Dissertation Abstract

Semantic Investigation of Proof Systems for Non-classical Logics

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Ever since the introduction of sequent calculi for classical and intuitionistic logic by Gentzen [20], sequent calculi have been widely applied in the fields of proof theory, computational logic, and automated deduction. These systems and their natural generalizations (such as many-sided sequent and hypersequent calculi) provide suitable proof-theoretic frameworks for a huge variety of non-classical logics, including intermediate logics [18, 16], modal logics [37], substructural logics [19], many-valued logics [23], fuzzy logics [26], and paraconsistent logics [6]. In many important cases they suggest an "algorithmic presentation" of a logic, which is particularly useful in automated reasoning tasks, as well as for studying its properties, such as decidability and computational complexity (mainly for propositional logics), consistency, interpolation, the Herbrand theorem (for first-order logics). Thus in the last decades Gentzen-type calculi have been frequently used for introducing, handling, comparing, investigating and using new and existing non-classical logics. Each such calculus usually requires a soundness and completeness theorem with respect to its corresponding logic, and its proof-theoretic properties should be verified. Traditionally, this is done each time from scratch. In many cases the fundamental theorem of *cut-elimination* is proved, i.e., the redundancy of the well-known cut rule, which usually ensures the usefulness of the calculus. Another desirable and computationally crucial property of Gentzen calculi is *analyticity*. Roughly speaking, a calculus is analytic if whenever a sequent s is provable in it, then s can be proved using only the syntactic material available inside s. Often analyticity is an immediate corollary of cut-elimination, but in many cases cut-elimination fails, and the calculus can still be shown to be analytic.

This thesis aims at a systematic investigation of Gentzen-type systems as mathematical objects in their own right. We study a wide variety of sequent

and hypersequent calculi for many logics of different natures. Our main contribution is a *semantic* analysis of several general families of Gentzen-type sequent and hypersequent calculi, that, generally speaking, consists of the following:

1. We provide a uniform, general and modular semantic characterization for the systems in the families we study. Thus each calculus $\mathbf{G}$ corresponds to a certain set of semantic structures $\mathcal{V}_{\mathbf{G}}$; and the consequence relation induced by $\mathcal{V}_{\mathbf{G}}$ (together with an appropriate definition of when a structure in $\mathcal{V}_{\mathbf{G}}$ is a *model* of a given sequent or hypersequent) is shown to be identical to $\vdash_{\mathbf{G}}$, the derivability relation of $\mathbf{G}$. For each family of calculi, we present a general method for extracting the set $\mathcal{V}_{\mathbf{G}}$ from a given system $\mathbf{G}$ in this family. Soundness and completeness of many known Gentzen-type systems with respect to their usual semantics is obtained as a particular instance of the proposed method. The semantics provides a complementary dual view on Gentzen systems, and for certain families of propositional calculi it is also *effective*, i.e. it naturally induces a semantic decision procedure for derivability in the corresponding calculus. Thus we derive new general decidability results for large families of Gentzen-type systems, and hence, also for the logics they induce.

2. We apply this semantic presentation of calculi (and extend and refine it, when needed) for investigating and characterizing crucial proof-theoretic properties of the systems we study. This includes general notions of *cut-admissibility*, *analyticity*, and *axiom-expansion*. Indeed, an illuminating contribution of a semantic study of proof systems is the ability to provide semantic proofs (or refutations) of syntactic properties. Even when a traditional syntactic proof exists, it is usually a tedious and error-prone inductive argument that is specifically tailored to a certain system. On the other hand, semantic proofs tend to be simpler and easier to check. Thus we characterize these properties from a semantic point of view, providing a general "semantic toolbox" that can be applied to prove (or refute) these properties. In some of the families we study, these characterizations naturally lead to simple and decidable exact criteria for the aforementioned proof-theoretic properties.

A crucial feature of a systematic procedure relating proof systems and semantics should be its *modularity* – the correspondence between semantics and proof systems should be based on local equivalences between semantic ingredients (requirements from the semantic structures) and their syntactic counterparts (derivation rules). Among others, such a correspondence allows one to predict the semantic impact of employing the same rule in different proof systems, or to provide an appropriate rule for a given semantic condition added to different logics. In particular, our semantic characterizations of cut-admissibility in each of the families of calculi that we study are based on identifying the semantic effect of the cut rule(s), and comparing the semantics of the calculi with and without the cut rule(s). These tasks are of course impossible when the proof system and its semantics are considered as a whole, and there is no possibility to separate between the semantic conditions imposed by each particular rule.

