(c') \leq v^{\mathtt{t}}(c')$, or $v^{\mathtt{f}}(\sigma(s)) \leq v^{\mathtt{t}}(c')$ for every $s \in \mathcal{S}_2$. Let $\varphi = \sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))$. Suppose for a contradiction that $v \not\models \{\mathtt{f}{:}\varphi\} \cup c \cup c'$. Then by definition we have $v^{\mathtt{f}}(\{\mathtt{f}{:}\varphi\} \cup c \cup c') > v^{\mathtt{t}}(c')$. Therefore: (3) $v^{\mathtt{t}}(c') < v^{\mathtt{f}}(c)$ and $v^{\mathtt{t}}(c') < v^{\mathtt{f}}(c')$; and (4) $v^{\mathtt{t}}(c') < v^{\mathtt{f}}(\varphi)$. From (2) and (3) we obtain (5): $v^{\mathtt{f}}(\sigma(s)) \leq v^{\mathtt{t}}(c')$ for every $s \in \mathcal{S}_2$. Now, since $\varphi \in \mathcal{F}$ and $\mathcal{F}$ is closed under subformulas, we have also $\sigma(\{p_1, \dots, p_n\}) \subseteq \mathcal{F}$. Since v is r-legal, $v^{\mathtt{f}}(\varphi) \leq \diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)})))$. Let $x = v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. By Lemma 7.3.15, $\diamond^r_{\mathcal{V}_v}(v(\sigma(p_1)), \dots, v(\sigma(p_{ar(\diamond)}))) = x$, and so $v^{\mathtt{f}}(\varphi) \leq x$. Together with (4), we have that $v^{\mathtt{t}}(c') < x$. By (5), we obtain that $v^{\mathtt{f}}(\sigma(s)) < x$ for every $s \in \mathcal{S}_2$. Let s_0 be a sequent in $\mathcal{S}_2$ such that $v^{\mathtt{f}}(s)$ obtains a maximal value (i.e. $v^{\mathtt{f}}(s_0) = v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$). In particular, $v^{\mathtt{f}}(s_0) < v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(s_0)$. This entails that $v(\sigma(\mathcal{S}_1)) \leq v^{\mathtt{f}}(s_0)$. Now, (3) and (5) imply that $v^{\mathtt{f}}(s_0) < v^{\mathtt{f}}(c)$. It then follows that $v(\sigma(\mathcal{S}_1)) < v^{\mathtt{f}}(c)$. Hence $v(\sigma(s)) < v^{\mathtt{f}}(c)$ for some $s \in \mathcal{S}_1$. Equivalently, $v^{\mathtt{f}}(\sigma(s)) \to v^{\mathtt{t}}(\sigma(s)) < v^{\mathtt{f}}(c)$. This implies that $v^{\mathtt{t}}(\sigma(s)) < v^{\mathtt{f}}(\sigma(s))$

and $v^{\mathtt{t}}(\sigma(s)) < v^{\mathtt{f}}(c)$. But this contradicts (1) above.

Completeness

Suppose that $\mathcal{H} \nvdash_{\mathbf{G}\restriction\rho} H_0$. We construct a $\mathbf{G}\restriction\rho$-legal partial $\mathcal{L}$-Gvaluation v, which is a model of $\mathcal{H}$, but not of H_0.

As the previous completeness proofs, v is constructed using a "maximal" hypersequent. For this purpose, we introduce *extended hypersequents*:

Definition 7.5.1. An *extended $\mathcal{L}$-hypersequent* is a (possibly infinite) set of extended $\mathcal{L}$-sequents.[7] Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \subseteq \mu_2$.

We shall use the same notations as above for extended hypersequents. For example, we write $\Omega \mid \mu$ instead of $\Omega \cup \{\mu\}$. An extended $\mathcal{F}$-hypersequent is also defined as expected (namely, an extended hypersequent that consists only of formulas from $\mathcal{F}$). In addition, for the rest of this proof call an extended $\mathcal{L}$-hypersequent Ω:

1. *finite* if $|\Omega|$ is finite, and so is $|\mu|$ for every $\mu \in \Omega$.
2. *provable* if $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$ for some (ordinary) $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$.
3. *unprovable* if it is not provable.
4. *maximal with respect to an $\mathcal{L}$-formula φ* if for every $\mu \in \Omega$ and $\mathtt{x} \in \mathscr{L}_{\mathbf{2}}$, the extended $\mathcal{L}$-hypersequent $\Omega \mid \mu \cup \{\mathtt{x}{:}\varphi\}$ is provable whenever $\mathtt{x}{:}\varphi \notin \mu$.
5. *internally maximal* if it is maximal with respect to any $\varphi \in \mathcal{F}$.
6. *maximal with respect to an $\mathcal{L}$-sequent s* if $\Omega \mid s$ is provable whenever $\{s\} \not\sqsubseteq \Omega$.
7. *externally maximal* if it is maximal with respect to any $\mathcal{F}$-sequent.
8. *maximal* if it is an extended $\mathcal{F}$-hypersequent, unprovable, internally maximal, and externally maximal.

Less formally, an extended hypersequent Ω is internally maximal if every formula added on some side of some component of Ω would make it provable. Similarly, Ω is externally maximal if every sequent added to Ω would make it provable.

Proposition 7.5.2. Let Ω be an extended $\mathcal{L}$-hypersequent.

- Assume that Ω is maximal with respect to a formula $\varphi \in \mathcal{F}$. For every $\mu \in \Omega$:
  - If $\mathtt{f}{:}\varphi \notin \mu$, then $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H \mid s_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}\varphi\}$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $s_1, \ldots, s_n \subseteq \mu$.

[7] Recall that extended $\mathcal{L}$-sequents are (possibly infinite) set of labelled $\mathcal{L}$-formulas. In particular, extended $\mathcal{L}$-sequents are not necessarily single-conclusion.

  - If $\mathtt{t}{:}\varphi \notin \mu$, then $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H \mid s \cup \{\mathtt{t}{:}\varphi\}$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$ and negative $\mathcal{L}$-sequent $s \subseteq \mu$.

- Assume that Ω is maximal with respect to an $\mathcal{F}$-sequent s. Then, if $\{s\} \not\sqsubseteq \Omega$, then $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H \mid s$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$.

Proof. Directly follow from our definitions and the availability of the weakening rules. □

Next, we show that H_0 can be extended to a maximal extended $\mathcal{L}$-hypersequent Ω.

Lemma 7.5.3. Suppose that $\Omega = \mu_1 \mid ... \mid \mu_n$ is an unprovable finite extended $\mathcal{F}$-hypersequent. Let $\varphi \in \mathcal{F}$, and let s be an $\mathcal{F}$-sequent. Then there exists an unprovable finite extended $\mathcal{F}$-hypersequent Ω', such that:

- $\Omega' = \mu'_1 \mid ... \mid \mu'_{n'}$, where $n' \in \{n, n+1\}$, $\mu_i \subseteq \mu'_i$ for every $1 \leq i \leq n$.
- Ω' is maximal with respect to φ.
- Ω' is maximal with respect to s.

Proof. First, if $\Omega \mid s$ is unprovable, let $n' = n+1$ and define $\mu_{n+1} = s$. Otherwise, let $n' = n$. We recursively define a finite sequence of finite extended $\mathcal{F}$-hypersequents, $\Omega_0 = \mu^0_1 \mid ... \mid \mu^0_{n'}, \dots, \Omega_{n'} = \mu^{n'}_1 \mid ... \mid \mu^{n'}_{n'}$, in which $\mu^i_j \subseteq \mu^{i+1}_j$ for every $1 \leq j \leq n'$ and $0 \leq i \leq n'-1$. First, define $\mu^0_j = \mu_j$ for every $1 \leq j \leq n'$. Let $0 \leq i \leq n'-1$. Assume that $\Omega_i = \mu^i_1 \mid ... \mid \mu^i_{n'}$ is defined. To construct $\Omega_{i+1} = \mu^{i+1}_1 \mid ... \mid \mu^{i+1}_{n'}$, we take some maximal set $\mathtt{X} \subseteq \mathcal{L}_\mathbf{2}$ (with respect to set inclusion) for which $\mu^i_1 \mid ... \mid \mu^i_{i+1} \cup (\mathtt{X}{:}\varphi) \mid ... \mid \mu^i_{n'}$ is unprovable, and define $\mu^{i+1}_{i+1} = \mu^i_{i+1} \cup (\mathtt{X}{:}\varphi)$, and $\mu^{i+1}_j = \mu^i_j$ for every $j \neq i+1$. It is easy to verify that $\Omega_{n'} = \mu^{n'}_1 \mid ... \mid \mu^{n'}_{n'}$ has all of the required properties. For example, we show that $\Omega_{n'}$ is maximal with respect to φ. Let $\mu^{n'}_i \in \Omega_{n'}$, and assume that $\mathtt{x}{:}\varphi \notin \mu^{n'}_i$ (for $x \in \mathcal{L}_\mathbf{2}$). This implies that $\mu^{i-1}_1 \mid ... \mid \mu^i_i \cup \{\mathtt{x}{:}\varphi\} \mid ... \mid \mu^{i-1}_{n'}$ is provable. Using weakenings, it easily follows that $\Omega_{n'} \mid \mu^{n'}_i \cup \{\mathtt{x}{:}\varphi\}$ is provable. □

Lemma 7.5.4. There is some maximal extended $\mathcal{L}$-hypersequent Ω that extends H_0.

Proof. Let $\varphi_0, \varphi_1, ...$ be an enumeration of all $\mathcal{F}$-formulas, in which every formula appears an infinite number of times. Let $s_0, s_1, ...$ be an enumeration of all $\mathcal{F}$-sequents (with repetitions, if there is only finite number of them). We recursively define an infinite sequence of unprovable finite extended $\mathcal{F}$-hypersequents, $H_0 = s^0_1 \mid ... \mid s^0_{n_0}, H_1 = s^1_1 \mid ... \mid s^1_{n_1}, ...$, in which: (*a*) $n_0 \leq n_1 \leq ...$ and (*b*) $s^i_j \subseteq s^{i+1}_j$ for every $i \geq 0$ and $1 \leq j \leq n_i$. First, let $n_0 = n$ and let $s^0_1 \mid ... \mid s^0_{n_0}$ be the original $\mathcal{F}$-hypersequent H_0. Let $i \geq 0$. Assume $H_i = s^i_1 \mid ... \mid s^i_{n_i}$ is defined. By Lemma 7.5.3, there exists an unprovable finite extended $\mathcal{F}$-hypersequent H' such that:

- $H' = s'_1 \mid ... \mid s'_{n'}$ where $n' \in \{n_i, n_i+1\}$, and $s_i \subseteq s'_i$ for every $1 \leq i \leq n_i$.

- H' is maximal with respect to φ_i.
- H' is maximal with respect to s_i.

Let $n_{i+1} = n'$, and $s^{i+1}_j = s'_j$ for every $1 \leq j \leq n_{i+1}$. Note that after every step we have an unprovable finite extended $\mathcal{F}$-hypersequent, so Lemma 7.5.3 can be applied. Finally, let $N = \max\{n_0, n_1, ...\} + 1$, if such a maximum exists, and infinity otherwise. Let $n(j) = \min\{i \mid j \leq n_i\}$ for every $1 \leq j < N$. Define $\mu_j = \cup_{i \geq n(j)} s^i_j$ for every $1 \leq j < N$. Let Ω be the extended $\mathcal{F}$-hypersequent $\mu_1 \mid \mu_2 \mid ....$ Obviously, Ω extends H_0. We prove that it is maximal:

Unprovability Suppose by way of contradiction that $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$ for an $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$. Assume that $H = s_1 \mid ... \mid s_n$. The construction of Ω ensures that for every $1 \leq i \leq n$, there exists $k_i \geq 1$ such that $s_i \subseteq \mu_{k_i}$. This entails that for every $1 \leq i \leq n$, there exists $m_i \geq 0$ such that $s_i \subseteq s^{m_i}_{k_i}$. By the construction of the s^i_j's, we have that for every $1 \leq i \leq n$ and $l \geq m_i$, $s_i \subseteq s^l_{k_i}$. Let $m = \max\{m_1, \dots, m_n\}$. Then, by definition $H \sqsubseteq H_m$. Since $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$, it follows that H_m is provable. But, this contradicts the fact that H_0 is unprovable, and that each application of Lemma 7.5.3 yields an unprovable extended $\mathcal{L}$-hypersequent.

Internal Maximality Let $\varphi \in \mathcal{F}$, $\mu_j \in \Omega$, and $\mathtt{x} \in \pounds_{\mathbf{2}}$. Since we included φ infinite number of times in the enumeration of $\mathcal{F}$-formulas, there exists some $i \geq n(j)$ such that $\varphi_i = \varphi$. Our construction ensures that H_{i+1} is maximal with respect to φ, and so if $\mathtt{x}{:}\varphi \notin s^{i+1}_j$ then $H_{i+1} \mid s^{i+1}_j \cup \{\mathtt{x}{:}\varphi\}$ is provable. Since $H_{i+1} \sqsubseteq \Omega$, it follows that if $\mathtt{x}{:}\varphi \notin \mu_j$ then $\Omega \mid \mu_j \cup \{\mathtt{x}{:}\varphi\}$ is provable.

External Maximality Let s be an $\mathcal{F}$-sequent. Assume that $s = s_i$ ($i \geq 0$), our construction ensures that H_{i+1} is maximal with respect to s. Hence, $H_{i+1} \mid s$ is provable whenever $\{s\} \not\sqsubseteq H_{i+1}$. Since $H_{i+1} \sqsubseteq \Omega$, we have that $\Omega \mid s$ is provable whenever $\{s\} \not\sqsubseteq \Omega$. $\square$

Using the maximal extended $\mathcal{L}$-hypersequent Ω (that extends H_0), we are not ready to construct a $\mathbf{G}\restriction\rho$-legal partial $\mathcal{L}$-Gvaluation v which is a model of $\mathcal{H}$, but not of H_0. For every $\varphi \in \mathcal{F}$, define $L(\varphi)$ and $R(\varphi)$ as follows:

$$L(\varphi) = \{\mu \in \Omega \mid \mathtt{f}{:}\varphi \in \mu\} \qquad \text{and} \qquad R(\varphi) = \{\mu \in \Omega \mid \mathtt{t}{:}\varphi \notin \mu\}.$$

v is defined as follows:

1. $\mathcal{V}_v = \langle V, \subseteq \rangle$, where: $V = \{L(\varphi) \mid \varphi \in \mathcal{F}\} \cup \{R(\varphi) \mid \varphi \in \mathcal{F}\} \cup \{\Omega, \emptyset\}$ ($\subseteq$ denotes set inclusion).
2. $Dom_v = \mathcal{F}$.
3. For every $\varphi \in \mathcal{F}$, $v(\varphi) = \langle L(\varphi), R(\varphi) \rangle$.

First, we show that $\mathcal{V}_v = \langle V, \subseteq \rangle$ is indeed a Gödel set (see Definition 7.3.1). Clearly, $\subseteq$ is a partial order on V, Ω is a maximal element, and $\emptyset$ is a minimal one. To see that V is linearly ordered by $\subseteq$, it suffices to prove the following:

1. $L(\varphi_1) \subseteq L(\varphi_2)$ or $L(\varphi_2) \subseteq L(\varphi_1)$ for every $\varphi_1, \varphi_2 \in \mathcal{F}$. To see this, suppose for a contradiction that $L(\varphi_1) \not\subseteq L(\varphi_2)$ and $L(\varphi_2) \not\subseteq L(\varphi_1)$ for some $\varphi_1, \varphi_2 \in \mathcal{F}$. Thus there exist $\mu_1, \mu_2 \in \Omega$, such that $\mu_1 \in L(\varphi_1) \setminus L(\varphi_2)$ and $\mu_2 \in L(\varphi_2) \setminus L(\varphi_1)$. Hence, $\mathtt{f}{:}\varphi_1 \in \mu_1$, $\mathtt{f}{:}\varphi_1 \notin \mu_2$, $\mathtt{f}{:}\varphi_2 \in \mu_2$ and $\mathtt{f}{:}\varphi_2 \notin \mu_1$. By Proposition 7.5.2, there exist an $\mathcal{L}$-hypersequent $H_1 \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $s_1, \dots, s_n \subseteq \mu_1$ such that
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid s_1 \cup \{\mathtt{f}{:}\varphi_2\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\varphi_2\}.$$
Similarly, there exist $H_2 \sqsubseteq \Omega$ and $s'_1, \dots, s'_m \subseteq \mu_2$, such that
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_2 \mid s'_1 \cup \{\mathtt{f}{:}\varphi_1\} \mid \dots \mid s'_m \cup \{\mathtt{f}{:}\varphi_1\}.$$
By Proposition 7.2.10:
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid H_2 \mid s_1 \cup \{\mathtt{f}{:}\varphi_1\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\varphi_1\} \mid s'_1 \cup \{\mathtt{f}{:}\varphi_2\} \mid \dots \mid s'_m \cup \{\mathtt{f}{:}\varphi_2\}.$$
But, Ω extends this hypersequent, and this contradicts the fact that Ω is unprovable.

2. $R(\varphi_1) \subseteq R(\varphi_2)$ or $R(\varphi_2) \subseteq R(\varphi_1)$ for every $\varphi_1, \varphi_2 \in \mathcal{F}$. To see this, suppose for a contradiction that $R(\varphi_1) \not\subseteq R(\varphi_2)$ and $R(\varphi_2) \not\subseteq R(\varphi_1)$ for some $\varphi_1, \varphi_2 \in \mathcal{F}$. Thus there are $\mu_1, \mu_2 \in \Omega$, such that $\mu_1 \in R(\varphi_1) \setminus R(\varphi_2)$ and $\mu_2 \in R(\varphi_2) \setminus R(\varphi_1)$. Hence, $\mathtt{t}{:}\varphi_1 \notin \mu_1$, $\mathtt{t}{:}\varphi_1 \in \mu_2$, $\mathtt{t}{:}\varphi_2 \in \mu_1$ and $\mathtt{t}{:}\varphi_2 \notin \mu_2$. By Proposition 7.5.2, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega$ and negative $\mathcal{L}$-sequents s_1, s_2 such that
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid s_1 \cup \{\mathtt{t}{:}\varphi_1\} \quad \text{and} \quad \mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_2 \mid s_2 \cup \{\mathtt{t}{:}\varphi_2\}.$$
By applying (com), we obtain
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid H_2 \mid s_2 \cup \{\mathtt{t}{:}\varphi_1\} \mid s_1 \cup \{\mathtt{t}{:}\varphi_2\}.$$
Again, since Ω extends this $\mathcal{L}$-hypersequent, this contradicts the fact that Ω is unprovable.

3. $L(\varphi_1) \subseteq R(\varphi_2)$ or $R(\varphi_2) \subseteq L(\varphi_1)$ for every $\varphi_1 \in \mathcal{F}$ and $\varphi_2 \in \mathcal{F}$. To see this, suppose for a contradiction that $L(\varphi_1) \not\subseteq R(\varphi_2)$ and $R(\varphi_2) \not\subseteq L(\varphi_1)$ for some $\varphi_1 \in \mathcal{F}$ and $\varphi_2 \in \mathcal{F}$. Let $\mu_1, \mu_2 \in \Omega$, such that $\mu_1 \in L(\varphi_1) \setminus R(\varphi_2)$ and $\mu_2 \in R(\varphi_2) \setminus L(\varphi_1)$. Hence, $\mathtt{f}{:}\varphi_1 \in \mu_1$, $\mathtt{f}{:}\varphi_1 \notin \mu_2$, $\mathtt{t}{:}\varphi_2 \in \mu_1$ and $\mathtt{t}{:}\varphi_2 \notin \mu_2$. By Proposition 7.5.2, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega$, $\mathcal{L}$-sequents $s_1, \dots, s_n \subseteq \mu_2$ and a negative $\mathcal{L}$-sequent $s' \subseteq \mu_2$, such that
$$\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid s_1 \cup \{\mathtt{f}{:}\varphi_1\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\varphi_1\} \quad \text{and} \quad \mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_2 \mid s' \cup \{\mathtt{t}{:}\varphi_2\}.$$
By Proposition 7.2.10, $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H_1 \mid H_2 \mid s_1 \cup s' \mid \dots \mid s_n \cup s' \mid \{\mathtt{f}{:}\varphi_1, \mathtt{t}{:}\varphi_2\}$. Again, this contradicts the fact that Ω is unprovable.

Now, it is easy to verify that v is a model of $\mathcal{H}$ but not of H_0:

- Let H be an $\mathcal{F}$-hypersequent in $\mathcal{H}$. Since Ω is unprovable, $H \not\sqsubseteq \Omega$. Thus there exists some $\mathcal{L}$-sequent $s \in H$ such that $s \not\subseteq \mu$ for every $\mu \in \Omega$. This implies that for every $\mu \in \Omega$, either $\mu \notin L(\varphi)$ for some φ such that $\mathtt{f}{:}\varphi \in s$, or $\mu \in R(\varphi)$ for some φ such that $\mathtt{t}{:}\varphi \in s$. It follows that for every $\mu \in \Omega$, we have $\mu \notin v^{\mathtt{f}}(s)$ or $\mu \in v^{\mathtt{t}}(s)$. Thus $v^{\mathtt{f}}(s) \subseteq v^{\mathtt{t}}(s)$, and so $v \models s$. Consequently, $v \models H$.
- Let $s \in H_0$. Since $H_0 \sqsubseteq \Omega$, there exists $\mu \in \Omega$ such that $s \subseteq \mu$. Hence $\mu \in L(\varphi)$ for every φ such that $\mathtt{f}{:}\varphi \in s$, and $\mu \notin R(\varphi)$ for every φ such that $\mathtt{t}{:}\varphi \in s$. It follows that $\mu \in v^{\mathtt{f}}(s)$ and $\mu \notin v^{\mathtt{t}}(s)$. Therefore, $v^{\mathtt{f}}(s) \not\subseteq v^{\mathtt{t}}(s)$, and so $v \not\models s$. Consequently, $v \not\models H_0$.

It remains to show that v is $\mathbf{G}{\restriction_\rho}$-legal. First, we show that it is ρ-legal. By definition $Dom_v = \mathcal{F}$. In addition:

- Let $\varphi \in \mathcal{C} \cap \mathcal{F}$. Assume for a contradiction that $v^{\mathtt{t}}(\varphi) \not\subseteq v^{\mathtt{f}}(\varphi)$, and thus there exists some $\mu \in \Omega$ such that $\mu \in R(\varphi)$ but $\mu \notin L(\varphi)$. Thus $\mathtt{t}{:}\varphi \notin \mu$ and $\mathtt{f}{:}\varphi \notin \mu$. By Proposition 7.5.2, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega$, $\mathcal{L}$-sequents $s_1, \dots, s_n \subseteq \mu$ and a negative $\mathcal{L}$-sequent $s' \subseteq \mu$, such that $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H_1 \mid s_1 \cup \{\mathtt{f}{:}\varphi\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\varphi\}$ and $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H_2 \mid s' \cup \{\mathtt{t}{:}\varphi\}$. By n consecutive applications of (cut) (on φ, which is an element of $\mathcal{C}$), we obtain $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H_1 \mid H_2 \mid s_1 \cup s' \mid \dots \mid s_n \cup s'$, but this contradicts the fact that Ω is unprovable.
- Let $\varphi \in \mathcal{A} \cap \mathcal{F}$. Thus $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} \{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}\}$ (by applying (id) with φ, which is an element of $\mathcal{A}$). Since Ω is unprovable, either $\mathtt{f}{:}\varphi \notin \mu$ or $\mathtt{t}{:}\varphi \notin \mu$ for every $\mu \in \Omega$. Equivalently, for every $\mu \in \Omega$, either $\mu \notin L(\varphi)$ or $\mu \in R(\varphi)$. It follows that $L(\varphi) \subseteq R(\varphi)$.

To show that v is $\mathbf{G}$-legal, we first prove that the following hold:

(**a**) For every $\mathcal{F}$-sequent s, if $v^{\mathtt{f}}(s) \subseteq v^{\mathtt{t}}(s)$ then there exists an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega$ such that $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H' \mid s$.

Proof. Suppose that there does not exist $H' \sqsubseteq \Omega$ such that $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H' \mid s$. Then Proposition 7.5.2 implies that $\{s\} \sqsubseteq \Omega$. Hence $s \subseteq \mu$ for some $\mu \in \Omega$. By definition, $\mu \in L(\varphi)$ for every φ such that $\mathtt{f}{:}\varphi \in s$, and $\mu \notin R(\varphi)$ for φ such that $\mathtt{t}{:}\varphi \in s$. It follows that $\mu \in v^{\mathtt{f}}(s)$ and $\mu \notin v^{\mathtt{t}}(s)$. □

(**b**) For every $\mathcal{F}$-sequent s, if $\mu \in v(s)$, then there exist an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega$, and a negative $\mathcal{L}$-sequent $s' \subseteq \mu$, such that $\mathcal{H} \vdash_{\mathbf{G}\restriction_\rho} H' \mid s \cup s'$.

Proof. Suppose that $\mu \in v(s) = v^{\mathtt{f}}(s) \to v^{\mathtt{t}}(s)$. Now, if $v^{\mathtt{f}}(s) \subseteq v^{\mathtt{t}}(s)$, then the claim follows by (**a**) (take $s' = \emptyset$). Otherwise, $\mu \in v^{\mathtt{t}}(s)$. Hence there is some

$\varphi \in \mathcal{L}$ such that $\mathtt{t}{:}\varphi \in s$ and $\mu \in R(\varphi)$. Thus $\mathtt{t}{:}\varphi \notin \mu$. By (possibly) using weakening, Proposition 7.5.2 implies that there exist an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega$, and a negative $\mathcal{L}$-sequent $s' \subseteq \mu$, such that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H' \mid s \cup s'$. □

(c) For every negative $\mathcal{F}$-sequent s, if $\mu \notin v^{\mathtt{f}}(s)$ then there exist an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $s_1, \dots, s_n \subseteq \mu$ such that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H' \mid s \cup s_1 \mid \dots \mid s \cup s_n$.

Proof. Suppose that $\mu \notin v^{\mathtt{f}}(s)$. Thus there exists some $\varphi \in \mathcal{L}$ such that $\mathtt{f}{:}\varphi \in s$ and $\mu \notin L(\varphi)$. Hence, $\mathtt{f}{:}\varphi \notin \mu$. The claim follows from Proposition 7.5.2 (possibly by using weakenings). □

Next, we show that v is r-legal for each canonical $\mathcal{L}$-rule r of $\mathbf{G}$:

- Let $r = \mathcal{S}/\{\mathtt{t}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ be a right canonical $\mathcal{L}$-rule of $\mathbf{G}$, and σ an $\mathcal{L}$-substitution, such that $\sigma(\{p_1, \dots, p_n\}) \cup \{\varphi\} \subseteq \mathcal{F}$, where $\varphi = \sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))$. Using Lemma 7.3.15, it suffices to prove that $v(\sigma(\mathcal{S})) \subseteq R(\varphi)$. Let $\mu \in v(\sigma(\mathcal{S}))$. Suppose that $\mathcal{S} = \{s_1, \dots, s_n\}$. Since $\mu \in v(\sigma(\mathcal{S}))$, **(b)** entails that for every $1 \leq i \leq n$, there exist an $\mathcal{L}$-hypersequent $H_i \sqsubseteq \Omega$ and a negative $\mathcal{L}$-sequent $s'_i \subseteq \mu$, such that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H_i \mid s'_i \cup \sigma(s_i)$. By applying the rule r, we obtain that
$$\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H_1 \mid \dots \mid H_n \mid s'_1 \cup \dots \cup s'_n \cup \{\mathtt{t}{:}\varphi\}.$$
The fact that Ω is unprovable then entails that $\mathtt{t}{:}\varphi \notin \mu$, and so $\mu \in R(\varphi)$.

