From Frame Properties to Hypersequent Rules in Modal Logics

Ori Lahav
School of Computer Science
Tel Aviv University
Tel Aviv, Israel
Email: orilahav@post.tau.ac.il

***Abstract*—We provide a general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics. The method applies to all modal logics characterized by Kripke frames, transitive Kripke frames, or symmetric Kripke frames satisfying some properties, given by first-order formulas of a certain simple form. This includes the logics KT, KD, S4, S5, K4D, K4.2, K4.3, KBD, KBT, and other modal logics, for some of which no Gentzen calculi was presented before. Cut-admissibility (or analyticity in the case of symmetric Kripke frames) is proved semantically in a uniform way for all constructed calculi. The decidability of each modal logic in this class immediately follows.**

***Keywords*-modal logic; frame properties; proof theory; hypersequent calculi; cut-admissibility;**

I. Introduction

Modal logics have important applications in many areas of computer science. Ever since Kripke and others developed relational semantics, the assembling of a new modal logic for particular applications often begins by locating relevant frame properties. The class of frames satisfying these properties can be then used to characterize a new logic. An accompanying proof-theoretic characterization of a given modal logic is usually required. First, it is essential for the development of automated reasoning methods. Second, it provides a complementary view on the logic, that may be useful in the theoretic investigation of the new logic in hand. Developing general methods for generating proof systems for modal logics, specified by frame properties, is the main aim of this paper.

Since their introduction by Gentzen, sequent calculi have been a preferred framework to define efficient well-behaved proof-theoretic characterizations for various logics. However, ordinary sequent systems do not suffice, as they are many useful non-classical logics (sometimes with simple relational semantics) for which no well-behaved useful sequent calculus seems to exist. This led to the development of frameworks employing higher-level objects, generalizing in different ways the standard sequent calculi. A particularly simple and straightforward generalization is the framework of *hypersequents*. In fact, hypersequents, which are nothing more than disjunctions of sequents, turned out to be applicable for a large variety of non-classical logics (see, e.g., [1], [2]). In particular, the modal logic $\mathbf{S5}$, that lacks a well-behaved simple cut-free sequent calculus, is naturally handled using hypersequents [3]. However, the full power of the hypersequent framework in the context of modal logics has not yet been exploited.

In this paper we show that the hypersequent framework can be easily used for many more normal modal logics in addition to $\mathbf{S5}$. Indeed, we identify an (infinite) set of so-called "simple" frame properties, given by first-order formulas of a certain general form, and show how to construct a corresponding hypersequent derivation rule for each simple property. The constructed rules are proved to be "well-behaved", in the sense that augmenting the (cut-free) basic hypersequent calculus for the fundamental modal logic $\mathbf{K}$ with any (finite) number of these rules do not harm cut-admissibility. In addition, we show how to generate rules for simple properties, to be added to the calculi for the modal logics $\mathbf{K4}$ (the logic of transitive frames) or $\mathbf{KB}$ (the logic of symmetric frames). In the case of $\mathbf{K4}$, cut-admissibility is again preserved. In the case of $\mathbf{KB}$, we only have analyticity (the subformula property), rather than full cut-admissibility.[1] This provides a systematic method to construct hypersequent systems for a variety of modal logics, including among others $\mathbf{KT}$, $\mathbf{KD}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{K4D}$, $\mathbf{K4.2}$, $\mathbf{K4.3}$, $\mathbf{KBD}$, $\mathbf{KBT}$, as well as logics characterized by frames of bounded cardinality, bounded width, and bounded top width. Soundness and completeness for the intended semantics, as well as cut-admissibility or analyticity are proved uniformly for all obtained systems.

Finally, the existence of such a well-behaved hypersequent calculus immediately implies the decidability of the corresponding logic. Hence the decidability of all modal logics characterized by simple properties is a corollary of our results.

Related works

A lot of effort has been invested to study and develop proof-theoretical frameworks in which modal logics are uniformly handled. Among others, this includes semantic tableaus, display calculi, nested sequents calculi, labelled calculi, and tree-hypersequent calculi (see, e.g., [4], [5], [6],

[1] Note that transitivity and symmetricity are not "simple" frame properties themselves, and we have to change the basic calculus in order to handle them.

[7], [8], [9]). It is beyond our scope to provide a general comparison between the various different formalisms. We can say that in general, there is a delicate balance between several aspects: the expressibility of a certain formalism (how many modal logics are captured?); its modularity (ideally, one-to-one correspondence between the ingredients of the logic specification and those of the proof system); its applicability (its usefulness in decision procedures and in the investigation of the logics); and its simplicity (how simple are the data-structures it employs?). We believe that the framework of hypersequents, studied in this paper, is still very close to Gentzen's approach, and yet it is modular, applicable, and it suffices for various important modal logics.

The formulas that specify our, so-called, simple frame properties, are a small portion of the family of *geometric formulas* (see [7]). Thus, the methods of [7] can be used to obtain a labelled sequent calculus for each of the logics that we handle in this paper. The main difference here is that these labelled sequent calculi internalize the Kripke semantics in the sequents themselves, while our hypersequent calculi are purely syntactic, and the semantics is not explicitly involved in the formal derivations. In addition, the calculi constructed by the method of this paper all have the subformula property, and therefore, we easily obtain a uniform decidability result for the corresponding logics (see Corollary 2). In contrast, the methods of [7] lead to calculi enjoying only weak notions of the subformula property, and consequently, decidability requires a separate proof of the termination of the proof-search algorithm for each calculus.

Several other works provide hypersequent calculi for different kinds of logics. Among others, this includes [2] where fuzzy logics are studied, [10] that considers intermediate logics, and [1] that studies various propositional substructural logics. In fact, [1] provides a general method to construct a hypersequent rule out of an (Hilbert) axiom of a certain sort. However, this method does not cover axioms that include modal operators. Concerning modal logics, we are aware (except for [3] mentioned above) of [11], that provides several hypersequent systems for extensions of $\mathbf{MTL}$ with modal operators and proves (syntactic) cut-elimination for them, and [12], that provides a cut-free hypersequent calculus for $\mathbf{S4.3}$. While $\mathbf{S4.3}$ is a particular instance of a logic that can be handled using our method, the logics studied in [11] are beyond our scope, as they require *substructural* hypersequent calculi.

Finally, we note that the large topic of "correspondence theory" in modal logic aims to study the connection between formulas in modal logics (modal axioms) and frame properties formulated in (higher-order) classical logic. Our work translates frame properties (given in first-order classical logic) *directly* to hypersequent rules. This provides a different sort of (one-directional) correspondence, that does not involve modal axioms per se. Investigating the correspondence between modal axioms and simple frame properties is left for further research.

II. Simple Frame Properties

Let $\mathcal{L}$ be a propositional language consisting of the propositional constant $\bot$, the binary connective $\supset$, and the modal operator $\Box$. For all modal logics considered below, $\wedge, \vee, \neg$ and $\diamond$ can be easily expressed in $\mathcal{L}$. As usual, given a set Γ of formulas, $\Box\Gamma$ denotes the set $\{\Box\varphi : \varphi \in \Gamma\}$. Similarly, $\Box^{-1}\Gamma$ is standing for $\{\varphi : \Box\varphi \in \Gamma\}$. We denote by $At_{\mathcal{L}}$ the set of atomic formula of $\mathcal{L}$, and by $Frm_{\mathcal{L}}$ the set of all $\mathcal{L}$-formulas. $\vdash^{l}_{\mathbf{K}}$ and $\vdash^{g}_{\mathbf{K}}$ denote (respectively) the local and the global consequence relations of the modal logic $\mathbf{K}$ for the language $\mathcal{L}$. Semantically, these relations can be characterized as follows:

Definition 1: A *Kripke frame* is a pair $\langle W, R\rangle$, where W is a non-empty set of elements (called *worlds*), and R is a binary relation on W (called *accessibility relation*). A *Kripke model* is a triple $\langle W, R, [\,]\rangle$ obtained by augmenting a Kripke frame $\langle W, R\rangle$ with a function $[\,]$ (called *valuation*), that assigns to every atomic formula $p \in At_{\mathcal{L}}$ a subset $[p] \subseteq W$ (the worlds in which p is *true*). $[\,]$ is recursively extended to $Frm_{\mathcal{L}}$ as follows:

1) $[\bot] = \emptyset$.
2) $[\varphi_1 \supset \varphi_2] = [\varphi_2] \cup (W \setminus [\varphi_1])$.
3) $[\Box\varphi] = \{w \in W : \forall u \in W.\ \textit{if}\ \ wRu\ \textit{then}\ \ u \in [\varphi]\}$.

Fact 1: Let $\mathcal{T}$ be a (possibly infinite) set of formulas, and φ be a formula.

1) $\mathcal{T} \vdash^{l}_{\mathbf{K}} \varphi$ iff for every Kripke model $\langle W, R, [\,]\rangle$, and every $w \in W$: either $w \notin [\psi]$ for some $\psi \in \mathcal{T}$, or $w \in [\varphi]$.
2) $\mathcal{T} \vdash^{g}_{\mathbf{K}} \varphi$ iff for every Kripke model $\langle W, R, [\,]\rangle$: if $[\psi] = W$ for every $\psi \in \mathcal{T}$, then $[\varphi] = W$.

