Kripke-Style Semantics for Normal Systems

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In this paper we describe a generalized framework of sequent calculi, called *normal systems*, which is suitable for a wide family of propositional logics. This family includes classical logic, several modal logics, intuitionistic logic, its dual, bi-intuitionistic logic, and various finite-valued logics. We present a general method for providing a non-deterministic finite-valued Kripke-style semantics for every normal system, and prove a corresponding general soundness and completeness theorem.

As observed in [3], there are two dual standpoints regarding signed calculi (or many-sided sequent calculi). In the "positive" interpretation a signed formula consisting of a sign i and a formula ψ is true, iff i is the truth value assigned to ψ. In the "negative" interpretation the same signed formula is true, iff i is *not* the truth value assigned to ψ (The two approaches practically coincide in the two-valued case). As in [4], it turns out that the "negative" interpretation is more suited for our purposes, since it leads to a natural Kripke-style semantics, that generalizes the well-known two-valued semantics for the above-mentioned logics. The definitions below (and the notation we use) reflect this choice.

Definition A *signed formula* for a propositional language $\mathcal{L}$ and a finite set of signs $\mathcal{I}$ is an expression of the form $i \div \psi$, where $i \in \mathcal{I}$ and ψ is an $\mathcal{L}$-formula. A *sequent* is a finite set of signed formulas. The usual two-sided notation $\Gamma \Rightarrow \Delta$ is interpreted as $\{t \div \psi \mid \psi \in \Gamma\} \cup \{f \div \psi \mid \psi \in \Delta\}$, i.e. a sequent over $\mathcal{I} = \{t, f\}$.

Definition

1. A *context-restriction* is a set of signed formulas. A *normal premise* is an expression of the form $\langle s, \pi \rangle$, where s is a sequent and π is a context-restriction.

2. A *normal rule* is an expression of the form: S/C, where S is a finite set normal premises, and C is a sequent. An *application* of the normal rule S/C is any inference step which derives $\sigma(C) \cup s'$ from the set of sequents $\{\sigma(s) \cup (s' \cap \pi) \mid \langle s, \pi \rangle \in S\}$, where s' is a sequent and σ is an $\mathcal{L}$-substitution.

3. The *identity axioms* are normal rules of the form $\emptyset / \{i \div p_1, j \div p_1\}$, for every $i, j \in \mathcal{I}$ such that $i \neq j$. The *cut rule* is the normal rule

*This research was supported by the Israel Science Foundation (grant No 809-06).

$\{\langle i \div p_1, \pi\rangle \mid i \in \mathcal{I}\}/\emptyset$, where π is the set of all signed formulas. The *weakening rules* are the normal rules of the form $\{\langle\emptyset, \pi\rangle\}/i \div p_1$ for every $i \in \mathcal{I}$, where π is the set of all signed formulas.

4. A sequential system is called *normal* iff it consists of normal rules only, and the identity axioms, cut, and weakening are among its rules.

We give some examples of the variety of derivation rules included in our normal rule definition:

Classical Implication. Assume $\mathcal{I} = \{t, f\}$. The following normal rules are the rules for implication in Gentzen's LK (see [6])[1] :

$$\{\langle\{t \div p_1, f \div p_2\}, \pi_1\rangle\}/f \div p_1 \supset p_2$$
$$\{\langle\{f \div p_1\}, \pi_2\rangle, \langle\{t \div p_2\}, \pi_2\rangle\}/t \div p_1 \supset p_2$$

where π_1 and π_2 are the sets of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

Intuitionistic Implication. Assume $\mathcal{I} = \{t, f\}$. Consider the previous two rules for $\supset$ where π_2 is the same, and π_1 is the set of all signed formulas of the form $t \div \psi$. This gives the rules for implication in LJ' (see [7]). Using a more usual notation, applications of the second rule have the form:

$$\frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi}$$

S4's Box. Assume $\mathcal{I} = \{t, f\}$. The following normal rules are the rules for box in $S4^*$ (see [5]):

$$\{\langle\{f \div p_1\}, \pi_1\rangle\}/f \div \Box p_1 \qquad \{\langle\{t \div p_1\}, \pi_2\rangle\}/t \div \Box p_1$$

where π_1 is the set of all signed formulas of the form $t \div \Box\psi$, and π_2 is the set of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Box\Gamma \Rightarrow \psi}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box\psi} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

