From Frame Properties to Hypersequent Rules in Modal Logics

Ori Lahav

Tel Aviv University

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Ever since the introduction of relational semantics for modal logics, the assembling of a new modal logic for particular applications often begins by locating relevant frame properties.

Examples:

- \( K4 = K + \) transitivity
- \( S4 = K4 + \) reflexivity
- \( S4.3 = S4 + \) linearity
- \( KDBC_8 = K + \text{seriality} + \text{cardinality} \leq 8 \)

and many more...

Our goal is to uniformly obtain proof-theoretic characterizations for modal logics defined by frame properties.
Sequent Calculi

- There is a well-studied strong correspondence between frame properties and Hilbert-style systems (correspondence theory).
- Hilbert-style systems are hardly useful for proof-search and proof-theoretic investigations.
- On the other hand, Gentzen-style calculi are particularly suitable for these purposes.
There is a well-studied strong correspondence between frame properties and Hilbert-style systems (correspondence theory).

Hilbert-style systems are hardly useful for proof-search and proof-theoretic investigations.

On the other hand, Gentzen-style calculi are particularly suitable for these purposes.

Definition

A sequent is an ordered pair of finite set of formulas, denoted by

\[ \Gamma \Rightarrow \Delta. \]

Intuitively,

\[ \Gamma \Rightarrow \Delta \iff \bigwedge \Gamma \supset \bigvee \Delta. \]
Gentzen Calculi for Modal Logics

The calculus $G_K$

\begin{align*}
(IW \Rightarrow) & \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} & (\Rightarrow IW) & \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \\
(id) & \frac{A}{A \Rightarrow A} & (cut) & \frac{\Gamma \Rightarrow A, \Delta, \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \\
(\supset \Rightarrow) & \frac{\Gamma \Rightarrow \Delta, A_1, \Gamma, A_2 \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A_1 \supset A_2 \Rightarrow \Delta} & (\Rightarrow \supset) & \frac{\Gamma, A_1 \Rightarrow A_2, \Delta}{\Gamma \Rightarrow A_1 \supset A_2, \Delta} \\
(\bot \Rightarrow) & \frac{\bot}{\bot \Rightarrow} & (\Rightarrow \Box) & \frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A}
\end{align*}

Facts

1. $A$ is a theorem of $K$ iff $\Rightarrow A$ is provable in $G_K$.
2. $(cut)$ is admissible.
Hypersequent Calculi

1. For many important modal logics (e.g. $\textbf{S}5 = \text{universal accessibility relation}$) there is no known (cut-free) sequent calculus.

2. It is possible to characterize $\textbf{S}5$ by going “one step further” — from sequents to hypersequents [Pottinger ‘83, Avron ‘87].

**Definition**

A *hypersequent* is a finite set of sequents, denoted by

$$
\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.
$$

Intuitively, a hypersequent is a disjunction of sequents:

$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is true in some Kripke model if *some* $\Gamma_i \Rightarrow \Delta_i$ is true in all worlds.
Hypersequent Calculus for $\text{S}5$

\[
(IW \Rightarrow) \quad \frac{H | \Gamma \Rightarrow \Delta}{H | \Gamma, A \Rightarrow \Delta} \quad (\Rightarrow IW) \quad \frac{H | \Gamma \Rightarrow \Delta}{H | \Gamma \Rightarrow \Delta, A} \quad (EW) \quad \frac{H}{H | \Gamma \Rightarrow \Delta}
\]

\[
(id) \quad \frac{H | A \Rightarrow A}{H | \Gamma \Rightarrow A \Rightarrow A} \quad (cut) \quad \frac{H | \Gamma \Rightarrow A, \Delta}{H | \Gamma \Rightarrow \Delta}
\]

\[
(\Rightarrow \Rightarrow) \quad \frac{H | \Gamma \Rightarrow \Delta, A_1 \quad H | \Gamma, A_2 \Rightarrow \Delta}{H | \Gamma, A_1 \Rightarrow \Delta \quad A_2 \Rightarrow \Delta} \quad (\Rightarrow \Rightarrow) \quad \frac{H | \Gamma, A_1 \Rightarrow A_2, \Delta}{H | \Gamma \Rightarrow A_1 \Rightarrow A_2, \Delta}
\]

\[
(\perp \Rightarrow) \quad \frac{H | \perp \Rightarrow}{H | \Gamma \Rightarrow \perp} \quad (\Rightarrow \Box) \quad \frac{H | \Gamma \Rightarrow A}{H | \Box \Gamma \Rightarrow \Box A} \quad (S5) \quad \frac{H | \Gamma, \Gamma' \Rightarrow \Delta}{H | \Box \Gamma' \Rightarrow \Gamma \Rightarrow \Delta}
\]

