Non-deterministic Matrices for Semi-canonical Deduction Systems

Ori Lahav

The Blavatnik School of Computer Science
Tel Aviv University

IEEE 42nd International Symposium on Multiple-Valued Logic ISMVL-2012
May 14-16, Victoria, BC, Canada
Using Multiple Values for Characterizing Sequent Systems
Sequent Systems

- **Sequent systems** are formal calculi that manipulate sequents.

- **Sequents** are objects of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite *sets* of formulas.

Intuition:

$$\Gamma \Rightarrow \Delta \quad \iff \quad \bigwedge \Gamma \supset \bigvee \Delta$$
# LK [Gentzen 1934]

## Identity Rules

### (id)

\[ A \Rightarrow A \]

### (cut)

\[
\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}
\]

## Weakening Rules

### (W ⇒)

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}
\]

### (⇒ W)

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}
\]

## Logical Rules

### (⊃ ⇒)

\[
\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}
\]

### (⇒ ⊃)

\[
\frac{\Gamma \Rightarrow A \supset B, \Delta}{\Gamma \Rightarrow A, \Delta}
\]

### (∧ ⇒)

\[
\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta}
\]

### (⇒ ∧)

\[
\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \land B, \Delta}
\]
## LK [Gentzen 1934]

### Identity Rules

**Identity Rule** (id) \[ A \vdash A \]

**Cut Rule** (cut) \[
\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}
\]

### Weakening Rules

**Weakening Rule** (W ⇒) \[
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}
\]

**Weakening Rule** (⇒ W) \[
\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}
\]

### Logical Rules

**Logical Implication** (⊃ ⇒) \[
\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}
\]

**Logical Implication** (⇒ ⊃) \[
\frac{\Gamma \vdash A \supset B, \Delta}{\Gamma \vdash A \vdash B, \Delta}
\]

**Logical Conjunction** (∧ ⇒) \[
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}
\]

**Logical Conjunction** (⇒ ∧) \[
\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta}
\]

### Theorem

\[ \mathcal{T} \vdash_{cl} A \iff \{ \vdash B \mid B \in \mathcal{T} \} \vdash_{LK} \Rightarrow A \]
Logical Rules

(⊃ ⇒) \[ \Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \]
\[ \Gamma, A \supset B \Rightarrow \Delta \]

(⇒ ⊃) \[ \Gamma, A \Rightarrow B, \Delta \]
\[ \Gamma \Rightarrow A \supset B, \Delta \]

(∧ ⇒) \[ \Gamma, A, B \Rightarrow \Delta \]
\[ \Gamma, A \land B \Rightarrow \Delta \]

(⇒ ∧) \[ \Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta \]
\[ \Gamma \Rightarrow A \land B, \Delta \]

Logical rules of an “ideal” type:

- Exactly one connective is introduced
- The active formulas of the premises are immediate subformulas of the principal formula
- Context-independence
Logical Rules

\[
\begin{align*}
\to \Rightarrow) & : \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \Delta \Rightarrow \Gamma} \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \\
\land \Rightarrow) & : \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, \Delta \Rightarrow \Gamma} \quad \frac{\Gamma, A \land B \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta \land B, \Delta}
\end{align*}
\]

Logical rules of an “ideal” type:

- Exactly one connective is introduced
- The active formulas of the premises are *immediate subformulas* of the principal formula
- Context-independence

\[
\begin{align*}
\rightarrow \Rightarrow) & : \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \\
\Rightarrow \rightarrow) & : \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}
\end{align*}
\]
A Canonical System =

The two identity rules
+

The two weakening rules
+

A (finite) set of canonical rules

Prominent Examples:

- LK
- Primal Logic [Gurevich, Neeman 2009] (\(\sim\) instead of \(\supset\))
A Canonical System $=$

The two identity rules

+ 

The two weakening rules

+ 

A (finite) set of canonical rules

Prominent Examples:

- LK
- Primal Logic [Gurevich, Neeman 2009] ($\leadsto$ instead of $\supset$)