To formulate frame properties, we use a first-order language, denoted by $\mathcal{L}^1$. Its variables are $u, w_1, w_2, \ldots$, and the binary predicate symbols R and $=$ are its only predicate symbols. Next, we identify a set of $\mathcal{L}^1$-formulas, for which we will be able to construct hypersequent calculi.

Definition 2: An $\mathcal{L}^1$-formula is called n*-simple* if it has the form $\forall w_1, \ldots, w_n \exists u \theta$, where θ consists of $\wedge$, $\vee$, and atomic $\mathcal{L}^1$-formulas of the form $w_i R u$ or $w_i = u$ where $1 \le i \le n$.

It turns out that various important and well-studied frame properties can be formulated by n-simple $\mathcal{L}^1$-formulas. These are called *simple* frame properties. Examples are given in Table I.

Given a set Θ of $\mathcal{L}^1$-formulas, a Kripke frame $\langle W, R\rangle$ is called a Θ*-frame* if the first-order $\mathcal{L}^1$-structure naturally induced by $\langle W, R\rangle$ is a model of every formula in Θ. A Θ-model is any Kripke model based on a Θ-frame. $\vdash^{l}_{\mathbf{K}+\Theta}$ and $\vdash^{g}_{\mathbf{K}+\Theta}$ denote the local and the global consequence relations induced by Θ-models. Formally, these are defined as follows:

Definition 3: Let Θ be a set of $\mathcal{L}^1$-formulas. For every set $\mathcal{T}$ of formulas and formula φ:

Table I
SOME n-SIMPLE $\mathcal{L}^1$-FORMULAS AND THEIR NORMAL DESCRIPTIONS

Seriality	$\forall w_1 \exists u\,(w_1 R u)$	$\{\langle\{1\},\emptyset\rangle\}$
Reflexivity	$\forall w_1 \exists u\,(w_1 R u \wedge w_1 = u)$	$\{\langle\{1\},\{1\}\rangle\}$
Directedness	$\forall w_1, w_2 \exists u (w_1 R u) \wedge (w_2 R u)$	$\{\langle\{1,2\},\emptyset\rangle\}$
Degenerateness	$\forall w_1, w_2 \exists u\,(w_1 = u \wedge w_2 = u)$	$\{\langle\emptyset,\{1,2\}\rangle\}$
Universality	$\forall w_1, w_2 \exists u\,(w_1 R u \wedge w_2 = u)$	$\{\langle\{1\},\{2\}\rangle\}$
Linearity	$\forall w_1, w_2 \exists u (w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)$	$\{\langle\{1\},\{2\}\rangle, \langle\{2\},\{1\}\rangle\}$
Bounded Cardinality	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \le i < j \le n} (w_i = u \wedge w_j = u)$	$\{\langle\emptyset,\{i,j\}\rangle : 1 \le i < j \le n\}$
Bounded Top Width	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \le i < j \le n} (w_i R u \wedge w_j R u)$	$\{\langle\{i,j\},\emptyset\rangle : 1 \le i < j \le n\}$
Bounded Acyclic Subgraph	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \le i < j \le n} (w_i R u \wedge w_j = u)$	$\{\langle\{i\},\{j\}\rangle : 1 \le i < j \le n\}$
Bounded Width	$\forall w_1, \ldots, w_n \exists u \bigvee_{1 \le i,j \le n; i \ne j} (w_i R u \wedge w_j = u)$	$\{\langle\{i\},\{j\}\rangle : 1 \le i,j \le n, i \ne j\}$

1) $\mathcal{T} \vdash^l_{\mathbf{K}+\Theta} \varphi$ iff for every Θ-model $\langle W, R, [\,]\rangle$, and every $w \in W$: either $w \notin [\psi]$ for some $\psi \in \mathcal{T}$, or $w \in [\varphi]$.
2) $\mathcal{T} \vdash^g_{\mathbf{K}+\Theta} \varphi$ iff for every Θ-model $\langle W, R, [\,]\rangle$: if $[\psi] = W$ for every $\psi \in \mathcal{T}$, then $[\varphi] = W$.

Example 1: Several well-known logics are particular instances of $\vdash^l_{\mathbf{K}+\Theta}$ and $\vdash^g_{\mathbf{K}+\Theta}$ where Θ is a set of n-simple $\mathcal{L}^1$-formulas. For example, classical logic (where $\Box$ is an "identity connective") is obtained by taking the formulas for "degenerateness" and reflexivity; $\mathbf{KT}$ is obtained by taking the formula for reflexivity; $\mathbf{KD}$ is obtained by taking the formula for seriality; and $\mathbf{S5}$ by the formula for universality.

Remark 1: To the best of our knowledge, the property called "Bounded Acyclic Subgraph" in Table I was not studied before in the context of modal logics. It is not hard to see that this property holds (for some $n \ge 2$) for $\langle W, R, [\,]\rangle$ iff R is reflexive and all acyclic directed subgraphs (of the naturally induced directed graph, without the self-loops) are of size $< n$.

Obviously, there might be more than one way to express some frame property using an n-simple $\mathcal{L}^1$-formula. To produce hypersequent rules out of simple frame properties, we use the *normal descriptions*:

Definition 4: A *normal description* of an n-simple $\mathcal{L}^1$-formula $\forall w_1, \ldots, w_n \exists u \theta$ is a non-empty finite set S of pairs of the form $\langle S_R, S_=\rangle$, where S_R and $S_=$ are subsets of $\{1, \ldots, n\}$, such that $S_R \cup \mathcal{S}_=$ is non-empty, and θ is equivalent to $\bigvee_{\langle S_R, S_=\rangle \in S} (\bigwedge_{i \in S_R} w_i R u \;\wedge\; \bigwedge_{i \in S_=} w_i = u)$.

Proposition 1: Every simple $\mathcal{L}^1$-formula has a normal description.

Examples of normal descriptions of n-simple $\mathcal{L}^1$-formulas are given in the right column of Table I. Henceforth, for every n-simple $\mathcal{L}^1$-formula θ, we denote by $S(\theta)$ some normal description of θ.

III. From Simple Frame Properties to Hypersequent Rules

For our purposes it is most convenient to define a *sequent* as an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas. This means in particular that the usual structural rules of exchange, contraction and expansion (the converse of contraction) are built-in in our calculi. A *hypersequent*, in turn, is a finite set of sequents (so, the external versions of the aforementioned rules are also implicit). We shall use the usual hypersequent notation $s_1 \mid \ldots \mid s_n$ (for $\{s_1, \ldots, s_n\}$). We also employ the standard abbreviations, e.g. $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $H \mid s$ instead of $H \cup \{s\}$.[2]

We use the hypersequent calculus $\mathbf{HK}$, given in Figure 1, as the basic calculus. As every hypersequent system, $\mathbf{HK}$

[2]Note that expressions like $\Gamma \Rightarrow \Delta$ are actually ambiguous between sequents and singleton hypersequents.

$$(IW \Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, \varphi \Rightarrow \Delta} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, \varphi} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow \Delta}$$

$$(id)\ \frac{}{\varphi \Rightarrow \varphi} \qquad (cut)\ \frac{H \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad H \mid \Gamma_2, \varphi \Rightarrow \Delta_2}{H \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$(\supset\Rightarrow)\ \frac{H \mid \Gamma_1 \Rightarrow \Delta_1, \varphi_1 \quad H \mid \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{H \mid \Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2, \Delta}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2, \Delta}$$

$$(\bot \Rightarrow)\ \frac{}{\bot \Rightarrow} \qquad (\Rightarrow \Box)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Box\Gamma \Rightarrow \Box\varphi}$$

Figure 1. The basic calculus **HK**

naturally induces two consequence relations between sets of formulas and formulas:

1) A "local" relation, denoted by $\vdash^{l}_{\mathbf{HK}}$, defined by: $\mathcal{T} \vdash^{l}_{\mathbf{HK}} \varphi$ if there exists a derivation of $\Gamma \Rightarrow \varphi$ in $\mathbf{HK}$ for some finite set $\Gamma \subseteq \mathcal{T}$.
2) A "global" relation, denoted by $\vdash^{g}_{\mathbf{HK}}$, defined by: $\mathcal{T} \vdash^{g}_{\mathbf{HK}} \varphi$ if there exists a derivation of $\Rightarrow \varphi$ from the assumptions $\{\Rightarrow \psi : \psi \in \mathcal{T}\}$ in $\mathbf{HK}$.

The following facts can be easily proved (in fact, they are particular instances of the results below, obtained by taking empty set of simple $\mathcal{L}^1$-formulas).

Fact 2: $\mathbf{HK}$ is sound and complete for the modal logic $\mathbf{K}$, i.e. $\vdash^{l}_{\mathbf{HK}} = \vdash^{l}_{\mathbf{K}}$ and $\vdash^{g}_{\mathbf{HK}} = \vdash^{g}_{\mathbf{K}}$.

