Cut-free Calculus for Second-Order Gödel Logic

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Context and Motivation

- Fuzzy logics have semantic origins.
- As *logics* they should have a proof theory.
- The same applies for higher-order fuzzy logics.
- This is essential for basing fuzzy mathematics on fuzzy logic.

G. Metcalfe, N. Olivetti, and D. Gabbay. **Proof Theory for Fuzzy Logics.** Volume 36 of Applied Logic. Springer, 2008.

Simplified Second-Order Language

Augment a first-order language with the following:

- Set variables and set constants.
- Second-order quantifiers.
- Inclusion predicate ε.

Example (Comprehension Scheme)

$$\exists X.\, \forall y.\, A(y) \leftrightarrow (y \varepsilon X) \qquad \text{where } X \text{ is not free in } A$$

Semantics

Structure

$\langle U, \leq, 0, 1\rangle$	Bounded complete linearly ordered set of truth values.
$\mathcal{D}_i$	Domain of individuals.
$\mathcal{D}_s$	Domain of sets.
$\mathcal{I}$	Interpretation of relation symbols and sets: • for any $D \in \mathcal{D}_s$, $\mathcal{I}(D)$ is a fuzzy subset of $\mathcal{D}_i$. • for any n-ary relation symbol R, $\mathcal{I}(R)$ is a fuzzy set of n-tuples of elements of $\mathcal{D}_i$.

Semantics

$$
\begin{aligned}
[\![R(t_1, \dots, t_n)]\!]_\sigma &= \mathcal{I}(R)([\![t_1]\!]_\sigma, \dots, [\![t_n]\!]_\sigma) \\
[\![t \varepsilon T]\!]_\sigma &= \mathcal{I}([\![T]\!]_\sigma)([\![t]\!]_\sigma) \\
[\![\bot]\!]_\sigma &= 0 \\
[\![A \wedge B]\!]_\sigma &= \min\{[\![A]\!]_\sigma, [\![B]\!]_\sigma\} \\
[\![A \vee B]\!]_\sigma &= \max\{[\![A]\!]_\sigma, [\![B]\!]_\sigma\} \\
[\![A \supset B]\!]_\sigma &= [\![A]\!]_\sigma \to [\![B]\!]_\sigma
\end{aligned}
$$

Semantics

$$
\begin{aligned}
[\![\forall x.\, A]\!]_\sigma &= \inf_{d \in \mathcal{D}_i} [\![A]\!]_{\sigma_{x:=d}} \\
[\![\exists x.\, A]\!]_\sigma &= \sup_{d \in \mathcal{D}_i} [\![A]\!]_{\sigma_{x:=d}} \\
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$$\llbracket \forall x.\, A \rrbracket_\sigma = \inf_{d \in \mathcal{D}_i} \llbracket A \rrbracket_{\sigma_{x:=d}}$$

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$$\llbracket \forall X.\, A \rrbracket_\sigma = \inf_{D \in \mathcal{D}_s} \llbracket A \rrbracket_{\sigma_{X:=D}}$$

$$\llbracket \exists X.\, A \rrbracket_\sigma = \sup_{D \in \mathcal{D}_s} \llbracket A \rrbracket_{\sigma_{X:=D}}$$

For every A, y, and σ, there exists some $D \in \mathcal{D}_s$ such that:

$$\mathcal{I}(D) = \lambda d \in \mathcal{D}_i.\, \llbracket A \rrbracket_{\sigma_{y:=d}}$$

$$\exists X.\, \forall y.\, A(y) \leftrightarrow (y \varepsilon X) \qquad \text{where } X \text{ is not free in } A$$

The Proof Theory of Gödel Logic (Propositional Level)

$$(\mathit{Linearity}) \qquad (A \supset B) \vee (B \supset A)$$

- "Syntactically", Gödel logic is obtained by adding (*Linearity*) to an axiomatization of intuitionistic logic.
- Various sequent systems have been introduced (e.g., [Sonobe '75], [Corsi '86], [Avellone et al. '99], [Dyckhoff '99], [Avron and Konikowska '01], [Dyckhoff and Negri '06]).
- Each of them has some ad-hoc logical rules of a nonstandard form.