The calculus $\text{HG}_{\text{S}5}$

**Facts**

1. $A$ is a theorem of $\text{S}5$ iff $\Rightarrow A$ is provable in $\text{HG}_{\text{S}5}$.
2. $(\text{cut})$ is admissible.
Derivation of $\Rightarrow \Diamond A \supset \Box \Diamond A$ in $HG_{S5}$

\[
\begin{array}{c}
\frac{A \supset \bot \Rightarrow A \supset \bot}{\Rightarrow A \supset \bot \ | \ \Box (A \supset \bot) \Rightarrow} \quad (id) \\
\frac{\Rightarrow A \supset \bot \ | \ \Box (A \supset \bot) \Rightarrow}{\Rightarrow \Box (A \supset \bot) \ | \ \Box (A \supset \bot) \Rightarrow} \quad (S5) \\
\vdots \\
\frac{\Box (A \supset \bot) \supset \bot \Rightarrow \Rightarrow \Box (A \supset \bot) \supset \bot}{\Box (A \supset \bot) \supset \bot \Rightarrow \Rightarrow \Box (\Box (A \supset \bot)) \supset \bot} \quad (\Rightarrow \Box) \\
\vdots \\
\frac{\Box (A \supset \bot) \supset \bot \Rightarrow \Rightarrow \Box (\Box (A \supset \bot)) \supset \bot}{\Box (A \supset \bot) \supset \bot \Rightarrow \Rightarrow \Box (\Box (A \supset \bot)) \supset \bot} \quad (\Rightarrow \Box) \\
\Rightarrow (\Box (A \supset \bot) \supset \bot) \supset \Box (\Box (A \supset \bot)) \supset \bot) \quad (\Rightarrow \Box)
\end{array}
\]
Main Contribution

Questions

- What is the full power of hypersequent calculi for modal logics?
- What frame properties can be characterized?

(Partial) Answers:

- We recognize a class of frame properties, called *simple frame properties*, for which it is possible to construct a hypersequent calculus.
- We provide the method to construct these calculi, and uniformly prove soundness and completeness, and cut-admissibility.
Remark

There are other proof-theoretical frameworks for modal logics. E.g.:

- Semantic tableaus
- Display calculi
- Tree-hypersequent calculi and nested sequent calculi
- Labelled calculi

We are interested in (fully-structural) hypersequent calculi:

- Very close to Gentzen’s approach.
- Kripke semantics is not explicitly involved in derivations.
- Useful for many other logics of different natures.
- Decidability follows from cut-admissibility (in the propositional level).
Simple Frame Properties

- We use classical first-order language to formulate the frame properties.
  - For example, $\forall w. wRw$ captures reflexivity.

**Simple frame properties** are formulated by formulas of the form

$$\forall w_1 \ldots \forall w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_i Ru$ or $w_i = u$.
- Conjunctions and disjunctions.

- Reflexivity is simple:
  $$\forall w_1 \exists u (w_1 Ru \land w_1 = u)$$
Examples

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td><strong>Seriality</strong></td>
<td>$\forall w_1 \exists u \ (w_1 Ru)$</td>
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<tr>
<td><strong>Directedness</strong></td>
<td>$\forall w_1 \forall w_2 \exists u \ (w_1 Ru) \land (w_2 Ru)$</td>
</tr>
<tr>
<td><strong>Degenerateness</strong></td>
<td>$\forall w_1 \forall w_2 \exists u \ (w_1 = u \land w_2 = u)$</td>
</tr>
<tr>
<td><strong>Universality</strong></td>
<td>$\forall w_1 \forall w_2 \exists u \ (w_1 Ru \land w_2 = u)$</td>
</tr>
<tr>
<td><strong>Linearity</strong></td>
<td>$\forall w_1 \forall w_2 \exists u \ (w_1 Ru \land w_2 = u) \lor (w_2 Ru \land w_1 = u)$</td>
</tr>
<tr>
<td><strong>Bounded Cardinality</strong></td>
<td>$\forall w_1 \ldots \forall w_n \exists u \ \bigvee_{1 \leq i &lt; j \leq n} (w_i = u \land w_j = u)$</td>
</tr>
<tr>
<td><strong>Bounded Top Width</strong></td>
<td>$\forall w_1 \ldots \forall w_n \exists u \ \bigvee_{1 \leq i &lt; j \leq n} (w_i Ru \land w_j Ru)$</td>
</tr>
<tr>
<td><strong>Bounded Width</strong></td>
<td>$\forall w_1 \ldots \forall w_n \exists u \ \bigvee_{1 \leq i, j \leq n; i \neq j} (w_i Ru \land w_j = u)$</td>
</tr>
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</table>
From Simple Frame Properties to Hypersequent Rules