**Theorem (Avron, Lev 2001)**

*Every (consistent) canonical system has two-valued non-deterministic semantics.*
Non-deterministic Matrices by Example

\[ \land \begin{array}{c|cc}
\land & F & T \\
F & F & F \\
T & F & T \\
\end{array} \]

\[ \nu(A \land B) = \land(\nu(A), \nu(B)) \]
Non-deterministic Matrices by Example

\[ \nu(A \land B) = \land(\nu(A), \nu(B)) \]

<table>
<thead>
<tr>
<th>∧</th>
<th>F</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Diagram:

- Inputs: `in_1`, `in_2`, `in_3`
- Operations: `\land`, `\lor`
- Outputs: `out_1`, `out_2`
\[ \nu(A \land B) = \land(\nu(A), \nu(B)) \]
Non-deterministic Matrices by Example

\[
\begin{array}{c|cc}
\wedge & F & T \\
\hline
F & F & F \\
T & F & T \\
\end{array}
\]

\[
\begin{array}{c|cc}
\wedge & F & T \\
\hline
F & \{F\} & \{F, T\} \\
T & \{F, T\} & \{T\} \\
\end{array}
\]

\[
v(A \land B) = \land(v(A), v(B))
\]

\[
v(A \land B) \in \land(v(A), v(B))
\]
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.
Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{align*}
\Gamma & \Rightarrow A, \Delta & \Gamma, B & \Rightarrow \Delta \\
\Gamma, A \supset B & \Rightarrow \Delta \\
\Gamma, A & \Rightarrow B, \Delta \\
\Gamma & \Rightarrow A \supset B, \Delta
\end{align*}
\]
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma, A \supset B \Rightarrow \Delta \\
\Gamma, A \Rightarrow B, \Delta \\
\hline
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\supset & F & T \\
\hline
\Gamma & \{F, T\} & \{F, T\} \\
\hline
\end{array}
\]
Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\Gamma, A \supset B \Rightarrow \Delta \\
\Gamma, A \Rightarrow B, \Delta \\
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

<table>
<thead>
<tr>
<th>\varnothing</th>
<th>F</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>{F, T}</td>
<td>{F, T}</td>
</tr>
<tr>
<td>T</td>
<td>{F, T}</td>
<td>{F, T}</td>
</tr>
</tbody>
</table>
Every (consistent) canonical system has two-valued non-deterministic matrix.
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{align*}
\Gamma \Rightarrow A, \Delta & \quad \Gamma, B \Rightarrow \Delta \\
\Gamma, A \supset B & \Rightarrow \Delta \\
\Gamma, A \Rightarrow B, \Delta & \\
\Gamma & \Rightarrow A \supset B, \Delta
\end{align*}
\]

<table>
<thead>
<tr>
<th>⊃</th>
<th>F</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>{F, T}</td>
<td>{F, T}</td>
</tr>
<tr>
<td>T</td>
<td>{F, T}</td>
<td>{F, T}</td>
</tr>
</tbody>
</table>
Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{align*}
\Gamma \Rightarrow A, \Delta & \quad \Gamma, B \Rightarrow \Delta \\
\Gamma, A \supset B \Rightarrow \Delta \\
\Gamma, A \Rightarrow B, \Delta & \quad \Gamma \Rightarrow A \supset B, \Delta
\end{align*}
\]

\[
\begin{array}{c|cc}
\supset & F & T \\
\hline 
F & \{F, T\} & \{F, T\} \\
T & \{F, T\} & \{F, T\}
\end{array}
\]

\[
\begin{array}{c|cc}
\supset & F & T \\
\hline 
F & \{T\} & \{T\} \\
T & \{F\} & \{T\}
\end{array}
\]
Non-deterministic Matrices for Canonical Systems

**Theorem (Avron, Lev 2001)**

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma, A \supset B \Rightarrow \Delta \\