The major key to have this modularity, as well as to provide semantics to *every* calculus in the families that we study, is the use of *non-deterministic semantics*. Thus, following [8, 9], we relax the principle of truth-functionality, and allow cases in which the truth value of a compound formula is not uniquely determined by the truth values of its subformulas. By allowing non truth-functional semantic structures, we are able to separately analyze the semantic effect of each component of the syntactic machinery (each derivation rule, and in fact also each ingredient of a rule). The full semantics of the calculus is then obtained by joining the semantic effects of all of its components. In fact, we show that non truth-functional semantics is unavoidable for characterizing the cut-free and the identity-axiom-free fragments of calculi, as needed in order to obtain semantic conditions for cut-admissibility and axiom-expansion. Pursuing this approach, we develop several frameworks of non-deterministic semantics, including generalized non-deterministic versions of finite-valued semantics, real-valued algebraic semantics, and Kripke semantics. The modular semantics sheds light on the syntactic properties of the corresponding calculus, reveals deep useful connections between semantics and proof theory, and turns out to be useful for proving these properties in particular examples.

The notion of *analyticity* of a proof system plays a central role in this thesis. Analyticity is perhaps the most important property of fully-structural propositional proof systems, as it usually implies its decidability and consistency (the fact that the empty sequent is not derivable). As mentioned above, a sequent calculus is *analytic* if whenever a sequent s is provable in it, then s can be proved using only the syntactic material available within the sequent s. However, there is more than one way to precisely define the "material available within some sequent". Usually, it is taken to consist of all subformulas occurring in the sequent, and then analyticity amounts to *the global subformula property* (i.e., if there exists a proof of a sequent s, then there exists a proof of s using only its subformulas). However, it is also possible (and sometimes necessary) to consider analyticity properties that are based on different relations defining the "material available within sequents". While these substitutes might be weaker than the global subformula property, they still suffice to imply the consistency and the decidability of many proof systems. Therefore, we define and study a generalized analyticity that is based on arbitrary partial orders whose properties ensure the usefulness of the corresponding analyticity property. In addition, the semantic methods allow us to study analyticity of general Gentzen-type systems, *regardless* of cut-admissibility, which leads in many cases to much simpler criteria for analyticity. Besides previous works on the particular family of canonical sequent calculi (see [9]), we are not aware of any investigation of analyticity in general Gentzen-type systems that was done independently of cut-admissibility, or any previously known semantic account for this crucial syntactic property.

Next, we briefly describe the families of Gentzen-type systems that are studied in this thesis, their corresponding developed semantic framework, and the main contributions obtained for each of them. Note that our scope includes only fully-structural calculi, i.e., systems that include the usual structural rules of exchange, contraction, and weakening.

Pure Sequent Calculi

The first family of calculi introduced in this thesis is the family of pure sequent calculi. These are propositional sequent calculi, whose derivation rules do not enforce any limitation on the side formulas (also known as: context formulas). This family of calculi provides a suitable proof-theoretic framework for several important propositional logics, including classical logic, many well-studied many-valued logics, and various paraconsistent logics. Our scope here is broader than what is usually considered as a sequent system:

- We consider *many-sided* sequents, rather than just ordinary *two-sided* ones. This allows us to naturally capture a large family of many-valued logics (see, e.g., [25]).
- We do not presuppose that all systems include identity axioms or cut rules of a given form. Instead, we allow arbitrary combinations of these rules. This plays a major role in the semantic characterizations of proof-theoretic properties of these systems. For example, it makes it possible to compare the semantics of a given system with cut, and the semantics of the same system without cut, in order to derive a semantic characterization of cut-admissibility.

On the semantic side, we define *many-valued systems* and use them to semantically characterize pure calculi. Many-valued systems provide a semantic framework for specifying sets of valuations – functions assigning truth values to formulas of a given propositional language. Each many-valued system includes a set of semantic conditions, that can be easily read off the derivation rules of a pure sequent calculi, and used to restrict its corresponding set of valuations (e.g. "If φ_1 has some truth value u_1, and $\neg\varphi_1$ has some truth value u_2, then $\neg(\varphi_1 \wedge \varphi_1)$ should have the truth value u_3"). This framework generalizes the "bivaluation semantics" [12, 15], many-valued matrices [35, 23], and non-deterministic many-valued matrices [8, 9], and is used here to provide semantics for pure sequent calculi.