- Let $r = \mathcal{S}_1, \mathcal{S}_2/\{\mathtt{f}{:} \diamond (p_1, \dots, p_{ar(\diamond)})\}$ be a left canonical $\mathcal{L}$-rule of $\mathbf{G}$, and σ an $\mathcal{L}$-substitution, such that $\sigma(\{p_1, \dots, p_n\}) \cup \{\varphi\} \subseteq \mathcal{F}$, where $\varphi = \sigma(\diamond(p_1, \dots, p_{ar(\diamond)}))$. Using Lemma 7.3.15, it suffices to prove that $L(\varphi) \subseteq v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. Let $\mu \in \Omega$ and suppose that $\mu \notin v(\sigma(\mathcal{S}_1)) \to v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. Hence, $v(\sigma(\mathcal{S}_1)) \not\subseteq v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$ and $\mu \notin v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. Let $\mu' \in v(\sigma(\mathcal{S}_1))$ such that $\mu' \notin v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$. Suppose that $\mathcal{S}_1 = \{s_1, \dots, s_n\}$ and $\mathcal{S}_2 = \{s'_1, \dots, s'_m\}$. We have the following:

  (1) Since $\mu' \in v(\sigma(\mathcal{S}_1))$, we have $\mu' \in v(\sigma(s_i))$ for every $1 \leq i \leq n$. **(b)** entails that for every $1 \leq i \leq n$, there exist an $\mathcal{L}$-hypersequent $H_i \sqsubseteq \Omega$, and a negative $\mathcal{L}$-sequent $c_i \subseteq \mu'$, such that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H_i \mid c_i \cup \sigma(s_i)$. The availability of weakenings entail that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H_1 \mid c \cup \sigma(s_i)$ for every $1 \leq i \leq n$, where $H = H_1 \mid \dots \mid H_n$ and $c = c_1 \cup \dots \cup c_n$.

  (2) Since $\mu' \notin v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$, $\mu' \notin v^{\mathtt{f}}(\sigma(\{\psi \mid \mathtt{f}{:}\psi \in s'_i\}))$ for every $1 \leq i \leq m$. **(c)** entails that for every $1 \leq i \leq m$, there exist an $\mathcal{L}$-hypersequent $H'_i \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $s^i_1, \dots, s^i_{n_i} \subseteq \mu'$ such that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H'_i \mid s^i_1 \cup \sigma(s'_i) \mid \dots \mid s^i_{n_i} \cup \sigma(s'_i)$. The availability of weakenings entail that $\mathcal{H} \vdash_{\mathbf{G}\upharpoonright\rho} H' \mid s^i_1 \cup \sigma(s'_i) \mid \dots \mid s^i_{n_i} \cup \sigma(s'_i)$ for every $1 \leq i \leq m$, where $H' = H'_1 \mid \dots \mid H'_m$.

(3) Similarly, the fact that $\mu \notin v^{\mathtt{f}}(\sigma(\mathcal{S}_2))$ entails that for every $1 \leq i \leq m$, there exist an $\mathcal{L}$-hypersequent $H''_i \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $c^i_1, \ldots, c^i_{n'_i} \subseteq \mu$ for which we have $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H''_i \mid c^i_1 \cup \sigma(s'_i) \mid \ldots \mid c^i_{n'_i} \cup \sigma(s'_i)$. The availability of weakenings entail that $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H''_i \mid c^i_1 \cup \sigma(s'_i) \mid \ldots \mid c^i_{n'_i} \cup \sigma(s'_i)$ for every $1 \leq i \leq m$, where $H'' = H''_1 \mid \ldots \mid H''_m$.

Let $H^* = H \mid H' \mid H''$. From (1) and (2), by Proposition 7.2.11 we obtain that the following $\mathcal{L}$-hypersequent has a ρ-proof in $\mathbf{G}$ of $\mathcal{H}$:

$$H^* \mid c \cup s^1_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup s^1_{n_1} \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup s^m_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup s^m_{n_m} \cup \{\mathtt{f}{:}\varphi\}.$$

Similarly, from (1) and (3), by Proposition 7.2.11 we obtain that the following $\mathcal{L}$-hypersequent has a ρ-proof in $\mathbf{G}$ of $\mathcal{H}$:

$$H^* \mid c \cup c^1_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup c^1_{n'_1} \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup c^m_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c \cup c^m_{n'_m} \cup \{\mathtt{f}{:}\varphi\}.$$

By Proposition 7.2.10, it follows that the following hypersequent has a ρ-proof in $\mathbf{G}$ of $\mathcal{H}$:

$$H^* \mid c \cup s^1_1 \mid \ldots \mid c \cup s^1_{n_1} \mid \ldots \mid c \cup s^m_1 \mid \ldots \mid c \cup s^m_{n_m} \mid$$
$$c^1_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c^1_{n'_1} \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c^m_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid c^m_{n'_m} \cup \{\mathtt{f}{:}\varphi\}.$$

Now, if $\mathtt{f}{:}\varphi \in \mu$, then Ω extends this $\mathcal{L}$-hypersequent, and this contradicts the fact that Ω is unprovable. Therefore, $\mathtt{f}{:}\varphi \notin \mu$, and consequently $\mu \notin L(\varphi)$.

Chapter 8

Calculus for First-Order Gödel Logic

So far we have considered logics and calculi only at the propositional level. However, the ideas and methods described in the previous chapters are applicable for first-order calculi as well. In this chapter we demonstrate our methods for the hypersequent calculus **HIF** for standard first-order Gödel logic (standard means that the real interval $[0, 1]$ can be used as the underlying set of truth values).[1] **HIF**, introduced in [30], is obtained from **HG** (the original hypersequent calculus for propositional Gödel logic, see Chapter 7) by adding standard (hypersequential versions of) rules for the quantifiers $\forall$ and $\exists$. It was proved in [30] that **HIF** is sound and complete for standard first-order Gödel logic by showing its equivalence to an Hilbert system for this logic (see [63]). Furthermore, it was shown in [30] and [22] that **HIF** admits cut-elimination.[2] As a corollary, one obtains Herbrand theorem for the prenex fragment of this logic [76].

In this chapter we obtain alternative semantic proofs for these facts, by using the ideas and techniques from Chapter 7. First, we briefly present standard first-order Gödel logic (from a many-valued semantic point of view), and the hypersequent calculus **HIF**. Then we prove the soundness of **HIF** for standard first-order Gödel logic. The completeness proof is tied together with cut-admissibility and involves two stages: (*i*) We present a non-deterministic semantics and show its completeness for the cut-free fragment of **HIF**; (*ii*) It is shown that from every non-deterministic counter-model, one can extract a usual counter-model. From these two facts together, it easily follows that **HIF** enjoys cut-admissibility, and that it is complete for standard first-order Gödel logic.

Note that unlike previous chapters, this chapters studies one specific calculus for a

[1] Note that Gödel logic is the only fundamental fuzzy logic whose first-order version is recursively axiomatizable [76].

[2] In fact, the first syntactic proof in [30] of cut-elimination was erroneous. A corrected syntactic proof appear in [22]. There has also been a gap in the proof given in [4] for **HG** in its handling of the case of disjunction.

particular logic, aiming to demonstrate the usefulness of semantic tools in studying first-order calculi. It should be also possible to define and study general canonical rules for quantifiers in this context as done in [18, 98]. This chapter also serves as a preparation for the next one, where we extend of **HIF** with rules for *second-order* quantifiers. Cut-admissibility for **HIF** itself can be derived as a corollary of the results of the next chapter. However, to make our presentation more accessible we first include here the full proof for **HIF**.

Remark 8.0.5. The method of proving cut-admissibility in this chapter is similar to what we did for (coherent) canonical Gödel hypersequent calculi in Chapter 7. Indeed, a general cut-admissibility theorem (Theorem 7.4.6) was proven in Chapter 7 by (uniformly) extracting a **G**-legal $\mathcal{L}$-Gvaluation from a $\mathbf{G}\lceil\langle\mathcal{L},\emptyset,\mathcal{L}\rangle$-legal $\mathcal{L}$-Gvaluation. The non-deterministic semantics of the cut-free fragment of **HIF** developed in this chapter is a natural first-order version of the semantics obtained in Chapter 7 for cut-free canonical Gödel hypersequent calculi (i.e., using the proof-specification $\langle\mathcal{L},\emptyset,\mathcal{L}\rangle$).

Publications Related to this Chapter

The results of this chapter were included in [16] and [72]. Nevertheless, our method here is completely different (and it allows generalization for second-order). In [16, 72] we extended the results of [10] concerning the propositional calculus **HG** to the first-order calculus **HIF**. As in [10] this was done by proving the completeness of the cut-free part of **HIF** for its Kripke-style semantics (thereby proving both completeness of the calculus and the admissibility of the cut rule in it). In this chapter we take a novel approach which is very close to the methods of Chapter 7. In particular, we consider many-valued semantics rather than Kripke-style one. Note also that [16] introduced a multiple-conclusion calculus for first-order Gödel logic that is not included here.

8.1 Preliminaries

In what follows, $\mathcal{L}^1$ denotes an arbitrary first-order language, defined by:

Definition 8.1.1. A *first-order language* consists of the following:

1. Infinitely many variables $\nu_1, \nu_2, ...$ We use the metavariables x, y, z (with or without subscripts) for variables.
2. A propositional constant $\bot$.
3. Binary connectives $\wedge, \vee, \supset$. We use $\diamond$ as a metavariable for the binary connectives.
4. Quantifiers $\forall$ and $\exists$. We use Q as a metavariable for the quantifiers.

5. An arbitrary set of constant symbols. The metavariable c is used to range over constant symbols.
6. An arbitrary set of function symbols. The metavariable f is used to range over them.
7. An arbitrary set of predicate symbols. The metavariable p is used to range over them.
8. Parentheses $'('$ and $')'$.

Definition 8.1.2. The set of *$\mathcal{L}^1$-terms* consists of: (i) all variables of $\mathcal{L}^1$; (ii) all constant symbols of $\mathcal{L}^1$; and (iii) if f is an n-ary function symbol of $\mathcal{L}^1$ and $t_1, \dots, t_n$ are $\mathcal{L}^1$-terms then $f(t_1, \dots, t_n)$ is an $\mathcal{L}^1$-term. We use t (with or without subscripts) as a metavariable for $\mathcal{L}^1$-terms. The set of variables occurring in an $\mathcal{L}^1$-term t is defined as usual, and denoted by $Fv[t]$.

Following the convention of Girard in [58], we define a first-order formula as an *equivalence class* of what we call *concrete* formulas, so that two formulas that differ only by the names of their bound variables are considered the same.[3] This is convenient for handling the bureaucracy of free and bound variables. Moreover, it simplifies the non-deterministic semantics below (see Remark 8.5.7).

Definition 8.1.3. *Concrete $\mathcal{L}^1$-formulas* are inductively defined as follows:

1. $p(t_1, \dots, t_n)$ is a concrete $\mathcal{L}^1$-formula for every predicate symbol p of arity n and $\mathcal{L}^1$-terms $t_1, \dots, t_n$.
2. $\perp$ is a concrete $\mathcal{L}^1$-formula.
3. If Φ_1 and Φ_2 are concrete $\mathcal{L}^1$-formulas, so are $(\Phi_1 \wedge \Phi_2)$, $(\Phi_1 \vee \Phi_2)$, and $(\Phi_1 \supset \Phi_2)$.
4. If Φ is a concrete $\mathcal{L}^1$-formula, and x is a variable of $\mathcal{L}^1$, then $(\forall x \Phi)$ and $(\exists x \Phi)$ are concrete $\mathcal{L}^1$-formulas.

We use Φ (with or without subscripts) as a metavariable for concrete $\mathcal{L}^1$-formulas. *Free and bound variables* in concrete $\mathcal{L}^1$-formulas are defined as usual. We denote by $Fv[\Phi]$, the set of variables occurring free in a concrete $\mathcal{L}^1$-formula Φ. *Alpha-equivalence* between concrete $\mathcal{L}^1$-formulas is defined as usual (renaming of bound variables). We denote by $[\Phi]_\alpha$ the set of all concrete $\mathcal{L}^1$-formulas which are alpha-equivalent to Φ (i.e. the equivalence class of Φ under alpha-equivalence). $cp[\Phi]$, the *complexity of a concrete $\mathcal{L}^1$-formula* Φ is the sum of the numbers of occurrences of quantifiers, connectives (including $\perp$), and atomic concrete formulas (formulas of the form $p(t_1, \dots, t_n)$) in Φ.

[3] Since [58] does not provide all the technical details for this convention, we do it here.

Definition 8.1.4. An *$\mathcal{L}^1$-formula* is an equivalence class of concrete $\mathcal{L}^1$-formulas under alpha-equivalence. We mainly use φ, ψ (with or without subscripts) as metavariables for $\mathcal{L}^1$-formulas. The set of free variables and the complexity of an $\mathcal{L}^1$-formula are defined using representatives, i.e. for an $\mathcal{L}^1$-formula φ, $Fv[\varphi] = Fv[\Phi]$ and $cp[\varphi] = cp[\Phi]$ for some $\Phi \in \varphi$.

In the last definition and henceforth, it is easy to verify that all notions defined using representatives are well-defined.

Definition 8.1.5. We define two operations on $\mathcal{L}^1$-formulas:

- For $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}^1$-formulas φ_1 and φ_2:
$$(\varphi_1 \diamond \varphi_2) = [(\Phi_1 \diamond \Phi_2)]_\alpha \text{ for some } \Phi_1 \in \varphi_1 \text{ and } \Phi_2 \in \varphi_2.$$
- For $Q \in \{\forall, \exists\}$, a variable x of $\mathcal{L}^1$, and an $\mathcal{L}^1$-formula φ:
$$(Qx\varphi) = [(Qx\Phi)]_\alpha \text{ for some } \Phi \in \varphi.$$

The next proposition allows us to use inductive definitions and to prove claims by induction on complexity of formulas:

Proposition 8.1.6. Exactly one of the following holds for every $\mathcal{L}^1$-formula φ:

- $cp[\varphi] = 1$ and exactly one of the following holds:
  - $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$.
  - $\varphi = \{\bot\}$.
- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and unique $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$.
- For every variable $x \notin Fv[\varphi]$, $\varphi = (Qx\psi)$ for some $Q \in \{\forall, \exists\}$, and unique $\mathcal{L}^1$-formula ψ such that $cp[\psi] < cp[\varphi]$.

Substitutions are defined as follows:

Definition 8.1.7. Let t be an $\mathcal{L}^1$-term and x a variable of $\mathcal{L}^1$.

1. Given an $\mathcal{L}^1$-term t', $t'\{^t/_x\}$ is inductively defined by:
$$t'\{^t/_x\} = \begin{cases} t & t' = x \\ t' & t' = y \text{ for } y \neq x, \text{ or } t' = c \\ f(t_1\{^t/_x\}, \dots, t_n\{^t/_x\}) & t' = f(t_1, \dots, t_n) \end{cases}$$

2. Given an $\mathcal{L}^1$-formula φ, $\varphi\{{}^t\!/\!_x\}$ is inductively defined by:

$$\varphi\{{}^t\!/\!_x\} = \begin{cases} \{p(t_1\{{}^t\!/\!_x\}, \dots, t_n\{{}^t\!/\!_x\})\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \varphi & \varphi = \{\bot\} \\ (\varphi_1\{{}^t\!/\!_x\} \diamond \varphi_2\{{}^t\!/\!_x\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Qy\psi\{{}^t\!/\!_x\}) & \varphi = (Qy\psi) \text{ for } y \notin Fv[t] \cup \{x\} \end{cases}$$

Note that the above substitution operations are well-defined. In particular, the choice of the variable y is immaterial.

8.2 Standard First-Order Gödel Logic

In this section we briefly present standard first-order Gödel logic from a (many-valued) semantic point of view (see, e.g., [63, 76] for more detailed presentations). The first component of the semantics is the set of truth values. These should form a Gödel set:

Definition 8.2.1. A *(standard) Gödel set* $\mathcal{V} = \langle V, \leq \rangle$ is defined just like a propositional Gödel set (see Definition 7.3.1), with the additional restriction that $\mathcal{V}$ is a *complete* totally ordered set. The operations $\inf_{\mathcal{V}}$ and $\sup_{\mathcal{V}}$ are defined as usual.

Next, the semantic structures include a domain and an interpretation function defined as follows:

Definition 8.2.2. A *domain* is a non-empty set $\mathcal{D}$. Given a domain $\mathcal{D}$, an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-interpretation is a function I assigning an element in $\mathcal{D}$ to every constant symbol of $\mathcal{L}^1$, and a function in $\mathcal{D}^n \to \mathcal{D}$ to every n-ary function symbol of $\mathcal{L}^1$.

To interpret predicate symbols we use *fuzzy subsets*:

Definition 8.2.3. Given a Gödel set $\mathcal{V}$, and some non-empty set $\mathcal{D}$, a function D from $\mathcal{D}$ to $\mathcal{V}$ is called a *fuzzy subset* of $\mathcal{D}$ over $\mathcal{V}$.

Predicate symbols are naturally interpreted as *fuzzy* subsets of tuples of elements of $\mathcal{D}$.

Definition 8.2.4. An *$\mathcal{L}^1$-structure* is a triple $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$, where:

1. $\mathcal{V}$ is a Gödel set.
2. $\mathcal{D}$ is a domain.
3. I is an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-interpretation.
4. P is a function assigning a fuzzy subset of $\mathcal{D}^n$ over $\mathcal{V}$ to every n-ary predicate symbol of $\mathcal{L}^1$.

As usual, an additional function is used for interpreting the *free* variables.

Definition 8.2.5. Let $\mathcal{D}$ be a domain.

1. An $\langle \mathcal{L}^1, \mathcal{D}\rangle$-*assignment* is a function assigning elements of $\mathcal{D}$ to the variables of $\mathcal{L}^1$.
2. Given an $\langle \mathcal{L}^1, \mathcal{D}\rangle$-interpretation I and an $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ, σ^I is the function assigning elements of $\mathcal{D}$ to all $\mathcal{L}^1$-terms, recursively defined by:
   - $\sigma^I[c] = I[c]$ for every constant symbol c of $\mathcal{L}^1$.
   - $\sigma^I[x] = \sigma[x]$ for every variable x of $\mathcal{L}^1$.
   - $\sigma^I[f(t_1, \dots, t_n)] = I[f](\sigma^I[t_1], \dots, \sigma^I[t_n])$ for every n-ary function symbol f of $\mathcal{L}^1$ and n $\mathcal{L}^1$-terms $t_1, \dots, t_n$.
3. Let σ be an $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment. Given a variable x of $\mathcal{L}^1$ and $d \in \mathcal{D}^I$, we denote by $\sigma_{x:=d}$ the $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment that is identical to σ except for $\sigma_{x:=d}[x] = d$. This notation is naturally extended to several distinct variables (e.g. $\sigma_{\nu_1:=d_1,\nu_2:=d_2}$).

Lemma 8.2.6. Let $\mathcal{D}$ be a domain, σ an $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment, t an $\mathcal{L}^1$-term, and x a variable of $\mathcal{L}^1$. For every $\mathcal{L}^1$-term t': $\sigma^I[t'\{^t/_x\}] = \sigma^I_{x:=\sigma^I[t]}[t']$.

Proof. By usual induction on the structure of t'. □

We can now define the truth value assigned by a given structure to an arbitrary formula with respect to some assignment. This definition generalizes in a natural way the usual recursive definition used in classical higher-order logics, where instead of the usual truth tables we use their counterparts of Gödel logic: $\wedge$ corresponds to min, $\vee$ to max, and the implication $\supset$ is interpreted as the $\to$ operation. For the quantifiers, we take inf and sup. Since the set of truth values is complete by definition, inf and sup are always defined.

Definition 8.2.7. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P\rangle$ be an $\mathcal{L}^1$-structure. For every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ, $\mathcal{W}[\varphi, \sigma]$ is the element of $\mathcal{V}$ inductively defined as follows:

$$\mathcal{W}[\varphi,\sigma] = \begin{cases} P[p][\sigma^I[t_1], \dots, \sigma^I[t_n]] & \varphi = \{p(t_1, \dots, t_n)\} \\ 0 & \varphi = \{\bot\} \\ \min\{\mathcal{W}[\varphi_1,\sigma], \mathcal{W}[\varphi_2,\sigma]\} & \varphi = (\varphi_1 \wedge \varphi_2) \\ \max\{\mathcal{W}[\varphi_1,\sigma], \mathcal{W}[\varphi_2,\sigma]\} & \varphi = (\varphi_1 \vee \varphi_2) \\ \mathcal{W}[\varphi_1,\sigma] \to \mathcal{W}[\varphi_2,\sigma] & \varphi = (\varphi_1 \supset \varphi_2) \\ \inf_{d\in\mathcal{D}} \mathcal{W}[\psi, \sigma_{x:=d}] & \varphi = (\forall x \psi) \\ \sup_{d\in\mathcal{D}} \mathcal{W}[\psi, \sigma_{x:=d}] & \varphi = (\exists x \psi) \end{cases}$$

It can be verified that the choice of x in the last definition is immaterial, and $\mathcal{W}[\varphi, \sigma]$ is well-defined. The following usual lemmas will be needed below:

Lemma 8.2.8. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^1$-structure, x a variable of $\mathcal{L}^1$, and d an element of $\mathcal{D}$. Then $\mathcal{W}[\varphi, \sigma_{x:=d}] = \mathcal{W}[\varphi, \sigma]$ for every $\mathcal{L}^1$-formula φ such that $x \notin Fv[\varphi]$, and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ.

Proof. After proving the claim for $\mathcal{L}^1$-terms (that $\sigma^I_{x:=d}[t] = \sigma^I[t]$ for every $\mathcal{L}^1$-term such that $x \notin Fv[t]$, and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ), the claim is obtained by usual induction on the complexity of φ. □

Lemma 8.2.9. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^1$-structure, t an $\mathcal{L}^1$-term, and x a variable of $\mathcal{L}^1$. For every $\mathcal{L}^1$-formula φ, and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ:

$$\mathcal{W}[\varphi, \sigma_{x:=\sigma^I[t]}] = \mathcal{W}[\varphi\{{}^t/_x\}, \sigma].$$

Proof. We prove the claim by induction on the complexity of φ. First, suppose that $cp[\varphi] = 1$, and let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = \{\bot\}$. In this case the claim obviously holds.
- $\varphi = \{p(t_1, \dots, t_n)\}$. In this case, $\varphi\{{}^t/_x\} = \{p(t_1\{{}^t/_x\}, \dots, t_n\{{}^t/_x\})\}$. Thus:
$$\mathcal{W}[\varphi\{{}^t/_x\}, \sigma] = P[p][\sigma^I[t_1\{{}^t/_x\}], \dots, \sigma^I[t_n\{{}^t/_x\}]].$$
By Lemma 8.2.6, $P[p][\sigma^I[t_1\{{}^t/_x\}], \dots, \sigma^I[t_n\{{}^t/_x\}] = P[p][\sigma^I_{x:=\sigma^I[t]}[t_1], \dots, \sigma^I_{x:=\sigma^I[t]}[t_n]]$. By definition, this is equal to $\mathcal{W}[\varphi, \sigma_{x:=\sigma^I[t]}]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^1$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. By definition, $\varphi\{{}^t/_x\} = (\varphi_1\{{}^t/_x\} \diamond \varphi_2\{{}^t/_x\})$. We continue with $\diamond = \supset$ (the proof is similar for $\wedge$ and $\vee$). By the induction hypothesis, $\mathcal{W}[\varphi\{{}^t/_x\}, \sigma] = \mathcal{W}[\varphi_1\{{}^t/_x\}, \sigma] \to \mathcal{W}[\varphi_2\{{}^t/_x\}, \sigma] = \mathcal{W}[\varphi_1, \sigma_{x:=\sigma^I[t]}] \to \mathcal{W}[\varphi_2, \sigma_{x:=\sigma^I[t]}]$. By definition, this is equal to $\mathcal{W}[\varphi, \sigma_{x:=\sigma^I[t]}]$.
- $\varphi = (Qy\psi)$ for some $Q \in \{\forall, \exists\}$, variable $y \notin \{x\} \cup Fv[t]$ of $\mathcal{L}^1$, and $\mathcal{L}^1$-formula ψ such that $cp[\psi] < cp[\varphi]$. By definition, $\varphi\{{}^t/_x\} = (Qy\psi\{{}^t/_x\})$. Continuing with $Q = \forall$ ($\exists$ is similar), we have $\mathcal{W}[\varphi\{{}^t/_x\}, \sigma] = \inf_{d \in \mathcal{D}} \mathcal{W}[\psi\{{}^t/_x\}, \sigma_{y:=d}]$. By the induction hypothesis, $\mathcal{W}[\psi\{{}^t/_x\}, \sigma_{y:=d}] = \mathcal{W}[\psi, \sigma_{y:=d,x:=\sigma^I[t]}]$ for every $d \in \mathcal{D}$ (note that $y \neq x$), and so $\mathcal{W}[\varphi\{{}^t/_x\}] = \inf_{d \in \mathcal{D}} \mathcal{W}[\psi, \sigma_{y:=d,x:=\sigma^I[t]}]$. By definition, $\inf_{d \in \mathcal{D}} \mathcal{W}[\psi, \sigma_{y:=d,x:=\sigma^I[t]}] = \mathcal{W}[\varphi, \sigma_{x:=\sigma^I[t]}]$. □

Finally, we define standard first-order Gödel logic. For simplicity, *unlike in the previous chapters*, we identify a logic with its set of theorems and do not consider consequence relations.

Definition 8.2.10. For an $\mathcal{L}^1$-formula φ, we write $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^1}} \varphi$ if $\mathcal{W}[\varphi, \sigma] = 1$ for every $\mathcal{L}^1$-structure $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ. $\mathbf{G\ddot{o}}_{\mathcal{L}^1}$ is the logic consisting of all formulas φ such that $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^1}} \varphi$.

8.3 The Hypersequent Calculus HIF

In this section we present the hypersequent calculus **HIF** for $\mathbf{G\ddot{o}}_{\mathcal{L}^1}$ from [30] (adapted to our definitions, where formulas are equivalence classes of concrete formulas). (single-conclusion) $\mathcal{L}^1$-sequents and $\mathcal{L}^1$-hypersequents are defined exactly as in Chapter 7. **HIF** is obtained by augmenting the calculus **HG** (see Chapter 7, and Table 7.1 in particular), with the following rules for first-order quantifiers:

$$(\mathtt{f}{:}\forall)\ \frac{H \mid \Gamma, \varphi\{{}^{t}/_{x}\} \Rightarrow E}{H \mid \Gamma, (\forall x \varphi) \Rightarrow E} \qquad (\mathtt{t}{:}\forall)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow (\forall x \varphi)}$$

$$(\mathtt{f}{:}\exists)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, (\exists x \varphi) \Rightarrow E} \qquad (\mathtt{t}{:}\exists)\ \frac{H \mid \Gamma \Rightarrow \varphi\{{}^{t}/_{x}\}}{H \mid \Gamma \Rightarrow (\exists x \varphi)}$$

Applications of the rules $(\mathtt{t}{:}\forall)$ and $(\mathtt{f}{:}\exists)$ must obey the eigenvariable condition: x is not a free variable in the lower hypersequent.

Below, we write $\vdash_{\mathbf{HIF}} H$ to denote that an $\mathcal{L}^1$-hypersequent H is provable in **HIF**, and $\vdash^{cf}_{\mathbf{HIF}} H$ to denote that H is provable in **HIF** without applying (*cut*). Several clarifications should be noted:

1. The above rules are formulated by schemes using metavariables. For example, an $\mathcal{L}^1$-hypersequent H_1 can be derived from an $\mathcal{L}^1$-hypersequent H_2 by applying the rule $(\mathtt{f}{:}\forall)$ iff $H_1 = H \cup \{\Gamma \Rightarrow (\forall x \varphi)\}$ and $H_2 = H \cup \{\Gamma \cup \varphi\{{}^{t}/_{x}\} \Rightarrow E\}$ for some $\mathcal{L}^1$-hypersequent H, finite set Γ of $\mathcal{L}^1$-formulas, variable x of $\mathcal{L}^1$, $\mathcal{L}^1$-formula φ, $\mathcal{L}^1$-term t, and singleton or empty set E of $\mathcal{L}^1$-formulas.