\Gamma, A \Rightarrow B, \Delta \\
\hline
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

\[
\begin{array}{c|cc}
\supset & F & T \\
\hline
F & \{F, T\} & \{F, T\} \\
T & \{F, T\} & \{F, T\}
\end{array}
\]

\[
\begin{array}{c|cc}
\supset & F & T \\
\hline
F & \{T\} & \{T\} \\
T & \{F\} & \{T\}
\end{array}
\]
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma, A \supset B \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma \Rightarrow A \bowtie B \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\sim \Rightarrow F \\ T
\end{array}
\]

\[
\begin{array}{c|c|c}
\supset & F & T \\
F & \{F, T\} & \{F, T\} \\
T & \{F, T\} & \{F, T\} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\bowtie & F & T \\
F & \{F, T\} & \{F, T\} \\
T & \{F, T\} & \{F, T\} \\
\end{array}
\]
Non-deterministic Matrices for Canonical Systems

Theorem (Avron, Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic matrix.

\[
\begin{array}{c}
\Gamma \nRightarrow A, \Delta & \Gamma, B \nRightarrow \Delta \\
\hline
\Gamma, A \supset B \nRightarrow \Delta \\
\Gamma, A \Rightarrow B, \Delta & \Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

\[
\begin{array}{ccc}
\nRightarrow & F & T \\
F & \{F, T\} & \{F, T\} \\
T & \{F, T\} & \{F, T\}
\end{array}
\]

\[
\begin{array}{ccc}
\nRightarrow & F & T \\
F & \{T\} & \{T\} \\
T & \{F\} & \{T\}
\end{array}
\]
What is the Role of the Identity Rules?

<table>
<thead>
<tr>
<th>Identity Rules</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(id)</strong> A ⇒ A</td>
</tr>
</tbody>
</table>
| **(cut)** \[
\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}
\] |

A Canonical System = The two identity rules + The two weakening rules + A (finite) set of canonical rules

A Semi-canonical System = Any subset of the identity rules + The two weakening rules + A (finite) set of canonical rules

Discarding (cut) is useful for the study of cut-admissibility

Discarding (id) is useful for proof-theoretic analysis of logic [Hölsli, Jäger 1994]
What is the Role of the Identity Rules?

**Identity Rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(id)</td>
<td>$A \Rightarrow A$</td>
</tr>
<tr>
<td>(cut)</td>
<td>$\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta$</td>
</tr>
</tbody>
</table>

**A Canonical System =**

- The two identity rules
- The two weakening rules
- A (finite) set of canonical rules

Discarding (cut) is useful for the study of cut-admissibility

Discarding (id) is useful for proof-theoretic analysis of logic

[H"osli, J"ager 1994]
What is the Role of the Identity Rules?

<table>
<thead>
<tr>
<th>Identity Rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>((id)) (A \Rightarrow A)</td>
</tr>
</tbody>
</table>

A Canonical System =
- The two identity rules
- The two weakening rules
- A (finite) set of canonical rules

A Semi-canonical System =
- Any subset of the identity rules
- The two weakening rules
- A (finite) set of canonical rules

Discarding (cut) is useful for the study of cut-admissibility
Discarding (id) is useful for proof-theoretic analysis of logic [H"osli, J"ager 1994]
What is the Role of the Identity Rules?

**Identity Rules**

\[(id) \quad A \Rightarrow A\]

\[(cut) \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}\]

**A Canonical System** =
- The two identity rules
- The two weakening rules
- A (finite) set of canonical rules

**A Semi-canonical System** =
- Any subset of the identity rules
- The two weakening rules
- A (finite) set of canonical rules

- Discarding (cut) is useful for the study of cut-admissibility
- Discarding (id) is useful for proof-theoretic analysis of logic programming [Hösli, Jäger 1994]
Semantics of Semi-canonical Systems

Identity Rules

(id) \[ A \Rightarrow A \]  
(cut) \[ \begin{array}{c} A \Rightarrow \\ \Rightarrow \end{array} A \]  