Fact 3: $\mathbf{HK}$ enjoys *strong cut-admissibility*, i.e. if there exists a derivation of a hypersequent H in $\mathbf{HK}$ from a set $\mathcal{H}$ of hypersequents, then there also exists a derivation of H from $\mathcal{H}$ in $\mathbf{HK}$, in which only formulas from $\mathcal{H}$ serve as cut-formulas. In particular, if there exists a derivation of H in $\mathbf{HK}$ with no assumptions (namely, from $\emptyset$), then there exists a cut-free derivation of H in $\mathbf{HK}$.

Now, given a normal description S of an n-simple $\mathcal{L}^1$-formula θ, the *hypersequent rule induced by S for* $\mathbf{HK}$, denoted by $r^{\mathbf{HK}}_{S}$, is given by the following scheme:

$$\frac{\{H \mid \bigcup_{i \in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \Rightarrow \bigcup_{i \in S_=} \Delta_i : \langle S_R, S_= \rangle \in S\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

This is a hypersequent rule that corresponds the the frame property formulated by θ. Examples of the rules obtained for the normal descriptions from Table I are given in Figure 2.

Remark 2: In all rule schemes, H is a metavariable for hypersequents, and Γ, Δ (with or without primes or subscripts) are metavariables for finite sets of formulas.

Given a set $\mathbf{R}$ of hypersequent rules, $\mathbf{HK}+\mathbf{R}$ denotes the system obtained by augmenting $\mathbf{HK}$ with the rules of $\mathbf{R}$. The relations $\vdash^{l}_{\mathbf{HK+R}}$ and $\vdash^{g}_{\mathbf{HK+R}}$ are defined as expected. We can now formulate our first main result:

Theorem 1: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HK}}_{S(\theta)} : \theta \in \Theta\}$. Then, (i) $\vdash^{l}_{\mathbf{HK+R}} = \vdash^{l}_{\mathbf{K}+\Theta}$; (ii) $\vdash^{g}_{\mathbf{HK+R}} = \vdash^{g}_{\mathbf{K}+\Theta}$; and (iii) $\mathbf{HK}+\mathbf{R}$ enjoys strong cut-admissibility.

Remark 3: The hypersequent calculus $\mathbf{GS5}$ for $\mathbf{S5}$ from [3] is different from the one constructed for $\mathbf{S5}$ by this general method (by taking the formula for universality of the accessibility relation). It is easy to see that the rules of our calculus are cut-free derivable in $\mathbf{GS5}$. On the other hand, cut is needed for deriving the rules of $\mathbf{GS5}$ in our calculus. Thus, our result is stronger, in some sense, as it implies that $\mathbf{GS5}$ enjoys cut-admissibility.

Theorem 1 is obtained as a corollary of Theorem 2 and Theorem 3 below. The first establishes the soundness of $\mathbf{HK}+\mathbf{R}$ for the logic of $\mathbf{K}+\Theta$, and the second simultaneously provide completeness and cut-admissibility. We shall use the following additional notion:

Definition 5: A Kripke model $\langle W, R, [\,] \rangle$ *validates*:

- a *sequent* $\Gamma \Rightarrow \Delta$ if for every $w \in W$, either $w \notin [\varphi]$ for some $\varphi \in \Gamma$, or $w \in [\varphi]$ for some $\varphi \in \Delta$.
- a *hypersequent* H if it validates some $\Gamma \Rightarrow \Delta \in H$.
- a *set* $\mathcal{H}$ of hypersequents if it validates every $H \in \mathcal{H}$.

Theorem 2: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HK}}_{S(\theta)} : \theta \in \Theta\}$. If a hypersequent H_0 is derivable in $\mathbf{HK}+\mathbf{R}$ from a set $\mathcal{H}_0$ of hypersequents, then every Θ-model which validates $\mathcal{H}_0$, also validates H_0.

$$\frac{H \mid \Gamma'_1 \Rightarrow}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1}\ \text{Seriality} \qquad \frac{H \mid \Gamma_1, \Gamma'_1 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1}\ \text{Reflexivity} \qquad \frac{H \mid \Gamma'_1, \Gamma'_2 \Rightarrow}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ \text{Directedness}$$

$$\frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ \text{Degenerateness} \qquad \frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ \text{Universality}$$

$$\frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2 \qquad H \mid \Gamma_1, \Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}\ \text{Linearity} \qquad \frac{\{H \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j : 1 \leq i < j \leq n\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}\ \text{Bounded Cardinality}$$

$$\frac{\{H \mid \Gamma'_i, \Gamma'_j \Rightarrow\ : 1 \leq i < j \leq n\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}\ \text{Bounded Top Width} \qquad \frac{\{H \mid \Gamma_j, \Gamma'_i \Rightarrow \Delta_j : 1 \leq i < j \leq n\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}\ \text{Bounded Acyclic Subgraph}$$

$$\frac{\{H \mid \Gamma_j, \Gamma'_i \Rightarrow \Delta_j : 1 \leq i, j \leq n, i \neq j\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}\ \text{Bounded Width}$$

Figure 2. Hypersequent rules induced by normal descriptions for **HK**

Proof: We show that the property of being validated by a Θ-model is preserved by applications of the rules of $\mathbf{HK}+\mathbf{R}$. For the rules of $\mathbf{HK}$, this is a routine matter. Let $r \in \mathbf{R}$, and let $\theta \in \Theta$ be an n-simple $\mathcal{L}^1$-formula, for which $r = r^{\mathbf{HK}}_{S(\theta)}$. Consider an application of r. Such an application derives a hypersequent H' of the form $H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n$ from the set $\{H \mid \bigcup_{i\in S_=} \Gamma_i, \bigcup_{i\in S_R} \Gamma'_i \Rightarrow \bigcup_{i\in S_=} \Delta_i : \langle S_R, S_= \rangle \in S(\theta)\}$ of hypersequents. Assume that a Θ-model $\mathcal{W} = \langle W, R, [\,]\rangle$ does not validate H'. We show that $\mathcal{W}$ does not validate at least one premise of this application. Our assumption entails that for every $1 \leq i \leq n$, there exists $w_i \in W$ such that $w_i \in [\varphi]$ for every $\varphi \in \Gamma_i \cup \Box\Gamma'_i$, and $w_i \notin [\varphi]$ for every $\varphi \in \Delta_i$. Since $\mathcal{W}$ is a Θ-model, there exist some $u \in W$ and $\langle S_R, S_= \rangle \in S(\theta)$ such that: $w_i R u$ for every $i \in S_R$, and $w_i = u$ whenever $i \in S_=$. Let $s_0 = \bigcup_{i\in S_=} \Gamma_i, \bigcup_{i\in S_R} \Gamma'_i \Rightarrow \bigcup_{i\in S_=} \Delta_i$. We show that $\mathcal{W}$ does not validate $H \mid s_0$ (which is a premise of this application). First, since $H \subseteq H'$, $\mathcal{W}$ does not validate any $s \in H$. We prove that $u \in [\varphi]$ for every φ on the left side of s_0, and $u \notin [\varphi]$ for every φ on the right side of s_0:

1) Let $\varphi \in \Gamma_i$ for some $i \in S_=$. Thus $w_i \in [\varphi]$. Since $u = w_i$, we have $u \in [\varphi]$.
2) Let $\varphi \in \Gamma'_i$ for some $i \in S_R$. Thus $w_i \in [\Box\varphi]$. Since $w_i R u$, we have $u \in [\varphi]$.
3) Let $\varphi \in \Delta_i$ for some $i \in S_=$. Thus $w_i \notin [\varphi]$. Since $u = w_i$, we have $u \notin [\varphi]$.

Consequently, $\mathcal{W}$ does not validate s_0. ∎

For the completeness and cut-admissibility proof, we use the following additional definitions and notations:

Definition 6: An *extended sequent* is an ordered pair of (possibly infinite) sets of formulas. Given two extended sequents $\mu_1 = \langle \mathcal{T}_1, \mathcal{U}_1 \rangle$ and $\mu_2 = \langle \mathcal{T}_2, \mathcal{U}_2 \rangle$, we write $\mu_1 \sqsubseteq \mu_2$ (and say that μ_2 *extends* μ_1) if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$. An *extended hypersequent* is a (possibly infinite) set of extended sequents. Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every $\mu_1 \in \Omega_1$, there is some $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$.

We shall use the same notations as above for extended sequents and extended hypersequents. For example, we write $\mathcal{T} \Rightarrow \varphi, \mathcal{U}$ instead of $\langle \mathcal{T}, \mathcal{U} \cup \{\varphi\}\rangle$, and $\Omega \mid \mu$ instead of $\Omega \cup \{\mu\}$. In addition, the notion of validating an extended sequent or an extended hypersequent is defined exactly as for ordinary sequents and hypersequents (see Definition 5).

Notation 1: Given a set $\mathbf{R}$ of hypersequent rules, we write $\mathcal{H} \vdash^{cf}_{\mathbf{HK}+\mathbf{R}} H$ if there exists a derivation of the hypersequent H from the set $\mathcal{H}$ of hypersequents in $\mathbf{HK}+\mathbf{R}$, in which only formulas from $\mathcal{H}$ serve as cut-formulas.