2. Since formulas are equivalence classes, the rules $(\mathtt{t}{:}\forall)$, $(\mathtt{f}{:}\exists)$ could be written as well as:

$$(\mathtt{t}{:}\forall)\ \frac{H \mid \Gamma \Rightarrow \varphi\{{}^{y}/_{x}\}}{H \mid \Gamma \Rightarrow (\forall x \varphi)} \qquad (\mathtt{f}{:}\exists)\ \frac{H \mid \Gamma, \varphi\{{}^{y}/_{x}\} \Rightarrow E}{H \mid \Gamma, (\exists x \varphi) \Rightarrow E}$$

   where y is not a free variable in the lower hypersequent.

The following standard lemma establishes the admissibility of substitution:

Lemma 8.3.1. If $\vdash_{\mathbf{HIF}} H$, then $\vdash_{\mathbf{HIF}} H\{y/x\}$ for every variables x, y of $\mathcal{L}^1$, such that $y \notin Fv[H]$. The same holds for $\vdash^{cf}_{\mathbf{HIF}}$.

Next, the rule $(\mathtt{f}{:}\exists)$ can be generalized as follows.

Proposition 8.3.2. For every $n \geq 0$, $\mathcal{L}^1$-hypersequent H, $\mathcal{L}^1$-sequents $s_1, \ldots, s_n$, $\mathcal{L}^1$-formula φ, and variable $x \notin Fv[H \mid s_1 \mid \ldots \mid s_n]$: if $\vdash_{\mathbf{HIF}} H \mid s_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}\varphi\}$ then $\vdash_{\mathbf{HIF}} H \mid s_1 \cup \{\mathtt{f}{:}(\exists x \varphi)\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}(\exists x \varphi)\}$. The same holds for $\vdash^{cf}_{\mathbf{HIF}}$.

Proof. We use induction on n. The claim is trivial for $n = 0$. Now assume that the claim holds for $n - 1$, we prove it for n. Let H be an $\mathcal{L}^1$-hypersequent, $s_1, \ldots, s_n$ $\mathcal{L}^1$-sequents, φ an $\mathcal{L}^1$-formula, and x a variable of $\mathcal{L}^1$ such that $x \notin Fv[H \mid s_1 \mid \ldots \mid s_n]$. Let $H_0 = H \mid s_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}\varphi\}$. Suppose that $\vdash_{\mathbf{HIF}} H_0$. Let y be a variable of $\mathcal{L}^1$ such that $y \notin Fv[H_0]$. By Lemma 8.3.1, $\vdash_{\mathbf{HIF}} H_0\{y/x\}$. Using a generalized version of (*com*) (see Proposition 7.2.10), the following $\mathcal{L}^1$-hypersequent is cut-free derivable from H_0 and $H_0\{y/x\}$: $\{H \mid s_n \cup \{\mathtt{f}{:}\varphi\} \mid s_1 \cup \{\mathtt{f}{:}\varphi\{y/x\}\} \mid \ldots \mid s_{n-1} \cup \{\mathtt{f}{:}\varphi\{y/x\}\}$ (to see this, take $H_1 = H \mid s_n \cup \{\mathtt{f}{:}\varphi\}$ and $H_2 = H \mid s_1 \cup \{\mathtt{f}{:}\varphi\{y/x\}\} \mid \ldots \mid s_{n-1} \cup \{\mathtt{f}{:}\varphi\{y/x\}\}$). By an application of $(\mathtt{f}{:}\exists)$, we obtain: $H \mid s_n \cup \{\mathtt{f}{:}(\exists x \varphi)\} \mid s_1 \cup \{\mathtt{f}{:}\varphi\{y/x\}\} \mid \ldots \mid s_{n-1} \cup \{\mathtt{f}{:}\varphi\{y/x\}\}$. The induction hypothesis entails that $\vdash_{\mathbf{HIF}} H \mid s_1 \cup \{\mathtt{f}{:}(\exists x \varphi)\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}(\exists x \varphi)\}$. Since cuts were not involved in this proof, the proof for $\vdash^{cf}_{\mathbf{HIF}}$ is exactly the same. $\square$

8.4 Soundness

In this section we prove the soundness of **HIF** for $\mathbf{G\ddot{o}}_{\mathcal{L}^1}$. The following definition is the first-order version of Definition 7.3.4.

Definition 8.4.1. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^1$-structure.

1. Given an $\mathcal{L}^1$-sequent s and an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ, $\mathcal{W}^{\mathtt{f}}[s, \sigma]$, $\mathcal{W}^{\mathtt{t}}[s, \sigma]$ and $\mathcal{W}[s, \sigma]$ are defined as follow:

   (a) $\mathcal{W}^{\mathtt{f}}[s, \sigma] = \min\{\mathcal{W}[\varphi, \sigma] \mid \mathtt{f}{:}\varphi \in s\}$.

   (b) $\mathcal{W}^{\mathtt{t}}[s, \sigma] = \max\{\mathcal{W}[\varphi, \sigma] \mid \mathtt{t}{:}\varphi \in s\}$.

   (c) $\mathcal{W}[s, \sigma] = \mathcal{W}^{\mathtt{f}}[s, \sigma] \to \mathcal{W}^{\mathtt{t}}[s, \sigma]$.

2. An $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ is a *model* (with respect to $\mathcal{W}$) of:

   (a) an *$\mathcal{L}^1$-sequent* s (denoted by: $\mathcal{W}, \sigma \models s$) if $\mathcal{W}[s, \sigma] = 1$.

   (b) an *$\mathcal{L}^1$-hypersequent* H (denoted by: $\mathcal{W}, \sigma \models H$) if $\mathcal{W}, \sigma \models s$ for some $s \in H$.

3. $\mathcal{W}$ is a *model* of an $\mathcal{L}^1$-hypersequent H if $\mathcal{W}, \sigma \models H$ for every $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ.

Theorem 8.4.2. Let H be an $\mathcal{L}^1$-hypersequent. If $\vdash_{\mathbf{HIF}} H$, then every $\mathcal{L}^1$-structure is a model of H.

Proof. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^1$-structure, where $\mathcal{V} = \langle V, \leq \rangle$. It suffices to prove soundness of each possible application of a rule of **HIF**. For the weakening rules, (*cut*), (*com*), and (*id*) this is as in Section 7.5. We do here several other cases and leave the rest to the reader:

($\mathtt{t}{:}\supset$) Suppose that $H \mid s \cup \{\mathtt{t}{:}(\varphi_1 \supset \varphi_2)\}$ is derived from $H \mid s \cup \{\mathtt{f}{:}\varphi_1, \mathtt{t}{:}\varphi_2\}$ using ($\mathtt{t}{:}\supset$) (here s must be a negative $\mathcal{L}^1$-sequent). Let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. If $\mathcal{W}, \sigma \models s'$ for some $s' \in H$, then we are done. Otherwise, $\mathcal{W}, \sigma \models s \cup \{\mathtt{f}{:}\varphi_1, \mathtt{t}{:}\varphi_2\}$. Thus, either $\mathcal{W}[\varphi_1, \sigma] \leq \mathcal{W}^{\mathtt{f}}[s, \sigma]$ and $\mathcal{W}[\varphi_1, \sigma] \leq \mathcal{W}[\varphi_2, \sigma]$, or $\mathcal{W}^{\mathtt{f}}[s, \sigma] \leq \mathcal{W}[\varphi_2, \sigma]$. In both cases, it follows that $\mathcal{W}^{\mathtt{f}}[s, \sigma] \leq \mathcal{W}[\varphi_1, \sigma] \to \mathcal{W}[\varphi_2, \sigma]$. By definition, $\mathcal{W}[\varphi_1, \sigma] \to \mathcal{W}[\varphi_2, \sigma] = \mathcal{W}[\varphi_1 \supset \varphi_2, \sigma]$. It follows that $\mathcal{W}, \sigma \models s \cup \{\mathtt{t}{:}(\varphi_1 \supset \varphi_2)\}$, and so $\mathcal{W}, \sigma \models H \mid s \cup \{\mathtt{t}{:}(\varphi_1 \supset \varphi_2)\}$.

($\mathtt{f}{:}\supset$) Suppose that $H \mid s' \cup s \cup \{\mathtt{f}{:}(\varphi_1 \supset \varphi_2 \Rightarrow E$ is derived from $H \mid s' \cup \{\mathtt{t}{:}\varphi_1\}$ and $H \mid s \cup \{\mathtt{f}{:}\varphi_2\}$ using ($\mathtt{f}{:}\supset$) (here s' must be a negative $\mathcal{L}^1$-sequent). Let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. If $\mathcal{W}, \sigma \models s''$ for some $s'' \in H$, then we are done. Otherwise, $\mathcal{W}, \sigma \models s' \cup \{\mathtt{t}{:}\varphi_1\}$ and $\mathcal{W}, \sigma \models s \cup \{\mathtt{f}{:}\varphi_2\}$. Let $u_1 = \mathcal{W}^{\mathtt{f}}[s' \cup s, \sigma]$ and $u_2 = \mathcal{W}^{\mathtt{t}}[s, \sigma]$. If $u_1 \leq u_2$, then $\mathcal{W}, \sigma \models \Gamma, (\varphi_1 \supset \varphi_2) \Rightarrow E$, and we are done. Otherwise, we have $u_1 \leq \mathcal{W}[\varphi_1, \sigma]$, $\mathcal{W}[\varphi_2, \sigma] \leq u_2$, and so $\mathcal{W}[\varphi_2, \sigma] < \mathcal{W}[\varphi_1, \sigma]$. It follows that

$$\mathcal{W}[(\varphi_1 \supset \varphi_2), \sigma] = \mathcal{W}[\varphi_1, \sigma] \to \mathcal{W}[\varphi_2, \sigma] = \mathcal{W}[\varphi_2, \sigma] \leq u_2.$$

Consequently, $\mathcal{W}, \sigma \models s' \cup s \cup \{\mathtt{f}{:}(\varphi_1 \supset \varphi_2)\}$ in this case as well.

($\mathtt{t}{:}\forall$) Suppose that $H = H' \mid s \cup \{\mathtt{t}{:}(\forall x \varphi)\}$ is derived from $H' \mid s \cup \{\mathtt{t}{:}\varphi\}$ using ($\mathtt{t}{:}\forall$) (where $x \notin Fv[H]$, and s is a negative $\mathcal{L}^1$-sequent). Assume that $\mathcal{W}$ is not a model of H. Thus there exists an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ, such that $\mathcal{W}, \sigma \not\models H$. Hence, $\mathcal{W}, \sigma \not\models s'$ for every $s' \in H'$, and $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{t}{:}(\forall x \varphi)\}$. Thus we have $\mathcal{W}^{\mathtt{f}}[s, \sigma] > \mathcal{W}[(\forall x \varphi), \sigma]$. It follows that $\mathcal{W}^{\mathtt{f}}[s, \sigma] > \mathcal{W}[\varphi, \sigma_{x:=d}]$ for some $d \in \mathcal{D}$. Since $x \notin Fv[s]$, by Lemma 8.2.8 we have that $\mathcal{W}[\psi, \sigma_{x:=d}] = \mathcal{W}[\psi, \sigma]$ for every ψ that occurs in s. It follows that $\mathcal{W}, \sigma_{x:=d} \not\models s \cup \{\mathtt{t}{:}\varphi\}$. Moreover, since $x \notin Fv[H']$, again using by Lemma 8.2.8, we obtain that $\mathcal{W}, \sigma_{x:=d} \not\models s'$ for every $s' \in H'$. Hence, $\mathcal{W}, \sigma_{x:=d} \not\models H' \mid s \cup \{\mathtt{t}{:}\varphi\}$, and consequently $\mathcal{W}$ is not a model of $H' \mid s \cup \{\mathtt{t}{:}\varphi\}$.

($\mathtt{f}{:}\forall$) Suppose that $H = H' \mid s \cup \{\mathtt{f}{:}(\forall x \varphi)\}$ is derived from $H' \mid s \cup \{\mathtt{f}{:}\varphi\{^t/_x\}\}$ using ($\mathtt{f}{:}\forall$). Assume that $\mathcal{W}, \sigma \not\models H$ for some $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ. Hence, $\mathcal{W}, \sigma \not\models s'$ for every $s' \in H'$, and $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{f}{:}(\forall x \varphi)\}$. Let $u = \mathcal{W}^{\mathtt{t}}[s, \sigma]$. The assumption that $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{f}{:}(\forall x \varphi)\}$ entails that $\mathcal{W}^{\mathtt{f}}[s, \sigma] > u$ and $\mathcal{W}[(\forall x \varphi), \sigma] > u$. By definition,

$\mathcal{W}[(\forall x\varphi),\sigma] = \inf_{d\in\mathcal{D}} \mathcal{W}[\varphi,\sigma_{x:=d}]$. Thus $\mathcal{W}[\varphi,\sigma_{x:=d}] > u$ for every $d \in \mathcal{D}$. In particular, $\mathcal{W}[\varphi,\sigma_{x:=\sigma^I[t]}] > u$. Lemma 8.2.9 implies that $\mathcal{W}[\varphi\{^t/_x\},\sigma] > u$. It follows that $\mathcal{W},\sigma \not\models s \cup \{\mathtt{f}{:}\varphi\{^t/_x\}\}$. Hence $\mathcal{W}$ is not a model of $H' \mid s \cup \{\mathtt{f}{:}\varphi\{^t/_x\}\}$. □

Soundness for $\mathbf{G\ddot{o}}_{\mathcal{L}^1}$ is an obvious corollary:

Corollary 8.4.3. For every $\mathcal{L}^1$-formula φ, if $\vdash_{\mathbf{HIF}} \{\{\mathtt{t}{:}\varphi\}\}$ then $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^1}} \varphi$.

Proof. Follows from Theorem 8.4.2, since for every $\mathcal{L}^1$-structure $\mathcal{W} = \langle \mathcal{V},\mathcal{D},I,P\rangle$ and $\langle \mathcal{L}^1,\mathcal{D}\rangle$-assignment σ: σ is a model of $\{\{\mathtt{t}{:}\varphi\}\}$ with respect to $\mathcal{W}$ iff $\mathcal{W}[\varphi,\sigma] = 1$. □

8.5 Complete Non-deterministic Semantics

In this section we present a non-deterministic semantics for which the cut-free fragment of **HIF** is complete. This semantics will be used in the next section, where we show that ordinary counter-models can be extracted out of non-deterministic ones. As corollaries, we will obtain the completeness of **HIF** for the (usual) semantics described above, and the fact that (*cut*) is admissible in **HIF**.

The non-deterministic semantics is based on *quasi*-$\mathcal{L}^1$-structures. The idea behind these structures is similar to what we had in Chapter 7 for accommodating systems in which (*cut*) is not available (i.e. for the proof-specification $\langle \mathcal{L},\emptyset,\mathcal{L}\rangle$). Thus quasi-$\mathcal{L}^1$-structures assign *two* truth values to each formula – one for its "f-labelled" occurrences, and one for its "t-labelled" ones (see Page 113).

Definition 8.5.1. Let $\mathcal{V} = \langle V,\leq\rangle$ be a Gödel set. Given some non-empty set $\mathcal{D}$, a function D from $\mathcal{D}$ to $\{\langle u^{\mathtt{f}},u^{\mathtt{t}}\rangle \in \mathcal{V}\times\mathcal{V} \mid u^{\mathtt{f}} \leq u^{\mathtt{t}}\}$ is called a *quasi fuzzy subset* of $\mathcal{D}$ over $\mathcal{V}$.

Definition 8.5.2. A *quasi-$\mathcal{L}^1$-structure* is a tuple $\mathcal{Q} = \langle \mathcal{V},\mathcal{D},I,P,v\rangle$, where:

1. $\mathcal{V}$, $\mathcal{D}$, and I are defined as in $\mathcal{L}^1$-structures (Definition 8.2.4).
2. P is a function assigning a *quasi* fuzzy subset of $\mathcal{D}^n$ over $\mathcal{V}$ to every n-ary predicate symbol of $\mathcal{L}^1$.
3. v is a function assigning a pair in $\{\langle u^{\mathtt{f}},u^{\mathtt{t}}\rangle \in \mathcal{V}\times\mathcal{V} \mid u^{\mathtt{f}} \leq u^{\mathtt{t}}\}$ to every ordered pair of the form $\langle\varphi,\sigma\rangle$, where φ is an $\mathcal{L}^1$-formula and σ is an $\langle \mathcal{L}^1,\mathcal{D}\rangle$-assignment, such that $v[\varphi,\sigma_{x:=d}] = v[\varphi\{^y/_x\},\sigma_{y:=d}]$ for every variable x, variable $y \notin Fv[\varphi]$, and $d \in \mathcal{D}$.

Note that quasi-structures include a function v that assigns truth values to *every* formula. This is related to the fact that the semantics is *non-deterministic.* Thus, in contrast to (ordinary) structures, in quasi structures the values of the atomic formulas do not uniquely determine the values of all compound formulas. The function v is then used to "store“ the values of the compound formulas. Obviously, in order to be able to extract an ordinary counter-model out of a quasi-structure, further conditions should be imposed:

Notation 8.5.3. For each function F whose range is $\{\langle u^{\mathtt{f}}, u^{\mathtt{t}}\rangle \in \mathcal{V} \times \mathcal{V} \mid u^{\mathtt{f}} \leq u^{\mathtt{t}}\}$ (e.g. the functions $P[p]$ for every predicate symbol p and v from Definition 8.5.2), we denote by $F^{\mathtt{f}}$ and $F^{\mathtt{t}}$ the functions obtained from F by taking only the left and the right components (respectively). For instance, for every φ and σ, $v^{\mathtt{f}}[\varphi, \sigma]$ is the left component of the pair $v[\varphi, \sigma]$.

Definition 8.5.4. Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$ be a quasi-$\mathcal{L}^1$-structure. For every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ, $\underline{v}_{\mathcal{Q}}[\varphi, \sigma]$ is the pair defined as follows:

$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = \begin{cases} P[p][\sigma^I[t_1], \dots, \sigma^I[t_n]] & \varphi = \{p(t_1, \dots, t_n)\} \\ \langle 0, 0\rangle & \varphi = \{\bot\} \\ \langle \min\{v^{\mathtt{f}}[\varphi_1,\sigma], v^{\mathtt{f}}[\varphi_2,\sigma]\}, \min\{v^{\mathtt{t}}[\varphi_1,\sigma], v^{\mathtt{t}}[\varphi_2,\sigma]\}\rangle & \varphi = (\varphi_1 \wedge \varphi_2) \\ \langle \max\{v^{\mathtt{f}}[\varphi_1,\sigma], v^{\mathtt{f}}[\varphi_2,\sigma]\}, \max\{v^{\mathtt{t}}[\varphi_1,\sigma], v^{\mathtt{t}}[\varphi_2,\sigma]\}\rangle & \varphi = (\varphi_1 \vee \varphi_2) \\ \langle v^{\mathtt{t}}[\varphi_1,\sigma] \to v^{\mathtt{f}}[\varphi_2,\sigma], v^{\mathtt{f}}[\varphi_1,\sigma] \to v^{\mathtt{t}}[\varphi_2,\sigma]\rangle & \varphi = (\varphi_1 \supset \varphi_2) \\ \langle \inf_{d\in\mathcal{D}} v^{\mathtt{f}}[\psi, \sigma_{x:=d}], \inf_{d\in\mathcal{D}} v^{\mathtt{t}}[\psi, \sigma_{x:=d}]\rangle & \varphi = (\forall x\psi) \\ \langle \sup_{d\in\mathcal{D}} v^{\mathtt{f}}[\psi, \sigma_{x:=d}], \sup_{d\in\mathcal{D}} v^{\mathtt{t}}[\psi, \sigma_{x:=d}]\rangle & \varphi = (\exists x\psi) \end{cases}$$

The condition on v in Definition 8.5.2 ensures that $\mathcal{Q}$ is well-defined, namely that the choice of x is immaterial. It is straightforward to verify that $\underline{v}_{\mathcal{Q}}[\varphi,\sigma]^{\mathtt{f}} \leq \underline{v}_{\mathcal{Q}}[\varphi,\sigma]^{\mathtt{t}}$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ (for $\supset$, note that if $u_1 \leq u_2$ and $u_3 \leq u_4$ then $u_2 \to u_3 \leq u_1 \to u_4$).

Notation 8.5.5. Let $\mathcal{V} = \langle V, \leq\rangle$ be a Gödel set. For $u \in \mathcal{V}$ and a pair $\langle u^{\mathtt{f}}, u^{\mathtt{t}}\rangle \in \mathcal{V} \times \mathcal{V}$ with $u^{\mathtt{f}} \leq u^{\mathtt{t}}$, we write $u \in \langle u^{\mathtt{f}}, u^{\mathtt{t}}\rangle$ if $u^{\mathtt{f}} \leq u \leq u^{\mathtt{t}}$. For two pairs $\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle, \langle u_2^{\mathtt{f}}, u_2^{\mathtt{t}}\rangle$ in $\{\langle u^{\mathtt{f}}, u^{\mathtt{t}}\rangle \in \mathcal{V} \times \mathcal{V} \mid u^{\mathtt{f}} \leq u^{\mathtt{t}}\}$, we write $\langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle \subseteq \langle u_2^{\mathtt{f}}, u_2^{\mathtt{t}}\rangle$ if $u_1^{\mathtt{f}} \geq u_2^{\mathtt{f}}$ and $u_1^{\mathtt{t}} \leq u_2^{\mathtt{t}}$.

Definition 8.5.6. A quasi-$\mathcal{L}^1$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$ is called *legal* if we have $\underline{v}_{\mathcal{Q}}[\varphi,\sigma] \subseteq v[\varphi,\sigma]$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ.

We can now demonstrate the non-deterministic nature of the semantics. For instance, consider an $\mathcal{L}^1$-formula of the form $(\varphi_1 \wedge \varphi_2)$. Suppose that $v[\varphi_1, \sigma] = \langle u_1^{\mathtt{f}}, u_1^{\mathtt{t}}\rangle$ and $v[\varphi_2, \sigma] = \langle u_2^{\mathtt{f}}, u_2^{\mathtt{t}}\rangle$. All we require from $v[(\varphi_1 \wedge \varphi_2), \sigma]$ is that

$$\langle \min\{u_1^{\mathtt{f}}, u_2^{\mathtt{f}}\}, \min\{u_1^{\mathtt{t}}, u_2^{\mathtt{t}}\}\rangle \subseteq v[(\varphi_1 \wedge \varphi_2), \sigma].$$

In other words, every pair $\langle u^{\mathtt{f}}, u^{\mathtt{t}} \rangle$ such that $u^{\mathtt{f}} \leq \min\{u_1^{\mathtt{f}}, u_2^{\mathtt{f}}\}$ and $\min\{u_1^{\mathtt{t}}, u_2^{\mathtt{t}}\} \leq u^{\mathtt{t}}$ can be chosen as a value for $v[(\varphi_1 \wedge \varphi_2), \sigma]$. Thus, in contrast to ordinary structures, here the values of $\langle \varphi_1, \sigma \rangle$ and $\langle \varphi_2, \sigma \rangle$ do not uniquely determine the value of $\langle (\varphi_1 \wedge \varphi_2), \sigma \rangle$.

Remark 8.5.7. Since formulas are defined to be alpha equivalence classes of concrete formulas, we do not have to explicitly enforce that two alpha-equivalent formulas obtain the same value. Previous works on non-deterministic semantics for languages with quantifiers, such as [18], studied structures in which truth values are non-deterministically assigned to concrete formulas. In this case, additional restrictions are needed.

The notion of *model* for quasi-$\mathcal{L}^1$-structures is a natural first-order version of what we had for Gvaluations in Chapter 7 (see Definition 7.3.4):

Definition 8.5.8. Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$ be a quasi-$\mathcal{L}^1$-structure.

1. Given an $\mathcal{L}^1$-sequent s and an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ, $\mathcal{Q}^{\mathtt{f}}[s, \sigma]$, $\mathcal{Q}^{\mathtt{t}}[s, \sigma]$ and $\mathcal{Q}[s, \sigma]$ are defined as follow:

   (a) $\mathcal{Q}^{\mathtt{f}}[s, \sigma] = \min\{v^{\mathtt{f}}[\varphi, \sigma] \mid \mathtt{f}{:}\varphi \in s\}$.
   (b) $\mathcal{Q}^{\mathtt{t}}[s, \sigma] = \max\{v^{\mathtt{t}}[\varphi, \sigma] \mid \mathtt{t}{:}\varphi \in s\}$.
   (c) $\mathcal{Q}[s, \sigma] = \mathcal{Q}^{\mathtt{f}}[s, \sigma] \rightarrow \mathcal{Q}^{\mathtt{t}}[s, \sigma]$.

2. An $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ is a *model* (with respect to $\mathcal{Q}$) of:

   (a) an *$\mathcal{L}^1$-sequent* s (denoted by: $\mathcal{Q}, \sigma \models s$) if $\mathcal{Q}[s, \sigma] = 1$.
   (b) an *$\mathcal{L}^1$-hypersequent* H (denoted by: $\mathcal{Q}, \sigma \models H$) if $\mathcal{Q}, \sigma \models s$ for some $s \in H$.

3. $\mathcal{Q}$ is a *model* of an $\mathcal{L}^1$-hypersequent H if $\mathcal{Q}, \sigma \models H$ for every $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ.

Now we can state the main completeness theorem.

Theorem 8.5.9. Suppose that $\not\vdash^{cf}_{\mathbf{HIF}} H_0$ for some $\mathcal{L}^1$-hypersequent H_0. Then there exists a legal quasi-$\mathcal{L}^1$-structure that is not a model of H_0.

The rest of this section is devoted to prove this theorem. First, we introduce the two main ingredients of this proof: maximal extended hypersequents and Herbrand domains.

Maximal Extended Hypersequents

As in Chapter 7, maximal extended $\mathcal{L}^1$-hypersequents will play a crucial role in the completeness proof below. These are defined exactly as in Section 7.5, but here we add one more requirement:

Definition 8.5.10. An extended $\mathcal{L}^1$-sequent μ admits *the witness property* if the following hold for every $\mathcal{L}^1$-formula φ, and variable x of $\mathcal{L}^1$:

1. If $\mathtt{t}{:}(\forall x \varphi) \in \mu$, then $\mathtt{t}{:}\varphi\{y/x\} \in \mu$ for some variable y of $\mathcal{L}^1$.
2. If $\mathtt{f}{:}(\exists x \varphi) \in \mu$, then $\mathtt{f}{:}\varphi\{y/x\} \in \mu$ for some variable y of $\mathcal{L}^1$.

An extended $\mathcal{L}^1$-hypersequent Ω admits *the witness property* if every $\mu \in \Omega$ admits the witness property. Ω is called *maximal* if it is unprovable, internally maximal, externally maximal, and it admits the witness property (see Section 7.5).[4]

As in Section 7.5, the following hold:

Proposition 8.5.11. Let Ω be an extended $\mathcal{L}^1$-hypersequent.

- Assume that Ω is maximal with respect to an $\mathcal{L}^1$-formula φ. For every $\mu \in \Omega$:
  - If $\mathtt{f}{:}\varphi \notin \mu$, then $\vdash^{cf}_{\mathbf{HIF}} H \mid s_1 \cup \{\mathtt{f}{:}\varphi\} \mid \ldots \mid s_n \cup \{\mathtt{f}{:}\varphi\}$ for some $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega$ and $\mathcal{L}^1$-sequents $s_1, \ldots, s_n \subseteq \mu$.
  - If $\mathtt{t}{:}\varphi \notin \mu$, then $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{t}{:}\varphi\}$ for some $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega$ and negative $\mathcal{L}^1$-sequent $s \subseteq \mu$.
- Assume that Ω is maximal with respect to an $\mathcal{L}^1$-sequent s. Then, if $\{s\} \not\sqsubseteq \Omega$, then there exists an $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega$ such that $\vdash^{cf}_{\mathbf{HIF}} H \mid s$.

Lemma 8.5.12. Every unprovable $\mathcal{L}^1$-hypersequent can be extended to a *maximal* extended $\mathcal{L}^1$-hypersequent.