- Using only \( T \) and \( F \):
  - Every valuation is a model of \( A \Rightarrow A \)
  - No valuation is a model of both \( A \Rightarrow \) and \( \Rightarrow A \)
Semantics of Semi-canonical Systems

### Identity Rules

<table>
<thead>
<tr>
<th>(id)</th>
<th>(cut)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \Rightarrow A$</td>
<td>$A \Rightarrow \Rightarrow A$</td>
</tr>
</tbody>
</table>

- Using only $T$ and $F$:
  - Every valuation is model of $A \Rightarrow A$
  - No valuation is a model both of $A \Rightarrow$ and of $\Rightarrow A$

- Idea:
  - Without (id) - add a value $\bot$
    
    $$v(A) = \bot \iff v \text{ is not a model of } A \Rightarrow A$$

  - Without (cut) - add a value $\top$
    
    $$v(A) = \top \iff v \text{ is a model of both } A \Rightarrow \text{ and } \Rightarrow A$$
The Truth-Values

\[ v(\text{A}) \geq f \text{ for some } A \in \Gamma \text{ or } v(\text{A}) \geq t \text{ for some } A \in \Delta \]

\[ v(\text{A}) = \bot \iff v \text{ is not a model of } A \]

\[ v(\text{A}) = \top \iff v \text{ is a model of both } A \text{ and } A \]

(id) = \text{omit} \top = \text{omit}
The Truth-Values

\[ \begin{array}{c|c|c}
F & T & \top \\
\top & \bot & F \\
\bot & \bot & \bot \\
\end{array} \]

\( \nu \) is a model of \( \Gamma \Rightarrow \Delta \) if \( \nu(A) \geq F \) for some \( A \in \Gamma \) or \( \nu(A) \geq T \) for some \( A \in \Delta \).
### The Truth-Values

<table>
<thead>
<tr>
<th>$\top$</th>
<th>$\bot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bot$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$\bot$</td>
</tr>
</tbody>
</table>

$v$ is a model of $\Gamma \Rightarrow \Delta$ if $v(A) \geq F$ for some $A \in \Gamma$ or $v(A) \geq T$ for some $A \in \Delta$

\[
\begin{align*}
v(A) = \bot & \iff v \text{ is not a model of } A \Rightarrow A \\
v(A) = T & \iff v \text{ is a model of both } A \Rightarrow \text{and } \Rightarrow A
\end{align*}
\]
Semantics of Semi-canonical Systems

The Truth-Values

\[ \begin{array}{c}
F \\
\perp \\
T
\end{array} \]

\( \nu \) is a model of \( \Gamma \Rightarrow \Delta \) if \( \nu(A) \geq F \) for some \( A \in \Gamma \) or \( \nu(A) \geq T \) for some \( A \in \Delta \)

\( \nu(A) = \perp \iff \nu \) is not a model of \( A \Rightarrow A \)

\( \nu(A) = T \iff \nu \) is a model of both \( A \Rightarrow \) and \( \Rightarrow A \)

(id) \( \Rightarrow \) omit \( \perp \)

(cut) \( \Rightarrow \) omit \( T \)
What are the Truth-Tables?
What are the Truth-Tables?

Theorem

Some simple semi-canonical systems (e.g. $LK - (id)$ or $LK - (cut)$) do not have a finite (ordinary) matrix.
What are the Truth-Tables?

Theorem

Some simple semi-canonical systems (e.g. LK − (id) or LK − (cut)) do not have a finite (ordinary) matrix.

Theorem

Every (consistent) semi-canonical system has a three or four valued non-deterministic matrix.