3-Valued Lukasiewicz's Implication. Assume $\mathcal{I} = \{t, I, f\}$. The following normal rules are the rules for implication in $NL3$ (see [8]):

[1] More precisely, we mean the version of LK, in which the two sides of a sequent consists of sets of formulas. The same applies to the other systems that we mention.

$$\{\langle\{f \div p_1\}, \pi\rangle, \langle\{f \div p_2, t \div p_1\}, \pi\rangle\}/f \div p_1 \supset p_2$$
$$\{\langle\{f \div p_1, I \div p_1\}, \pi\rangle, \langle\{I \div p_2, t \div p_1\}, \pi\rangle\}/I \div p_1 \supset p_2$$
$$\{\langle\{f \div p_1\}, \pi\rangle, \langle\{t \div p_2\}, \pi\rangle, \langle\{I \div p_1, I \div p_2\}, \pi\rangle\}/I \div p_1 \supset p_2$$

where π is the set of all signed formulas for $\mathcal{L}$ and $\mathcal{I}$. Using a more usual notation, applications of these rules have the form:

$$\frac{\Gamma|\Delta|\Sigma, \psi \quad \Gamma, \varphi|\Delta|\Sigma, \psi}{\Gamma, \psi \supset \varphi|\Delta|\Sigma} \quad \frac{\Gamma, \psi|\Delta, \psi|\Sigma \quad \Gamma|\Delta, \varphi|\Sigma, \psi}{\Gamma|\Delta, \psi \supset \varphi|\Sigma} \quad \frac{\Gamma, \psi|\Delta|\Sigma \quad \Gamma|\Delta|\Sigma, \varphi \quad \Gamma|\Delta|\Sigma}{\Gamma|\Delta, \psi, \varphi|\Sigma, \psi \supset \varphi}$$

Now we define the Kripke semantics induced by any given normal system. Non-deterministic Kripke-frames for single-conclusion proper sequential calculi were first introduced in [2].

Definition Let $\mathbf{G}$ be a normal system. Denote by $\Pi_{\mathbf{G}}$ the set of context-restrictions which appear in $\mathbf{G}$.

1. A $\Pi_{\mathbf{G}}$-*frame* is a tuple $\langle W, \mathcal{R}, v\rangle$, where:
   (a) W is a set of elements, and $\mathcal{R}$ is a set of preorders on W, consisting of a preorder $\lesssim_\pi$ for every $\pi \in \Pi_{\mathbf{G}}$. If π is the set of all signed formulas, then $\lesssim_\pi$ is the identity relation.
   (b) $v : W \times Frm_{\mathcal{L}} \to \mathcal{I}$ is a *persistent* function, i.e. for every $\pi \in \Pi_{\mathbf{G}}$: if $i \div \psi \in \pi$ and $v(a, \psi) = i$, then $v(b, \psi) = i$ for every $b \in W$ such that $a \lesssim_\pi b$.
2. Let $\mathcal{W} = \langle W, \mathcal{R}, v\rangle$ be a $\Pi_{\mathbf{G}}$-frame.
   (a) A sequent s is *true* in $a \in W$ iff there exists $i \div \psi \in s$ such that $v(a, \psi) \neq i$. $\mathcal{W}$ is a *model* of s iff s is true in every $a \in W$.
   (b) $\mathcal{W}$ *respects* a normal rule of $\mathbf{G}$, $r = S/C$, iff for every $a \in W$ and every $\mathcal{L}$-substitution σ: If $\sigma(s)$ is true in every $b \in W$ such that $a \lesssim_\pi b$ for every $\langle s, \pi\rangle \in S$, then $\sigma(C)$ is true in a.
   (c) $\mathcal{W}$ is $\mathbf{G}$-*legal* iff it respects all the rules of $\mathbf{G}$.

We give some examples of the sets of all $\mathbf{G}$-legal frames for important logics:

Classical Logic. Consider the normal system LK. LK includes one sort of context-restriction, π, which is the set of all signed formulas. Thus we consider frames with one preorder $\lesssim_\pi$, which is the identity relation. The persistence condition then vacuously holds. LK-legality reduces to usual Tarski-style semantics in every element of W.