Proof. The proof proceeds similarly to the proof of Lemma 7.5.4. To obtain an extended $\mathcal{L}^1$-hypersequent that admits the witness property, we add another step in the definition of H_{i+1}, based on the following claim:

Let $\Omega = \mu_1 \mid \ldots \mid \mu_n$ be an unprovable finite extended $\mathcal{L}^1$-hypersequent. Then there exists an unprovable finite extended $\mathcal{L}^1$-hypersequent Ω' of the form $\mu'_1 \mid \ldots \mid \mu'_n$, such that $\mu_i \subseteq \mu'_i$ for every $1 \leq i \leq n$, and Ω' admits the witness property.

To prove this claim describe an extension of Ω to Ω'. This extension is done in steps.[5] In every step, we take some extended $\mathcal{L}^1$-sequent $\mu \in \Omega$, and proceed as follows:

- If μ contains a labelled formula of the form $\mathtt{t}{:}(\forall x \varphi)$, we take a variable y of $\mathcal{L}^1$, which is not free in the current hypersequent, and add $\mathtt{t}{:}\varphi\{y/x\}$ to μ.

[4] Obviously, instead of $\mathcal{H} \vdash_{\mathbf{G}\restriction\rho} H$ that we had in the definition of "provable extended hypersequent" in Section 7.5 , we should have $\vdash^{cf}_{\mathbf{HIF}} H$ for the present case.

[5] Formally, this extension should be defined inductively, but the intention should be clear.

- If μ contains a labelled formula of the form $\texttt{f}{:}(\exists x \varphi)$, we take a variable y of $\mathcal{L}^1$, which is not free in the current hypersequent, and add $\texttt{f}{:}\varphi\{y/x\}$ to μ.

We continue this procedure until the obtained extended $\mathcal{L}^1$-hypersequent admits the witness property. Note that since the number of formulas in Ω is finite, and the complexity of the formulas which are added is decreasing, this procedure would terminate after a finite number of steps. Ω' is the finite extended $\mathcal{L}^1$-hypersequent obtained from Ω by this procedure. We show that every such extension keeps the extended $\mathcal{L}^1$-hypersequent unprovable (and thus Ω' is unprovable):

- Suppose that an unprovable extended $\mathcal{L}^1$-hypersequent Ω_1 contains an extended $\mathcal{L}^1$-sequent μ, that contains a labelled formula of the form $\texttt{t}{:}(\forall x \varphi)$. Let Ω_2 be the extended $\mathcal{L}^1$-hypersequent obtained from Ω_1 by adding $\texttt{t}{:}\varphi\{y/x\}$ to μ, where y is a variable which does not occur in $Fv[\Omega_1]$. Assume for contradiction that Ω_2 is provable. Hence there exist an $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega_2$, and a negative $\mathcal{L}^1$-sequent $s' \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H \mid s' \cup \{\texttt{t}{:}\varphi\{y/x\}\}$. By applying $(\texttt{t}{:}\forall)$, we obtain $\vdash^{cf}_{\mathbf{HIF}} H \mid s' \cup \{\texttt{t}{:}(\forall x \varphi)\}$. This contradicts the fact the Ω_1 is unprovable.

- Suppose that an unprovable extended $\mathcal{L}^1$-hypersequent Ω_1 contains an extended $\mathcal{L}^1$-sequent μ, that contains a labelled formula of the form $\texttt{f}{:}(\exists x \varphi)$. Let Ω_2 be the extended $\mathcal{L}^1$-hypersequent obtained from Ω_1 by adding $\texttt{f}{:}\varphi\{y/x\}$ to μ, where y is a variable which does not occur in $Fv[\Omega_1]$. Assume for contradiction that Ω_2 is provable. Hence $\vdash^{cf}_{\mathbf{HIF}} H \mid s'_1 \cup \{\texttt{f}{:}\varphi\{y/x\}\} \mid \ldots \mid s'_n \cup \{\texttt{f}{:}\varphi\{y/x\}\}$ for some $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega_2$, and $\mathcal{L}^1$-sequents $s'_1, \ldots, s'_n \subseteq \mu$. Proposition 8.3.2 entails that $\vdash^{cf}_{\mathbf{HIF}} H \mid s'_1 \cup \{\texttt{f}{:}(\exists x \varphi)\} \mid \ldots \mid s'_n \cup \{\texttt{f}{:}(\exists x \varphi)\}$. This contradicts the fact the Ω_1 is unprovable. □

The Herbrand Domain

Definition 8.5.13. The *Herbrand domain for $\mathcal{L}^1$*, denoted by $\mathcal{D}^{\mathcal{L}^1}$, is the domain consisting of all $\mathcal{L}^1$-terms. The *Herbrand interpretation for $\mathcal{L}^1$*, denoted by $I^{\mathcal{L}^1}$, is the $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-interpretation defined by: $I[c] = c$ for every constant symbol c of $\mathcal{L}^1$, and $I[f] = \lambda t_1, \ldots, t_n \in \mathcal{D}^{\mathcal{L}^1}.f(t_1, \ldots, t_n)$ for every n-ary function symbol f of $\mathcal{L}^1$.

Below, given an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment σ and an $\mathcal{L}^1$-term t, we write $\sigma[t]$ instead of $\sigma^{I_{\mathcal{L}^1}}[t]$ (see Definition 8.2.5). In addition, $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignments are extended to apply on $\mathcal{L}$-formulas. Roughly speaking, every occurrence of a free variable x in a formula φ is replaced in $\sigma[\varphi]$ by $\sigma[x]$. Formally, this is defined as follows.

Definition 8.5.14. Let φ be an $\mathcal{L}^1$-formula, and σ an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment. The set of free variables of the pair $\langle \varphi, \sigma \rangle$ (denoted by $Fv[\langle \varphi, \sigma \rangle]$) consists of the variables of $\sigma[x]$ for every variable $x \in Fv[\varphi]$.

Definition 8.5.15. $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignments are extended to $\mathcal{L}^1$-formulas, according to the following inductive definition:

$$\sigma[\varphi] = \begin{cases} \{p(\sigma[t_1], \dots, \sigma[t_n])\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \{\bot\} & \varphi = \{\bot\} \\ (\sigma[\varphi_1] \diamond \sigma[\varphi_2]) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Qx\sigma_{x:=x}[\psi]) & \varphi = (Qx\psi) \text{ for } x \notin Fv[\langle \varphi, \sigma \rangle] \end{cases}$$

Note that the choice of x in the last definition is immaterial, and thus $\sigma[\varphi]$ is well-defined. The following technical lemmas are needed in the completeness proof.

Lemma 8.5.16. Let t be an $\mathcal{L}^1$-term.

1. $\sigma_{x:=t}[t'] = \sigma_{y:=t}[t'\{{}^y/_x\}]$ for every $\mathcal{L}^1$-term t', $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment σ, and variables x, y such that $y \notin Fv[t']$.
2. $\sigma_{x:=t}[\varphi] = \sigma_{y:=t}[\varphi\{{}^y/_x\}]$ for every $\mathcal{L}^1$-formula φ, $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment σ, and variables x, y such that $y \notin Fv[\varphi]$.

Proof. This first claim is proved by induction on the structure of t':

- Suppose that $t' = c$ for some constant symbol c of $\mathcal{L}^1$, or $t' = z$ for some variable $z \notin \{x, y\}$. Then $t'\{{}^y/_x\} = t'$, and $\sigma_{x:=t}[t'] = \sigma_{y:=t}[t']$.
- Suppose that $t' = x$. Then $\sigma_{x:=t}[t'] = t$, and $\sigma_{y:=t}[t'\{{}^y/_x\}] = \sigma_{y:=t}[y] = t$.
- Suppose that $t' = f(t_1, \dots, t_n)$ for some n-ary function symbol f of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. Then, $\sigma_{x:=t}[t'] = f(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])$. By the induction hypotheses this term equals $f(\sigma_{y:=t}[t_1\{{}^y/_x\}], \dots, \sigma_{y:=t}[t_n\{{}^y/_x\}])$, which in turn equals $\sigma_{y:=t}[f(t_1, \dots, t_n)\{{}^y/_x\}]$.

Next, we prove the second claim in the lemma by induction on the complexity of φ. First, suppose that $cp[\varphi] = 1$. Let σ be an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment, and x, y variables such that $y \notin Fv[\varphi]$. Exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^1$ and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. Then, by definition $\sigma_{x:=t}[\varphi] = \{p(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])\}$. Since $y \notin Fv[t_i]$ for every $1 \leq i \leq n$, this formula equals $\{p(\sigma_{y:=t}[t_1\{{}^y/_x\}], \dots, \sigma_{y:=t}[t_n\{{}^y/_x\}])\}$, which is, by definition, $\sigma_{y:=t}[\varphi\{{}^y/_x\}]$.
- $\varphi = \{\bot\}$. Then, $\sigma_{x:=t}[\varphi] = \{\bot\} = \sigma_{y:=t}[\varphi\{{}^y/_x\}]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^1$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment, and x, y variables such that $y \notin Fv[\varphi]$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. Then, $\sigma_{x:=t}[\varphi] = (\sigma_{x:=t}[\varphi_1] \diamond \sigma_{x:=t}[\varphi_2])$. By the induction hypothesis, this $\mathcal{L}^1$-formula is equal to $(\sigma_{y:=t}[\varphi_1\{{}^{y}/_{x}\}] \diamond \sigma_{y:=t}[\varphi_2\{{}^{y}/_{x}\}])$. And, by definition, this is equal to $\sigma_{y:=t}[\varphi\{{}^{y}/_{x}\}]$.
- $\varphi = (Qz\psi)$ for $Q \in \{\forall, \exists\}$, $\mathcal{L}^1$-formula ψ such that $cp[\psi] < cp[\varphi]$, and variable $z \notin \{x, y\} \cup \sigma[\varphi] \cup Fv[t]$. Then, $\sigma_{x:=t}[\varphi] = (Qz\sigma_{x:=t,z:=z}[\psi])$. By the induction hypothesis, this $\mathcal{L}^1$-formula is equal to $(Qz\sigma_{y:=t,z:=z}[\psi\{{}^{y}/_{x}\}])$. And this is (by definition) equal to $\sigma_{y:=t}[(Qz\psi\{{}^{y}/_{x}\})]$, which is equal to $\sigma_{y:=t}[\varphi\{{}^{y}/_{x}\}]$. □

Lemma 8.5.17. Let t be an $\mathcal{L}^1$-term.

1. $\sigma_{x:=z}[t']\{{}^{t}/_{z}\} = \sigma_{x:=t}[t']$ for every $\mathcal{L}^1$-term t', $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment σ, and variables x, z such that $z \notin Fv[\sigma[t']]$.
2. $\sigma_{x:=z}[\varphi]\{{}^{t}/_{z}\} = \sigma_{x:=t}[\varphi]$ for every $\mathcal{L}^1$-formula φ, $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment σ, and variables x, z such that $z \notin Fv[\sigma[\varphi]]$.

Proof. The first claim is proved by induction on the structure of t':

- Suppose that $t' = c$ for some constant symbol c of $\mathcal{L}^1$. Then:
$$\sigma_{x:=z}[t']\{{}^{t}/_{z}\} = I^{\mathcal{L}^1}[c]\{{}^{t}/_{z}\} = c\{{}^{t}/_{z}\} = c = I^{\mathcal{L}^1}[c] = \sigma_{x:=t}[t'].$$
- Suppose that $t' = y$ for a variable $y \neq x$. Then $\sigma_{x:=z}[t']\{{}^{t}/_{z}\} = \sigma[y]\{{}^{t}/_{z}\}$. Since $z \notin Fv[\sigma[y]]$, we have $\sigma[y]\{{}^{t}/_{z}\} = \sigma[y]$. The claim follows since $\sigma[y] = \sigma_{x:=t}[y]$.
- Suppose that $t' = x$. Then, $\sigma_{x:=z}[t']\{{}^{t}/_{z}\} = z\{{}^{t}/_{z}\} = t = \sigma_{x:=t}[x]$.
- Suppose that $t' = f(t_1, \dots, t_n)$ for some n-ary function symbol f of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. Then,
$$\sigma_{x:=z}[t']\{{}^{t}/_{z}\} = f(\sigma_{x:=z}[t_1], \dots, \sigma_{x:=z}[t_n])\{{}^{t}/_{z}\} = f(\sigma_{x:=z}[t_1]\{{}^{t}/_{z}\}, \dots, \sigma_{x:=z}[t_n]\{{}^{t}/_{z}\}).$$
By the induction hypotheses this term equals $f(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])$, which in turn equals $\sigma_{x:=t}[t']$.

Next, we prove the second claim in the lemma by induction on the complexity of φ. First, suppose that $cp[\varphi] = 1$. Let σ be an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment, and x, z variables such that $z \notin Fv[\sigma[\varphi]]$. Exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. Then, by definition
$$\begin{aligned}\sigma_{x:=z}[\varphi]\{{}^{t}/_{z}\} &= \{p(\sigma_{x:=z}[t_1], \dots, \sigma_{x:=z}[t_n])\}\{{}^{t}/_{z}\} = \\ &= \{p(\sigma_{x:=z}[t_1]\{{}^{t}/_{z}\}, \dots, \sigma_{x:=z}[t_n]\{{}^{t}/_{z}\})\}.\end{aligned}$$

Since $z \notin Fv[\sigma[t_i]]$ for every $1 \leq i \leq n$, the claim above for terms entails that this formula equals $\{p(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])\}$, which is, by definition, $\sigma_{x:=t}[\varphi]$.

- $\varphi = \{\bot\}$. Then, $\sigma_{x:=z}[\varphi]\{{}^t/_z\} = \{\bot\} = \sigma_{x:=t}[\varphi]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^1$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^1, \mathcal{D}^{\mathcal{L}^1} \rangle$-assignment, and x, z variables such that $z \notin Fv[\sigma[\varphi]]$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. Then, $\sigma_{x:=x}[\varphi]\{{}^t/_x\} = (\sigma_{x:=z}[\varphi_1]\{{}^t/_z\} \diamond \sigma_{x:=z}[\varphi_2]\{{}^t/_z\})$. By the induction hypothesis, this $\mathcal{L}^1$-formula is equal to $(\sigma_{x:=t}[\varphi_1] \diamond \sigma_{x:=t}[\varphi_2])$. And, by definition, this is equal to $\sigma_{x:=t}[\varphi]$.
- $\varphi = (Qy\psi)$ for $Q \in \{\forall, \exists\}$, $\mathcal{L}^1$-formula ψ such that $cp[\psi] < cp[\varphi]$, and variable $y \notin Fv[t] \cup \{x, z\} \cup Fv[\sigma[\varphi]]$. Then,
$$\sigma_{x:=z}[\varphi]\{{}^t/_z\} = (Qy\sigma_{x:=z,y:=y}[\psi])\{{}^t/_z\} = (Qy\sigma_{x:=z,y:=y}[\psi]\{{}^t/_z\}).$$
By the induction hypothesis, this $\mathcal{L}^1$-formula is equal to $(Qy\sigma_{x:=t,y:=y}[\psi])$. And this is (by definition) equal to $\sigma_{x:=t}[\varphi]$. □

Proof of Theorem 8.5.9

Suppose that $\not\vdash^{cf}_{\mathbf{HIF}} H_0$. The availability of external and internal weakenings ensures that H_0 is unprovable. By Lemma 8.5.12, there exists a maximal extended $\mathcal{L}^1$-hypersequent Ω^* such that $H_0 \sqsubseteq \Omega^*$. We use Ω^* to construct a counter-model for H_0 in the form of a quasi-$\mathcal{L}^1$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$.

First, we define a bounded linearly ordered set $\mathcal{V}_0$, that will be used to construct (using the Dedekind-MacNeille completion) the Gödel set $\mathcal{V}$. For every $\mathcal{L}^1$-formula φ we define:

$$L[\varphi] = \{\mu \in \Omega^* \mid \mathtt{f}{:}\varphi \in \mu\}, \quad R[\varphi] = \{\mu \in \Omega^* \mid \mathtt{t}{:}\varphi \notin \mu\}.$$

Let $\mathcal{V}_0 = \langle V_0, \subseteq \rangle$, where

$$V_0 = \{L(\varphi) \mid \varphi \text{ is an } \mathcal{L}^1\text{-formula}\} \cup \{R(\varphi) \mid \varphi \text{ is an } \mathcal{L}^1\text{-formula}\} \cup \{\Omega^*, \emptyset\}.$$

Clearly, $\mathcal{V}$ is partially ordered set, bounded by $0 = \emptyset$ and $1 = \Omega^*$. The proof that V is linearly ordered by $\subseteq$ proceeds exactly as in Section 7.5. Now, since $\mathcal{V}_0$ might not be complete, we consider its Dedekind-MacNeille completion $\mathcal{V} = \langle V, \subseteq \rangle$ defined by:

$$V = \{\Pi \subseteq V_0 \mid (\Pi^\uparrow)^\downarrow = \Pi\}$$

where $\Pi^\uparrow = \{\Omega \in V_0 \mid \Omega' \subseteq \Omega \text{ for all } \Omega' \in \Pi\}$ and $\Pi^\downarrow = \{\Omega \in V_0 \mid \Omega \subseteq \Omega' \text{ for all } \Omega' \in \Pi\}$. $\mathcal{V}$ is a bounded complete linearly ordered set (see [76]), and thus it forms a Gödel set. Note that using $\subseteq$ as the order relation, min and max are sets intersection and sets union (respectively). In addition, the function $\eta : V_0 \to V$ defined by $\eta(\Omega) = \{\Omega\}^{\downarrow}$ is injective and it satisfies the following properties:[6]

- $\{\emptyset\} = \eta(\emptyset)$.
- For every $\Omega, \Omega' \in V_0$:
  - $\Omega \subseteq \Omega'$ iff $\eta(\Omega) \subseteq \eta(\Omega')$.
  - $\eta(\Omega) \cap \eta(\Omega') = \eta(\Omega \cap \Omega')$.
  - $\eta(\Omega) \cup \eta(\Omega') = \eta(\Omega \cup \Omega')$.
  - $\eta(\Omega) \to \eta(\Omega') = \eta(\Omega \to \Omega')$
- For every $\Omega \in V_0$ and $\Pi \subseteq V_0$:
  - If $\Omega \subseteq \bigcap_{\Omega' \in \Pi} \Omega'$, then $\eta(\Omega) \subseteq \inf_{\Omega' \in \Pi} \eta(\Omega')$.
  - If $\bigcap_{\Omega' \in \Pi} \Omega' \subseteq \Omega$, then $\inf_{\Omega' \in \Pi} \eta(\Omega') \subseteq \eta(\Omega)$.
  - If $\Omega \subseteq \bigcup_{\Omega' \in \Pi} \Omega'$, then $\eta(\Omega) \subseteq \sup_{\Omega' \in \Pi} \eta(\Omega')$.
  - If $\bigcup_{\Omega' \in \Pi} \Omega' \subseteq \Omega$, then $\sup_{\Omega' \in \Pi} \eta(\Omega') \subseteq \eta(\Omega)$.

The proofs of these properties are straightforward (note that the linearity of V_0 is needed in some of them).
Henceforth, we will identify the elements of V_0 of the form $\{\Omega\}^{\downarrow}$ with the (unique) corresponding element Ω, and freely use the properties above.

Next, for every formula φ, let $\Omega^*[\varphi]$ be the pair defined by: $\Omega^*[\varphi] = \langle L[\varphi], R[\varphi] \rangle$. Note that $\Omega^*[\varphi] \in \{\langle u^{\mathtt{f}}, u^{\mathtt{t}} \rangle \in \mathcal{V} \times \mathcal{V} \mid u^{\mathtt{f}} \subseteq u^{\mathtt{t}}\}$ for every $\mathcal{L}^1$-formula φ. Indeed, in the presence of (id), either $\mathtt{f}{:}\varphi \notin \mu$ or $\mathtt{t}{:}\varphi \notin \mu$ for every $\mu \in \Omega$ and $\mathcal{L}^1$-formula φ (otherwise, $\{\{\mathtt{f}{:}\varphi, \mathtt{t}{:}\varphi\}\} \sqsubseteq \Omega$, contradicting the fact that Ω is unprovable), and consequently, $L[\varphi] \subseteq R[\varphi]$. Let $\mathcal{D}$ be the Herbrand domain for $\mathcal{L}^1$, I the Herbrand interpretation for $\mathcal{L}^1$, and define P and v as follows:

- For every n-ary predicate symbol p of $\mathcal{L}^1$, $P[p] = \lambda t_1, \dots, t_n \in \mathcal{D}.\Omega^*[\{p(t_1, \dots, t_n)\}]$.
- For every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ, $v[\varphi, \sigma] = \Omega^*[\sigma[\varphi]]$.

It is easy to verify the condition on v from Definition 8.5.2. Indeed, Lemma 8.5.16 ensures that if $y \notin Fv[\varphi]$, then for every $\mathcal{L}^1$-term we have $\sigma_{x:=t}[\varphi] = \sigma_{y:=t}[\varphi\{{}^{y}/_{x}\}]$. This implies that $v[\varphi, \sigma_{x:=t}] = v[\varphi\{{}^{y}/_{x}\}, \sigma_{y:=t}]$ for every $t \in \mathcal{D}$.

We show that $\mathcal{Q}$ is not a model of H_0. Consider the $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ_{id} defined by $\sigma_{id}[x] = x$ for every variable x of $\mathcal{L}^1$. Let $s \in H_0$. Since $H_0 \sqsubseteq \Omega^*$, there exists some $\mu \in \Omega^*$, such that $s \subseteq \mu$. We claim that $\mu \in v^{\mathtt{f}}[\varphi, \sigma_{id}]$ whenever $\mathtt{f}{:}\varphi \in s$, and $\mu \notin v^{\mathtt{t}}[\varphi, \sigma_{id}]$ whenever $\mathtt{t}{:}\varphi \in s$. To see this, it suffices to note that $\sigma_{id}[\varphi] = \varphi$ for every $\mathcal{L}^1$-formula φ. This fact follows from the definition of $\sigma_{id}[\varphi]$. Consequently, $v^{\mathtt{f}}[\varphi, \sigma_{id}] \not\subseteq v^{\mathtt{t}}[\varphi, \sigma_{id}]$, and so $\mathcal{Q}, \sigma_{id} \not\models s$.

[6] All operations notations from Definition 8.2.1 are adopted to the set $\mathcal{V}_0$ in the obvious way.

It remains to prove that $\mathcal{Q}$ is legal, namely that $\underline{v}_{\mathcal{Q}}[\varphi,\sigma] \subseteq v[\varphi,\sigma]$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment σ. Let φ be an $\mathcal{L}^1$-formula, and σ an $\langle \mathcal{L}^1, \mathcal{D}\rangle$-assignment. Then, exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. Then, by definition:
$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = P[p][\sigma[t_1], \dots, \sigma[t_n]] = \Omega^*[\{p(\sigma[t_1], \dots, \sigma[t_n])\}] = \Omega^*[\sigma[\varphi]] = v[\varphi,\sigma].$$

- $\varphi = \{\bot\}$. Then, $\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = \langle \emptyset, \emptyset\rangle$. To see that $\underline{v}_{\mathcal{Q}}[\varphi,\sigma] \subseteq v[\varphi,\sigma]$, it suffices to note that $v^{\mathtt{f}}[\varphi,\sigma] = \emptyset$. This follows from the fact that $\mathtt{f}{:}\sigma[\varphi] = \mathtt{f}{:}\{\bot\} \notin \mu$ for every $\mu \in \Omega^*$. (Otherwise, $\{\{\mathtt{f}{:}\{\bot\}\}\} \subseteq \Omega^*$, but $\vdash^{cf}_{\mathbf{HIF}} \{\{\mathtt{f}{:}\{\bot\}\}\}$ by applying the rule $(\mathtt{f}{:}\,\bot)$.)

- $\varphi = (\varphi_1 \wedge \varphi_2)$ for some $\mathcal{L}^1$-formulas φ_1 and φ_2. Then:
$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = \langle v^{\mathtt{f}}[\varphi_1,\sigma] \cap v^{\mathtt{f}}[\varphi_2,\sigma], v^{\mathtt{t}}[\varphi_1,\sigma] \cap v^{\mathtt{t}}[\varphi_2,\sigma]\rangle.$$
We first prove that $v^{\mathtt{f}}[\varphi,\sigma] \subseteq v^{\mathtt{f}}[\varphi_1,\sigma] \cap v^{\mathtt{f}}[\varphi_2,\sigma]$. Let $\mu \notin v^{\mathtt{f}}[\varphi_1,\sigma]$ ($\mu \notin v^{\mathtt{f}}[\varphi_2,\sigma]$ is symmetric). Thus $\mathtt{f}{:}\sigma[\varphi_1] \notin \mu$. We show that $\mu \notin v^{\mathtt{f}}[\varphi,\sigma]$. By Proposition 8.5.11, since $\mathtt{f}{:}\sigma[\varphi_1] \notin \mu$, there exist an $\mathcal{L}^1$-hypersequent $H_1 \sqsubseteq \Omega^*$, and $\mathcal{L}^1$-sequents $s_1, \dots, s_n \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H_1 \mid s_1 \cup \{\mathtt{f}{:}\sigma[\varphi_1]\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\sigma[\varphi_1]\}$. The availability of weakening and $(\mathtt{f}{:}\wedge)$ entails that $\vdash^{cf}_{\mathbf{HIF}} H$ for
$$H = H_1 \mid s_1 \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \wedge \sigma[\varphi_2])\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \wedge \sigma[\varphi_2])\}.$$
Since Ω^* is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\mathtt{f}{:}(\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \notin \mu$. By definition, $(\sigma[\varphi_1] \wedge \sigma[\varphi_2]) = \sigma[\varphi]$. It follows that $\mu \notin v^{\mathtt{f}}[\varphi,\sigma]$.

  Next, we prove that $v^{\mathtt{t}}[\varphi_1,\sigma] \cap v^{\mathtt{t}}[\varphi_2,\sigma] \subseteq v^{\mathtt{t}}[\varphi,\sigma]$. Let $\mu \in v^{\mathtt{t}}[\varphi_1,\sigma] \cap v^{\mathtt{t}}[\varphi_2,\sigma]$. Then we have $\mathtt{t}{:}\sigma[\varphi_1] \notin \mu$ and $\mathtt{t}{:}\sigma[\varphi_2] \notin \mu$. By Proposition 8.5.11, there exist $\mathcal{L}^1$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, and negative $\mathcal{L}^1$-sequents $s_1, s_2 \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H_1 \mid s_1 \cup \{\mathtt{t}{:}\sigma[\varphi_1]\}$ and $\vdash^{cf}_{\mathbf{HIF}} H_2 \mid s_2 \cup \{\mathtt{t}{:}\sigma[\varphi_2]\}$. The availability of $(\mathtt{t}{:}\wedge)$ entails that $\vdash^{cf}_{\mathbf{HIF}} H$ for $H = H_1 \mid H_2 \mid s_1 \cup s_2 \cup \{\mathtt{t}{:}(\sigma[\varphi_1] \wedge \sigma[\varphi_2])\}$. Since Ω^* is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\mathtt{t}{:}\sigma[\varphi] = \mathtt{t}{:}(\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \notin \mu$. It follows that $\mu \in v^{\mathtt{t}}[\varphi,\sigma]$.