Definition 7: Let Ω be an extended hypersequent, $\mathbf{R}$ be a set of hypersequent rules, and $\mathcal{H}$ be a set of (ordinary) hypersequents. Ω is called:

1) $\langle \mathbf{R}, \mathcal{H}\rangle$*-provable* if $\mathcal{H} \vdash^{cf}_{\mathbf{HK}+\mathbf{R}} H$ for some (ordinary) hypersequent $H \sqsubseteq \Omega$. Otherwise, it is called $\langle \mathbf{R}, \mathcal{H}\rangle$-*unprovable*.
2) $\langle \mathbf{R}, \mathcal{H}\rangle$*-maximal w.r.t. a formula* φ if the following hold for every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:
   - If $\varphi \notin \mathcal{T}$ then $\Omega \mid \mathcal{T}, \varphi \Rightarrow \mathcal{U}$ is $\langle \mathbf{R}, \mathcal{H}\rangle$-provable.
   - If $\varphi \notin \mathcal{U}$ then $\Omega \mid \mathcal{T} \Rightarrow \varphi, \mathcal{U}$ is $\langle \mathbf{R}, \mathcal{H}\rangle$-provable.
3) $\langle \mathbf{R}, \mathcal{H}\rangle$*-maximal w.r.t. an extended sequent* μ if either $\{\mu\} \sqsubseteq \Omega$, or $\Omega \mid \mu$ is $\langle \mathbf{R}, \mathcal{H}\rangle$-provable.
4) $\langle \mathbf{R}, \mathcal{H}\rangle$*-maximal* if it is $\langle \mathbf{R}, \mathcal{H}\rangle$-unprovable, and $\langle \mathbf{R}, \mathcal{H}\rangle$-maximal w.r.t. any formula and any extended sequent.

Proposition 2: Let Ω be an $\langle \mathbf{R}, \mathcal{H}\rangle$-maximal extended hypersequent for some set $\mathbf{R}$ of hypersequent rules and set $\mathcal{H}$ of hypersequents.

1) For every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ and formula φ: if $\varphi \notin \mathcal{T}$ (respectively, $\varphi \notin \mathcal{U}$) then $\mathcal{H} \vdash^{cf}_{\mathbf{HK}+\mathbf{R}} H \mid \Gamma, \varphi \Rightarrow \Delta$ ($\mathcal{H} \vdash^{cf}_{\mathbf{HK}+\mathbf{R}} H \mid \Gamma \Rightarrow \varphi, \Delta$) for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.
2) For every $\mu_1, \mu_2 \in \Omega$: if $\mu_1 \sqsubseteq \mu_2$, then $\mu_1 = \mu_2$.
3) For every extended sequent μ: if $\{\mu\} \not\sqsubseteq \Omega$, then $\mathcal{H} \vdash^{cf}_{\mathbf{HK}+\mathbf{R}} H \mid s$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $s \sqsubseteq \mu$.

Proof: Immediately follows from the definitions, using the availability of external and internal weakenings. ∎

The following lemma plays a central role in the completeness proof. Due to space limitations, the proof is omitted.

Lemma 1: Let $\mathbf{R}$ be a set of hypersequent rules, and $\mathcal{H}$ be a set of hypersequents. Every $\langle \mathbf{R}, \mathcal{H}\rangle$-unprovable extended hypersequent Ω can be extended to an $\langle \mathbf{R}, \mathcal{H}\rangle$-maximal extended hypersequent.

Theorem 3: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HK}}_{S(\theta)} : \theta \in \Theta\}$. Let $\mathcal{H}_0$ be a set of hypersequents, and Ω_0 be an $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable extended hypersequent. Then, there exists a Θ-model $\mathcal{W}$ which validates $\mathcal{H}_0$, but not Ω_0.

Proof: By Lemma 1, there exists an $\langle \mathbf{R}, \mathcal{H}_0\rangle$-maximal hypersequent Ω that extends Ω_0. Define $\mathcal{W} = \langle W, R, [\,]\rangle$, as follows:

- $W = \{\mathcal{T} \subseteq Frm_{\mathcal{L}} : \exists \mathcal{U} \subseteq Frm_{\mathcal{L}}.\ \mathcal{T} \Rightarrow \mathcal{U} \in \Omega\}$.
- $\mathcal{T}_1 R \mathcal{T}_2$ iff $\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- For every $p \in At_{\mathcal{L}}$: $[p] = \{\mathcal{T} \in W : p \in \mathcal{T}\}$.

We show that $\mathcal{W}$ is a Θ-model. Let θ be an n-simple $\mathcal{L}^1$-formula in Θ. We show that for every $\mathcal{T}_1, \ldots, \mathcal{T}_n \in W$, there exists some $\mathcal{T} \in W$, and $\langle S_R, S_= \rangle \in S(\theta)$, such that $\mathcal{T}_i R \mathcal{T}$ for every $i \in S_R$ and $\mathcal{T}_i = \mathcal{T}$ for every $i \in S_=$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be elements of W. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be sets of formulas such that $\mathcal{T}_i \Rightarrow \mathcal{U}_i \in \Omega$ for every $1 \leq i \leq n$. We claim that there exists some $\langle S^0_R, S^0_= \rangle \in S(\theta)$ such that $\Omega \mid \bigcup_{i\in S^0_=} \mathcal{T}_i, \bigcup_{i\in S^0_R} \Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i\in S^0_=} \mathcal{U}_i$ is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable. To see this, assume for a contradiction that $\Omega \mid \bigcup_{i\in S_=} \mathcal{T}_i, \bigcup_{i\in S_R} \Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i\in S_=} \mathcal{U}_i$ is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-provable for every $\langle S_R, S_= \rangle \in S(\theta)$. By Proposition 2 (and using weakenings), there exist a hypersequent $H \sqsubseteq \Omega$, and finite sets

$\Gamma_1 \subseteq \mathcal{T}_1,\dots,\Gamma_n \subseteq \mathcal{T}_n$, $\Gamma'_1 \subseteq \Box^{-1}\mathcal{T}_1,\dots,\Gamma'_n \subseteq \Box^{-1}\mathcal{T}_n$, and $\Delta_1 \subseteq \mathcal{U}_1,\dots,\Delta_n \subseteq \mathcal{U}_n$, such that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \bigcup_{i\in S_=}\Gamma_i, \bigcup_{i\in S_R}\Gamma'_i \Rightarrow \bigcup_{i\in S_=}\Delta_i$ for every $\langle S_R, S_=\rangle \in S(\theta)$. By applying the rule $r^{\mathbf{HK}}_{S(\theta)}$, we obtain that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n$. But Ω extends this hypersequent, and this contradicts the fact Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable. Now, let μ_0 be the extended sequent $\bigcup_{i\in S^0_=}\mathcal{T}_i, \bigcup_{i\in S^0_R}\Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i\in S^0_=}\mathcal{U}_i$. Since Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-maximal, $\{\mu_0\} \sqsubseteq \Omega$. Thus there exists some $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\mu_0 \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$. For every $i \in S^0_R$, we have $\Box^{-1}\mathcal{T}_i \subseteq \mathcal{T}$, and so $\mathcal{T}_i R \mathcal{T}$. In addition, for every $i \in S^0_=$, $\mathcal{T}_i \subseteq \mathcal{T}$ and $\mathcal{U}_i \subseteq \mathcal{U}$. By Proposition 2, $\mathcal{T}_i = \mathcal{T}$ in this case.

Next, we show that for every $\mathcal{T} \in W$: **(a)** $\mathcal{T} \in [\varphi]$ for every $\varphi \in \mathcal{T}$; and **(b)** $\mathcal{T} \notin [\varphi]$ for every formula φ for which $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. **(a)** and **(b)** are proved together by a simultaneous induction on the complexity of φ. Let $\mathcal{T} \in W$.