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- Each of them has some ad-hoc logical rules of a nonstandard form.
- In contrast, standard logical rules are used in **HG** [Avron '91], the system obtained by "lifting" (propositional) **LJ** to the hypersequent level, and adding the communication rule.

Hypersequents

A *hypersequent* is a finite set of sequents denoted by:

$$\Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n$$

The Communication Rule

$$\frac{H \mid \Gamma, \Delta \Rightarrow E_1 \qquad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2}$$

Semantics of Hypersequents

$$[\![\Gamma \Rightarrow E]\!]_\sigma = \begin{cases} 1 & \min_{A \in \Gamma} [\![A]\!]_\sigma \leq \max_{A \in E} [\![A]\!]_\sigma \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\ s_1 \mid \ldots \mid s_n\ ]\!]_\sigma = \max_{1 \leq i \leq n} [\![s_i]\!]_\sigma$$

$$[\![\ s_1 \mid \ldots \mid s_n\ ]\!] = \min_\sigma [\![s_1 \mid \ldots \mid s_n]\!]_\sigma$$

The System **HG**

Structural Rules:

$$(IW \Rightarrow) \quad \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, A \Rightarrow E} \qquad (\Rightarrow IW) \quad \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow A} \qquad (EW) \quad \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com) \quad \frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2}$$

Identity Rules:

$$(id) \quad \frac{}{A \Rightarrow A} \qquad (cut) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma \Rightarrow E}$$

Logical Rules:

$$(\supset\Rightarrow) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma, B \Rightarrow E}{H \mid \Gamma, A \supset B \Rightarrow E} \qquad (\Rightarrow\supset) \quad \frac{H \mid \Gamma, A \Rightarrow B}{H \mid \Gamma \Rightarrow A \supset B}$$

$$(\wedge \Rightarrow) \quad \frac{H \mid \Gamma, A, B \Rightarrow E}{H \mid \Gamma, A \wedge B \Rightarrow E} \qquad (\Rightarrow \wedge) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B}$$

The System **HIF**

- Augment **HG** with the usual rules for first-order quantifiers:

$$(\forall \Rightarrow)\ \frac{H \mid \Gamma, A\{{}^{t}/_{x}\} \Rightarrow E}{H \mid \Gamma, \forall x.\, A \Rightarrow E} \qquad (\Rightarrow \forall)\ \frac{H \mid \Gamma \Rightarrow A}{H \mid \Gamma \Rightarrow \forall x.\, A}$$

$$(\exists \Rightarrow)\ \frac{H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma, \exists x.\, A \Rightarrow E} \qquad (\Rightarrow \exists)\ \frac{H \mid \Gamma \Rightarrow A\{{}^{t}/_{x}\}}{H \mid \Gamma \Rightarrow \exists x.\, A}$$

  x is not free in the lower hypersequent in $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$.
- The resulting calculus is sound and complete for standard first-order Gödel logic.
- It enjoys cut-admissibility ([Baaz, Zach '00], [Avron, L. '13]).

The System $\mathbf{HIF}^2$

- Augment **HIF** with the usual rules for second-order quantifiers.

$$(\forall \Rightarrow)\ \frac{H \mid \Gamma, A\{\tau/x\} \Rightarrow E}{H \mid \Gamma, \forall X.\, A \Rightarrow E} \qquad (\Rightarrow \forall)\ \frac{H \mid \Gamma \Rightarrow A}{H \mid \Gamma \Rightarrow \forall X.\, A}$$

$$(\exists \Rightarrow)\ \frac{H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma, \exists X.\, A \Rightarrow E} \qquad (\Rightarrow \exists)\ \frac{H \mid \Gamma \Rightarrow A\{\tau/x\}}{H \mid \Gamma \Rightarrow \exists X.\, A}$$

X is not free in the lower hypersequent in $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$.