- $\varphi = (\varphi_1 \vee \varphi_2)$ for some $\mathcal{L}^1$-formulas φ_1 and φ_2. Then:
$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = \langle v^{\mathtt{t}}[\varphi_1,\sigma] \cup v^{\mathtt{t}}[\varphi_2,\sigma], v^{\mathtt{f}}[\varphi_1,\sigma] \cup v^{\mathtt{f}}[\varphi_2,\sigma]\rangle.$$

  We first prove that $v^{\mathtt{f}}[\varphi,\sigma] \subseteq v^{\mathtt{f}}[\varphi_1,\sigma] \cup v^{\mathtt{f}}[\varphi_2,\sigma]$. Let $\mu \notin v^{\mathtt{f}}[\varphi_1,\sigma] \cup v^{\mathtt{f}}[\varphi_2,\sigma]$. We prove that $\mu \notin v^{\mathtt{f}}[\varphi,\sigma]$. Our assumption entails that $\mathtt{f}{:}\sigma[\varphi_1] \notin \mu$ and $\mathtt{f}{:}\sigma[\varphi_2] \notin \mu$. By Proposition 8.5.11, there exist $\mathcal{L}^1$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, and $\mathcal{L}^1$-sequents $s_1, \dots, s_n, s'_1, \dots, s'_m \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H_1 \mid s_1 \cup \{\mathtt{f}{:}\sigma[\varphi_1]\} \mid \dots \mid s_n \cup \{\mathtt{f}{:}\sigma[\varphi_1]\}$ and

$\vdash^{cf}_{\mathbf{HIF}} H_2 \mid s'_1 \cup \{\mathtt{f}{:}\sigma[\varphi_2]\} \mid ... \mid s'_m \cup \{\mathtt{f}{:}\sigma[\varphi_2]\}$. As in the proof of Proposition 7.2.11, it is possible to use (*com*) and $(\mathtt{f}{:}\vee)$ to obtain that $\vdash^{cf}_{\mathbf{HIF}} H$ for

$$H = H_1 \mid H_2 \mid s_1 \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2])\} \mid ... \mid s_n \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2])\}$$
$$\mid s'_1 \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2])\} \mid ... \mid s'_m \cup \{\mathtt{f}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2])\}.$$

Since Ω^* is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\mathtt{f}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2]) \notin \mu$. By definition, $(\sigma[\varphi_1] \vee \sigma[\varphi_2]) = \sigma[\varphi]$. It follows that $\mu \notin v^{\mathtt{f}}[\varphi, \sigma]$.

Next, we prove that $v^{\mathtt{t}}[\varphi_1, \sigma] \cup v^{\mathtt{t}}[\varphi_2, \sigma] \subseteq v^{\mathtt{t}}[\varphi, \sigma]$. Let $\mu \in v^{\mathtt{t}}[\varphi_1, \sigma]$, and so $\mathtt{t}{:}\sigma[\varphi_1] \notin \mu$ (the case that $\mu \in v^{\mathtt{t}}[\varphi_2, \sigma]$ is symmetric). By Proposition 8.5.11, there exist an $\mathcal{L}^1$-hypersequent $H_1 \sqsubseteq \Omega^*$, and a negative $\mathcal{L}^1$-sequent $s \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H_1 \mid s \cup \{\mathtt{t}{:}\sigma[\varphi_1]\}$. The availability of weakening and $(\mathtt{t}{:}\vee)$ entails that $\vdash^{cf}_{\mathbf{HIF}} H$ for $H = H_1 \mid s \cup \{\mathtt{t}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2])\}$. Since Ω^* is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\mathtt{t}{:}\sigma[\varphi] = \mathtt{t}{:}(\sigma[\varphi_1] \vee \sigma[\varphi_2]) \notin \mu$. It follows that $\mu \in v^{\mathtt{t}}[\varphi, \sigma]$.

- $\varphi = (\varphi_1 \supset \varphi_2)$ for some $\mathcal{L}^1$-formulas φ_1 and φ_2. Then:

$$\underline{v}_{\mathcal{Q}}[\varphi, \sigma] = \langle v^{\mathtt{f}}[\varphi_1, \sigma] \to v^{\mathtt{t}}[\varphi_2, \sigma], v^{\mathtt{t}}[\varphi_1, \sigma] \to v^{\mathtt{f}}[\varphi_2, \sigma] \rangle.$$

We first prove that $v^{\mathtt{f}}[\varphi, \sigma] \subseteq v^{\mathtt{t}}[\varphi_1, \sigma] \to v^{\mathtt{f}}[\varphi_2, \sigma]$. Let $\mu \notin v^{\mathtt{t}}[\varphi_1, \sigma] \to v^{\mathtt{f}}[\varphi_2, \sigma]$. Then, $v^{\mathtt{t}}[\varphi_1, \sigma] \not\subseteq v^{\mathtt{f}}[\varphi_2, \sigma]$ and $\mu \notin v^{\mathtt{f}}[\varphi_2, \sigma]$. Let $\mu' \in \Omega^*$ such that $\mu' \in v^{\mathtt{t}}[\varphi_1, \sigma]$, and $\mu' \notin v^{\mathtt{f}}[\varphi_2, \sigma]$. Hence, $\mathtt{t}{:}\sigma[\varphi_1] \notin \mu'$ and $\mathtt{f}{:}\sigma[\varphi_2] \notin \mu'$. By Proposition 8.5.11, there exist $\mathcal{L}^1$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, a negative $\mathcal{L}^1$-sequent $s' \subseteq \mu'$, and $\mathcal{L}^1$-sequents $s'_1, \dots, s'_n \subseteq \mu'$, such that $\vdash^{cf}_{\mathbf{HIF}} H_1 \mid s' \cup \{\mathtt{t}{:}\sigma[\varphi_1]\}$, and

$$\vdash^{cf}_{\mathbf{HIF}} H_2 \mid s'_1 \cup \{\mathtt{f}{:}\sigma[\varphi_2]\} \mid ... \mid s'_n \cup \{\mathtt{f}{:}\sigma[\varphi_2]\}.$$

By n consecutive applications of $(\mathtt{f}{:}\supset)$ (note that $(\sigma[\varphi_1] \supset \sigma[\varphi_2]) = \sigma[\varphi]$), we obtain that

$$\vdash^{cf}_{\mathbf{HIF}} H_1 \mid H_2 \mid s' \cup s'_1 \cup \{\mathtt{f}{:}\sigma[\varphi]\} \mid ... \mid s' \cup s'_n \cup \{\mathtt{f}{:}\sigma[\varphi]\}. \tag{8.1}$$

Since $\mu \notin v^{\mathtt{f}}[\varphi_2, \sigma]$, we also have $\mathtt{f}{:}\sigma[\varphi_2] \notin \mu$. Proposition 8.5.11 entails that there also exist $\mathcal{L}^1$-hypersequent $H_3 \sqsubseteq \Omega^*$, and $\mathcal{L}^1$-sequents $s_1, \dots, s_m \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H_3 \mid s_1 \cup \{\mathtt{f}{:}\sigma[\varphi_2]\} \mid ... \mid s_m \cup \{\mathtt{f}{:}\sigma[\varphi_2]\}$. By another m applications of $(\mathtt{f}{:}\supset)$, we obtain that

$$\vdash^{cf}_{\mathbf{HIF}} H_1 \mid H_3 \mid s' \cup s_1 \cup \{\mathtt{f}{:}\sigma[\varphi]\} \mid ... \mid s' \cup s_m \cup \{\mathtt{f}{:}\sigma[\varphi]\}. \tag{8.2}$$

Using a generalized version of (*com*) (see Proposition 7.2.10) we obtain from (8.1) and (8.2) above:

$$\vdash^{cf}_{\mathbf{HIF}} H_1 \mid H_2 \mid H_3 \mid s' \cup s'_1 \mid ... \mid s' \cup s'_n \mid s_1 \cup \{\mathtt{f}{:}\sigma[\varphi]\} \mid ... \mid s_m \cup \{\mathtt{f}{:}\sigma[\varphi]\}.$$

Now, if $\mathtt{f}{:}\sigma[\varphi] \in \mu$, then Ω^* extends this hypersequent, and this contradicts the fact that Ω^* is unprovable. Therefore, $\mathtt{f}{:}\sigma[\varphi] \notin \mu$, and consequently $\mu \notin v^{\mathtt{f}}[\sigma[\varphi]]$.

Next, we prove that $v^{\mathtt{f}}[\varphi_1, \sigma] \to v^{\mathtt{t}}[\varphi_2, \sigma] \subseteq v^{\mathtt{t}}[\varphi, \sigma]$. Suppose that $\mu \notin v^{\mathtt{t}}[\varphi, \sigma]$, and so $\mathtt{t}{:}\sigma[\varphi] \in \mu$. To show that $\mu \notin v^{\mathtt{f}}[\varphi_1, \sigma] \to v^{\mathtt{t}}[\varphi_2, \sigma]$, we first show that

$\mu \notin v^{\mathtt{t}}[\varphi_2, \sigma]$ and then we show that $v^{\mathtt{f}}[\varphi_1, \sigma] \not\subseteq v^{\mathtt{t}}[\varphi_2, \sigma]$:

1. Assume for contradiction that $\mu \in v^{\mathtt{t}}[\varphi_2, \sigma]$, and thus $\mathtt{t}{:}\sigma[\varphi_2] \notin \mu$. Then by Proposition 8.5.11, there exist an $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega^*$, and a negative $\mathcal{L}^1$-sequent $s \subseteq \mu$, such that $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{t}{:}\sigma[\varphi_2]\}$. By applying internal weakening we obtain $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{f}{:}\sigma[\varphi_1], \mathtt{t}{:}\sigma[\varphi_2]\}$. Using $(\mathtt{t}{:} \supset)$ we obtain $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{t}{:}\sigma[\varphi]\}$. This contradicts the fact that Ω^* is unprovable (because $H \mid s \cup \{\mathtt{t}{:}\sigma[\varphi]\} \sqsubseteq \Omega^*$).
2. Note that the fact that Ω^* is unprovable and the availability of $(\mathtt{t}{:} \supset)$ also entail that $\not\vdash^{cf}_{\mathbf{HIF}} H \mid \{\mathtt{f}{:}\sigma[\varphi_1], \mathtt{t}{:}\sigma[\varphi_2]\}$. Therefore, Proposition 8.5.11 entails that $\{\mathtt{f}{:}\sigma[\varphi_1], \mathtt{t}{:}\sigma[\varphi_2]\} \sqsubseteq \Omega^*$. Thus there is an extended $\mathcal{L}^1$-sequent $\mu' \in \Omega^*$, such that $\mathtt{f}{:}\sigma[\varphi_1] \in \mu'$ and $\mathtt{t}{:}\sigma[\varphi_2] \in \mu'$. Consequently, $\mu' \in v^{\mathtt{f}}[\varphi_1, \sigma]$ and $\mu' \notin v^{\mathtt{t}}[\varphi_2, \sigma]$. Hence $v^{\mathtt{f}}[\varphi_1, \sigma] \not\subseteq v^{\mathtt{t}}[\varphi_2, \sigma]$.

- $\varphi = (\exists x \psi)$ for some variable $x \notin Fv[\sigma[\varphi]]$ and $\mathcal{L}^1$-formula ψ. Then:
$$\underline{v}_{\mathcal{Q}}[\varphi, \sigma] = \langle \sup_{t \in \mathcal{D}} v^{\mathtt{t}}[\psi, \sigma_{x:=t}], \sup_{t \in \mathcal{D}} v^{\mathtt{f}}[\psi, \sigma_{x:=t}] \rangle.$$
We first prove that $v^{\mathtt{f}}[\varphi, \sigma] \subseteq \sup_{t \in \mathcal{D}} v^{\mathtt{f}}[\psi, \sigma_{x:=t}]$. Suppose that $\mu \in v^{\mathtt{f}}[\varphi, \sigma]$. Thus $\mathtt{f}{:}\sigma[\varphi] \in \mu$. By definition, $\sigma[\varphi] = (\exists x \sigma_{x:=x}[\psi])$. Since Ω^* admits the witness property, there exists a variable y of $\mathcal{L}^1$, such that $\mathtt{f}{:}\sigma_{x:=x}[\psi]\{y/x\} \in \mu$. By Lemma 8.5.17, $\sigma_{x:=x}[\psi]\{y/x\} = \sigma_{x:=y}[\psi]$. It follows that $\mu \in v^{\mathtt{f}}[\psi, \sigma_{x:=y}]$, and therefore $\mu \in \bigcup_{t \in \mathcal{D}} v^{\mathtt{f}}[\psi, \sigma_{x:=t}]$.

  Next, we prove that $\sup_{t \in \mathcal{D}} v^{\mathtt{t}}[\psi, \sigma_{x:=t}] \subseteq v^{\mathtt{t}}[\varphi, \sigma]$. Let $\mu \in \bigcup_{t \in \mathcal{D}} v^{\mathtt{t}}[\psi, \sigma_{x:=t}]$. Thus $\mu \in v^{\mathtt{t}}[\psi, \sigma_{x:=t}]$ for some $t \in \mathcal{D}$. By definition, $\mathtt{t}{:}\sigma_{x:=t}[\psi] \notin \mu$. By Lemma 8.5.17, $\sigma_{x:=t}[\psi] = \sigma_{x:=x}[\psi]\{t/x\}$. By Proposition 8.5.11, $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{t}{:}\sigma_{x:=x}[\psi]\{t/x\}\}$ for some $\mathcal{L}^1$-hypersequent $H \sqsubseteq \Omega^*$, and negative $\mathcal{L}^1$-sequent $s \subseteq \mu$. By an application of $(\mathtt{t}{:}\exists)$, we obtain $\vdash^{cf}_{\mathbf{HIF}} H \mid s \cup \{\mathtt{t}{:}(\exists x \sigma_{x:=x}[\psi])\}$. Since Ω^* is unprovable, $\mathtt{t}{:}(\exists x \sigma_{x:=x}[\psi]) \notin \mu$. By definition, $(\exists x \sigma_{x:=x}[\psi]) = \sigma[\varphi]$. It follows that $\mu \in v^{\mathtt{t}}[\varphi, \sigma]$.

- The case $\varphi = (\forall x \psi)$ is handled similarly. □

8.6 Completeness for the Ordinary Semantics

In this section we use the complete semantics of quasi-structures to prove the completeness of **HIF** for the (ordinary) structures of first-order Gödel logic. To do so, we show that from every legal quasi-structure which is a counter-model of some hypersequent H, it is possible to extract an (ordinary) structure, which is also not a model of H.

Theorem 8.6.1. Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$ be a legal quasi-$\mathcal{L}^1$-structure. There exists an $\mathcal{L}^1$-structure $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P' \rangle$, such that $\mathcal{W}[\varphi, \sigma] \in v[\varphi, \sigma]$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ.

Proof. Define P' by $P'[p] = P[p]^{\mathsf{f}}$ for every predicate symbol p. We prove that the $\mathcal{L}^1$-structure $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P' \rangle$ satisfies the requirement in the theorem. Let $\mathcal{V} = \langle V, \leq \rangle$. We use induction on the complexity of φ to show that $\mathcal{W}[\varphi, \sigma] \in v[\varphi, \sigma]$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ. Note that since $\mathcal{Q}$ is legal, it suffices to show that $\mathcal{W}[\varphi, \sigma] \in \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$ for every $\mathcal{L}^1$-formula φ and $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ. First, suppose that $cp[\varphi] = 1$, and let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^1$, and $\mathcal{L}^1$-terms $t_1, \dots, t_n$. By definition, $\mathcal{W}[\varphi, \sigma] = P'[p][\sigma^I[t_1], \dots, \sigma^I[t_n]] = P[p]^{\mathsf{f}}[\sigma^I[t_1]], \dots, \sigma^I[t_n]]] \in \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$.
- $\varphi = \{\bot\}$. Then by definition, $\mathcal{W}[\varphi, \sigma] = 0 \in \langle 0, 0 \rangle = \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^1$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = (\varphi_1 \wedge \varphi_2)$ for $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. By the induction hypothesis,
$$\mathcal{W}[\varphi_1, \sigma] \in \langle v^{\mathsf{f}}[\varphi_1, \sigma], v^{\mathsf{t}}[\varphi_1, \sigma] \rangle \quad \text{and} \quad \mathcal{W}[\varphi_2, \sigma] \in \langle v^{\mathsf{f}}[\varphi_2, \sigma], v^{\mathsf{t}}[\varphi_2, \sigma] \rangle.$$
Hence,
$$\min\{\mathcal{W}[\varphi_1, \sigma], \mathcal{W}[\varphi_2, \sigma]\} \in \langle \min\{v^{\mathsf{f}}[\varphi_1, \sigma], v^{\mathsf{f}}[\varphi_2, \sigma]\}, \min\{v^{\mathsf{t}}[\varphi_1, \sigma], v^{\mathsf{t}}[\varphi_2, \sigma]\} \rangle,$$
and so $\mathcal{W}[\varphi, \sigma] \in \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$.
- $\varphi = (\varphi_1 \vee \varphi_2)$ for $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. This case is similar to the previous case (replace min by max).
- $\varphi = (\varphi_1 \supset \varphi_2)$ for $\mathcal{L}^1$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. By the induction hypothesis,
$$\mathcal{W}[\varphi_1, \sigma] \in \langle v^{\mathsf{f}}[\varphi_1, \sigma], v^{\mathsf{t}}[\varphi_1, \sigma] \rangle \quad \text{and} \quad \mathcal{W}[\varphi_2, \sigma] \in \langle v^{\mathsf{f}}[\varphi_2, \sigma], v^{\mathsf{t}}[\varphi_2, \sigma] \rangle.$$
Since $u_1 \leq u' \leq u_2$ and $u_3 \leq u'' \leq u_4$ imply that $u_2 \to u_3 \leq u' \to u'' \leq u_1 \to u_4$, we obtain that: $\mathcal{W}[\varphi_1, \sigma] \to \mathcal{W}[\varphi_2, \sigma] \in \langle v^{\mathsf{t}}[\varphi_1, \sigma] \to v^{\mathsf{f}}[\varphi_2, \sigma], v^{\mathsf{f}}[\varphi_1, \sigma] \to v^{\mathsf{t}}[\varphi_2, \sigma] \rangle$, and so $\mathcal{W}[\varphi, \sigma] \in \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$.
- $\varphi = (Qx\psi)$ for some $Q \in \{\forall, \exists\}$, variable x of $\mathcal{L}^1$, and $\mathcal{L}^1$-formula ψ such that $cp[\psi] < cp[\varphi]$. We continue with $Q = \forall$ (the proof is similar for $\exists$). By the induction hypothesis, for every $d \in \mathcal{D}$, $\mathcal{W}[\psi, \sigma_{x:=d}] \in v[\psi, \sigma_{x:=d}]$. Hence,
$$\mathcal{W}[\varphi, \sigma] = \inf_{d \in \mathcal{D}} \mathcal{W}[\psi, \sigma_{x:=d}] \in \langle \inf_{d \in \mathcal{D}} v^{\mathsf{f}}[\psi, \sigma_{x:=d}], \inf_{d \in \mathcal{D}} v^{\mathsf{t}}[\psi, \sigma_{x:=d}] \rangle = \underline{v}_{\mathcal{Q}}[\varphi, \sigma]. \qquad \square$$

Corollary 8.6.2. If $\not\vdash^{cf}_{\mathbf{HIF}} H$, then there exists an $\mathcal{L}^1$-structure which is not a model of H.

Proof. Suppose that $\nvdash^{cf}_{\mathbf{HIF}} H$. Then, by Theorem 8.5.9, there exists a legal quasi-$\mathcal{L}^1$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$, which is not a model of H. This implies that there exists an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-assignment σ, such that $\mathcal{Q}, \sigma \not\models s$ for every $s \in H$. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P' \rangle$ be an $\mathcal{L}^1$-structure satisfying the requirement in Theorem 8.6.1. We show that $\mathcal{W}, \sigma \not\models H$. Let $s \in H$. Since $\mathcal{Q}, \sigma \not\models s$, we have $\mathcal{Q}^{\mathtt{f}}[s, \sigma] > \mathcal{Q}^{\mathtt{t}}[s, \sigma]$. The fact that $\mathcal{W}[\varphi, \sigma] \in v[\varphi, \sigma]$ for every φ entails that $\mathcal{W}^{\mathtt{f}}[\varphi, \sigma] \geq v^{\mathtt{f}}[\varphi, \sigma]$ and $\mathcal{W}^{\mathtt{t}}[\varphi, \sigma] \leq v^{\mathtt{t}}[\varphi, \sigma]$ for every φ. It follows that $\mathcal{W}^{\mathtt{f}}[s, \sigma] > \mathcal{W}^{\mathtt{t}}[s, \sigma]$, and so $\mathcal{W}, \sigma \not\models s$. □

Corollary 8.6.3. For every $\mathcal{L}^1$-formula φ, if $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^1}} \varphi$ then $\vdash_{\mathbf{HIF}} \{\{\mathtt{t}{:}\varphi\}\}$.

Finally, we automatically obtain the admissibility of the cut rule:

Corollary 8.6.4. If an $\mathcal{L}^1$-hypersequent is provable in **HIF** then it is provable in **HIF** without applying (*cut*).

Proof. Note that if an $\mathcal{L}^1$-hypersequent H is provable in **HIF**, then every $\mathcal{L}^1$-structure is a model of H. By Corollary 8.6.2, this implies that $\vdash^{cf}_{\mathbf{HIF}} H$. □

Remark 8.6.5. While we allowed any Gödel set to serve as the set of truth values in $\mathcal{L}^1$-structures, we could equivalently take the real interval $[0, 1]$. Obviously, soundness for $[0, 1]$ is a particular instance. Completeness for $[0, 1]$ can be obtained by embedding the set V_0 in the proof of Theorem 8.5.9 into the rational numbers in $[0, 1]$, and continuing the proof with $\mathcal{V} = [0, 1]$.

Chapter 9

Calculus for Second-Order Gödel Logic

Fuzzy logics, and Gödel logic in particular, have a wide variety of applications, as they provide a reasonable model of certain very common vagueness phenomena. Both their propositional and first-order versions are well-studied by now (see, e.g., [63]). Clearly, for many interesting applications (see, e.g., [38] and Section 5.5.2 in Chapter I of [45]), propositional and first-order fuzzy logics do not suffice, and one has to use higher-order versions. These are much less developed (see, e.g., [95] and [45]), especially from the proof-theoretic perspective. Evidently, higher-order fuzzy logics deserve a proof-theoretic study, with the aim of providing a basis for automated deduction methods, as well as a complimentary point of view in the investigation of these logics.

In this chapter we study the extension of **HIF** with usual rules for *second-order* quantifiers. These consist of the single-conclusion hypersequent version of the rules for introducing the second-order quantifiers in the ordinary sequent calculus for classical logic (see, e.g., [58, 90]). We denote by $\mathbf{HIF}^2$ the extension of **HIF** with these rules. To the best of our knowledge, this system is studied here for the first time. Our main results is that $\mathbf{HIF}^2$ is sound and complete for second-order Gödel logic, and that (*cut*) is admissible in $\mathbf{HIF}^2$. It should be noted that like in the case of second-order classical logic, the obtained calculus characterizes *Henkin-style* second-order Gödel logic. Thus second-order quantifiers range over a domain (of fuzzy sets) that is directly specified in the second-order structure, and this domain should admit full comprehension. This is in contrast to what is called the *standard semantics*, where second-order quantifiers range over *all* subsets of the universe. Hence $\mathbf{HIF}^2$ is practically a system for two-sorted first-order Gödel logic together with the comprehension axiom (see also [37]).

Our approach in proving cut-admissibility for $\mathbf{HIF}^2$ is (of-course) semantic, and it is similar to the one taken in Chapter 8 for **HIF**. Note that unlike in first-order calculi, usual syntactic arguments for cut-elimination dramatically fail for the rules of second-order quantifiers. Thus the first proof of cut-admissibility for the extension of **LK** with rules for second-order quantifiers was also semantic (see also the discussion in Chapter 1, page 8).

Publications Related to this Chapter

The material in this chapter was not published before.

9.1 Preliminaries

For the simplicity of the presentation, we follow [58] and restrict ourselves to simplified second-order languages, in which the second-order part of the signature consists only of one predicate symbol ε (with the intuitive meaning of set inclusion). This is formulated in the next definition.

Definition 9.1.1. A *simple second-order language* is obtained by augmenting a first-order language (see Definition 8.1.1) with the following:

1. Infinitely many *set variables* $\chi_1, \chi_2, \dots$. We use the metavariables X, Y, Z (with or without subscripts) for set variables. To avoid confusion, we shall refer the variables $\nu_1, \nu_2, \dots$ of the underlying first-order language as *individual variables*.
2. *Set quantifiers* $\forall^s$ and $\exists^s$. We use Q^s as a metavariable for the set quantifiers. We shall refer the quantifiers $\forall, \exists$ of the underlying first-order language as *individual quantifiers*, and usually denote them by $\forall^i$ and $\exists^i$.
3. An arbitrary set of *set constant symbols*. The metavariable C are is to range over set constant symbols. We shall refer the constant symbols of the underlying first-order language as *individual constant symbols*.
4. A predicate symbol ε with two places, the first for individuals and the second for sets.

In what follows, $\mathcal{L}^2$ denotes an arbitrary simple second-order language.

Definition 9.1.2. The set of $\mathcal{L}^2$*-terms* consists of *first-order* $\mathcal{L}^2$*-terms* and *second-order* $\mathcal{L}^2$*-terms*. First-order $\mathcal{L}^2$-terms are defined as in Definition 8.1.2, while second-order $\mathcal{L}^2$-terms consists of all set variables of $\mathcal{L}^2$ and all set constant symbols of $\mathcal{L}^2$. We use T (with or without subscripts) as a metavariable for second-order $\mathcal{L}^2$-terms. The set of (individual) variables occurring in a first-order $\mathcal{L}^2$-term t is defined as usual, and denoted by $Fv[t]$. Similarly, the set of set variables occurring in a second-order $\mathcal{L}^2$-term

T is denoted by $Fv[T]$. Substitutions in first-order $\mathcal{L}^2$-terms (denoted by $t'\{^t/_x\}$) are defined as in Definition 8.1.7.

Definition 9.1.3. *Concrete $\mathcal{L}^2$-formulas* are defined as for a first-order language (Definition 8.1.3), with the following additions:

1. $(t\varepsilon T)$ is a concrete $\mathcal{L}^2$-formula for every first-order $\mathcal{L}^2$-term t and second-order $\mathcal{L}^2$-term T.
2. If Φ is a concrete $\mathcal{L}^2$-formula, and X is a set variable of $\mathcal{L}^2$, then $(\forall^s X\Phi)$ and $(\exists^s X\Phi)$ are concrete $\mathcal{L}^2$-formulas.

$Fv[\Phi]$, $[\Phi]_\alpha$, and $cp[\Phi]$ for a concrete $\mathcal{L}^2$-formula Φ, are defined as in first-order languages (Definition 8.1.3), where concrete $\mathcal{L}^2$-formulas of the form $(t\varepsilon T)$ are also considered as atomic concrete formulas, whose complexity is 1. As for first-order language, $\mathcal{L}^2$-formulas are defined as equivalence classes of concrete $\mathcal{L}^2$-formulas (see Definition 8.1.4), and $Fv[\varphi]$ and $cp[\varphi]$ for an $\mathcal{L}^2$-formula φ are defined exactly as for $\mathcal{L}^1$-formulas (using representatives). Similarly, $(\varphi_1 \diamond \varphi_2)$ for $\mathcal{L}^2$-formulas φ_1, φ_2 and $\diamond \in \{\wedge, \vee, \supset\}$, as well as $(Q^i x\varphi)$ for $\mathcal{L}^2$-formula φ and $Q^i \in \{\forall^i, \exists^i\}$ are defined as in Definition 8.1.5. In addition, we define the following:

Definition 9.1.4. For $Q^s \in \{\forall^s, \exists^s\}$, a set variable X of $\mathcal{L}^2$, and an $\mathcal{L}^2$-formula φ:

$$(Q^s X\varphi) = [(Q^s X\Phi)]_\alpha \text{ for some } \Phi \in \varphi.$$

Proposition 9.1.5. Exactly one of the following holds for every $\mathcal{L}^2$-formula φ:

- $cp[\varphi] = 1$ and exactly one of the following holds:
  - $\varphi = \{p(t_1, \dots, t_n)\}$ for some n-ary predicate symbol p of $\mathcal{L}^2$,and first-order $\mathcal{L}^2$-terms $t_1, \dots, t_n$.
  - $\varphi = \{(t\varepsilon T)\}$ for some first-order $\mathcal{L}^2$-term t, and second-order $\mathcal{L}^2$-term T.
  - $\varphi = \{\bot\}$.
- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and unique $\mathcal{L}^2$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$.
- For every individual variable $x \notin Fv[\varphi]$, $\varphi = (Q^i x\psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, and unique $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$.
- For every set variable $X \notin Fv[\varphi]$, $\varphi = (Q^s X\psi)$ for some $Q^s \in \{\forall^s, \exists^s\}$, and unique $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$.