- Suppose that $\varphi = p$ is an atomic formula. Our construction ensures that $\mathcal{T} \in [\varphi]$ iff $\varphi \in \mathcal{T}$. This proves **(a)**. For **(b)**, note that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} \varphi \Rightarrow \varphi$ (for every formula φ), and since Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable, $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$ implies that $\varphi \notin \mathcal{T}$.
- Suppose that $\varphi = \bot$. Then **(b)** is trivially true. On the other hand, **(a)** is vacuously true. To see this, note that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} \bot \Rightarrow$. Hence $\{\bot \Rightarrow\ \} \not\sqsubseteq \Omega$, and so $\bot \notin \mathcal{T}$.
- Suppose that $\varphi = \varphi_1 \supset \varphi_2$. Assume that $\varphi \in \mathcal{T}$. Let $\mathcal{U} \subseteq \mathit{Frm}_{\mathcal{L}}$ such that $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$. Then $\varphi_1 \in \mathcal{U}$ or $\varphi_2 \in \mathcal{T}$. To see this, suppose that $\varphi_1 \notin \mathcal{U}$ and $\varphi_2 \notin \mathcal{T}$. Then by Proposition 2 (and using weakening), $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma \Rightarrow \Delta, \varphi_1$ and $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma, \varphi_2 \Rightarrow \Delta$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$. By applying $(\supset\Rightarrow)$, we obtain $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma, \varphi \Rightarrow \Delta$. But since $\varphi \in \mathcal{T}$, Ω extends $H \mid \Gamma, \varphi \Rightarrow \Delta$, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable. By the induction hypothesis, $\mathcal{T} \notin [\varphi_1]$ or $\mathcal{T} \in [\varphi_2]$. Consequently, $\mathcal{T} \in [\varphi]$.
  Next, assume that $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. Let $\mathcal{U} \subseteq \mathit{Frm}_{\mathcal{L}}$ such that $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ and $\varphi \in \mathcal{U}$. Then $\varphi_1 \in \mathcal{T}$ and $\varphi_2 \in \mathcal{U}$. To see this, suppose that $\varphi_1 \notin \mathcal{T}$ or $\varphi_2 \notin \mathcal{U}$. Then by Proposition 2, either $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma, \varphi_1 \Rightarrow \Delta$ or $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma \Rightarrow \varphi_2, \Delta$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$. By (possibly) applying internal weakening and $(\Rightarrow\supset)$, we obtain $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma \Rightarrow \varphi, \Delta$. But since $\varphi \in \mathcal{U}$, Ω extends $H \mid \Gamma \Rightarrow \varphi, \Delta$, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable. By the induction hypothesis, $\mathcal{T} \in [\varphi_1]$ and $\mathcal{T} \notin [\varphi_2]$. Consequently, $\mathcal{T} \notin [\varphi]$.
- Suppose that $\varphi = \Box\psi$. Assume that $\varphi \in \mathcal{T}$, and let $\mathcal{T}' \in W$ such that $\mathcal{T} R \mathcal{T}'$. Thus $\psi \in \mathcal{T}'$, and by the induction hypothesis, $\mathcal{T}' \in [\psi]$. It follows that $\mathcal{T} \in [\varphi]$. Assume now that $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. Let $\mu = \Box^{-1}\mathcal{T} \Rightarrow \psi$. Then $\{\mu\} \sqsubseteq \Omega$. (Otherwise, by Proposition 2 we have $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma \Rightarrow \psi$ for some hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \Box^{-1}\mathcal{T}$. By applying $(\Rightarrow\Box)$, we obtain that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Box\Gamma \Rightarrow \varphi$. But since $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$, Ω extends $H \mid \Box\Gamma \Rightarrow \varphi$, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable.) Thus there exists $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\mu \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$. By the induction hypothesis, $\mathcal{T}' \notin [\psi]$. In addition, the definition of R ensures that $\mathcal{T} R \mathcal{T}'$. Thus $\mathcal{T} \notin [\varphi]$.

Using **(a)** and **(b)**, we show that $\mathcal{W}$ validates $\mathcal{H}_0$, but not Ω_0. Since $\Omega_0 \sqsubseteq \Omega$, it immediately follows that $\mathcal{W}$ does not validate Ω_0. On the other hand, let $H \in \mathcal{H}_0$. Obviously $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H$, and so $H \not\sqsubseteq \Omega$. Thus, there exists some sequent $\Gamma \Rightarrow \Delta \in H$ such that $\{\Gamma \Rightarrow \Delta\} \not\sqsubseteq \Omega$. We show that $\mathcal{W}$ validates $\Gamma \Rightarrow \Delta$. Let $\mathcal{T} \in W$, and let $\mathcal{U} \subseteq \mathit{Frm}_{\mathcal{L}}$ such that $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$. Since $\{\Gamma \Rightarrow \Delta\} \not\sqsubseteq \Omega$, either $\Gamma \not\subseteq \mathcal{T}$ or $\Delta \not\subseteq \mathcal{U}$. We consider only the first case (the second case is similar). Thus there is some $\varphi \in \Gamma$ such that $\varphi \notin \mathcal{T}$. By Proposition 2, there exist a hypersequent $H' \sqsubseteq \Omega$, and sequent $\Gamma' \Rightarrow \Delta' \subseteq \mathcal{T} \Rightarrow \mathcal{U}$ such that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H' \mid \Gamma', \varphi \Rightarrow \Delta'$. It follows that we must have $\varphi \in \mathcal{U}$. (Otherwise, there exist a hypersequent $H'' \sqsubseteq \Omega$, and sequent $\Gamma'' \Rightarrow \Delta'' \subseteq \mathcal{T} \Rightarrow \mathcal{U}$ such that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H'' \mid \Gamma'' \Rightarrow \Delta'', \varphi$. Using weakenings and cut on φ we would obtain that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H' \mid H'' \mid \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''$. But, Ω extends this hypersequent.) Therefore, $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$, and by **(b)**, we have that $\mathcal{T} \notin [\varphi]$. ∎

Finally, Theorem 1 is a simple corollary of Theorem 2 and Theorem 3 using our definitions. We leave the details to the reader.

IV. Transitive Modal Logics

Unfortunately, several important frame properties are not *simple* in the sense defined above. This includes, first and foremost, transitivity, defined by the $\mathcal{L}^1$-formula $\theta_{tr} = \forall w_1, w_2, w_3((w_1 R w_2 \wedge w_2 R w_3) \supset w_1 R w_3)$, which is a property widely studied and used in applications. To see that transitivity cannot be captured by an n-simple $\mathcal{L}^1$-formula, note that, unlike transitivity, simple properties are *monotone increasing*, namely they are preserved under the addition of accessibilities (enrichment of R). To handle transitive modal logics, we change our basic calculus to a one that characterizes $\mathbf{K4}$, the basic transitive modal logic, and reproduce the above results for transitive modal logics.

We denote by $\vdash^l_{\mathbf{K4}}$ and $\vdash^g_{\mathbf{K4}}$ the local and global consequence relations of the modal logic $\mathbf{K4}$ for the language $\mathcal{L}$. Note that $\vdash^l_{\mathbf{K4}} = \vdash^l_{\mathbf{K}+\{\theta_{tr}\}}$, and $\vdash^g_{\mathbf{K4}} = \vdash^g_{\mathbf{K}+\{\theta_{tr}\}}$. By enforcing more restrictions formulated by n-simple $\mathcal{L}^1$-formulas, we are able to handle more well-known modal logics. Among others, this includes: $\mathbf{S4}$, $\mathbf{S4.3}$, $\mathbf{K4D}$, $\mathbf{K4.2}$, and $\mathbf{K4.3}$. We denote by $\vdash^l_{\mathbf{K4}+\Theta}$ and $\vdash^g_{\mathbf{K4}+\Theta}$ the local and global consequence relations (respectively) induced by transitive Θ-models. As the basic hypersequent calculus, we use the calculus **HK4**, obtained from **HK** by replacing $(\Rightarrow \Box)$ with the rule:

$$(\Rightarrow \Box_4) \qquad \frac{H \mid \Gamma, \Box\Gamma \Rightarrow \varphi}{H \mid \Box\Gamma \Rightarrow \Box\varphi}$$

All notations used for **HK** are adapted for **HK4** in the obvious way.

Fact 4: **HK4** is sound and complete for the modal logic **K4**, i.e. $\vdash^l_{\mathbf{HK4}} = \vdash^l_{\mathbf{K4}}$ and $\vdash^g_{\mathbf{HK4}} = \vdash^g_{\mathbf{K4}}$.

Fact 5: **HK4** enjoys strong cut-admissibility.

Given a normal description S of an n-simple $\mathcal{L}^1$-formula, the *hypersequent rule induced by S for* **HK4**, denoted by $r_S^{\mathbf{HK4}}$, is given by the first scheme in Figure 3. Examples of the rules obtained for some of the normal descriptions from Table I are also given in Figure 3.

Theorem 4: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HK4}}_{S(\theta)} : \theta \in \Theta\}$. Then, (i) $\vdash^l_{\mathbf{HK4+R}} = \vdash^l_{\mathbf{K4}+\Theta}$; (ii) $\vdash^g_{\mathbf{HK4+R}} = \vdash^g_{\mathbf{K4}+\Theta}$; and (iii) $\mathbf{HK4}+\mathbf{R}$ enjoys strong cut-admissibility.