- τ is a set abstraction of the form $\{\!\!\{y \mid \psi(y)\}\!\!\}$
- $A\{\tau/x\}$ is the formula obtained from A by substituting each atomic formula of the form $t \varepsilon X$ by $\psi(t)$, e.g.

$$(f(c)\varepsilon X \vee g(c)\varepsilon X)\{\{\!\!\{y \mid R(y,y)\}\!\!\}/x\} = R(f(c), f(c)) \vee R(g(c), g(c))$$

Main Results

Theorem (Soundness and Completeness)

$$\vdash H \quad \textit{iff} \quad [\![H]\!] = 1 \textit{ for every structure}$$

Theorem

Cut is admissible.

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Cut is admissible.

Major Obstacle

As in $\mathbf{LK}^2$, the usual syntactic approach dramatically fails.

Semantic Approach to Cut-Elimination

History of $\mathbf{LK}^2$

- 1954 Takeuti's conjecture (aimed to prove consistency of analysis)
- 1960 Schütte presented three-valued semantics for the cut-free fragment
- 1965 Tait proved the conjecture using Schütte's semantics

Semantic Approach to Cut-Elimination

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Basically, we take the same approach.

- Develop complete semantics for $\mathbf{HIF}^2$ without (*cut*).
  - "More" truth values.
  - Non-deterministic.
- Show that from every countermodel in this semantics, it is possible to extract an ordinary countermodel.

The Semantic Role of the Identity Rules

$$(id) \quad \frac{}{A \Rightarrow A} \qquad (cut) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma, A \Rightarrow E}{H \mid \Gamma \Rightarrow E}$$

- These rules bind together the two sides of the sequent.
- Without them each formula can have different values on the left side and on the right side.

$$(id) \quad \textit{left side} \leq \textit{right side} \qquad (cut) \quad \textit{right side} \leq \textit{left side}$$

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$$(id)\quad \textit{left side} \leq \textit{right side} \qquad (cut)\quad \cancel{\textit{right side} \leq \textit{left side}}$$

$$[\![A]\!]_\sigma = [\![\mathtt{L}|\mathtt{R}]\!] \qquad \mathtt{L} \leq \mathtt{R}$$

$$[\![A|_\sigma = \mathtt{L} \qquad |A]\!]_\sigma = \mathtt{R}$$

(cut)-Free Semantics

Quasi-Structure

$\langle U, \leq, 0, 1\rangle$	Bounded complete linearly ordered set of truth values.
$\mathcal{D}_i$	Domain of individuals.
$\mathcal{D}_s$	Domain of sets.
$\mathcal{I}^{\mathtt{L}}$	Left interpretation of relation symbols and sets.
$\mathcal{I}^{\mathtt{R}}$	Right interpretation of relation symbols and sets.

Non-determinism

The rules do not uniquely determine truth values of compound formulas.

$$(\wedge \Rightarrow) \quad \frac{H \mid \Gamma, A, B \Rightarrow E}{H \mid \Gamma, A \wedge B \Rightarrow E} \qquad (\Rightarrow \wedge) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B}$$

$$[\![A \wedge B|_\sigma \leq \min\{[\![A|_\sigma, [\![B|_\sigma\} \qquad |A \wedge B]\!]_\sigma \geq \min\{|A]\!]_\sigma, |B]\!]_\sigma\}$$

Non-determinism

The rules do not uniquely determine truth values of compound formulas.

$$(\wedge \Rightarrow) \quad \frac{H \mid \Gamma, A, B \Rightarrow E}{H \mid \Gamma, A \wedge B \Rightarrow E} \qquad (\Rightarrow \wedge) \quad \frac{H \mid \Gamma \Rightarrow A \quad H \mid \Gamma \Rightarrow B}{H \mid \Gamma \Rightarrow A \wedge B}$$

$$[\![A \wedge B |_\sigma \leq \min\{[\![A|_\sigma, [\![B|_\sigma\} \qquad |A \wedge B]\!]_\sigma \geq \min\{|A]\!]_\sigma, |B]\!]_\sigma\}$$

The semantics is non-deterministic.