Substitution operations are defined as follows:

Definition 9.1.6. Let t be a first-order $\mathcal{L}^2$-term, and x an individual variable of $\mathcal{L}^2$. Given an $\mathcal{L}^2$-formula φ, $\varphi\{{}^t/x\}$ is inductively defined by:

$$\varphi\{{}^t/x\} = \begin{cases} \{p(t_1\{{}^t/x\}, \dots, t_n\{{}^t/x\})\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \{(t'\{{}^t/x\}\varepsilon T)\} & \varphi = \{(t'\varepsilon T)\} \\ \varphi & \varphi = \{\bot\} \\ (\varphi_1\{{}^t/x\} \diamond \varphi_2\{{}^t/x\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{{}^t/x\}) & \varphi = (Q^i y \psi) \text{ for } y \notin Fv[t] \cup \{x\} \\ (Q^s Y \psi\{{}^t/x\}) & \varphi = (Q^s Y \psi) \end{cases}$$

Definition 9.1.7. Let T be a second-order $\mathcal{L}^2$-term, and X a set variable of $\mathcal{L}^2$. Given an $\mathcal{L}^2$-formula φ, $\varphi\{{}^T/X\}$ is inductively defined by:

$$\varphi\{{}^T/X\} = \begin{cases} \varphi & \varphi = \{p(t_1, \dots, t_n)\} \\ \varphi & \varphi = \{(t\varepsilon T')\} \text{ for } T' \neq X \\ \{(t\varepsilon T)\} & \varphi = \{(t\varepsilon X)\} \\ \varphi & \varphi = \{\bot\} \\ (\varphi_1\{{}^T/X\} \diamond \varphi_2\{{}^T/X\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{{}^T/X\}) & \varphi = (Q^i y \psi) \\ (Q^s Y \psi\{{}^T/X\}) & \varphi = (Q^s Y \psi) \text{ for } Y \notin Fv[T] \cup \{X\} \end{cases}$$

Note that the above substitution operations are well-defined. In particular, the choice of the variables y and Y is immaterial.

9.2 Henkin-style Second-Order Gödel Logic

In this section we precisely define Henkin-style second-order Gödel logic, via a semantic presentation. These definitions naturally extend the usual definitions of Henkin-style second-order classical logic, by replacing the usual two truth values $True$ and $False$ by any bounded complete linearly ordered set of truth values. From a different angle, these definitions naturally extend (standard) first-order Gödel logic (presented in Chapter 8) by adding an additional collection of fuzzy sets, over which the set quantifiers range.

Definition 9.2.1. A *domain* $\mathcal{D}$ for a Gödel set $\mathcal{V}$ consists of:

- A non-empty set, called *individuals domain* and denoted by $\mathcal{D}_i$.
- A non-empty collection of fuzzy subsets of $\mathcal{D}_i$ over $\mathcal{V}$ (see Definition 8.2.3), called *sets domain* and denoted by $\mathcal{D}_s$.

Given a domain $\mathcal{D}$, an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-interpretation I consists of:

- A function assigning an element in $\mathcal{D}_i$ to every individual constant symbol of $\mathcal{L}^2$, and a function in ${\mathcal{D}_i}^n \to \mathcal{D}_i$ to every n-ary function symbol of $\mathcal{L}^2$. We call this function *individuals interpretation*, and denote it by I_i.
- A function assigning a fuzzy subset in $\mathcal{D}_s$ to every set constant symbol of $\mathcal{L}^2$. We call this function *sets interpretation*, and denote it by I_s.

Note that $\mathcal{D}_i$ is a (first-order) domain, and I_i is an $\langle \mathcal{L}^1, \mathcal{D} \rangle$-interpretation (see Definition 8.2.2). Next, we define $\mathcal{L}^2$*-structures* exactly like we defined $\mathcal{L}^1$-structures (Definition 8.2.4), based on the new notions of domain and interpretation.

Definition 9.2.2. An $\mathcal{L}^2$*-structure* is a triple $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$, where:

1. $\mathcal{V}$ is a Gödel set.
2. $\mathcal{D}$ is a domain for $\mathcal{V}$.
3. I is an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-interpretation.
4. P is a function assigning a fuzzy subset of $\mathcal{D}_i^n$ over $\mathcal{V}$ to every n-ary predicate symbol of $\mathcal{L}^2$.

Assignments are also defined as their first-order counterparts (Definition 8.2.5):

Definition 9.2.3. Let $\mathcal{D}$ be a domain.

1. An $\langle \mathcal{L}^2, \mathcal{D} \rangle$*-assignment* is a function assigning an element of $\mathcal{D}_i$ to every individual variable of $\mathcal{L}^2$, and an element of $\mathcal{D}_s$ to every set variable of $\mathcal{L}^2$.

2. Given an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-interpretation I and an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ, σ^I is the function assigning elements of $\mathcal{D}_i$ and $\mathcal{D}_s$ to $\mathcal{L}^2$-terms, recursively defined by:

   - $\sigma^I[c] = I_i[c]$ for every individual constant symbol c of $\mathcal{L}^2$.
   - $\sigma^I[x] = \sigma[x]$ for every individual variable x of $\mathcal{L}^2$.
   - $\sigma^I[f(t_1, \dots, t_n)] = I_i[f](\sigma^I[t_1], \dots, \sigma^I[t_n])$ for every n-ary function symbol f of $\mathcal{L}^2$ and n first-order $\mathcal{L}^2$-terms $t_1, \dots, t_n$.
   - $\sigma^I[C] = I_s[C]$ for every set constant symbol C of $\mathcal{L}^2$.
   - $\sigma^I[X] = \sigma[X]$ for every set variable X of $\mathcal{L}^2$.

3. Let σ be an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment. Given an individual variable x of $\mathcal{L}^2$ and $d \in \mathcal{D}_i$, we denote by $\sigma_{x:=d}$ the $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment that is identical to σ except for $\sigma_{x:=d}[x] = d$. Similarly, given a set variable X of $\mathcal{L}^2$, and $D \in \mathcal{D}_s$, we denote by $\sigma_{X:=D}$ the $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment that is identical to σ except for $\sigma_{X:=D}[X] = D$. These notations are naturally extended to several distinct variables (e.g. $\sigma_{\nu_1:=d_1, \nu_2:=d_2, \chi_1:=D}$).

Next, we generalize Definition 8.2.7 for $\mathcal{L}^2$-structures.

Definition 9.2.4. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^2$-structure. For every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ, $\mathcal{W}[\varphi, \sigma]$ is the element of $\mathcal{V}$ inductively defined as follows:

$$\mathcal{W}[\varphi, \sigma] = \begin{cases} P[p][\sigma^I[t_1], \dots, \sigma^I[t_n]] & \varphi = \{p(t_1, \dots, t_n)\} \\ \sigma^I[T][\sigma^I[t]] & \varphi = \{(t \varepsilon T)\} \\ 0 & \varphi = \{\bot\} \\ \min\{\mathcal{W}[\varphi_1, \sigma], \mathcal{W}[\varphi_2, \sigma]\} & \varphi = (\varphi_1 \wedge \varphi_2) \\ \max\{\mathcal{W}[\varphi_1, \sigma], \mathcal{W}[\varphi_2, \sigma]\} & \varphi = (\varphi_1 \vee \varphi_2) \\ \mathcal{W}[\varphi_1, \sigma] \to \mathcal{W}[\varphi_2, \sigma] & \varphi = (\varphi_1 \supset \varphi_2) \\ \inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi, \sigma_{x:=d}] & \varphi = (\forall^i x \psi) \\ \sup_{d \in \mathcal{D}_i} \mathcal{W}[\psi, \sigma_{x:=d}] & \varphi = (\exists^i x \psi) \\ \inf_{D \in \mathcal{D}_s} \mathcal{W}[\psi, \sigma_{X:=D}] & \varphi = (\forall^s X \psi) \\ \sup_{D \in \mathcal{D}_s} \mathcal{W}[\psi, \sigma_{X:=D}] & \varphi = (\exists^s X \psi) \end{cases}$$

Again, it can be verified that the choice of x and X in the last definition is immaterial. Note that the last definition establishes the connection between the predicate symbol ε, and the (fuzzy) set inclusion. The truth value assigned to a formula of the form $\{(t \varepsilon T)\}$ with respect to an assignment σ is equal to the membership degree of $\sigma^I[t]$ in the fuzzy subset $\sigma^I[T]$.

The following usual lemma will be needed below (the proof is similar to the proof of Lemma 8.2.8).

Lemma 9.2.5. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^2$-structure.

1. Let x be an individual variable of $\mathcal{L}^2$ and d an element of $\mathcal{D}_i$. For every $\mathcal{L}^2$-formula φ such that $x \notin Fv[\varphi]$, and $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ: $\mathcal{W}[\varphi, \sigma_{x:=d}] = \mathcal{W}[\varphi, \sigma]$.
2. Let X be a set variable of $\mathcal{L}^2$ and D an element of $\mathcal{D}_s$. For every $\mathcal{L}^2$-formula φ such that $X \notin Fv[\varphi]$, and $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ: $\mathcal{W}[\varphi, \sigma_{X:=D}] = \mathcal{W}[\varphi, \sigma]$.

Next, we define Henkin-style second-order Gödel logic. This amounts to the set of tautologies induced by the structures defined above with the additional restriction of *comprehension*. Thus, as done in Henkin-style classical second-order logic, we require that all (fuzzy) subsets of the universe that can be captured by some formula, are indeed included in the domain of (fuzzy) subsets. Structures satisfying this property (namely, admit the comprehension axiom) are called *comprehensive*.

Definition 9.2.6. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^2$-structure. Given an $\mathcal{L}^2$-formula φ, an individual variable x of $\mathcal{L}^2$, and an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ, we denote by $\mathcal{W}[\varphi, \sigma, x]$ the fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ defined by $\lambda d \in \mathcal{D}_i.\mathcal{W}[\varphi, \sigma_{x:=d}]$. $\mathcal{W}$ is called *comprehensive* if $\mathcal{W}[\varphi, \sigma, x] \in \mathcal{D}_s$ for every φ, x, and σ.

Definition 9.2.7. For an $\mathcal{L}^2$-formula φ, we write $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^2}} \varphi$ if $\mathcal{W}[\varphi, \sigma] = 1$ for every comprehensive $\mathcal{L}^2$-structure $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ and $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ. $\mathbf{G\ddot{o}}_{\mathcal{L}^2}$ is the logic consisting of all formulas φ such that $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^2}} \varphi$.

Example 9.2.8. It is easy to see that the comprehension axiom scheme is valid in $\mathbf{G\ddot{o}}_{\mathcal{L}^2}$, i.e.

$$\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^2}} (\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi))))$$

for every $\mathcal{L}^2$-formula φ, set variable $X \notin Fv[\varphi]$, and individual variable x. Indeed, let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be a comprehensive $\mathcal{L}^2$-structure, and σ be an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment. By definition, $\mathcal{W}[\varphi, \sigma, x] \in \mathcal{D}_s$. Thus

$$\mathcal{W}[(\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)))), \sigma] \geq \mathcal{W}[(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi))), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x]}].$$

By definition, for every $d \in \mathcal{D}_i$ we have:

$$\mathcal{W}[\{(x\varepsilon X)\}), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x],x:=d}] = \mathcal{W}[\varphi, \sigma, x][d] = \mathcal{W}[\varphi, \sigma_{x:=d}].$$

Since $X \notin Fv[\varphi]$,

$$\mathcal{W}[(\varphi \supset \{(x\varepsilon X)\}), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x],x:=d}] = \mathcal{W}[\varphi, \sigma_{x:=d}] \to \mathcal{W}[\varphi, \sigma_{x:=d}] = 1,$$

and similarly,

$$\mathcal{W}[(\{(x\varepsilon X)\} \supset \varphi), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x],x:=d}] = 1.$$

It follows that

$$\mathcal{W}[((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x],x:=d}] = \min\{1, 1\} = 1.$$

Since this holds for every $d \in \mathcal{D}_i$, we have:

$$\inf_{d \in \mathcal{D}_i} \mathcal{W}[((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)), \sigma_{X:=\mathcal{W}[\varphi,\sigma,x],x:=d}] = 1.$$

Consequently,

$$\mathcal{W}[(\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)))), \sigma] = 1.$$

9.3 The Hypersequent Calculus HIF^2

In this section we present a hypersequent calculus **HIF**^{2} for $\mathbf{G\ddot{o}}_{\mathcal{L}^2}$. **HIF**^{2} is obtained by augmenting the hypersequent calculus **HIF** for standard first-order Gödel logic (presented in Section 8.3) with rules for second-order quantifiers. These are the hypersequent versions of the sequent rules used for classical logic (see the calculus $\mathbf{L}^2\mathbf{K}$ in [58]). They have the same structure of the rules for individual quantifiers, where instead of using first-order terms in (`f`:$\forall$) and (`t`:$\exists$), one uses *abstraction terms* (*abstracts* for short). Abstracts are syntactic objects of the form $\{\!\!\{x \mid \varphi\}\!\!\}$ that intuitively represent sets of individuals. Note that abstracts are just a syntactic tool for formulating the rules of the set quantifiers. Derivations in the calculus still consists solely of hypersequents, and no abstracts are mentioned in them. As we did for formulas, we first define *concrete abstracts*, and abstracts are defined as alpha-equivalence classes of concrete ones.

Definition 9.3.1. A *concrete $\mathcal{L}^2$-abstract* is an expression of the form $\{\!\!\{x \mid \Phi\}\!\!\}$, where x is an individual variable of $\mathcal{L}^2$, and Φ is a concrete $\mathcal{L}^2$-formula. Alpha-equivalence between concrete $\mathcal{L}^2$-abstracts is defined as usual (where x is considered bound in $\{\!\!\{x \mid \Phi\}\!\!\}$), and $[\{\!\!\{x \mid \Phi\}\!\!\}]_\alpha$ is standing for the set of all concrete $\mathcal{L}^2$-abstracts which are alpha-equivalent to $\{\!\!\{x \mid \Phi\}\!\!\}$. An *$\mathcal{L}^2$-abstract* is an equivalence class of concrete $\mathcal{L}^2$-abstracts under alpha-equivalence. We mainly use τ as a metavariable for $\mathcal{L}^2$-abstracts. The *set of free variables* of an $\mathcal{L}^2$-abstract is defined using representatives, i.e. for an $\mathcal{L}^2$-abstract τ, $Fv[\tau] = Fv[\{\!\!\{x \mid \Phi\}\!\!\}]$ for some $\{\!\!\{x \mid \Phi\}\!\!\} \in \tau$.

Definition 9.3.2. Given an individual variable x of $\mathcal{L}^2$ and an $\mathcal{L}^2$-formula φ, $\{\!\!\{x \mid \varphi\}\!\!\}$ is the $\mathcal{L}^2$-abstract $[\{\!\!\{x \mid \Phi\}\!\!\}]_\alpha$ for some $\Phi \in \varphi$.

Proposition 9.3.3. For every $\mathcal{L}^2$-abstract τ and individual variable $x \notin Fv[\tau]$, there exists a unique $\mathcal{L}^2$-formula φ, such that $\tau = \{\!\!\{x \mid \varphi\}\!\!\}$.

Definition 9.3.4. Let τ be an $\mathcal{L}^2$-abstract, and t a first-order $\mathcal{L}^2$-term. $\tau[t]$ is defined to be the $\mathcal{L}^2$-formula $\varphi\{{}^{t}/_{x}\}$ for some individual variable x and $\mathcal{L}^2$-formula φ, such that $\tau = \{\!\!\{x \mid \varphi\}\!\!\}$.

It is easy to see that $\tau[t]$ is well-defined, as it does not depend on the choice of x.

Definition 9.3.5. Let τ be an $\mathcal{L}^2$-abstract and X a set variable of $\mathcal{L}^2$. Given an $\mathcal{L}^2$-formula φ, $\varphi\{{}^{\tau}/X\}$ is inductively defined by:

$$\varphi\{{}^{\tau}/X\} = \begin{cases} \varphi & \varphi = \{p(t_1, \dots, t_n)\}, \varphi = \{(t\varepsilon T)\} \text{ for } T \neq X, \varphi = \{\bot\} \\ \tau[t] & \varphi = \{(t\varepsilon X)\} \\ (\varphi_1\{{}^{\tau}/X\} \diamond \varphi_2\{{}^{\tau}/X\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{{}^{\tau}/X\}) & \varphi = (Q^i y \psi) \text{ for } y \notin Fv[\tau] \\ (Q^s Y \psi\{{}^{\tau}/X\}) & \varphi = (Q^s Y \psi) \text{ for } Y \notin Fv[\tau] \cup \{X\} \end{cases}$$

Note that the this substitution operation is well-defined. In particular, the choice of the variables y and Y is immaterial.

Example 9.3.6. For $\varphi = (\forall^i \nu_1(\{(\nu_1 \varepsilon \chi_1)\} \supset (\exists^s \chi_2\{(\nu_1 \varepsilon \chi_2)\})))$, and $\tau = \{\!\!\{\nu_2 \mid \{p(\nu_2, \nu_2)\}\}\!\!\}$, we have $\varphi\{{}^{\tau}/\chi_1\} = (\forall^i \nu_1(\{p(\nu_1, \nu_1)\} \supset (\exists^s \chi_2\{(\nu_1 \varepsilon \chi_2)\})))$.

The following lemmas will be useful in the sequel.

Notation 9.3.7. For a second-order $\mathcal{L}^2$-term T, the $\mathcal{L}^2$-abstract $\lbrace\!\!\lbrace \nu_1 \mid \{(\nu_1 \varepsilon T)\} \rbrace\!\!\rbrace$ is denoted by T_{abs}.

Lemma 9.3.8. Let T be a second-order $\mathcal{L}^2$-term. For every $\mathcal{L}^2$-formula φ and set variable X of $\mathcal{L}^2$, $\varphi\{{}^{T_{abs}}/X\} = \varphi\{{}^{T}/X\}$.

Proof. By usual induction on the complexity of φ. □

Lemma 9.3.9. Let τ be an $\mathcal{L}^2$-abstract, t, t' first-order $\mathcal{L}^2$-terms, and x an individual variable such that $x \notin Fv[\tau]$. Then, $\tau[t']\{{}^{t}/x\} = \tau[t'\{{}^{t}/x\}]$.

Proof. It is straightforward to prove that $\varphi\{{}^{t'}/y\}\{{}^{t}/x\} = \varphi\{{}^{t'\{{}^{t}/x\}}/y\}$ for every $\mathcal{L}^2$-formula φ, first-order $\mathcal{L}^2$-terms t and t', and individual variables x and y, such that $x \notin Fv[\varphi]$. The claim then easily follows from our definitions. □

Using abstracts, the rule schemes for the second-order quantifiers in **HIF**^{2} are given by:

$$(\mathtt{f}{:}\forall^s)\ \frac{H \mid \Gamma, \varphi\{{}^{\tau}/X\} \Rightarrow E}{H \mid \Gamma, (\forall^s X \varphi) \Rightarrow E} \qquad (\mathtt{t}{:}\forall^s)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow (\forall^s X \varphi)}$$

$$(\mathtt{f}{:}\exists^s)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, (\exists^s X \varphi) \Rightarrow E} \qquad (\mathtt{t}{:}\exists^s)\ \frac{H \mid \Gamma \Rightarrow \varphi\{{}^{\tau}/X\}}{H \mid \Gamma \Rightarrow (\exists^s X \varphi)}$$

where X must not be a free variable in the lower hypersequent in applications of the rules $(\mathtt{t}{:}\forall^s)$ and $(\mathtt{f}{:}\exists^s)$.

Below, we write $\vdash_{\mathbf{HIF}^2} H$ to denote that an $\mathcal{L}^2$-hypersequent H is provable in **HIF**^{2}, and $\vdash^{cf}_{\mathbf{HIF}^2} H$ to denote that H is provable in **HIF**^{2} without applying (cut).

Since formulas are equivalence classes, the rules $(\mathtt{t}{:}\forall^s)$, and $(\mathtt{f}{:}\exists^s)$ could be written as well as:

$$(\mathtt{t}{:}\forall^s)\ \frac{H \mid \Gamma \Rightarrow \varphi\{{}^{Y}/X\}}{H \mid \Gamma \Rightarrow (\forall^s X \varphi)} \qquad (\mathtt{f}{:}\exists^s)\ \frac{H \mid \Gamma, \varphi\{{}^{Y}/X\} \Rightarrow E}{H \mid \Gamma, (\exists^s X \varphi) \Rightarrow E}$$

where Y must not be a free variable in the lower hypersequent.

Remark 9.3.10. Note that rules given by the schemes

$$\frac{H \mid \Gamma, \varphi\{{}^{T}/X\} \Rightarrow E}{H \mid \Gamma, (\forall^s X \varphi) \Rightarrow E} \qquad \frac{H \mid \Gamma \Rightarrow \varphi\{{}^{T}/X\}}{H \mid \Gamma \Rightarrow (\exists^s X \varphi)},$$

where T is a second-order $\mathcal{L}^2$-term, are particular instances of $(\mathtt{f}{:}\forall^s)$ and $(\mathtt{t}{:}\exists^s)$, obtained by choosing $\tau = T_{abs}$ (see Lemma 9.3.8).

9.4 Soundness

In this section we prove the soundness of $\mathbf{HIF}^2$ for $\mathbf{G\ddot{o}}_{\mathcal{L}^2}$. Definition 8.4.1 (defining when a $\mathcal{L}^1$-structure is a model of an $\mathcal{L}^1$-hypersequent etc.) is adopted for $\mathcal{L}^2$-structures as is.

Theorem 9.4.1. Let H be an $\mathcal{L}^2$-hypersequent. If $\vdash_{\mathbf{HIF}^2} H$, then every comprehensive $\mathcal{L}^2$-structure is a model of H.

Soundness for $\mathbf{G\ddot{o}}_{\mathcal{L}^2}$ is an obvious corollary (see Corollary 8.4.3):

Corollary 9.4.2. For every $\mathcal{L}^2$-formula φ, if $\vdash_{\mathbf{HIF}^2} \{\{\mathtt{t}{:}\varphi\}\}$, then $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^2}} \varphi$.

Theorem 9.4.1 is proved in the usual way, by induction on the length of the derivation in $\mathbf{HIF}^2$. We use the following technical lemmas:

Lemma 9.4.3. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P\rangle$ be an $\mathcal{L}^2$-structure, t a first-order $\mathcal{L}^2$-term, and x an individual variable of $\mathcal{L}^2$. For every $\mathcal{L}^2$-formula φ, and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ:

$$\mathcal{W}[\varphi, \sigma_{x:=\sigma^I[t]}] = \mathcal{W}[\varphi\{^t/_x\}, \sigma].$$

Proof. The claim is proved by induction on the complexity of φ, similarly to the proof of Lemma 8.2.9 for the first-order case. We do here the case $\varphi = \{(t'\varepsilon T)\}$. Let σ be an $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment. Then $\varphi\{^t/_x\} = \{(t'\{^t/_x\}\varepsilon T)\}$. Thus $\mathcal{W}[\varphi\{^t/_x\}, \sigma] = \sigma^I[T][\sigma^I[t'\{^t/_x\}]]$. By Lemma 8.2.6 (adapted to second-order languages), $\sigma^I[t'\{^t/_x\}] = \sigma_{x:=\sigma^I[t]}[t']$. Now $\sigma^I[T] = \sigma^I_{x:=\sigma[t]}[T]$, and so $\sigma^I[T][\sigma^I[t'\{^t/_x\}]] = \sigma^I_{x:=\sigma[t]}[T][\sigma^I_{x:=\sigma[t]}[t']]$. By definition, this is equal to $\mathcal{W}[\varphi, \sigma_{x:=\sigma[t]}]$. □

Lemma 9.4.4. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P\rangle$ be an $\mathcal{L}^2$-structure, τ an $\mathcal{L}^2$-abstract, $x \notin Fv[\tau]$ an individual variable, and X a set variable. For every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ: if $\mathcal{W}[\tau[x], \sigma, x] \in \mathcal{D}_s$ then $\mathcal{W}[\varphi\{^\tau/X\}, \sigma] = \mathcal{W}[\varphi, \sigma_{X:=\mathcal{W}[\tau[x],\sigma,x]}]$.

Proof. If $X \notin Fv[\varphi]$, then $\varphi\{^\tau/X\} = \varphi$ and in this case the claim follows by Lemma 9.2.5. Suppose otherwise. We prove the claim by induction on the complexity of φ. Suppose that $cp[\varphi] = 1$. Let σ be an $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment, and let $D_0 = \mathcal{W}[\tau[x], \sigma, x]$. Suppose that $D_0 \in \mathcal{D}_s$. Since $X \in Fv[\varphi]$, $\varphi = \{(t\varepsilon X)\}$ for some first-order $\mathcal{L}^2$-term t. In this case, $\varphi\{^\tau/X\} = \tau[t]$. By Lemma 9.3.9, $\tau[t] = \tau[x\{^t/_x\}] = \tau[x]\{^t/_x\}$. Thus $\mathcal{W}[\varphi\{^\tau/X\}, \sigma] = \mathcal{W}[\tau[x]\{^t/_x\}, \sigma]$. By Lemma 9.4.3, $\mathcal{W}[\tau[x]\{^t/_x\}, \sigma] = \mathcal{W}[\tau[x], \sigma_{x:=\sigma^I[t]}]$. By definition, $\mathcal{W}[\tau[x], \sigma_{x:=\sigma^I[t]}] = D_0[\sigma^I[t]] = \mathcal{W}[\varphi, \sigma_{X:=D_0}]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^2$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment, and again let $D_0 = \mathcal{W}[\tau[x], \sigma, x]$. Suppose that $D_0 \in \mathcal{D}_s$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}^2$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. By definition, $\varphi\{{}^{\tau}/X\} = (\varphi_1\{{}^{\tau}/X\} \diamond \varphi_2\{{}^{\tau}/X\})$. We continue with $\diamond = \supset$ (the proof is similar for $\wedge$ and $\vee$). Thus,
$$\mathcal{W}[\varphi\{{}^{\tau}/X\}, \sigma] = \mathcal{W}[\varphi_1\{{}^{\tau}/X\}, \sigma] \to [\varphi_2\{{}^{\tau}/X\}, \sigma].$$
By the induction hypothesis (and the case in which $X \notin Fv[\varphi]$), we have both $\mathcal{W}[\varphi_1\{{}^{\tau}/X\}, \sigma] = \mathcal{W}[\varphi_1, \sigma_{X:=D_0}]$ and $\mathcal{W}[\varphi_2\{{}^{\tau}/X\}, \sigma] = \mathcal{W}[\varphi_2, \sigma_{X:=D_0}]$. By definition, $\mathcal{W}[\varphi_1, \sigma_{X:=D_0}] \to \mathcal{W}[\varphi_2, \sigma_{X:=D_0}] = \mathcal{W}[\varphi, \sigma_{X:=D_0}]$.

- $\varphi = (Q^i y \psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, individual variable $y \notin \{x\} \cup Fv[\tau]$ of $\mathcal{L}^2$, and $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$. By definition, $\varphi\{{}^{\tau}/X\} = (Q^i y \psi\{{}^{\tau}/X\})$. We continue with $Q^i = \forall^i$ (the proof is similar for $\exists^i$). Thus,
$$\mathcal{W}[\varphi\{{}^{\tau}/X\}, \sigma] = \inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi\{{}^{\tau}/X\}, \sigma_{y:=d}].$$
Now, using Lemma 9.2.5, we have $D_0 = \mathcal{W}[\tau[x], \sigma_{y:=d}, x]$ for every $d \in \mathcal{D}_i$ (since $y \notin Fv[\tau[x]]$). Therefore, $\inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi\{{}^{\tau}/X\}, \sigma_{y:=d}] = \inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi, \sigma_{y:=d, X:=D_0}]$ by the induction hypothesis. By definition, $\inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi, \sigma_{y:=d, X:=D_0}] = \mathcal{W}[\varphi, \sigma_{X:=D_0}]$.