Proof (Outline): The proof is similar to the proof of Theorem 1. The soundness proof is easily adapted for the transitive case using the fact that $w \in [\Box\varphi]$ implies that $u \in [\Box\varphi]$ for every world u such that wRu. The completeness and cut-admissibility proof requires several modifications. The construction of an $\langle \mathbf{R}, \mathcal{H}_0\rangle$-maximal hypersequent is the same, but it is done with **HK4** (instead of **HK**). The refuting model $\mathcal{W}$ is also the same, except for the accessibility relation which is now defined by: $\mathcal{T}_1 R \mathcal{T}_2$ iff $\Box^{-1}\mathcal{T}_1 \cup \Box\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$ (note that $\Box\Box^{-1}\mathcal{T}_1$ is the set of all $\Box$-formulas in $\mathcal{T}_1$). It is easy to see that R is transitive. We show that $\mathcal{W}$ is a Θ-model. Let θ be an n-simple $\mathcal{L}^1$-formula in Θ. We show that for every $\mathcal{T}_1, \ldots, \mathcal{T}_n \in W$, there exists some $\mathcal{T} \in W$, and $\langle S_R, S_=\rangle \in S(\theta)$, such that $\mathcal{T}_i R \mathcal{T}$ for every $i \in S_R$ and $\mathcal{T}_i = \mathcal{T}$ for every $i \in S_=$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be elements of W. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be sets of formulas such that $\mathcal{T}_i \Rightarrow \mathcal{U}_i \in \Omega$ for every $1 \leq i \leq n$. As in the proof for $\mathbf{HK}+\mathbf{R}$, it is possible to show that for some $\langle S_R, S_=\rangle \in S(\theta)$, the extended hypersequent $\Omega \mid \bigcup_{i\in S_=} \mathcal{T}_i, \bigcup_{i \in S_R} \Box^{-1}\mathcal{T}_i \cup \Box\Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i\in S_=} \mathcal{U}_i$ is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable. Let μ_0 be the extended sequent $\bigcup_{i\in S_=} \mathcal{T}_i, \bigcup_{i\in S_R} \Box^{-1}\mathcal{T}_i \cup \Box\Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i\in S_=}\mathcal{U}_i$. Since Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-maximal, $\{\mu_0\} \sqsubseteq \Omega$. Thus there exists some $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\mu_0 \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$. Now, for every $i \in S_R$, we have $\Box^{-1}\mathcal{T}_i \cup \Box\Box^{-1}\mathcal{T}_i \subseteq \mathcal{T}$, and so $\mathcal{T}_i R \mathcal{T}$. In addition, for every $i \in S_=$, $\mathcal{T}_i \subseteq \mathcal{T}$ and $\mathcal{U}_i \subseteq \mathcal{U}$. By Proposition 2, we have $\mathcal{T}_i = \mathcal{T}$ in this case.

Finally, the proof that $\mathcal{W}$ validates $\mathcal{H}_0$, but not Ω_0 is identical, except for the case of $\varphi = \Box\psi$ in the inductive proof of property **(b)**. This case is now handled as follows:

Assume that $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. Let $\mu = \Box^{-1}\mathcal{T}, \Box\Box^{-1}\mathcal{T} \Rightarrow \psi$. Then $\{\mu\} \sqsubseteq \Omega$. (Otherwise, by Proposition 2 and possibly using weakening we have $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Gamma, \Box\Gamma \Rightarrow \psi$ for some hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \Box^{-1}\mathcal{T}$. By applying $(\Rightarrow \Box_4)$, we obtain that $\mathcal{H}_0 \vdash^{cf}_{\mathbf{HK+R}} H \mid \Box\Gamma \Rightarrow \varphi$. But since $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$, Ω extends $H \mid \Box\Gamma \Rightarrow \varphi$, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0\rangle$-unprovable.) Thus there exists $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\mu \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$. By the induction hypothesis, $\mathcal{T}' \notin [\psi]$. In addition, the definition of R ensures that $\mathcal{T} R \mathcal{T}'$. Thus $\mathcal{T} \notin [\varphi]$. ∎

V. Symmetric Modal Logics

The property of symmetricity, defined by the $\mathcal{L}^1$-formula $\theta_{sym} = \forall w_1, w_2 (w_1 R w_2 \supset w_2 R w_1)$, is another frame property which is widely studied and used in applications. As transitivity, symmetricity is not captured by any n-simple $\mathcal{L}^1$-formula, and again, we have to change our basic calculus. In this section we switch to a basic calculus that characterizes **KB**, the basic symmetric modal logic, and provide a general construction of hypersequent calculi for symmetric modal logics satisfying an arbitrary finite collection of simple properties. The main difference with respect to the previous sections is that here we cannot construct cut-free calculi. In fact, even for **KB** itself, there is no known (standard) cut-free hypersequent calculus. However, we do guarantee that all generated calculi still have the *subformula property*, that is: if there exists a derivation of a hypersequent H from a set $\mathcal{H}$ of hypersequents, then there also exists a derivation of H from $\mathcal{H}$, consisting solely of the subformulas

$$\frac{\{H \mid \bigcup_{i\in S_=}\Gamma_i, \bigcup_{i\in S_R}\Gamma'_i \cup \Box\Gamma'_i \Rightarrow \bigcup_{i\in S_=}\Delta_i : \langle S_R, S_=\rangle \in S\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

General Form of the Hypersequent Rule Induced by S for **HK4**

$$\frac{H \mid \Gamma'_1, \Box\Gamma'_1 \Rightarrow}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1} \qquad \frac{H \mid \Gamma_1, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1} \qquad \frac{H \mid \Gamma'_1, \Box\Gamma'_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}$$

Seriality   Reflexivity   Directedness

$$\frac{H \mid \Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2} \qquad \frac{\{H \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j : 1 \leq i < j \leq n\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

Linearity   Bounded Cardinality

Figure 3. Hypersequent rules induced by normal descriptions for **HK4**

of the formulas in $\mathcal{H}$ and H. This property, called also "analyticity", is of-course weaker than cut-admissibility, but might serve as a sufficient substitute for performing proof-search (see, e.g., [13]). In particular, in the cases that we study, the fact that some logic has an analytic calculus automatically entails its decidability.

We denote by $\vdash^l_{\mathbf{KB}}$ and $\vdash^g_{\mathbf{KB}}$ the local and global consequence relations of the modal logic $\mathbf{KB}$ for the language $\mathcal{L}$. Note that $\vdash^l_{\mathbf{KB}}=\vdash^l_{\mathbf{K}+\{\theta_{sym}\}}$, and $\vdash^g_{\mathbf{KB}}=\vdash^g_{\mathbf{K}+\{\theta_{sym}\}}$. By enforcing more restrictions formulated by n-simple $\mathcal{L}^1$-formulas, we are able to handle more well-known modal logics. Among others, this includes $\mathbf{KTB}$ and $\mathbf{KDB}$. $\vdash^l_{\mathbf{KB}+\Theta}$ and $\vdash^g_{\mathbf{KB}+\Theta}$ denote the local and global consequence relations (respectively) induced by symmetric Θ-models. The hypersequent calculus $\mathbf{HKB}$ is obtained from $\mathbf{HK}$ by replacing $(\Rightarrow \Box)$ with the following rule:

$$(\Rightarrow \Box_B) \qquad \frac{H \mid \Gamma \Rightarrow \varphi, \Box\Delta}{H \mid \Box\Gamma \Rightarrow \Box\varphi, \Delta}$$

All notations used for $\mathbf{HK}$ are adapted for $\mathbf{HKB}$ in the obvious way.

Fact 6: $\mathbf{HKB}$ is sound and complete for the modal logic $\mathbf{KB}$, i.e. $\vdash^l_{\mathbf{HKB}}=\vdash^l_{\mathbf{KB}}$ and $\vdash^g_{\mathbf{HKB}}=\vdash^g_{\mathbf{KB}}$.

Fact 7: $\mathbf{HKB}$ is analytic (namely, it has the subformula property).

Given a normal description S of an n-simple $\mathcal{L}^1$-formula, the *hypersequent rule induced by S for* $\mathbf{HKB}$, denoted by $r_S^{\mathbf{HKB}}$, is given by the first scheme in Figure 4. Examples of the rules obtained for some of the normal descriptions from Table I are also given in Figure 4.

The following soundness theorem is easily proved similarly to the proof of Theorem 2 (using the fact that in a symmetric Kripke model $\langle W, R, [\,]\rangle$, if $w \in [\Box\varphi]$ and uRw then $u \in [\varphi]$).

Theorem 5: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HKB}}_{S(\theta)} : \theta \in \Theta\}$. If a hypersequent H_0 is derivable in $\mathbf{HKB} + \mathbf{R}$ from a set $\mathcal{H}_0$ of hypersequents, then every symmetric Θ-model, which validates $\mathcal{H}_0$, also validates H_0.

For completeness, together with analyticity, we use the following notation and adapt the notions and methods of Section III:

Notation 2: Given a set $\mathbf{R}$ of hypersequent rules, and a set $\mathcal{F}$ of formulas, we write $\mathcal{H} \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H$ if there exists a derivation of the hypersequent H from the set $\mathcal{H}$ of hypersequents in $\mathbf{HKB} + \mathbf{R}$, consisting solely of formulas from $\mathcal{F}$.

Definition 8: Let $\mathbf{R}$ be a set of hypersequent rules, $\mathcal{H}$ be a set of (ordinary) hypersequents, and $\mathcal{F}$ be a set of formulas. The notions of $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-(un)provability, $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-maximality w.r.t. a formula or an extended sequent are defined as $\langle \mathbf{R}, \mathcal{H}\rangle$-(un)provability and $\langle \mathbf{R}, \mathcal{H}\rangle$-maximality (see Definition 7) where $\vdash^{\mathcal{F}}_{\mathbf{HKB+R}}$ is used instead of $\vdash^{cf}_{\mathbf{HK+R}}$. In addition, an extended hypersequent Ω is called $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$*-maximal* if it consists only of formulas from $\mathcal{F}$, it is $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-unprovable, and $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-maximal w.r.t. any formula in $\mathcal{F}$ and any extended sequent consisting only of formulas from $\mathcal{F}$.

Proposition 3: Let Ω be an $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-maximal extended hypersequent for some set $\mathbf{R}$ of hypersequent rules, set $\mathcal{H}$ of hypersequents, and set $\mathcal{F}$ of formulas.