Intuitionistic Logic. Consider the full normal system LJ' from [7]. It has two sorts of context-restriction: π_1 and π_2 from the example of intuitionistic implication given above. π_1 (the set of all signed formulas of the form $t \div \psi$) is used in the right rule for implication, π_2 (the set of all signed formulas) in all the rest. Accordingly, the persistence condition means here that if $v(a, \psi) = t$ then $v(b, \psi) = t$ for every $b \in W$ such that $a \lesssim_{\pi_1} b$.

Now it is easy to see (using persistence) that a frame is LJ'-legal accoding to our definitions iff: (1) $v(a, \psi \supset \varphi) = t$ iff there does not exist an element $b \in W$ such that $a \lesssim_{\pi_1} b$, $v(b, \psi) = t$ and $v(b, \varphi) = f$; and (2) it respects the usual truth-tables of the other connectives ($\wedge$,$\vee$ and $\perp$) in every element of W.

S4. Consider the normal system $S4^*$ from [5]. $S4^*$ again has two sorts of context-restriction: π_1 and π_2 from the example of $S4$'s Box given above. Now $\lesssim_{\pi_2}$ is again the identity relation, so the persistence condition means that if $v(a, \Box\psi) = t$ then $v(b, \Box\psi) = t$ for every $b \in W$ such that $a \lesssim_{\pi_1} b$. Using the fact that $\lesssim_{\pi_1}$ is a preorder, it is easy to see that a frame is $S4^*$-legal iff: (1) $v(a, \Box\psi) = t$ iff $v(b, \psi) = t$ for every b such that $a \lesssim_{\pi_1} b$; and (2) it respects the usual truth-tables of the other connectives in every element of W.

Our main result is the following:

Theorem (Strong Soundness and Completeness) Let $\mathbf{G}$ be a normal system. There exists a proof in $\mathbf{G}$ of a sequent s from a set of sequents $\mathcal{S}$, iff every $\mathbf{G}$-legal $\Pi_{\mathbf{G}}$-frame which is a model of $\mathcal{S}$ is also a model of s.

A Compactness property of our semantic consequence relation (for any given normal system) is an easy corollary of this theorem. Moreover, the proof of this theorem shows that it is sufficient to consider a narrower class of frames. We point out two properties of the frames in this class:

1. If $\{j \div \psi \mid j \neq i\} \subseteq \pi_1$ whenever $i \div \psi \in \pi_2$, then $\lesssim_{\pi_2}$ is the inverse of $\lesssim_{\pi_1}$. In particular, if $i \div \psi \in \pi$ for every $i \in \mathcal{I}$ whenever $i \div \psi \in \pi$ for some $i \in \mathcal{I}$, then $\lesssim_{\pi}$ is symmetric.

2. If for every formula ψ, $i \div \psi \notin \pi$ for at most one $i \in \mathcal{I}$, then $\lesssim_{\pi}$ is anti-symmetric.

In particular:

1. For $S5^*$ (a calculus for $S5$, see [5]), it suffices to consider frames with one equivalence relation.

2. For LJ', it suffices to consider frames with an order relation. The same is true for dual-intuitionistic logic and for bi-intuitionistic logic.

Following this method, one obtains a finite-valued Kripke-style semantics for any normal system. The semantics is modular, allowing to separately investigate the effect of every normal rule. Note that for many normal systems, the resulting semantics is *non-deterministic*, and the truth-functionality principle does not hold. This happens, for example, for any normal system whose set of rules is a proper subset of any of the above-mentioned normal systems.

We believe that the current work is a good starting point for developing and investigating sequent calculi for various sorts of combinations of many valued, intuitionistic, modal, and multi-modal logics. However, it should be noted that in order to obtain a decision procedure for a normal system using this semantics, one have to ensure *analycity* ([1]) of the semantics, i.e. that every legal frame which is defined on some set of formulas can be extended to a legal frame defined on all formulas. It is interesting to look for a general characterization of normal systems which admit this property. Another important question is how (and in what cases) can we extract a semantic proof of cut-elimination from our completeness proof. We leave these questions for a future work.

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