(1) Extract the normal form of $\forall w_1 \ldots \forall w_n \exists u \varphi$

A set $\{\langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle\}$ such that

$$\varphi \equiv \bigvee_{1 \leq i \leq m} \left( \bigwedge_{j \in R_i} w_j Ru \land \bigwedge_{j \in E_i} w_j = u \right)$$

$$\forall w_1 \forall w_2 \exists u (w_1 Ru) \land (w_2 Ru) \quad \{\langle\{1, 2\}, \emptyset\rangle\}$$

$$\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u) \quad \{\langle\{1\}, \{2\}\rangle\}$$

$$\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u) \lor (w_2 Ru \land w_1 = u) \quad \{\langle\{1\}, \{2\}\rangle, \langle\{2\}, \{1\}\rangle\}$$

$$\forall w_1 \ldots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \land w_j = u) \quad \{\langle\emptyset, \{i, j\}\rangle \mid 1 \leq i < j \leq n\}$$
(2) For a normal form \( \{ \langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle \} \) construct the following rule and add it to \( \mathcal{HG}_K \):

\[
\begin{align*}
\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \ldots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H \mid \Gamma_1, \Box \Gamma'_{1} \Rightarrow \Delta_1 \quad \ldots \quad H \mid \Gamma_n, \Box \Gamma'_{n} \Rightarrow \Delta_n}
\end{align*}
\]

Notation: \( \prod_{\{i_1, \ldots, i_k\}} := \prod_{i_1, \ldots, \prod_{i_k}} \)
(2) For a normal form \( \{\langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle\} \) construct the following rule and add it to \( \mathbf{HG}_K \):

\[
\frac{H | \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \ldots H | \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H | \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n, \Box \Gamma'_n \Rightarrow \Delta_n}
\]

Notation: \( \Pi_{\{i_1, \ldots, i_k\}} := \Pi i_1, \ldots, \Pi i_k \)

In the presence of the weakening rules, \( \Gamma_i, \Gamma'_i, \Delta_i \)'s that appear only in the conclusion can be discarded.

\[
\frac{H | \Gamma'_1, \Gamma'_2 \Rightarrow}{H | \Box \Gamma'_1 \Rightarrow | \Box \Gamma'_2 \Rightarrow}
\text{Directedness}
\]

\[
\frac{H | \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2}{H | \Box \Gamma'_1 \Rightarrow | \Gamma_2 \Rightarrow \Delta_2}
\text{Universality}
\]

\[
\frac{H | \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2 \quad H | \Gamma_1, \Gamma'_2 \Rightarrow \Delta_1}{H | \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1 | \Gamma_2, \Box \Gamma'_2 \Rightarrow \Delta_2}
\text{Linearity}
\]

\[
\left\{H | \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j \mid 1 \leq i < j \leq n\right\}
\frac{H | \Gamma_1, \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n}{H | \Gamma_1, \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n}
\text{Bounded Cardinality}
\]
Main Result

<table>
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<th>Theorem</th>
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| **Theorem**  
The constructed hypersequent calculus is *sound and complete* for the modal logic, and it enjoys *cut-admissibility*. |

- Uniform semantic proof for all calculi of this form.
Main Result

**Theorem**

The constructed hypersequent calculus is sound and complete for the modal logic, and it enjoys cut-admissibility.

- Uniform semantic proof for all calculi of this form.

**Strong Soundness and Completeness**

\[
\Gamma \vdash_{Local} A \quad \text{A holds in every world in which } \Gamma \text{ holds} \\
\Gamma \vdash_{Global} A \quad \text{A holds in every world if } \Gamma \text{ holds in every world} \\
\{ \Rightarrow B \mid B \in \Gamma \} \vdash \Rightarrow A
\]

**Strong Cut-Admissibility**

(cut) can be confined to apply only on formulas that appear in the assumptions.
Decidability

Corollary

All modal logics characterized by finite sets of simple frame properties are decidable.

Proof.

Cut-admissibility $\rightarrow$ Subformula property $\rightarrow$

We can check one by one all possible proofs candidates.
Simple frame properties are formulated by formulas of the form

$$\forall w_1 \ldots \forall w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_i Ru$ or $w_i = u$.
- Conjunctions and disjunctions.