1) For every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ and formula $\varphi \in \mathcal{F}$: if $\varphi \notin \mathcal{T}$ (respectively, $\varphi \notin \mathcal{U}$) then $\mathcal{H} \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma, \varphi \Rightarrow \Delta$ ($\mathcal{H} \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma \Rightarrow \varphi, \Delta$) for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.
2) For every $\mu_1, \mu_2 \in \Omega$: if $\mu_1 \sqsubseteq \mu_2$, then $\mu_1 = \mu_2$.
3) For every extended sequent μ consisting only of formulas from $\mathcal{F}$: if $\{\mu\} \not\sqsubseteq \Omega$, then $\mathcal{H} \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid s$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $s \sqsubseteq \mu$.

Lemma 2: Let $\mathbf{R}$ be a set of hypersequent rules, $\mathcal{H}$ be a set of hypersequents, and $\mathcal{F}$ be a set of formulas. Let Ω be an $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-unprovable extended hypersequent consisting only of formulas from $\mathcal{F}$. Then, Ω can be extended to an $\langle \mathbf{R}, \mathcal{H}, \mathcal{F}\rangle$-maximal extended hypersequent.

Theorem 6: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HKB}}_{S(\theta)} : \theta \in \Theta\}$. Let $\mathcal{H}_0$ be a set of hypersequents, $\mathcal{F}$ be a set of formulas closed under subformula containing the subformulas of $\mathcal{H}_0$, and Ω_0 be an $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F}\rangle$-unprovable extended hypersequent consisting only of formulas from $\mathcal{F}$. Then, there exists a symmetric Θ-model $\mathcal{W}$ which validates $\mathcal{H}_0$, but not Ω_0.

Proof: By Lemma 2, there exists an $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F}\rangle$-maximal hypersequent Ω that extends Ω_0. Define $\mathcal{W} = \langle W, R, [\,]\rangle$, as follows:

- $W = \{\mathcal{T} \subseteq \mathcal{F} : \exists \mathcal{U} \subseteq \mathcal{F}.\ \mathcal{T} \Rightarrow \mathcal{U} \in \Omega\}$.
- $\mathcal{T}_1 R \mathcal{T}_2$ iff $\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\Box^{-1}\mathcal{T}_2 \subseteq \mathcal{T}_1$.
- For every $p \in At_{\mathcal{L}}$: $[p] = \{\mathcal{T} \in W : p \in \mathcal{T}\}$.

$$\frac{\{H \mid \bigcup_{i\in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \Rightarrow \bigcup_{i\in S_=} \Delta_i, \bigcup_{i\in S_R} \Box\Delta'_i : \langle S_R, S_= \rangle \in S\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}$$

General Form of the Hypersequent Rule Induced by S for $\mathbf{HKB}$

$$\frac{H \mid \Gamma'_1 \Rightarrow \Box\Delta'_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1}\ \text{Seriality} \qquad \frac{H \mid \Gamma_1, \Gamma'_1 \Rightarrow \Delta_1, \Box\Delta'_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1}\ \text{Reflexivity} \qquad \frac{\{H \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j : 1 \le i < j \le n\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}\ \text{Bounded Cardinality}$$

Figure 4. Hypersequent rules induced by normal descriptions for **HKB**

Clearly, R is symmetric. We further claim that for every two elements $\mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $\mathcal{T}_2 \Rightarrow \mathcal{U}_2$ of Ω, if $\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{F} \cap \Box\mathcal{U}_1 \subseteq \mathcal{U}_2$ then $\mathcal{T}_1 R \mathcal{T}_2$. Thus we prove that $\mathcal{F} \cap \Box\mathcal{U}_1 \subseteq \mathcal{U}_2$ implies that $\Box^{-1}\mathcal{T}_2 \subseteq \mathcal{T}_1$. Let $\varphi \in \Box^{-1}\mathcal{T}_2$. Hence $\Box\varphi \in \mathcal{T}_2$ (and, in particular, $\Box\varphi \in \mathcal{F}$), and so $\Box\varphi \notin \mathcal{U}_2$ (since $\vdash^{\mathcal{F}}_{\mathbf{HKB+R}} \Box\varphi \Rightarrow \Box\varphi$). The assumption that $\mathcal{F} \cap \Box\mathcal{U}_1 \subseteq \mathcal{U}_2$ entails that $\varphi \notin \mathcal{U}_1$. This implies that $\varphi \in \mathcal{T}_1$. Indeed, otherwise, by Proposition 3 (and using weakenings), there exist a hypersequent $H \sqsubseteq \Omega$, and sets of formulas $\Gamma \subseteq \mathcal{T}_1$ and $\Delta \subseteq \mathcal{U}_1$, such that $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma \Rightarrow \varphi, \Delta$ and $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma, \varphi \Rightarrow \Delta$ (note that $\varphi \in \mathcal{F}$ since $\mathcal{F}$ is closed under subformula). By applying cut on φ, we obtain that $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma \Rightarrow \Delta$. But this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}, \mathcal{F} \rangle$-unprovable.

We show that $\mathcal{W}$ is a Θ-model. Let θ be an n-simple $\mathcal{L}^1$-formula in Θ, and let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be elements of W. We show that there exists some $\mathcal{T} \in W$, and $\langle S_R, S_= \rangle \in S(\theta)$, such that $\mathcal{T}_i R \mathcal{T}$ for every $i \in S_R$ and $\mathcal{T}_i = \mathcal{T}$ for every $i \in S_=$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \subseteq \mathcal{F}$ such that $\mathcal{T}_i \Rightarrow \mathcal{U}_i \in \Omega$ for every $1 \le i \le n$. We claim that there exists some $\langle S^0_R, S^0_= \rangle \in S(\theta)$ such that $\Omega \mid \bigcup_{i \in S^0_=} \mathcal{T}_i, \bigcup_{i \in S^0_R} \Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i \in S^0_=} \mathcal{U}_i, \bigcup_{i \in S^0_R} \Box\mathcal{U}_i$ is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-unprovable. To see this, assume for a contradiction that $\Omega \mid \bigcup_{i \in S_=} \mathcal{T}_i, \bigcup_{i \in S_R} \Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i \in S_=} \mathcal{U}_i, \bigcup_{i \in S_R} \Box\mathcal{U}_i$ is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-provable for every $\langle S_R, S_= \rangle \in S(\theta)$. By Proposition 3 (and using weakenings), there exist a hypersequent $H \sqsubseteq \Omega$, and finite sets $\Gamma_1 \subseteq \mathcal{T}_1, \ldots, \Gamma_n \subseteq \mathcal{T}_n$, $\Gamma'_1 \subseteq \Box^{-1}\mathcal{T}_1, \ldots, \Gamma'_n \subseteq \Box^{-1}\mathcal{T}_n$, $\Delta_1 \subseteq \mathcal{U}_1, \ldots, \Delta_n \subseteq \mathcal{U}_n$, and $\Delta'_1 \subseteq \mathcal{U}_1, \ldots, \Delta'_n \subseteq \mathcal{U}_n$ such that $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \bigcup_{i \in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \Rightarrow \bigcup_{i \in S_=} \Delta_i, \bigcup_{i \in S_R} \Box\Delta'_i$ for every pair $\langle S_R, S_= \rangle$ in $S(\theta)$. Now, by applying the rule $r^{\mathbf{HKB}}_{S(\theta)}$, we obtain that $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n$. But Ω extends this hypersequent, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-unprovable. Let μ_0 be the extended sequent $\bigcup_{i \in S^0_=} \mathcal{T}_i, \bigcup_{i \in S^0_R} \Box^{-1}\mathcal{T}_i \Rightarrow \bigcup_{i \in S^0_=} \mathcal{U}_i, \bigcup_{i \in S^0_R} \mathcal{F} \cap \Box\mathcal{U}_i$. Note that μ_0 consists only of formulas from $\mathcal{F}$, and we have that $\Omega \mid \mu_0$ is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-unprovable. Thus, since Ω is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-maximal, we have $\{\mu_0\} \sqsubseteq \Omega$. Hence there exists some $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\mu_0 \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$. Now, for every $i \in S^0_R$, we have $\Box^{-1}\mathcal{T}_i \subseteq \mathcal{T}$ and $\mathcal{F} \cap \Box\mathcal{U}_i \subseteq \mathcal{U}$. It follows that $\mathcal{T}_i R \mathcal{T}$ for every $i \in S^0_R$. In addition, for every $i \in S^0_=$, $\mathcal{T}_i \subseteq \mathcal{T}$ and $\mathcal{U}^0_i \subseteq \mathcal{U}$. By Proposition 3, we have $\mathcal{T}_i = \mathcal{T}$ in this case.