- $\varphi = (Q^s Y \psi)$ for some $Q^s \in \{\forall^s, \exists^s\}$, set variable $Y \notin \{X\} \cup Fv[\tau]$ of $\mathcal{L}^2$, and $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$. By definition, $\varphi\{{}^{\tau}/X\} = (Q^s Y \psi\{{}^{\tau}/X\})$. We continue with $Q^s = \forall^s$ (the proof is similar for $\exists^s$). Thus,
$$\mathcal{W}[\varphi\{{}^{\tau}/X\}, \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{W}[\psi\{{}^{\tau}/X\}, \sigma_{Y:=D}].$$
Now, using Lemma 9.2.5, we have $D_0 = \mathcal{W}[\tau[x], \sigma_{Y:=D}, x]$ for every $D \in \mathcal{D}_s$ (since $Y \notin Fv[\tau[x]]$). Therefore, $\inf_{D \in \mathcal{D}_s} \mathcal{W}[\psi\{{}^{\tau}/X\}, \sigma_{Y:=D}] = \inf_{D \in \mathcal{D}_s} \mathcal{W}[\psi, \sigma_{Y:=D, X:=D_0}]$ by the induction hypothesis (note that $Y \neq X$). By definition,
$$\inf_{D \in \mathcal{D}_s} \mathcal{W}[\psi, \sigma_{Y:=D, X:=D_0}] = \mathcal{W}[\varphi, \sigma_{X:=D_0}]. \qquad \square$$

Proof of Theorem 9.4.1. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}, I, P \rangle$ be an $\mathcal{L}^2$-structure, where $\mathcal{V} = \langle V, \leq \rangle$. It suffices to prove soundness of each possible application of a rule of $\mathbf{HIF}^2$. For the rules of **HIF**, this is done as in the proof of Theorem 8.4.2. We prove here the soundness of $(\mathtt{f}{:}\forall^s)$, and leave the other three new rules to the reader:

Suppose that $H = H' \mid s \cup \{\mathtt{f}{:}(\forall^s X \varphi)\}$ is derived from $H' \mid s \cup \{\mathtt{f}{:}\{{}^{\tau}/X\}\}$ using $(\mathtt{f}{:}\forall^s)$. Assume that $\mathcal{W}, \sigma \not\models H$ for some $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ. Hence, $\mathcal{W}, \sigma \not\models s'$ for every $s' \in H'$, and $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{f}{:}(\forall^s X \varphi)\}$. Let $u = \mathcal{W}^{\mathtt{t}}[s, \sigma]$. The assumption that $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{f}{:}(\forall^s X \varphi)\}$ entails that $\mathcal{W}^{\mathtt{f}}[s, \sigma] > u$, and $\mathcal{W}[(\forall^s X \varphi), \sigma] > u$. By definition, $\mathcal{W}[(\forall^s X \varphi), \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{W}[\varphi, \sigma_{X:=D}]$. Thus $\mathcal{W}[\varphi, \sigma_{X:=D}] > u$ for every $D \in \mathcal{D}_s$. Let x be an individual variable such that $x \notin Fv[\tau]$, and let $D_0 = \mathcal{W}[\tau[x], \sigma, x]$. Since $\mathcal{W}$ is comprehensive, $D_0 \in \mathcal{D}_s$, and in particular, $\mathcal{W}[\varphi, \sigma_{X:=D_0}] > u$. Lemma 9.4.4 implies that $\mathcal{W}[\varphi\{{}^{\tau}/X\}, \sigma] > u$. It follows that $\mathcal{W}, \sigma \not\models s \cup \{\mathtt{f}{:}\varphi\{{}^{\tau}/X\}\}$. Consequently, $\mathcal{W}$ is not a model of $H' \mid s \cup \{\mathtt{f}{:}\varphi\{{}^{\tau}/X\}\}$. $\square$

9.5 Complete Non-deterministic Semantics

In this section we present a non-deterministic semantics for which the cut-free fragment of $\mathbf{HIF}^2$ is complete. As for **HIF** in the previous chapter, this semantics will be used in the next section, where we show that ordinary counter-models can be extracted out of non-deterministic ones.

Definition 9.5.1. A *quasi-domain* $\mathcal{D}$ consists of:

- A non-empty set, called *individuals domain* and denoted by $\mathcal{D}_i$.
- A non-empty set, called *sets domain* and denoted by $\mathcal{D}_s$.

Given a quasi-domain $\mathcal{D}$, an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-interpretation I consists of:

- A function assigning an element in $\mathcal{D}_i$ to every individual constant symbol of $\mathcal{L}^2$, and a function in ${\mathcal{D}_i}^n \to \mathcal{D}_i$ to every n-ary function symbol of $\mathcal{L}^2$. We call this function *individuals interpretation*, and denote it by I_i.
- A function assigning an element of $\mathcal{D}_s$ to every set constant symbol of $\mathcal{L}^2$. We call this function *sets interpretation*, and denote it by I_s.

$\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignments are defined for quasi-domains exactly as for domains (see Definition 9.2.3). Note that the elements of $\mathcal{D}_s$ in quasi-domains may not be fuzzy subsets. This allows us to compose $\mathcal{D}_s$ out of abstracts (as done in the completeness proof). Instead, as defined below, the interpretation function P of a quasi-$\mathcal{L}^2$-structure assigns a (quasi) fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ to every element of $\mathcal{D}_s$.

Definition 9.5.2. A *quasi-$\mathcal{L}^2$-structure* is a tuple $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$, where:

1. $\mathcal{V}$ is a Gödel set.
2. $\mathcal{D}$ is a quasi-domain.
3. I is an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-interpretation.
4. P is a function assigning a quasi fuzzy subset of $\mathcal{D}_i^n$ over $\mathcal{V}$ to every n-ary predicate symbol of $\mathcal{L}^2$, and a quasi fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ to every element of $\mathcal{D}_s$ (see Definition 8.5.1).
5. v is a function assigning a pair in $\{\langle u^{\mathtt{f}}, u^{\mathtt{t}} \rangle \in \mathcal{V} \times \mathcal{V} \mid u^{\mathtt{f}} \leq u^{\mathtt{t}}\}$ to every ordered pair of the form $\langle \varphi, \sigma \rangle$, where φ is an $\mathcal{L}^2$-formula and σ is an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment, such that the following hold:
   (a) For every two individual variables x, y such that $y \notin Fv[\varphi]$, and every $d \in \mathcal{D}_i$: $v[\varphi, \sigma_{x:=d}] = v[\varphi\{{}^{y}/_{x}\}, \sigma_{y:=d}]$.
   (b) For every two set variables X, Y such that $Y \notin Fv[\varphi]$, and every $D \in \mathcal{D}_s$: $v[\varphi, \sigma_{X:=D}] = v[\varphi\{{}^{Y}/_{X}\}, \sigma_{Y:=D}]$.

Next, we define which quasi-$\mathcal{L}^2$-structures are considered *legal*.

Definition 9.5.3. Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$ be a quasi-$\mathcal{L}^2$-structure. For every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ, $\underline{v}_{\mathcal{Q}}[\varphi, \sigma]$ is the pair defined as follows:

$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma] = \begin{cases} P[p][\sigma^I[t_1], \dots, \sigma^I[t_n]] & \varphi = \{p(t_1, \dots, t_n)\} \\ P[\sigma^I[T]][\sigma^I[t]] & \varphi = \{(t \varepsilon T)\} \\ \langle 0, 0\rangle & \varphi = \{\bot\} \\ \langle \min\{v^{\mathtt{f}}[\varphi_1,\sigma], v^{\mathtt{f}}[\varphi_2,\sigma]\}, \min\{v^{\mathtt{t}}[\varphi_1,\sigma], v^{\mathtt{t}}[\varphi_2,\sigma]\}\rangle & \varphi = (\varphi_1 \wedge \varphi_2) \\ \langle \max\{v^{\mathtt{f}}[\varphi_1,\sigma], v^{\mathtt{f}}[\varphi_2,\sigma]\}, \max\{v^{\mathtt{t}}[\varphi_1,\sigma], v^{\mathtt{t}}[\varphi_2,\sigma]\}\rangle & \varphi = (\varphi_1 \vee \varphi_2) \\ \langle v^{\mathtt{t}}[\varphi_1,\sigma] \to v^{\mathtt{f}}[\varphi_2,\sigma], v^{\mathtt{f}}[\varphi_1,\sigma] \to v^{\mathtt{t}}[\varphi_2,\sigma]\rangle & \varphi = (\varphi_1 \supset \varphi_2) \\ \langle \inf_{d\in\mathcal{D}_i} v^{\mathtt{f}}[\psi, \sigma_{x:=d}], \inf_{d\in\mathcal{D}_i} v^{\mathtt{t}}[\psi, \sigma_{x:=d}]\rangle & \varphi = (\forall^i x \psi) \\ \langle \sup_{d\in\mathcal{D}_i} v^{\mathtt{f}}[\psi, \sigma_{x:=d}], \sup_{d\in\mathcal{D}_i} v^{\mathtt{t}}[\psi, \sigma_{x:=d}]\rangle & \varphi = (\exists^i x \psi) \\ \langle \inf_{D\in\mathcal{D}_s} v^{\mathtt{f}}[\psi, \sigma_{X:=D}], \inf_{D\in\mathcal{D}_s} v^{\mathtt{t}}[\psi, \sigma_{X:=D}]\rangle & \varphi = (\forall^s X \psi) \\ \langle \sup_{D\in\mathcal{D}_s} v^{\mathtt{f}}[\psi, \sigma_{X:=D}], \sup_{D\in\mathcal{D}_s} v^{\mathtt{t}}[\psi, \sigma_{X:=D}]\rangle & \varphi = (\exists^s X \psi) \end{cases}$$

Conditions (a) and (b) in Definition 9.5.2 ensure that $\mathcal{Q}$ is well-defined, namely that the choice of x and X is immaterial. It is straightforward to verify that $\underline{v}^{\mathtt{f}}_{\mathcal{Q}}[\varphi,\sigma] \leq \underline{v}^{\mathtt{t}}_{\mathcal{Q}}[\varphi,\sigma]$ for every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ.

Definition 9.5.4. A quasi-$\mathcal{L}^2$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$ is called *legal* if we have $\underline{v}_{\mathcal{Q}}[\varphi,\sigma] \subseteq v[\varphi,\sigma]$ for every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ.

We adapt the definition of comprehensive structures to quasi-structures, keeping in mind that the function P interprets the elements of $\mathcal{D}_s$ as quasi fuzzy sets.

Definition 9.5.5. A quasi-$\mathcal{L}^2$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$ is called *comprehensive* if for every $\mathcal{L}^2$-formula φ, individual variable x, and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ, there exist some $D \in \mathcal{D}_s$ such that $P[D] = \lambda d \in \mathcal{D}_i . v[\varphi, \sigma_{x:=d}]$.

The notion of *model* for quasi-$\mathcal{L}^2$-structures is defined exactly as in the first-order case (Definition 8.5.8). In turn, the main completeness theorem is given by:

Theorem 9.5.6. Suppose that $\not\vdash^{cf}_{\mathbf{HIF}^2} H_0$, for some $\mathcal{L}^2$-hypersequent H_0. Then there exists a legal comprehensive quasi-$\mathcal{L}^2$-structure which is not a model of H_0.

The rest of this section is devoted to prove this theorem.

The Herbrand Quasi-Domain

A main ingredient in the completeness proof below is the *Herbrand quasi-domain*. It is defined as follows:

Definition 9.5.7. The *Herbrand quasi-domain for $\mathcal{L}^2$*, denoted by $\mathcal{D}^{\mathcal{L}^2}$, is the quasi-domain whose individuals domain $\mathcal{D}_i^{\mathcal{L}^2}$ consists of all first-order $\mathcal{L}^2$-terms, and sets domain $\mathcal{D}_s^{\mathcal{L}^2}$ consists of all $\mathcal{L}^2$-abstracts. The *Herbrand interpretation for $\mathcal{L}^2$*, denoted by $I_{\mathcal{L}^2}$, is the $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-interpretation defined by: $I_i[c] = c$ for every individual constant symbol c of $\mathcal{L}^2$, $I_i[f] = \lambda t_1, \dots, t_n \in \mathcal{D}_i . f(t_1, \dots, t_n)$ for every n-ary function symbol f of $\mathcal{L}^2$, and $I_s[C] = C_{abs} = \{\!\!\{\nu_1 \mid \{(\nu_1 \varepsilon C)\}\}\!\!\}$ for every set constant symbol C of $\mathcal{L}^2$.

Below, given an $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment σ and a first-order (second-order) $\mathcal{L}^2$-term t (T), we write $\sigma[t]$ ($\sigma[T]$) instead of $\sigma^{I_{\mathcal{L}^2}}[t]$ ($\sigma^{I_{\mathcal{L}^2}}[T]$). In addition, $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignments are extended to apply on $\mathcal{L}$-formulas. Roughly speaking, every occurrence of a free variable x or X in a formula φ is replaced in $\sigma[\varphi]$ by $\sigma[x]$ or $\sigma[X]$. Formally, this is defined as follows.

Definition 9.5.8. Let φ be an $\mathcal{L}^2$-formula, and σ an $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment. The set of free variables of the pair $\langle \varphi, \sigma \rangle$ (denoted by $Fv[\langle \varphi, \sigma \rangle]$) consists of the variables of $\sigma[x]$ for every individual variable $x \in Fv[\varphi]$, and the free variables of $\sigma[X]$ for every set variable $X \in Fv[\varphi]$.

Definition 9.5.9. $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignments are extended to $\mathcal{L}^2$-formulas, according to the following inductive definition:

$$\sigma[\varphi] = \begin{cases} \{p(\sigma[t_1], \dots, \sigma[t_n])\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \sigma[T][\sigma[t]] & \varphi = \{(t \varepsilon T)\} \\ \{\bot\} & \varphi = \{\bot\} \\ (\sigma[\varphi_1] \diamond \sigma[\varphi_2]) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i x \sigma_{x:=x}[\psi]) & \varphi = (Q^i x \psi) \text{ for } x \notin Fv[\langle \varphi, \sigma \rangle] \\ (Q^s X \sigma_{X:=X_{abs}}[\psi]) & \varphi = (Q^s X \psi) \text{ for } X \notin Fv[\langle \varphi, \sigma \rangle] \end{cases}$$

Note that the choice of x and X in the last definition is immaterial, and thus $\sigma[\varphi]$ is well-defined. The following properties of the Herbrand quasi-domain are needed in the completeness proof.

Lemma 9.5.10. Let t be a first-order $\mathcal{L}^2$-term and τ an $\mathcal{L}^2$-abstract.

1. For every $\mathcal{L}^2$-formula φ, $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment σ, and individual variables x, y such that $y \notin Fv[\varphi]$, $\sigma_{x:=t}[\varphi] = \sigma_{y:=t}[\varphi\{{}^{y}/_{x}\}]$.

2. For every $\mathcal{L}^2$-formula φ, $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment σ, and set variables X, Y such that $Y \notin Fv[\varphi]$, $\sigma_{X:=\tau}[\varphi] = \sigma_{Y:=\tau}[\varphi\{Y/X\}]$.

Proof. First, as in Lemma 8.5.16, it is straightforward to show that for every first-order $\mathcal{L}^2$-terms t' and t, $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment σ, and individual variables x, y such that $y \notin Fv[t']$, $\sigma_{x:=t}[t'] = \sigma_{y:=t}[t'\{y/x\}]$.

Next, we prove the first claim in the lemma by induction on the complexity of φ. We include here only the cases that are special for second-order language (the other cases are handled as in the proof of Lemma 8.5.16). Let σ be an $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment, and x, y individual variables such that $y \notin Fv[\varphi]$. Consider the following cases:

- $\varphi = \{(t' \varepsilon T)\}$ for some first-order $\mathcal{L}^2$-term t', and second-order $\mathcal{L}^2$-term T. Then, $\sigma_{x:=t}[\varphi] = \sigma_{x:=t}[T][\sigma_{x:=t}[t']]$. Since $y \notin Fv[t']$, this equals $\sigma_{x:=t}[T][\sigma_{y:=t}[t'\{y/x\}]]$. Since x and y does not occur in T, this is equal to $\sigma_{y:=t}[T][\sigma_{y:=t}[t'\{y/x\}]]$. By definition, this formula is equal to $\sigma_{y:=t}[\{(t'\{y/x\} \varepsilon T)\}]$, which is $\sigma_{y:=t}[\{(t' \varepsilon T)\}\{y/x\}]$.
- $\varphi = (Q^s X \psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and set variable $X \notin Fv[\sigma[\varphi]]$. Then, $\sigma_{x:=t}[\varphi] = (Q^s X \sigma_{x:=t, X:=X_{abs}}[\psi])$. By the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^s X \sigma_{y:=t, X:=X_{abs}}[\psi\{y/x\}])$. And this is (by definition) equal to $\sigma_{y:=t}[(Q^s X \psi\{y/x\})]$, which in turn equals $\sigma_{y:=t}[(\varphi\{y/x\}]$.

Next, we prove the second claim in the lemma. Suppose that $X \in Fv[\varphi]$ (otherwise, we have $\sigma_{X:=\tau}[\varphi] = \sigma[\varphi] = \sigma_{Y:=\tau}[\varphi\{Y/X\}]$). We use induction on the complexity of φ. First, suppose that $cp[\varphi] = 1$. Let σ be an $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment, and X, Y set variables such that $Y \notin Fv[\varphi]$. Assume that $X \in Fv[\varphi]$. Thus we have $\varphi = \{(t \varepsilon X)\}$ for some first-order $\mathcal{L}^2$-term t. Then, $\sigma_{X:=\tau}[\varphi] = \tau[\sigma_{X:=\tau}[t]]$. Since X and Y do not occur in t, this is equal to $\tau[\sigma_{Y:=\tau}[t]]$, which in turn equals $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$. Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^2$-formulas of lower complexity. Let σ be an $\langle \mathcal{L}^2, \mathcal{D}^{\mathcal{L}^2} \rangle$-assignment, and X, Y set variables such that $Y \notin Fv[\varphi]$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}^2$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. Then, $\sigma_{X:=\tau}[\varphi] = (\sigma_{X:=\tau}[\varphi_1] \diamond \sigma_{X:=\tau}[\varphi_2])$. By the induction hypothesis (and the case in which $X \notin Fv[\varphi]$), this $\mathcal{L}^2$-formula is equal to $(\sigma_{Y:=\tau}[\varphi_1\{Y/X\}] \diamond \sigma_{Y:=\tau}[\varphi_2\{Y/X\}])$. And, by definition, this is equal to $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$.
- $\varphi = (Q^i x \psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and individual variable $x \notin Fv[\tau] \cup Fv[\sigma[\varphi]]$. Then, $\sigma_{X:=\tau}[\varphi] = (Q^i x \sigma_{X:=\tau, x:=x}[\psi])$. By the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^i x \sigma_{Y:=\tau, x:=x}[\psi\{Y/X\}])$. And this is (by definition) equal to $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$.
- $\varphi = (Q^s Z \psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and set variable $Z \notin Fv[\tau] \cup \{X, Y\} \cup Fv[\sigma[\varphi]]$. Then, $\sigma_{X:=\tau}[\varphi] = (Q^s Z \sigma_{X:=\tau, Z:=Z_{abs}}[\psi])$. By

the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^s Z\sigma_{Y:=\tau,Z:=Z_{abs}}[\psi\{{}^{Y}\!/\!{}_{X}\}])$. And this is (by definition) equal to $\sigma_{Y:=\tau}[\varphi\{{}^{Y}\!/\!{}_{X}\}]$. □

Lemma 9.5.11. Let t be a first-order $\mathcal{L}^2$-term and τ an $\mathcal{L}^2$-abstract.

1. For every $\mathcal{L}^2$-formula φ, $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment σ, and individual variables x, z such that $z \notin Fv[\sigma[\varphi]]$, $\sigma_{x:=z}[\varphi]\{{}^{t}\!/\!{}_{z}\} = \sigma_{x:=t}[\varphi]$.
2. For every $\mathcal{L}^2$-formula φ, $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment σ, and set variable $X \notin Fv[\sigma[\varphi]]$, $\sigma_{X:=X_{abs}}[\varphi]\{{}^{\tau}\!/\!{}_{X}\} = \sigma_{X:=\tau}[\varphi]$.

Proof. First, as in Lemma 8.5.17, it is straightforward to show that for every first-order $\mathcal{L}^2$-terms t' and t, $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment σ, and individual variables x, z such that $z \notin Fv[\sigma[t']]$, $\sigma_{x:=z}[t']\{{}^{t}\!/\!{}_{z}\} = \sigma_{x:=t}[t']$.

Next, we prove the first claim in the lemma by induction on the complexity of φ. We include here only the cases that are special for second-order language (the other cases are handled as in the proof of Lemma 8.5.17). Let σ be an $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment, and x, z individual variables such that $z \notin Fv[\sigma[\varphi]]$. Consider the following cases:

- $\varphi = \{(t'\varepsilon T)\}$ for some first-order $\mathcal{L}^2$-term t', and second-order $\mathcal{L}^2$-term T. Then, $\sigma_{x:=z}[\varphi]\{{}^{t}\!/\!{}_{z}\} = \sigma_{x:=z}[T][\sigma_{x:=z}[t']]\{{}^{t}\!/\!{}_{z}\} = \sigma[T][\sigma_{x:=z}[t']]\{{}^{t}\!/\!{}_{z}\}$. By Lemma 9.3.9, since $z \notin Fv[\sigma[T]]$, this formula equals $\sigma[T][\sigma_{x:=z}[t']\{{}^{t}\!/\!{}_{z}\}]$. Since $z \notin Fv[\sigma[t']]$, the proof above for terms entails that this formula equals $\sigma[T][\sigma_{x:=t}[t']]$. Since x does not occur in T, this is equal to $\sigma_{x:=t}[T][\sigma_{x:=t}[t']]$, which is, by definition, $\sigma_{x:=t}[\varphi]$.
- $\varphi = (Q^s X\psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and set variable $X \notin Fv[\sigma[\varphi]]$. Then,
$$\sigma_{x:=z}[\varphi]\{{}^{t}\!/\!{}_{z}\} = (Q^s X\sigma_{x:=z,X:=X_{abs}}[\psi])\{{}^{t}\!/\!{}_{z}\} = (Q^i X\sigma_{x:=z,X:=X_{abs}}[\psi]\{{}^{t}\!/\!{}_{z}\}).$$
By the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^i y\sigma_{x:=t,X:=X_{abs}}[\psi])$. And this is (by definition) equal to $\sigma_{x:=t}[\varphi]$.

Next, we prove the second claim in the lemma. First, if we have $X \notin Fv[\varphi]$, then $\sigma_{X:=X_{abs}}[\varphi] = \sigma[\varphi] = \sigma_{X:=\tau}[\varphi]$. Since $X \notin Fv[\sigma[\varphi]]$, we also have $\sigma[\varphi]\{{}^{\tau}\!/\!{}_{X}\} = \sigma[\varphi]$ as well. Suppose now that $X \in Fv[\varphi]$. We use induction on the complexity of φ. First, suppose that $cp[\varphi] = 1$. Let σ be an $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment, and let $X \notin Fv[\sigma[\varphi]]$. Since $X \in Fv[\varphi]$, we must have $\varphi = \{(t\varepsilon X)\}$ for some first-order $\mathcal{L}^2$-term t. Then,
$$\sigma_{X:=X_{abs}}[\varphi]\{{}^{\tau}\!/\!{}_{X}\} = X_{abs}[\sigma[t]]\{{}^{\tau}\!/\!{}_{X}\} = \{(\sigma[t]\varepsilon X)\}\{{}^{\tau}\!/\!{}_{X}\} = \tau[\sigma[t]] = \sigma_{X:=\tau}[\varphi].$$

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^2$-formulas of lower complexity. Let σ be an $\langle\mathcal{L}^2,\mathcal{D}^{\mathcal{L}^2}\rangle$-assignment, and let $X \notin Fv[\sigma[\varphi]]$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}^2$-formulas φ_1, φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. Then, $\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (\sigma_{X:=X_{abs}}[\varphi_1]\{\tau/X\} \diamond \sigma_{X:=X_{abs}}[\varphi_2]\{\tau/X\})$. By the induction hypothesis (and the case in which $X \notin Fv[\varphi]$), this $\mathcal{L}^2$-formula is equal to $(\sigma_{X:=\tau}[\varphi_1] \diamond \sigma_{X:=\tau}[\varphi_2])$. And, by definition, this is equal to $\sigma_{X:=\tau}[\varphi]$.
- $\varphi = (Q^i x \psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and individual variable $x \notin Fv[\tau] \cup Fv[\sigma[\varphi]]$. Then,
$$\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (Q^i x \sigma_{X:=X_{abs},x:=x}[\psi])\{\tau/X\} = (Q^i x \sigma_{X:=X_{abs},x:=x}[\psi]\{\tau/X\}).$$
By the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^i x \sigma_{X:=\tau,x:=x}[\psi])$. And this is (by definition) equal to $\sigma_{X:=\tau}[\varphi]$.
- $\varphi = (Q^s Y \psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$, and set variable $Y \notin Fv[\tau] \cup \{X\} \cup Fv[\sigma[\varphi]]$. Then,
$$\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (Q^s Y \sigma_{X:=X_{abs},Y:=Y_{abs}}[\psi])\{\tau/X\} = (Q^i Y \sigma_{X:=X_{abs},Y:=Y_{abs}}[\psi]\{\tau/X\}).$$
By the induction hypothesis, this $\mathcal{L}^2$-formula is equal to $(Q^i Y \sigma_{X:=\tau,Y:=Y_{abs}}[\psi])$. And this is (by definition) equal to $\sigma_{X:=\tau}[\varphi]$. □

Proof of Theorem 9.5.6

Suppose that $\not\vdash^{cf}_{\mathbf{HIF^2}} H_0$. The availability of external and internal weakenings ensures that H_0 is unprovable. As in Lemma 8.5.12, it is possible to extend H_0 to a maximal extended $\mathcal{L}^2$-hypersequent Ω^* such that $H_0 \sqsubseteq \Omega^*$.[1] We use Ω^* to construct a counter-model for H_0 in the form of a quasi-$\mathcal{L}^2$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v\rangle$. First, the notations $L[\varphi], R[\varphi], \Omega^*[\varphi]$ are defined exactly in the completeness proof for **HIF** (Theorem 8.5.9). Similarly, the bounded linearly ordered set $\mathcal{V}_0$, the Gödel set $\mathcal{V}$ are constructed using Ω^* exactly as in the first-order proof. Now, let $\mathcal{D} = \mathcal{D}^{\mathcal{L}^2}$ be the Herbrand quasi-domain for $\mathcal{L}^2$, $I = I^{\mathcal{L}^2}$ the Herbrand interpretation for $\mathcal{L}^2$, and define P and v as follows:

- For every n-ary predicate symbol p of $\mathcal{L}^2$, $P[p] = \lambda t_1, \dots, t_n \in \mathcal{D}_i.\Omega^*[\{p(t_1, \dots, t_n)\}]$.
- For every $\mathcal{L}^2$-abstract $\tau \in \mathcal{D}_s$, $P[\tau] = \lambda t \in \mathcal{D}_i.\Omega^*[\tau[t]]$.
- For every $\mathcal{L}^2$-formula φ and $\langle \mathcal{L}^2, \mathcal{D}\rangle$-assignment σ, $v[\varphi, \sigma] = \Omega^*[\sigma[\varphi]]$.

It is easy to verify, using Lemma 9.5.10, that conditions (a) and (b) from Definition 9.5.2 hold.

[1] A maximal extended $\mathcal{L}^2$-hypersequent is defined just like a maximal extended $\mathcal{L}^1$-hypersequent, with the following natural additional requirements in "the witness property":

For every $\mathcal{L}^2$-formula φ and set variable X of $\mathcal{L}^2$:

1. If $\mathtt{t}{:}(\forall^s X \varphi) \in \mu$, then $\mathtt{t}{:}\varphi\{Y/X\} \in \mu$ for some set variable Y of $\mathcal{L}^2$.
2. If $\mathtt{f}{:}(\exists^s X \varphi) \in \mu$, then $\mathtt{f}{:}\varphi\{Y/X\} \in \mu$ for some set variable Y of $\mathcal{L}^2$.