$[\![\cdot |$ and $| \cdot]\!]$ are also included in each quasi-structure.

(cut)-Free Semantics

$$
\begin{aligned}
\llbracket R(t_1 \dots t_n)|_\sigma &= \mathcal{I}^{\mathtt{L}}(R)(\llbracket t_1\rrbracket_\sigma, \dots, \llbracket t_n\rrbracket_\sigma) & |R(t_1 \dots t_n)\rrbracket_\sigma &= \mathcal{I}^{\mathtt{R}}(R)(\llbracket t_1\rrbracket_\sigma, \dots, \llbracket t_n\rrbracket_\sigma) \\
\llbracket t\varepsilon T|_\sigma &= \mathcal{I}^{\mathtt{L}}(\llbracket T\rrbracket_\sigma)(\llbracket t\rrbracket_\sigma) & |t\varepsilon T\rrbracket_\sigma &= \mathcal{I}^{\mathtt{R}}(\llbracket T\rrbracket_\sigma)(\llbracket t\rrbracket_\sigma) \\
\llbracket \bot |_\sigma &\leq 0 & | \bot\rrbracket_\sigma &\geq 0 \\
\llbracket A \wedge B|_\sigma &\leq \min\{\llbracket A|_\sigma, \llbracket B|_\sigma\} & |A \wedge B\rrbracket_\sigma &\geq \min\{|A\rrbracket_\sigma, |B\rrbracket_\sigma\} \\
\llbracket A \vee B|_\sigma &\leq \max\{\llbracket A|_\sigma, \llbracket B|_\sigma\} & |A \vee B\rrbracket_\sigma &\geq \max\{|A\rrbracket_\sigma, |B\rrbracket_\sigma\} \\
\llbracket A \supset B|_\sigma &\leq |A\rrbracket_\sigma \to \llbracket B|_\sigma & |A \supset B\rrbracket_\sigma &\geq \llbracket A|_\sigma \to |B\rrbracket_\sigma
\end{aligned}
$$

(cut)-Free Semantics

$$
\begin{aligned}
\llbracket \forall x.\, A|_\sigma &\le \inf_{d \in \mathcal{D}_i} \llbracket A|_{\sigma_{x:=d}} & \qquad |\forall x.\, A\rrbracket_\sigma &\ge \inf_{d \in \mathcal{D}_i} |A\rrbracket_{\sigma_{x:=d}} \\
\llbracket \exists x.\, A|_\sigma &\le \sup_{d \in \mathcal{D}_i} \llbracket A|_{\sigma_{x:=d}} & \qquad |\exists x.\, A\rrbracket_\sigma &\ge \sup_{d \in \mathcal{D}_i} |A\rrbracket_{\sigma_{x:=d}} \\
\llbracket \forall X.\, A|_\sigma &\le \inf_{D \in \mathcal{D}_s} \llbracket A|_{\sigma_{X:=D}} & \qquad |\forall X.\, A\rrbracket_\sigma &\ge \inf_{D \in \mathcal{D}_s} |A\rrbracket_{\sigma_{X:=D}} \\
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\end{aligned}
$$

For every A, y, and σ, there exists some $D \in \mathcal{D}_s$ such that:

$$
\mathcal{I}^{\mathrm{L}}(D) = \lambda d \in \mathcal{D}_i.\, \llbracket A|_{\sigma_{y:=d}} \qquad \mathcal{I}^{\mathrm{R}}(D) = \lambda d \in \mathcal{D}_i.\, |A\rrbracket_{\sigma_{y:=d}}
$$