In fact, we provide a method to obtain such a matrix.
Example: Implication without (id) and (cut)

\[
\begin{align*}
\Gamma \Rightarrow A, \Delta & \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma, A \supset B & \Rightarrow \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma, A \Rightarrow B, \Delta & \quad \Gamma \Rightarrow \Delta
\end{align*}
\]
Example: Implication without (id) and (cut)

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \\
\Gamma, B \Rightarrow \Delta \\
\hline
\Gamma, A \supset B \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma, A \Rightarrow B, \Delta \\
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

<table>
<thead>
<tr>
<th>⊃</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>⊤</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>⊤</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
</tbody>
</table>
Example: Implication without (id) and (cut)

\[
\Gamma \Rightarrow A, \Delta \\ \Gamma, B \Rightarrow \Delta \\
\frac{}{\Gamma, A \supset B \Rightarrow \Delta}
\]

\[
\frac{}{\Gamma, A \Rightarrow B, \Delta} \\
\frac{}{\Gamma \Rightarrow A \supset B, \Delta}
\]

<table>
<thead>
<tr>
<th>⊢</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>⊤</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>⊤</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
</tbody>
</table>
Example: Implication without (id) and (cut)

\[
\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}
\]

\[
\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}
\]

<table>
<thead>
<tr>
<th>⊃</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>⊤</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, T}</td>
<td>\⊥, F, T, T }</td>
<td>{⊥, F, T, T}</td>
<td>\⊥, F, T, T }</td>
</tr>
<tr>
<td>⊤</td>
<td>{⊥, F, T, T}</td>
<td>\⊥, F, T, T }</td>
<td>{⊥, F, T, T}</td>
<td>\⊥, F, T, T }</td>
</tr>
</tbody>
</table>
Example: Implication without (id) and (cut)

\[
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}
\]

\[
\Gamma, A \Rightarrow B, \Delta \quad \Gamma \Rightarrow A \supset B, \Delta
\]

<table>
<thead>
<tr>
<th>⊃</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>√</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
<tr>
<td>√</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
<td>{⊥, F, T, T}</td>
</tr>
</tbody>
</table>
Example: Implication without (id) and (cut)

\[
\begin{array}{c}
\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta \\
\hline
\Gamma \Rightarrow A \supset B \Rightarrow \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma, A \Rightarrow B, \Delta \\
\hline
\Gamma \Rightarrow A \supset B, \Delta
\end{array}
\]

<table>
<thead>
<tr>
<th>⊃</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>⊤</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>⊤</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
</tbody>
</table>
Example: Implication without (id) and (cut)

\[
\begin{align*}
\Gamma &\Rightarrow A, \Delta & \Gamma, B &\Rightarrow \Delta \\
\hline
\Gamma, A &\supset B &\Rightarrow \Delta \\
\Gamma, A &\Rightarrow B, \Delta
\end{align*}
\]

<table>
<thead>
<tr>
<th>⊤</th>
<th>⊥</th>
<th>F</th>
<th>T</th>
<th>⊤</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊥</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>F</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>T</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
<tr>
<td>⊤</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
<td>{⊥, F, T, ⊤}</td>
</tr>
</tbody>
</table>
Semantics of (id)-free and/or (cut)-free systems makes it possible to obtain simple semantic proofs of proof-theoretic properties, e.g.:

1. Cut-admissibility

\[ \vdash \Gamma \Rightarrow \Delta \quad \implies \quad \vdash^{cf} \Gamma \Rightarrow \Delta \]

2. Axiom Expansion

\[ A \Rightarrow A ; \ B \Rightarrow B \quad \vdash \quad A \And B \Rightarrow A \And B \]
Related Works

- [Hösli, Jäger 1994] - semantics of $LK - (id)$.

Our framework is a **unified** approach: both logics can be defined by finite valued non-deterministic matrices.
Conclusions and Extensions

- We provided simple and modular semantic characterization for a natural family of sequent calculi.
- Two essential components: multiple values non-determinism
- Application: semantic proofs of proof-theoretic properties
Conclusions and Extensions

We provided simple and **modular** semantic characterization for a natural family of sequent calculi.

Two essential components:
- **multiple values**
- **non-determinism**

Application: semantic proofs of proof-theoretic properties

Similar ideas can be applied for:
- Many-sided systems
- Sequent systems for intuitionistic logic and modal logics
Thank you!