It remains to show that $\mathcal{W}$ validates $\mathcal{H}_0$, but not Ω_0. As in the proof of Theorem 3, this follows from the fact that for every $\mathcal{T} \in W$: **(a)** $\mathcal{T} \in [\varphi]$ for every $\varphi \in \mathcal{T}$; and **(b)** $\mathcal{T} \notin [\varphi]$ for every formula φ for which $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. **(a)** and **(b)** are proved together by a simultaneous induction on the complexity of φ. We do here only the case $\varphi = \Box\psi$ (the other cases are handled as in the proof of Theorem 3). Suppose that $\varphi = \Box\psi$. Let $\mathcal{T} \in W$. Assume that $\varphi \in \mathcal{T}$, and let $\mathcal{T}' \in W$ such that $\mathcal{T} R \mathcal{T}'$. Thus $\psi \in \mathcal{T}'$. By the induction hypothesis, $\mathcal{T}' \in [\psi]$. It follows that $\mathcal{T} \in [\varphi]$. Next, assume that $\{\mathcal{T} \Rightarrow \varphi\} \sqsubseteq \Omega$. Thus, there exists some $\mathcal{U} \subseteq \mathcal{F}$, such that $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ and $\varphi \in \mathcal{U}$. Let $\mu = \Box^{-1}\mathcal{T} \Rightarrow \psi, \mathcal{F} \cap \Box\mathcal{U}$. Then $\{\mu\} \sqsubseteq \Omega$. (Otherwise, by Proposition 3 and possibly using weakening we have $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Gamma \Rightarrow \psi, \Box\Delta$ for some hypersequent $H \sqsubseteq \Omega$ and finite sets $\Gamma \subseteq \Box^{-1}\mathcal{T}$ and $\Delta \subseteq \mathcal{U}$. By applying $(\Rightarrow \Box_B)$, we obtain that $\mathcal{H}_0 \vdash^{\mathcal{F}}_{\mathbf{HKB+R}} H \mid \Box\Gamma \Rightarrow \varphi, \Delta$. But since $\{\mathcal{T} \Rightarrow \mathcal{U}\} \sqsubseteq \Omega$, Ω extends $H \mid \Box\Gamma \Rightarrow \varphi, \Delta$, and this contradicts the fact that Ω is $\langle \mathbf{R}, \mathcal{H}_0, \mathcal{F} \rangle$-unprovable.) Thus there exists $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\mu \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$. By the induction hypothesis, $\mathcal{T}' \notin [\psi]$. In addition, since $\Box^{-1}\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{F} \cap \Box\mathcal{U} \subseteq \mathcal{U}'$, we have $\mathcal{T} R \mathcal{T}'$. Consequently, $\mathcal{T} \notin [\varphi]$. ■

Theorems 5 and 6 together entail the following corollary.

Corollary 1: Let Θ be a set of simple $\mathcal{L}^1$-formulas, and let $\mathbf{R} = \{r^{\mathbf{HKB}}_{S(\theta)} : \theta \in \Theta\}$. Then, (i) $\vdash^{l}_{\mathbf{HKB+R}} = \vdash^{l}_{\mathbf{KB}+\Theta}$; (ii) $\vdash^{g}_{\mathbf{HKB+R}} = \vdash^{g}_{\mathbf{KB}+\Theta}$; and (iii) $\mathbf{HKB+R}$ is analytic.

Finally, we note that extending the results to modal logics characterized by *both* transitive and symmetric frames (and any finite set of simple properties) is straightforward. In this case, the following rule is used instead of $(\Rightarrow \Box)$:

$$(\Rightarrow \Box_{4B}) \qquad \frac{H \mid \Gamma, \Box\Gamma \Rightarrow \varphi, \Box\Delta, \Box\Box^{-1}\Delta}{H \mid \Box\Gamma \Rightarrow \Box\varphi, \Delta}$$

Denote by $\mathbf{HK4B}$ the obtained calculus. A scheme of the hypersequent rule induced by a given normal description S of an n-simple $\mathcal{L}^1$-formula for $\mathbf{HK4B}$ is presented in Figure 5. Soundness, completeness and analyticity are proved in a similar manner as the corresponding results for symmetric modal logics. In particular, in the completeness proof, the following transitive and symmetric accessibility relation can be used: $\mathcal{T}_1 R \mathcal{T}_2$ iff $\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$, $\Box^{-1}\mathcal{T}_2 \subseteq \mathcal{T}_1$, $\Box\Box^{-1}\mathcal{T}_1 \subseteq \mathcal{T}_2$, and $\Box\Box^{-1}\mathcal{T}_2 \subseteq \mathcal{T}_1$.

VI. Concluding Remarks and Further Research

In this work we identified a family of frame properties that can be translated into hypersequent rules. We provided the necessary definitions to construct a hypersequent calculus for every modal logic characterized by a finite set of properties of this family, and proved a general cut-admissibility theorem, applicable for every calculus produced by these definitions. The results were extended to transitive modal

$$\frac{\{H \mid \bigcup_{i \in S_=} \Gamma_i, \bigcup_{i \in S_R} \Gamma'_i \cup \Box\Gamma'_i \Rightarrow \bigcup_{i \in S_=} \Delta_i, \bigcup_{i \in S_R} \Box\Delta'_i \cup \Box\Box^{-1}\Delta'_i : \langle S_R, S_= \rangle \in S\}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \ldots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}$$

Figure 5. General Form of the Hypersequent Rule Induced by S for $\mathbf{HK4B}$

logics and symmetric ones (in the latter case, we have analyticity rather than full cut-admissibility). The results of this papers may be used to develop new and improved automated reasoning algorithms for modal logics. In addition, since the produced hypersequent calculi are relatively simple and well-behaved, they may play an important role in the investigation of modal logics, alongside with the usual semantic tools. In particular, decidability is an immediate corollary:

Corollary 2: The local and global consequence relations (and theoremhood in particular) of every modal logic characterized by a finite set simple frame properties are decidable. The same holds for transitive modal logics and for symmetric modal logics.

Proof: All hypersequent calculi obtained by our method have the subformula property. Now, since there is a (computable) bound on the number of hypersequents consisting of a given finite set of formulas, derivability in any of these calculi is obviously decidable (simply, by checking one by one all possible candidates for proofs). The decidability of the corresponding consequence relations follows directly. ■

While we only considered modal logics with one modal operator, we believe that some of the methods of this paper can be extended to multimodal logics as well. This should allow a much wider variety of properties, involving more than one accessibility relation. Another natural question for further research is whether this method can be extended to *non-simple* frame properties. More generally, it is interesting to understand which modal logics can be handled by hypersequent calculi. Negative results, showing that some modal logic has no (either cut-free or analytic) hypersequent calculus, would be a major breakthrough. For that, one has to precisely define the general structure of well-behaved hypersequent rules. At this point we can only conjecture that dealing with non-simple properties, if possible, would require much more complicated hypersequent rules, than those needed for simple properties.

Acknowledgement

This research was supported by The Israel Science Foundation (grant 280-10) and by the FWF project START Y544-N23.

References

[1] A. Ciabattoni, N. Galatos, and K. Terui, "From axioms to analytic rules in nonclassical logics," in *Proceedings of LICS'08, IEEE*, 2008, pp. 229–240.

[2] G. Metcalfe, N. Olivetti, and D. M. Gabbay, *Proof Theory for Fuzzy Logics*, ser. Applied Logic Series. Springer, 2009.

[3] A. Avron, "The method of hypersequents in the proof theory of propositional non-classical logics," in *Logic: from foundations to applications*, W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, Eds. New York, NY, USA: Clarendon Press, 1996, pp. 1–32.

[4] M. Fitting, "Modal proof theory," in *Handbook of Modal Logic*, ser. Studies in Logic and Practical Reasoning, J. V. B. Patrick Blackburn and F. Wolter, Eds. Elsevier, 2007, vol. 3, pp. 85 – 138.

[5] H. Wansing, "Sequent systems for modal logics," in *Handbook of Philosophical Logic, 2nd edition*, D. M. Gabbay and F. Guenthner, Eds. Springer, 2002, vol. 8, pp. 61–145.

[6] K. Brünnler, "Deep sequent systems for modal logic," *Archive for Mathematical Logic*, vol. 48, no. 6, pp. 551–577, 2009.

[7] S. Negri, "Proof analysis in modal logic," *Journal of Philosophical Logic*, vol. 34, no. 5-6, pp. 507–544, 2005.

[8] F. Poggiolesi, *Gentzen Calculi for Modal Propositional Logic*, ser. Trends in Logic. Springer, 2011.

[9] R. P. Goré, "Cut-free sequent and tableau systems for propositional normal modal logics," University of Cambridge, Computer Laboratory, Tech. Rep. UCAM-CL-TR-257, May 1992.

[10] A. Ciabattoni and M. Ferrari, "Hypersequent calculi for some intermediate logics with bounded kripke models," *Journal of Logic and Computation*, vol. 11, no. 2, pp. 283–294, 2001.

[11] A. Ciabattoni, G. Metcalfe, and F. Montagna, "Algebraic and proof-theoretic characterizations of truth stressers for mtl and its extensions," *Fuzzy Sets and Systems*, vol. 161, no. 3, pp. 369 – 389, 2010.

[12] A. Indrzejczak, "Cut-free hypersequent calculus for S4.3," *Bulletin of the Section of Logic*, vol. 41, no. 1-2, pp. 89–104, 2012.

[13] M. Takano, "Subformula property as a substitute for cut-elimination in modal propositional logics," *Mathematica Japonica*, vol. 37, pp. 1129–1145, 1992.