We further show that many-valued systems suggest a semantic account for analyticity of pure calculi. Indeed, given a pure calculus **G**, its corresponding many-valued system $\mathbf{M_G}$ is not only sound and complete for **G**, but also enjoys the following stronger property: derivations in **G** that consist only of formulas from some set $\mathcal{F}$ precisely correspond to the semantics given by partial valuations in $\mathbf{M_G}$ whose domain is this set $\mathcal{F}$. It then follows that analyticity of **G** is equivalent to the fact that certain partial valuations in $\mathbf{M_G}$ can be extended to full ones. The most simple and well-known example here is (the propositional fragment of) Gentzen's calculus **LK** for classical logic. The fact that this calculus admits the subformula property was originally obtained as a direct consequence of cut-elimination. Using our method, one can derive the subformula property for **LK** from the fact that usual classical two-valued partial valuation functions that are defined on some set of formulas that is closed under subformulas can be extended to full classical valuations. Proving the latter fact is trivial. In fact, it is so trivial, so that many logic textbooks take it for granted, and do not even bother to mention it!

As a running example we take the sequent calculus for da Costa's historical paraconsistent logic $\mathbf{C_1}$ that was introduced in [5], present the many-valued semantics obtained for it by using the general method, show its effectiveness for deciding this logic, and use the developed general semantic criteria to reprove cut-admissibility for this calculus, as well as independently prove that it enjoys a certain generalized subformula property.

Canonical Calculi

The family of canonical calculi is a subfamily of pure sequent calculi, that was introduced in [8]. The idea behind canonical systems implicitly underlies a long tradition in the philosophy of logic, established by Gentzen in his seminal paper [20]. According to this tradition, the meaning of a connective $\diamond$ is determined by the derivation rules which are associated with it. For that matter, one should have rules of some "ideal" type, in each of which $\diamond$ is mentioned exactly once, and no other connective is involved. Formulating this idea, [8] introduced the notion of a "canonical (introduction) rule", and, in turn, "canonical propositional Gentzen-type systems" were defined as two-sided sequent systems in which: (i) all logical rules are canonical rules; and (ii) the usual cut rule, identity-axiom and all structural rules are included. As we did for pure sequent calculi, our work relaxes (ii) and includes systems without cut and identity. More generally, we study many-sided canonical sequent systems with arbitrary combinations of cut rules and identity axioms.

Since this family of calculi is a subfamily of pure sequent calculi, the results concerning the semantics of pure sequent calculi and the semantic characterizations of their proof-theoretic properties can be applied for canonical calculi as well. However, we show that for this more restricted family of calculi we are always able to obtain much simpler semantics. This semantics takes the form of a *partial non-deterministic matrix*, a special case of a many-valued system, in which the semantic conditions for specifying restrictions on valuation functions can be arranged in *generalized truth tables*. Usual logical matrices are particular instances, while non-determinism is introduced as done in *non-deterministic matrices* (see [8, 9]) by possibly allowing several options in some entries of the truth tables (thus the value of $\diamond(\varphi_1, \dots, \varphi_n)$ is restricted, but not uniquely determined, by the values of $\varphi_1, \dots, \varphi_n$). However, to handle arbitrary canonical calculi we slightly extend the framework of non-deterministic matrices by allowing one also to have empty sets of options in some entries of the truth tables. This intuitively mean that certain combinations of truth values are disallowed.

We show that the semantics of partial non-deterministic matrix is always effective, and thus canonical sequent calculi are all decidable. Furthermore, from the constructed matrix one can easily decide whether the calculus is analytic, whether it enjoys cut-admissibility, and what connectives admit axiom-expansion. In other words, partial non-deterministic matrices provide simple *decidable* characterizations of these proof-theoretic properties of canonical calculi. An interesting corollary of this semantic study is that the subformula property is actually equivalent to cut-admissibility in this family of calculi.