(cut)-Free Semantics

$$[\![\Gamma \Rightarrow E]\!]^{cf}_{\sigma} = \begin{cases} 1 & \min_{A\in\Gamma} [\![A|_{\sigma} \leq \max_{A\in E} |A]\!]_{\sigma} \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\ s_1 \mid \ldots \mid s_n\ ]\!]^{cf}_{\sigma} = \max_{1\leq i\leq n} [\![s_i]\!]^{cf}_{\sigma}$$

$$[\![\ s_1 \mid \ldots \mid s_n\ ]\!]^{cf} = \min_{\sigma} [\![s_1 \mid \ldots \mid s_n]\!]^{cf}_{\sigma}$$

Theorem (Completeness)

If H is not provable without (cut) then $[\![H]\!]^{cf} = 0$ for some non-deterministic structure.

(cut)-Free Completeness

Given Quasi-structure $\langle \mathcal{U}, \mathcal{D}_i, \mathcal{D}_s, \mathcal{I}^{\mathrm{L}}, \mathcal{I}^{\mathrm{R}}, [\![\cdot|, |\cdot]\!] \rangle$ and assignment σ, s.t. $[\![H]\!]_\sigma^{cf} = 0$.

Goal Structure $\langle \mathcal{U}', \mathcal{D}'_i, \mathcal{D}'_s, \mathcal{I} \rangle$ and assignment ρ, s.t. $[\![H]\!]_\rho = 0$.

- $\mathcal{U}' := \mathcal{U}$, $\mathcal{D}'_i := \mathcal{D}_i$, $\rho(x) := \sigma(x)$ for individual variables.
- $\mathcal{I}(R) := \mathcal{I}^{\mathrm{L}}(R)$ for relation symbols.
- $\mathcal{D}'_s$ includes a member $\langle D, S \rangle$ for any $D \in \mathcal{D}_s$ and fuzzy set S s.t. $\mathcal{I}^{\mathrm{L}}(D) \subseteq S \subseteq \mathcal{I}^{\mathrm{R}}(D)$.
- $\mathcal{I}(\langle D, S \rangle) := S$ for any $\langle D, S \rangle \in \mathcal{D}'_s$.
- For any set variable X, $\rho(X) := \langle \sigma(X), S \rangle$ for some fuzzy set S as above (we refer to all these assignments as *σ-suitable* assignments).

It remains to prove: $[\![H]\!]_\rho = 0$; comprehensiveness.

We show: $[\![A|_\sigma \leq [\![A]\!]_\rho \leq |A]\!]_\sigma$ for any formula A and σ-suitable assignment ρ.

(*cut*)-Free Completeness

One step in the (inductive) proof:

IH1 $[\![A|_\sigma \leq [\![A]\!]_\rho \leq |A]\!]_\sigma$

IH2 $[\![B|_\sigma \leq [\![B]\!]_\rho \leq |B]\!]_\sigma$

H1 $[\![A \supset B]\!]_\rho = [\![A]\!]_\rho \rightarrow [\![B]\!]_\rho$

H2 $[\![A \supset B|_\sigma \leq |A]\!]_\sigma \rightarrow [\![B|_\sigma$

H3 $|A \supset B]\!]_\sigma \geq [\![A|_\sigma \rightarrow |B]\!]_\sigma$

==

$[\![A \supset B|_\sigma \leq [\![A \supset B]\!]_\rho \leq |A \supset B]\!]_\sigma$

(using the fact that if $u_1 \leq u' \leq u_2$ and $u_3 \leq u'' \leq u_4$, then $u_2 \rightarrow u_3 \leq u' \rightarrow u'' \leq u_1 \rightarrow u_4$).

Conclusions and Further Work

- **HIF**^{2} is sound and complete with respect to the Henkin-style semantics for second-order Gödel logic.
- **HIF**^{2} enjoys cut-admissibility.

- Further work:
  - * Globalization ("Baaz Delta")
  - * Equality
  - * Richer signatures
  - * Full type theory
  - *** Other fuzzy logics

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Thank you!