We show that $\mathcal{Q}$ is not a model of H_0. Consider the $\langle\mathcal{L}^2,\mathcal{D}\rangle$-assignment σ_{id} defined by $\sigma_{id}[x]=x$ for every individual variable x of $\mathcal{L}^2$, and $\sigma_{id}[X]=X_{abs}$ for every set variable X of $\mathcal{L}^2$. Let $s\in H_0$. Since $H_0\sqsubseteq\Omega^*$, there exists some $\mu\in\Omega^*$, such that $s\subseteq\mu$. We claim that $\mu\in v^{\mathtt{f}}[\varphi,\sigma_{id}]$ whenever $\mathtt{f}{:}\varphi\in s$, and $\mu\notin v^{\mathtt{t}}[\varphi,\sigma_{id}]$ whenever $\mathtt{t}{:}\varphi\in s$. To see this, it suffices to note that $\sigma_{id}[\varphi]=\varphi$ for every $\mathcal{L}^2$-formula φ. This fact follows from the definition of $\sigma_{id}[\varphi]$. Consequently, $v^{\mathtt{f}}[\varphi,\sigma_{id}]\not\subseteq v^{\mathtt{t}}[\varphi,\sigma_{id}]$, and so $\mathcal{Q},\sigma_{id}\not\models s$.

It remains to prove that $\mathcal{Q}$ is legal and comprehensive. We first show that it is legal, namely that $\underline{v}_{\mathcal{Q}}[\varphi,\sigma]\subseteq v[\varphi,\sigma]$ for every $\mathcal{L}^2$-formula φ and $\langle\mathcal{L}^2,\mathcal{D}\rangle$-assignment σ. Let φ be an $\mathcal{L}^2$-formula, and σ an $\langle\mathcal{L}^2,\mathcal{D}\rangle$-assignment. If φ is a first-order formula ($\mathcal{L}^1$-formula) the proof proceeds as in the proof of Theorem 8.5.9. We consider here only the new possible cases:

- $\varphi=\{(t\varepsilon T)\}$ for some first-order $\mathcal{L}^2$-term t, and second-order $\mathcal{L}^2$-term T. Then
$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma]=P[\sigma[T]][\sigma[t]]=\Omega^*[\sigma[T][\sigma[t]]]=\Omega^*[\sigma[\varphi]]=v[\varphi,\sigma].$$
- $\varphi=(\forall^s X\psi)$ for some set variable $X\notin Fv[\sigma[\varphi]]$ and $\mathcal{L}^2$-formula ψ. Then:
$$\underline{v}_{\mathcal{Q}}[\varphi,\sigma]=\langle\inf_{\tau\in\mathcal{D}_s}v^{\mathtt{t}}[\psi,\sigma_{X:=\tau}],\inf_{\tau\in\mathcal{D}_s}v^{\mathtt{f}}[\psi,\sigma_{X:=\tau}]\rangle.$$
  We first prove that $v^{\mathtt{f}}[\varphi,\sigma]\subseteq\inf_{\tau\in\mathcal{D}_s}v^{\mathtt{f}}[\psi,\sigma_{X:=\tau}]$. Thus we show that for every $\tau\in\mathcal{D}_s$ we have $v^{\mathtt{f}}[\varphi,\sigma]\subseteq v^{\mathtt{f}}[\psi,\sigma_{X:=\tau}]$. Suppose that $\mu\notin v^{\mathtt{f}}[\psi,\sigma_{X:=\tau}]$ for some $\tau\in\mathcal{D}_s$. By definition, we have $\mathtt{f}{:}\sigma_{X:=\tau}[\psi]\notin\mu$. By Lemma 9.5.11, we have $\sigma_{X:=\tau}[\psi]=\sigma_{X:=X_{abs}}[\psi]\{\tau/X\}$. The maximality of Ω^* ensures that there exist an $\mathcal{L}^2$-hypersequent $H\sqsubseteq\Omega^*$, and $\mathcal{L}^2$-sequents $s_1,\dots,s_n\subseteq\mu$, such that
  $$\vdash^{cf}_{\mathbf{HIF}^2}H\mid s_1\cup\{\mathtt{f}{:}\sigma_{X:=X_{abs}}[\psi]\{\tau/X\}\}\mid\dots\mid s_n\cup\{\mathtt{f}{:}\sigma_{X:=X_{abs}}[\psi]\{\tau/X\}\}.$$
  By n consecutive applications of ($\mathtt{f}{:}\forall^s$), we obtain that
  $$\vdash^{cf}_{\mathbf{HIF}^2}H\mid s_1\cup\{\mathtt{f}{:}(\forall^s X\sigma_{X:=X_{abs}}[\psi])\}\mid\dots\mid s_n\cup\{\mathtt{f}{:}(\forall^s X\sigma_{X:=X_{abs}}[\psi])\}.$$
  Since Ω^* is unprovable, we must have $\mathtt{f}{:}(\forall^s X\sigma_{X:=X_{abs}}[\psi])\notin\mu$. By definition, $(\forall^s X\sigma_{X:=X_{abs}}[\psi])=\sigma[\varphi]$. It follows that $\mu\notin v^{\mathtt{f}}[\varphi,\sigma]$.

  Next, we prove that $\inf_{\tau\in\mathcal{D}_s}v^{\mathtt{t}}[\psi,\sigma_{X:=\tau}]\subseteq v^{\mathtt{t}}[\varphi,\sigma]$. Suppose that $\mu\notin v^{\mathtt{t}}[\varphi,\sigma]$. Thus $\mathtt{t}{:}\sigma[\varphi]\in\mu$. By definition, $\sigma[\varphi]=(\forall^s X\sigma_{X:=X_{abs}}[\psi])$. Since Ω^* admits the witness property, there exists a set variable Y of $\mathcal{L}^2$, such that $\mathtt{t}{:}\sigma_{X:=X_{abs}}[\psi]\{Y/X\}\in\mu$. By Lemma 9.3.8, $\sigma_{X:=X_{abs}}[\psi]\{Y/X\}=\sigma_{X:=X_{abs}}[\psi]\{Y_{abs}/X\}$. By Lemma 9.5.11, we have $\sigma_{X:=X_{abs}}[\psi]\{Y_{abs}/X\}=\sigma_{X:=Y_{abs}}[\psi]$. Thus, $\mathtt{t}{:}\sigma_{X:=Y_{abs}}[\psi]\in\mu$. It follows that $\mu\notin v^{\mathtt{t}}[\psi,\sigma_{X:=\tau}]$, for $\tau=Y_{abs}\in\mathcal{D}_s$, and therefore $\mu\notin\bigcap_{\tau\in\mathcal{D}_s}v^{\mathtt{t}}[\psi,\sigma_{X:=\tau}]$.
- The case $\varphi=(\exists^s X\psi)$ is handled similarly.

Finally, we show that $\mathcal{Q}$ is comprehensive. Let φ be an $\mathcal{L}^2$-formula, x an individual variable, and σ an $\langle\mathcal{L}^2,\mathcal{D}\rangle$-assignment. Let $y\notin Fv[\varphi]\cup Fv[\sigma[\varphi]]$ be an individual variable, and let $\tau = \{\!\!\{y \mid \sigma_{y:=y}[\varphi\{^y/_x\}]\}\!\!\}$. Then $\tau \in \mathcal{D}_s$. We show that $P[\tau] = \lambda t \in \mathcal{D}_i.v[\varphi, \sigma_{x:=t}]$. Let $t \in \mathcal{D}_i$. Then, $P[\tau][t] = \Omega^*[\tau[t]] = \Omega^*[\sigma_{y:=y}[\varphi\{^y/_x\}]\{^t/_y\}]$. By Lemma 9.5.10, $\sigma_{y:=y}[\varphi\{^y/_x\}] = \sigma_{x:=y}[\varphi]$. By Lemma 9.5.11, $\sigma_{x:=y}[\varphi]\{^t/_y\} = \sigma_{x:=t}[\varphi]$. Thus, $P[\tau][t] = \Omega^*[\sigma_{x:=t}[\varphi]] = v[\varphi, \sigma_{x:=t}]$. □

9.6 Completeness for the Ordinary Semantics

In this section we use the complete semantics of quasi-structures to prove the completeness of $\mathbf{HIF}^2$ for the (ordinary) structures of Henkin-style second-order Gödel logic. To do so, we show that from every legal quasi-structure which is a counter-model of some hypersequent H, it is possible to extract an (ordinary) structure, which is also not a model of H, without losing comprehension.

Definition 9.6.1. Let $\mathcal{D}$ be a quasi-domain, and $\mathcal{D}'$ a domain for $\mathcal{L}^2$, such that $\mathcal{D}_i = \mathcal{D}'_i$. Let δ be a function from $\mathcal{D}_s$ to $2^{\mathcal{D}'_s} \setminus \{\emptyset\}$. A pair $\langle \sigma, \sigma' \rangle$ of an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment and an $\langle \mathcal{L}^2, \mathcal{D}' \rangle$-assignment (respectively) is called a *δ-pair* if (i) $\sigma[x] = \sigma'[x]$ for every individual variable; and (ii) $\sigma'[X] \in \delta[\sigma[X]]$ for every set variable.

Theorem 9.6.2. Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, I, P, v \rangle$ be a legal and comprehensive quasi-$\mathcal{L}^2$-structure. Then there exists a comprehensive $\mathcal{L}^2$-structure $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}', I', P' \rangle$, where $\mathcal{D}'_i = \mathcal{D}_i$ and $I'_i = I_i$, and a function $\delta : \mathcal{D}_s \to 2^{\mathcal{D}'_s} \setminus \{\emptyset\}$, such that $\mathcal{W}[\varphi, \sigma'] \in v[\varphi, \sigma]$ for every $\mathcal{L}^2$-formula φ and δ-pair $\langle \sigma, \sigma' \rangle$ (of an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment and an $\langle \mathcal{L}^2, \mathcal{D}' \rangle$-assignment).

Proof. First, we define $\mathcal{D}'_s$. For every $D \in \mathcal{D}_s$, denote by F_D the set of fuzzy subsets D' of $\mathcal{D}_i$ over $\mathcal{V}$ (i.e. $D' : \mathcal{D}_i \to \mathcal{V}$), such that $D'[d] \in P[D][d]$ for every $d \in \mathcal{D}_i$. Note that for every $D \in \mathcal{D}_s$, F_D is non-empty, since $P[D][d]$ is non-empty for every $d \in \mathcal{D}_i$. Define $\mathcal{D}'_s$ to be $\bigcup_{D \in \mathcal{D}_s} F_D$. Next, I'_s and P' are defined as follows:

- For every set constant symbol C of $\mathcal{L}^2$, $I'_s[C]$ is an arbitrary element in $F_{I_s[C]}$.
- For every predicate symbol p of $\mathcal{L}^2$, $P'[p] = P[p]^{\mathfrak{f}}$.

Let $\delta : \mathcal{D}_s \to 2^{\mathcal{D}'_s} \setminus \{\emptyset\}$ be defined by $\delta = \lambda D \in \mathcal{D}_s.F_D$. We prove that $\mathcal{W}$ and δ satisfy the requirement in the theorem: $\mathcal{W}[\varphi, \sigma'] \in v[\varphi, \sigma]$ for every $\mathcal{L}^2$-formula φ and δ-pair $\langle \sigma, \sigma' \rangle$. Let $\mathcal{V} = \langle V, \leq \rangle$. We use induction on the complexity of φ. Note that since $\mathcal{Q}$ is legal, it suffices to show that $\mathcal{W}[\varphi, \sigma'] \in \underline{v}_{\mathcal{Q}}[\varphi, \sigma]$ for every δ-pair $\langle \sigma, \sigma' \rangle$.

First, suppose that $cp[\varphi] = 1$, and let $\langle \sigma, \sigma' \rangle$ be a δ-pair of an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment and an $\langle \mathcal{L}^2, \mathcal{D}' \rangle$-assignment. Exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ or $\varphi = \{\bot\}$. These cases are handled as in the proof of Theorem 8.6.1.

- $\varphi = \{(t\varepsilon T)\}$ for some first-order $\mathcal{L}^2$-term t, and second-order $\mathcal{L}^2$-term T. By definition, $\mathcal{W}[\varphi, \sigma'] = \sigma'^{I'}[T][\sigma'^{I'}[t]]$. Clearly, we have $\sigma'^{I'}[t] = \sigma^I[t]$. In addition, $\sigma'^{I'}[T] \in F_{\sigma^I[T]}$ (in case T is a variable, this holds since $\langle \sigma, \sigma' \rangle$ is a δ-pair, and if T is a constant then it holds by definition). Therefore,
$$\sigma'^{I'}[T][\sigma'^{I'}[t]] = \sigma'^{I'}[T][\sigma^I[t]] \in P[\sigma^I[T]][\sigma[t]] = \underline{v}_{\mathcal{Q}}[\varphi, \sigma].$$

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}^2$-formulas of lower complexity. Let $\langle \sigma, \sigma' \rangle$ be a δ-pair. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}^2$-formulas φ_1 and φ_2 such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$. This case is handled as in the proof of Theorem 8.6.1.

- $\varphi = (Q^i x \psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, individual variable x of $\mathcal{L}^2$, and $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$. We continue with $Q^i = \forall^i$ (the proof is similar for $\exists^i$). Clearly, for every $d \in \mathcal{D}_i$, $\langle \sigma_{x:=d}, \sigma'_{x:=d} \rangle$ is a δ-pair. Thus by the induction hypothesis, for every $d \in \mathcal{D}_i$, $\mathcal{W}[\psi, \sigma'_{x:=d}] \in v[\psi, \sigma_{x:=d}]$. Hence,
$$\mathcal{W}[\varphi, \sigma'] = \inf_{d \in \mathcal{D}_i} \mathcal{W}[\psi, \sigma'_{x:=d}] \in \langle \inf_{d \in \mathcal{D}_i} v^{\mathsf{f}}[\psi, \sigma_{x:=d}], \inf_{d \in \mathcal{D}_i} v^{\mathsf{t}}[\psi, \sigma_{x:=d}] \rangle = \underline{v}_{\mathcal{Q}}[\varphi, \sigma].$$

- $\varphi = (Q^s X \psi)$ for some $Q^s \in \{\forall^i, \exists^i\}$, set variable X of $\mathcal{L}^2$, and $\mathcal{L}^2$-formula ψ such that $cp[\psi] < cp[\varphi]$. We continue with $Q^s = \forall^s$ (the proof is similar for $\exists^s$). In this case, we should prove that:
$$\inf_{D' \in \mathcal{D}'_s} \mathcal{W}[\psi, \sigma'_{X:=D'}] \in \langle \inf_{D \in \mathcal{D}_s} v^{\mathsf{f}}[\psi, \sigma_{X:=D}], \inf_{D \in \mathcal{D}_s} v^{\mathsf{t}}[\psi, \sigma_{X:=D}] \rangle.$$
First, we show that $\inf_{D \in \mathcal{D}_s} v^{\mathsf{f}}[\psi, \sigma_{X:=D}] \leq \inf_{D' \in \mathcal{D}'_s} \mathcal{W}[\psi, \sigma'_{X:=D'}]$, by showing that $\inf_{D \in \mathcal{D}_s} v^{\mathsf{f}}[\psi, \sigma_{X:=D}] \leq \mathcal{W}[\psi, \sigma'_{X:=D'}]$ for every $D' \in \mathcal{D}'_s$. Let $D' \in \mathcal{D}'_s$, and let D be an arbitrary element in $\mathcal{D}_s$ such that $D' \in F_D$. Then $\langle \sigma_{X:=D}, \sigma'_{X:=D'} \rangle$ is a δ-pair. By the induction hypothesis, $v^{\mathsf{f}}[\psi, \sigma_{X:=D}] \leq \mathcal{W}[\psi, \sigma'_{X:=D'}]$. Thus, $\inf_{D \in \mathcal{D}_s} v^{\mathsf{f}}[\psi, \sigma_{X:=D}] \leq \mathcal{W}[\psi, \sigma'_{X:=D'}]$.

  Next, we show that $\inf_{D' \in \mathcal{D}'_s} \mathcal{W}[\psi, \sigma'_{X:=D'}] \leq \inf_{D \in \mathcal{D}_s} v^{\mathsf{t}}[\psi, \sigma_{X:=D}]$, by proving that $\inf_{D' \in \mathcal{D}'_s} \mathcal{W}[\psi, \sigma'_{X:=D'}] \leq v^{\mathsf{t}}[\psi, \sigma_{X:=D}]$ for every $D \in \mathcal{D}_s$. Let $D \in \mathcal{D}_s$. Take some $D' \in F_D$. Then $D' \in \mathcal{D}'_s$, and $\langle \sigma_{X:=D}, \sigma'_{X:=D'} \rangle$ is a δ-pair. By the induction hypothesis, $\mathcal{W}[\psi, \sigma'_{X:=D'}] \leq v^{\mathsf{t}}[\psi, \sigma_{X:=D}]$. Thus, $\inf_{D' \in \mathcal{D}'_s} \mathcal{W}[\psi, \sigma'_{X:=D'}] \leq v^{\mathsf{t}}[\psi, \sigma_{X:=D}]$.

Finally, we show that $\mathcal{W}$ is comprehensive. Let φ be an $\mathcal{L}^2$-formula, x an individual variable, and σ' an $\langle \mathcal{L}^2, \mathcal{D}' \rangle$-assignment. We show that $\mathcal{W}[\varphi, \sigma', x] \in \mathcal{D}'_s$. Define an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ as follows: (i) $\sigma[x] = \sigma'[x]$ for every individual variable x; and (ii) for every set variable X, $\sigma[X]$ is an (arbitrary) element of $\mathcal{D}_s$ such that $\sigma'[X] \in F_{\sigma[X]}$. Since $\mathcal{Q}$ is comprehensive, there exists some $D \in \mathcal{D}_s$ such that $P[D] = \lambda d \in \mathcal{D}_i . v[\varphi, \sigma_{x:=d}]$. We claim that $\mathcal{W}[\varphi, \sigma', x] \in F_D$ (and so, $\mathcal{W}[\varphi, \sigma', x] \in \mathcal{D}'_s$). By definition, we should show that $\mathcal{W}[\varphi, \sigma', x][d] \in P[D][d]$ for every $d \in \mathcal{D}_i$. Let

$d \in \mathcal{D}_i$. Obviously, $\langle \sigma_{x:=d}, \sigma'_{x:=d} \rangle$ is a δ-pair, and thus by the claim proved above, we have $\mathcal{W}[\varphi, \sigma', x][d] = \mathcal{W}[\varphi, \sigma'_{x:=d}] \in v[\varphi, \sigma_{x:=d}] = P[D][d]$. □

Corollary 9.6.3. If $\not\vdash^{cf}_{\mathbf{HIF}^2} H$, then there exists a comprehensive $\mathcal{L}^2$-structure which is not a model of H.

Proof. Suppose that $\not\vdash^{cf}_{\mathbf{HIF}^2} H$. Then, by Theorem 9.5.6, there exists a legal and comprehensive quasi-$\mathcal{L}^2$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$, which is not a model of H. This implies that there exists an $\langle \mathcal{L}^2, \mathcal{D} \rangle$-assignment σ, such that $\mathcal{Q}, \sigma \not\models s$ for every $s \in H$. Let $\mathcal{W} = \langle \mathcal{V}, \mathcal{D}', P' \rangle$ be a comprehensive $\mathcal{L}^2$-structure and δ a function, satisfying the requirement in Theorem 9.6.2. Let σ' be an $\langle \mathcal{L}^2, \mathcal{D}' \rangle$-assignment such that $\langle \sigma, \sigma' \rangle$ is a δ-pair (there exists such an assignment since the range of δ does not include the empty set). We show that $\mathcal{W}, \sigma' \not\models H$. Let $s \in H$. Since $\mathcal{Q}, \sigma \not\models s$, we have $\mathcal{Q}^{\mathtt{f}}[s, \sigma] > \mathcal{Q}^{\mathtt{t}}[s, \sigma]$. The fact that $\mathcal{W}[\varphi, \sigma'] \in v[\varphi, \sigma]$ for every φ entails that $\mathcal{W}^{\mathtt{f}}[s, \sigma'] > \mathcal{W}^{\mathtt{t}}[s, \sigma']$, and so $\mathcal{W}, \sigma' \not\models s$. □

Just like in the first-order case, we immediately obtain the following:

Corollary 9.6.4. For every $\mathcal{L}^2$-formula φ, if $\Vdash^{\mathbf{G\ddot{o}}_{\mathcal{L}^2}} \varphi$ then $\vdash_{\mathbf{HIF}^2} \{\{\mathtt{t}{:}\varphi\}\}$.

Corollary 9.6.5. If an $\mathcal{L}^2$-hypersequent is provable in $\mathbf{HIF}^2$, then it is provable in $\mathbf{HIF}^2$ without applying (*cut*).

Remark 9.6.6. Note that cut-admissibility for **HIF** (the original system for first-order Gödel logic) is obtained as a corollary, for the reason that second-order quantifiers cannot be involved in a cut-free proof of a first-order formula.

Chapter 10

Summary and Further Work

In this thesis, we studied Gentzen-type calculi from a semantic point of view. This study encompassed several general abstract families of sequent calculi: pure calculi, canonical calculi, quasi-canonical calculi, and basic calculi, as well as the family of canonical Gödel hypersequent calculi. For each of these families, a corresponding denotational semantic framework was identified, based on certain (possibly non-deterministic) semantic structures: many-valued valuations for pure calculi (including canonical and quasi-canonical ones); Kripke valuations for basic calculi; and Gödel valuations for canonical Gödel calculi. It was shown that each calculus in these families induces a set of semantic structures for which it is sound and complete. Moreover, we provided general, modular and uniform methods to obtain such a set for a given calculus, of which many important soundness and completeness theorems for known calculi are particular instances. In the case of canonical calculi (both canonical sequent calculi, and canonical Gödel hypersequent calculi), the resulting semantics was proven to be effective, leading to a semantic decision procedure for each such calculus. In addition, for each of the families of proof systems mentioned above, we derived semantic characterizations of analyticity, cut-admissibility and axiom-expansion. This provides a "semantic toolbox", intended to complement the usual proof-theoretic methods. Indeed, as was demonstrated in many examples, the proofs (or refutations) of these properties based on the semantic characterizations is straightforward in many cases, and less tedious and error-prone than the usual inductive syntactic arguments.

Finally, we demonstrated these methods on **HIF** and $\mathbf{HIF}^2$, two hypersequent calculi for first-order and second-order languages. Based on the ideas from the semantic analysis of canonical Gödel propositional calculi in Chapter 7, we showed that these calculi are sound and (cut-free) complete for standard first-order Gödel logic, and Henkin-style second-order Gödel logic (respectively). To the best of our knowledge, $\mathbf{HIF}^2$ is the first (cut-free) proof system introduced for Henkin-style second-order Gödel logic.

We believe that the semantic approach in the analysis of proof systems has demonstrated its potential in this thesis, as a useful methodology, that may be applied whenever a new Gentzen-type system (of one of the families mentioned above) is encountered. Evidently, this methodology is still in its early stages of its development, and there are various open questions, and possible promising extensions. The main directions for further research include the following:

Substructural Calculi All sequent and hypersequent calculi studied in this thesis are *fully-structural*, as they include (both internal and external, in the case of hypersequent calculi) the exchange rules, weakening rules, contraction rules, and expansion rules (recall that we defined sequents and hypersequents as *sets*). However, many important logics have only substructural sequent systems (in particular, contraction-free or weakening-free calculi), that cannot be treated in our framework. Extending some of the results above for families of substructural systems is an important future goal. This will require the development of new frameworks of non-deterministic semantics. In particular, an interesting question arises as to whether the semantic proof technique of cut-admissibility is applicable for substructural calculi. Two particularly important cases are the (sub-structural) hypersequent systems for the fundamental propositional fuzzy logics – Łukasiewicz's logic, and product logic (see [76]). In addition, the current work deals only with multiple-conclusion systems, while it can be useful to derive similar results for single-conclusion ones. For two-sided *canonical* single-conclusion systems, this was done in [14].

First-Order and Higher-Order Calculi All general families of sequent and hypersequent calculi studied above handle only *propositional* logics. Extending the methods and results of this thesis to languages that include quantifiers is evidently an important future goal. We believe that such an extension should be possible. Indeed, the original three-valued non-deterministic semantics for the (*cut*)-free fragment of **LK** applies also to the first-order quantifiers [58], as well as the three-valued non-deterministic semantics for its (*id*)-fragment [64]. Our results for $\mathbf{HIF}^2$ in Chapter 9 demonstrate that similar ideas can be applied in different higher-order calculi. Note that families of canonical sequent calculi with first-order quantifiers were studied in [18, 98].

Extending the Realm of PNmatrices In Chapters 4 and 5 the semantic framework of finite PNmatrices has shown to be adequate for canonical sequent calculi and quasi-canonical ones. Besides the fact that this framework provides intuitive semantics, its main attractive property is its *effectiveness*. Thus every calculus that has a sound and complete PNmatrix is decidable. A natural question is whether this semantic framework suffices for more than canonical and quasi-canonical calculi. In particular, consider the following schemes:

$$\frac{\Gamma \Rightarrow \neg^i\varphi, \Delta}{\Gamma, \neg^i\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \neg(\varphi \wedge \neg\varphi) \Rightarrow \Delta}$$

The first one is an example of a family of rules used in proof systems that combine classical and paraconsistent negations [66], and the second is taken from [12] for handling certain paraconsistent logics. Note that both schemes (for $i > 1$ in the first one) cannot be presented as canonical or quasi-canonical rules, and thus our semantic methods do not apply to them. Nevertheless, we believe that the ideas in Chapter 5 can be extended to handle also rules as the first one above, in which more than one unary connective may precede the formulas in the premises and the conclusion. For the second rule above, [12] suggests an infinite-valued non-deterministic matrix, and it remains open whether other systems with rules of this kind can be characterized by non-deterministic matrices, and in what cases this infinite-valued semantics is effective.

General Canonical Hypersequent Calculi. In Chapter 7, we studied a family of canonical hypersequent calculi that are based on the communication rule. It is interesting to study canonical hypersequent calculi that employ other structural hypersequential rules. For example, we believe that a similar methodology, using Kripke semantics of directed frames, should work for canonical calculi based on the hypersequent rule (lq). This rule is used in systems for logics of weak excluded middle [42]. Developing a general theory of structural hypersequential rules in canonical hypersequent calculi is another interesting direction for a future work.

Different Proof-Theoretic Formalisms In this thesis we investigated sequent and hypersequent systems. It is interesting to pursue similar investigations in other proof-theoretic frameworks, that handle different families of non-classical logics. This includes: semantic and prefixed tableaux systems [52], display calculi [33], nested sequents calculi [53, 60, 36], labelled calculi as in [54, 77], and sequents of relations [23].

Second-Order Gödel Logic In [72], we proved that **HIF** (a calculus for *first-order* Gödel logic) enjoys cut-admissibility. In fact, for **HIF** we proved slightly stronger properties than those shown in this paper for $\mathbf{HIF^2}$. Obtaining these stronger results for $\mathbf{HIF^2}$ seems to be straightforward. This includes:

- In [72] we considered also derivations from non-empty sets of hypersequents, and proved *strong cut-admissibility*.

- For applications, it is sometimes useful to enrich Gödel logic with a globalization connective (also known as Baaz Delta connective, see [22]). [72] studies the extension of **HIF** with rules for this connective, and the same can be done for $\mathbf{HIF}^2$.

In addition, the following extensions of the current result are left for a future work:

- It is interesting to consider equality, both between first-order terms and second-order ones. In this case, rules for extensionality should be added.
- Extending the calculus for richer second order signatures and also for full type theory seem to be possible. In the case of classical logic, cut-free completeness for the extended system was proved shortly after Tait's proof for the second-order one by Takahashi and Prawitz, [88, 82]. This extension is necessary in order to obtain a proof system for (the Gödel fragment) of fuzzy set theory (see [37]).

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