Quasi-Canonical Calculi

The family of quasi-canonical calculi is another subfamily of pure sequent calculi, that extends the family of canonical calculi. Here we allow also logical rules in which unary connectives precede the connective to be introduced in conclusions of logical rules (allowing, e.g., the introduction of a formula of the form $\neg(\varphi_1 \wedge \varphi_2)$), as well as the formulas in the premises. Calculi of this family are particularly useful for many-valued logics (e.g. for the relevance logic of first degree entailment [1]) and paraconsistent logics (particularly, for *C-systems* [15], [6]). Our investigation of these calculi is indirect: instead of studying the semantics of quasi-canonical calculi, we show how to translate each quasi-canonical calculus into an equivalent canonical one, and then utilize the results concerning canonical calculi. In particular, this translation is used to show that quasi-canonical calculi are all decidable. The idea behind this translation is to use sequents with more "sides" to encode the information related to the unary connectives that violate the canonicity requirements. We see this as a generalization of the original idea behind two-sided sequents in classical logic. Indeed, often one-sided sequents are translated to two-sided ones by differentiating the negated formulas from the non-negated ones by employing two different sides.

Basic Calculi

For various important propositional non-classical logics, such as modal logics and intuitionistic logic, there is no known (analytic or cut-free) pure calculus. Indeed, a major restriction in pure calculi is that unlimited context formulas may be used in all inference steps. Well-known sequent calculi for modal logics and intuitionistic logic do not meet this requirement, and thus they do not belong to the family of pure calculi. For example, consider the following schemes of rules written in the usual notation of two-sided sequents:

$$(1)\ \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \supset \varphi_2} \qquad (2)\ \frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad (3)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

These schemes demonstrate different possibilities regarding context formulas, and non of them can be presented as a rule of a pure calculus:

1. Scheme (1) allows only left context formulas and is employed in the multiple-conclusion sequent calculus for intuitionistic logic [33].

2. Scheme (2) again allows only left context formulas, but all of them should begin with $\Box$. This scheme is employed in the usual sequent calculus for the modal logic $S4$ [37].

3. Scheme (3) exhibits more complicated treatment of the context formulas: each formula φ on the left-hand side of the premise "becomes" $\Box\varphi$ in the conclusion. This scheme is employed in the usual sequent calculus for the modal logic K.

Basic calculi provide a general framework of sequent calculi, generalizing pure calculi, that allow certain context restrictions (including those demonstrated above). Unlike in the previous cases, we restrict our attention only to two-sided sequent calculi. Various sequent calculi that seem to have completely different natures can be directly presented as basic calculi. This includes all standard sequent calculi for modal logics, as well as the usual multiple-conclusion systems for intuitionistic logic, its dual, and bi-intuitionistic logic.

We carry out a general and uniform semantic study of these systems, that provide generalized Kripke-style semantics for them. As Kripke models, these semantic structures consist of a set of possible worlds and accessibility relations, and certain conditions connect the truth value assigned to a formula in each world w with values assigned to other formulas in the worlds accessible from w. The main idea is to formally differentiate between the *context* part and the *non-context* part of a rule application (see [34]), and separately analyze their semantic effects. It is shown that each syntactic ingredient imposes a certain constraint on Kripke models, that restricts the interpretation of the connectives and modalities or the properties of the accessibility relations. By taking all of these constraints together, we get a set of models for which the calculus is sound and complete. Furthermore, by using three or four truth values we characterize basic sequent systems with restricted cut rule and/or identity axiom. Then, we provide semantic criteria for the proof-theoretic properties of basic calculi, and demonstrate their usefulness in a variety of important cases. One interesting example is the (straightforward) application of these criteria to prove that a natural sequent system for bi-intuitionistic logic admits the subformula property. This answers a question raised in [27].[1]

Canonical Hypersequent Calculi

Hypersequents constitute a natural generalization of ordinary sequents, introduced independently by Pottinger [29] and Avron [3]. Hypersequents are defined to be finite sets (or multisets) of usual sequents (usually denoted as $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$), that intuitively stand for disjunction of sequents. This simple and straightforward modification significantly increases the expressive power of ordinary Gentzen apparatus, and shown to be very useful in building cut-free formalization of many non-classical logics, including modal, relevant, many-valued, and intermediate logics (see, e.g., [4, 10]). In particular, replacing ordinary sequents with hypersequents made possible obtaining different cut-free systems for the modal logic S5 (see, e.g., Poggiolesi [28]), as well as for Gödel logic (known also as Gödel-Dummett logic), that is perhaps the most prominent intermediate logic, and one of the three fundamental fuzzy logics (see, e.g., [24]). Avron [2] introduced a simple hypersequent calculus, called **HG**, for propositional Gödel logic, whose logical rules are practically the same rules as in **LJ**, the well-known single-conclusion sequent calculus for intuitionistic logic. In addition, this calculus includes the *communication rule* that allows "exchange of information" between two hypersequents [4]. More recently, hypersequents became the main proof-theoretic framework for fuzzy logics [26].