- Simple properties are *monotone increasing* (preserved under enrichment of $R$).
- Transitivity and symmetry are not simple.
Transitivity and Symmetry

**Simple frame properties** are formulated by formulas of the form

\[ \forall w_1 \ldots \forall w_n \exists u \varphi \]

where \( \varphi \) consists of:

- Atomic formulas of the form \( w_iRu \) or \( w_i = u \).
- Conjunctions and disjunctions.

- Simple properties are **monotone increasing** (preserved under enrichment of \( R \)).
- Transitivity and symmetry are not simple.
- We have to change the basic calculus:

\[
\begin{align*}
H \mid \Gamma & \Rightarrow A \\
\frac{H \mid \Box \Gamma \Rightarrow \Box A}{H \mid \Box \Gamma \Rightarrow \Box A} & \text{K} \\
\frac{H \mid \Gamma, \Box \Gamma \Rightarrow A}{H \mid \Box \Gamma \Rightarrow \Box A} & \text{K4} \\
\frac{H \mid \Gamma \Rightarrow A, \Box \Delta}{H \mid \Box \Gamma \Rightarrow \Box A, \Delta} & \text{KB}
\end{align*}
\]
Transitivity

For a normal form \{\langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle \} construct the rule:

\[
\begin{align*}
H \mid \Gamma_{E_1}, \Gamma'_{R_1}, \Box \Gamma'_{R_1} &\Rightarrow \Delta_{E_1} & \ldots & \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m}, \Box \Gamma'_{R_m} &\Rightarrow \Delta_{E_m} \\
H \mid \Gamma_1, \Box \Gamma'_{1} &\Rightarrow \Delta_1 & \ldots & \quad \Gamma_n, \Box \Gamma'_{n} &\Rightarrow \Delta_n
\end{align*}
\]

For example:

\[
\begin{align*}
H \mid \Gamma_{2}, \Gamma'_{1}, \Box \Gamma'_{1} &\Rightarrow \Delta_2 & H \mid \Gamma_{1}, \Gamma'_{2}, \Box \Gamma'_{2} &\Rightarrow \Delta_1 \\
H \mid \Gamma_1, \Box \Gamma'_{1} &\Rightarrow \Delta_1 & \Gamma_2, \Box \Gamma'_{2} &\Rightarrow \Delta_2
\end{align*}
\]

Linearity

Again, we show:

- Strong soundness and completeness.
- Strong cut-admissibility.
- Decidability.
Symmetry

For a normal form \( \{ \langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle \} \) construct the rule:

\[
\begin{align*}
H \mid \Gamma E_1, \Gamma' R_1 &\Rightarrow \Delta E_1, \Box \Delta' R_1 \\
&\ldots \\
H \mid \Gamma E_m, \Gamma' R_m &\Rightarrow \Delta E_m, \Box \Delta' R_m \\
H \mid \Gamma_1, \Box \Gamma' &\Rightarrow \Delta_1, \Delta'_1 \\
&\ldots \\
H \mid \Gamma_n, \Box \Gamma' &\Rightarrow \Delta_n, \Delta'_n
\end{align*}
\]

For example:

\[
\begin{align*}
H \mid \Gamma' &\Rightarrow \Box \Delta'_1 \\
H \mid \Box \Gamma' &\Rightarrow \Delta'_1 \\
H \mid \Gamma_1, \Gamma' &\Rightarrow \Delta_1, \Box \Delta'_1 \\
H \mid \Gamma_1, \Box \Gamma' &\Rightarrow \Delta_1, \Delta'_1
\end{align*}
\]

- Cut-admissibility does not hold (even for the basic calculus).
- All constructed calculi still enjoy the subformula property.
- Decidability still follows.
Conclusions and Further Research

- **Correspondence** between Kripke semantics and Gentzen-type calculi:
  
  \[ \text{simple frame property} \Leftrightarrow \text{simple hypersequent rule} \]

- Well-behaved Gentzen-type calculi can be constructed for all (transitive) (symmetric) modal logics characterized by simple frame properties.
  
  *E.g.* \( \text{KT, KD, S4, S5, K4D, K4.2, K4.3, S4.3, KBD, KBT, KBC}_n, \text{KBW}_n, \text{KBTW}_n \).

- These calculi may be helpful for investigating and using the logics (e.g. decidability).

**Further work and open questions:**

- Proof-search.
- Multi-modal logics.
- Non-simple properties. *E.g.* \( \text{K5} \) (euclidean), \( \text{BD}_n \) (bounded depth).
- Negative results?

Thank you!