Our work on hypersequent calculi generalizes the investigation of the calculus **HG** for propositional Gödel logic. We import the ideas behind canonical sequent calculi to hypersequent calculi, and define a general structure of a canonical hypersequential logical rule. The idea, just like in canonical sequent calculi, is to allow any "ideal" logical rules for introducing the logical connectives. Here there are many options concerning the additional

[1] Note that the system presented in [27] does not admit cut-elimination. Cut-free calculi for bi-intuitionistic logic were devised in [22] and [27]. These systems do not employ the standard notion of a sequent, but more complicated data-structures.

hypersequential structural rules. To demonstrate our methods, we choose to study single-conclusion canonical hypersequent calculi that are based on the communication rule (of which the prototype example is **HG**).

As in the other cases, we study canonical Gödel calculi from a *semantic* point of view. First and foremost, our study includes a general method to obtain a sound and complete semantics for every canonical Gödel calculus. This semantics is based on totally ordered algebraic structures with (possibly) non-deterministic interpretations of the different connectives. Thus the truth value of each compound formula of the form $\diamond(\varphi_1, \dots, \varphi_n)$ must lie within a certain interval whose edges are computed from the values assigned to $\varphi_1, \dots, \varphi_n$. The logical rules of $\diamond$ induce the two functions (from $\mathcal{V}^n$ to $\mathcal{V}$, where $\mathcal{V}$ is the linearly ordered set of truth values) that determine these intervals. We provide a general construction of these functions given some (canonical) rules for introducing $\diamond$. In particular, we show that it is possible to augment Gödel logic with new non-deterministic connectives. For example, one may introduce a new connective ⩙, that combines the left introduction rule of disjunction and the right introduction rule of conjunction. The resulting semantics would force the truth value of a formula of the form $\varphi_1 ⩙ \varphi_2$ to lie between the minimum of the values of φ_1 and φ_2 and their maximum. When the truth values of φ_1 and φ_2 are different, this requires a non-deterministic choice.

We also consider the semantic effect of the cut rule and the identity axiom, and obtain semantics for canonical Gödel calculi in which these rules are restricted to apply only on some given set of formulas. For this matter, we introduce new semantic structures called *Gödel valuations*. In these structures, the valuation functions assigns *a pair* of truth values (from a linearly ordered set) to each formula of the propositional language. Intuitively, the first element in the pair of truth values assigned to some formula φ is used for occurrences of φ on the left sides of sequents, while the second element in this pair is used for occurrences of φ on the right sides. The cut rule and the identity axiom relate the two elements. Intuitively, the (cut) and the identity axiom (id) have opposite semantic roles – if (cut) is allowed on φ (i.e., φ may serve as a cut formula) then the left value of φ should be greater than or equal to the right value; and if (id) is allowed on φ (i.e., $\varphi \Rightarrow \varphi$ may serve as an initial hypersequent) then the left value of φ should be lower than or equal to the right value. If they are both available for some formula φ, then the two elements in the pair of truth values of φ must be equal. The usual algebraic semantics of Gödel logic is a particular instance, in which all of these pairs are degenerate, as well as the intervals that bound the interpretations of compound formulas. This semantics is then used to characterize proof-theoretic properties of canonical Gödel calculi, and particularly to identify the "good" ones, namely those that enjoy cut-admissibility. We further show that the simple *coherence* criterion of [8, 7] characterizes (strong) cut-admissibility in canonical Gödel calculi as well.

Hypersequent Calculi for First and Second Order Gödel Logic

So far we have discussed logics and calculi only at the propositional level. However, the ideas and methods described above are applicable for first-order and higher-order calculi as well. To demonstrate this, we further study two specific hypersequent calculi: a calculus for standard first-order Gödel logic [11], and its extension for Henkin-style second-order Gödel logic. This contribution is of a completely different nature, as it is devoted to particular calculi for particular logics.

First, we demonstrate the applicability of our approach for the hypersequent calculus **HIF** for standard first-order Gödel logic (standard means that the real interval $[0, 1]$ can be used as the underlying set of truth values).[2] **HIF**, introduced in [11], is obtained from **HG** (the original hypersequent calculus for propositional Gödel logic) by adding standard (hypersequential versions of) rules for the quantifiers $\forall$ and $\exists$. It was proved in [11] that **HIF** is sound and complete for standard first-order Gödel logic by showing its equivalence to an Hilbert system for this logic. Furthermore, it was shown in [11] that **HIF** admits cut-elimination. In fact, the first (syntactic) proof in [11] of cut-elimination was erroneous. A corrected (still syntactic) proof appear in [10]. As a corollary, one obtains Herbrand theorem for the prenex fragment of this logic [26]. Using the same technique developed for canonical hypersequent calculi, we obtain alternative semantic proofs for completeness and cut-admissibility. These proofs are tied together and involve two stages: (*i*) We present a non-deterministic semantics and show its completeness for the *cut-free* fragment of **HIF**; (*ii*) It is shown that from every non-deterministic counter-model, one can extract a usual counter-model. From these two facts together, it easily follows that **HIF** enjoys cut-admissibility, and that it is complete for standard first-order Gödel logic.

Next, we study $\mathbf{HIF}^2$, namely the extension of **HIF** with usual rules for *second-order* quantifiers. These are single-conclusion hypersequent version of the rules for introducing the second-order quantifiers in the ordinary sequent calculus for classical logic (see, e.g., [21, 33]). To the best of our knowledge, this system is studied in this thesis for the first time. Our main results is that $\mathbf{HIF}^2$ is sound and complete for second-order Gödel logic, and that the cut rule is admissible in $\mathbf{HIF}^2$. It should be noted that like in the case of second-order classical logic, the obtained calculus characterizes *Henkin-style* second-order Gödel logic. Thus second-order quantifiers range over a domain (of fuzzy sets) that is directly specified in the second-order structure, and this domain should admit full comprehension. This is in contrast to what

[2] Note that Gödel logic is the only fundamental fuzzy logic whose first-order version is recursively axiomatizable.

is called the *standard semantics*, where second-order quantifiers range over *all* subsets of the universe. Hence $\mathbf{HIF}^2$ is practically a system for two-sorted first-order Gödel logic together with the comprehension axiom (see also [13]).

Our approach in proving cut-admissibility for $\mathbf{HIF}^2$ is (of course) semantic, and it is similar to the one taken for **HIF**. Note that unlike in first-order calculi, usual syntactic arguments for cut-elimination dramatically fail for the rules of second-order quantifiers. Thus the first proof of cut-admissibility for the extension of **LK** (Gentzen's original sequent calculus for first-order classical logic) with rules for second-order quantifiers was also semantic, and, in fact, it also used non-deterministic semantics. Indeed, in the quest to verify *Takeuti's conjecture* [32] (that was open for several years) regarding cut-admissibility in the calculus for second-order classical logic,[3] Schütte developed a three-valued non-deterministic semantics for the cut-free fragment of this calculus [30]. This provided a semantic equivalent to Takeuti's conjecture, that was verified by Tait a few years later [31], when it was shown that it is possible to extract a usual (two-valued) counter-model from every three-valued non-deterministic Schütte's counter-model. As a simple consequence, one obtains that if there is no cut-free proof of a certain sequent, then there is no proof at all (see also [21]). Our proof of cut-admissibility for $\mathbf{HIF}^2$ (as well as the proof for **HIF**) has a similar basic general structure.

From a different perspective, our results concerning $\mathbf{HIF}^2$ provide initial steps in the proof-theoretic study of higher-order fuzzy logics. Fuzzy logics, and Gödel logic in particular, provide a reasonable model of certain very common vagueness phenomena. Both their propositional and first-order versions are well-studied by now (see, e.g., [24]). Clearly, for many interesting applications (see, e.g., [14] and Section 5.5.2 in Chapter I of [17]), propositional and first-order fuzzy logics do not suffice, and one has to use higher-order versions. These are much less developed (see, e.g., [36] and [17]), especially from the proof-theoretic point of view. We believe that a proof-theoretic study of higher-order fuzzy logics is a prerequisite for the development of automated deduction methods for these logics. Furthermore, a semantic approach to perform this study seems to be adequate and even necessary.

Publications Related to this Dissertation

Most of the contributions included in this thesis have been separately presented and published in conference proceedings and/or journals:

[3]More precisely, Takeuti's conjecture concerned full type-theory, namely, the completeness of the cut-free sequent calculus that includes rules for quantifiers of any finite order. However, the proof for second-order fragment was the main breakthrough.

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