Non-deterministic Matrices for Semi-canonical Deduction Systems

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Using Multiple Values for Characterizing Sequent Systems

Sequent Systems

- Sequent systems are formal calculi that manipulate sequents.
- Sequents are objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *sets* of formulas.

  Intuition:

$$\Gamma \Rightarrow \Delta \qquad \leftrightsquigarrow \qquad \bigwedge \Gamma \supset \bigvee \Delta$$

LK [Gentzen 1934]

Identity Rules

$(id) \quad A \Rightarrow A$

$$(cut) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

Weakening Rules

$$(W \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Logical Rules

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset) \quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

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Theorem

$$\mathcal{T} \vdash_{cl} A \;\; iff \;\; \{ \Rightarrow B \mid B \in \mathcal{T}\} \vdash_{LK} \Rightarrow A$$

Canonical Rules [Avron,Lev 2001]

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Logical rules of an “ideal” type:

- *Exactly one* connective is introduced
- The active formulas of the premises are *immediate subformulas* of the principal formula
- Context-independence

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Canonical Systems

A Canonical System =

The two identity rules

$+$

The two weakening rules

$+$

A (finite) set of canonical rules

Prominent Examples:

- LK
- Primal Logic [Gurevich,Neeman 2009] ($\leadsto$ instead of $\supset$)

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Theorem (Avron,Lev 2001)

Every (consistent) canonical system has two-valued non-deterministic semantics.

Non-deterministic Matrices by Example

$\wedge$	F	T
F	F	F
T	F	T

$$v(A \wedge B) = \wedge(v(A), v(B))$$

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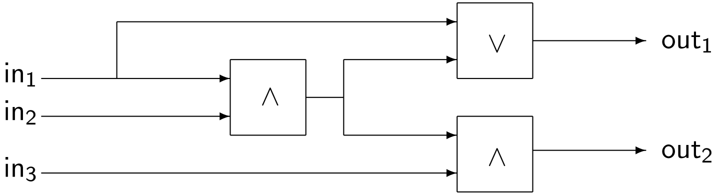

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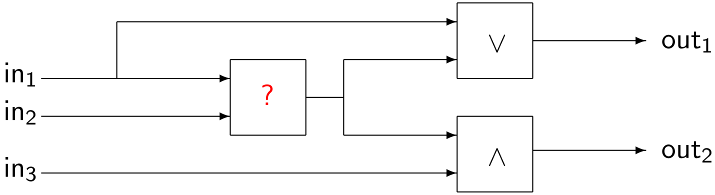

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F	$\{F\}$	$\{F, T\}$
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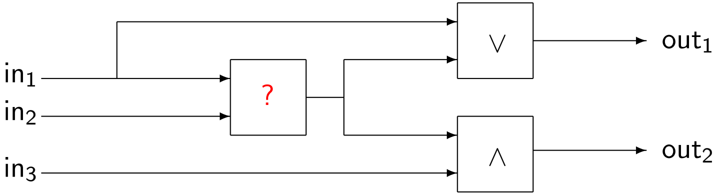

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What is the Role of the Identity Rules?

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Any subset of the identity rules

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- Discarding (cut) is useful for the study of cut-admissibility
- Discarding (id) is useful for proof-theoretic analysis of logic programming [Hösli, Jäger 1994]

Semantics of Semi-canonical Systems

Identity Rules

$$(id) \quad A \Rightarrow A \qquad\qquad (cut) \quad \frac{A \Rightarrow \quad \Rightarrow A}{\Rightarrow}$$

- Using only T and F:
  - Every valuation is model of $A \Rightarrow A$
  - No valuation is a model both of $A \Rightarrow$ and of $\Rightarrow A$

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$$(id) \quad A \Rightarrow A \qquad\qquad (cut) \quad \frac{A \Rightarrow \qquad \Rightarrow A}{\Rightarrow}$$

- Using only T and F:
  - Every valuation is model of $A \Rightarrow A$
  - No valuation is a model both of $A \Rightarrow$ and of $\Rightarrow A$
- Idea:
  - Without (id) - add a value $\bot$

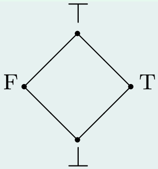
$$v(A) = \bot \qquad \Longleftrightarrow \qquad v \text{ is not a model of } A \Rightarrow A$$

  - Without (cut) - add a value $\top$

$$v(A) = \top \qquad \Longleftrightarrow \qquad v \text{ is a model of both } A \Rightarrow \text{ and } \Rightarrow A$$

Semantics of Semi-canonical Systems

The Truth-Values



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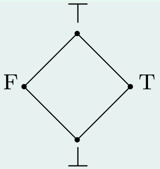

v is a model of $\Gamma \Rightarrow \Delta$ if $v(A) \geq \mathrm{F}$ for some $A \in \Gamma$ or $v(A) \geq \mathrm{T}$ for some $A \in \Delta$

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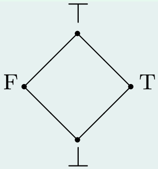

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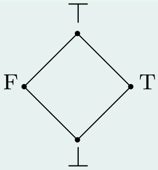

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$v(A) = \bot \iff$ *v is not a model of $A \Rightarrow A$*

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(id) $\Longrightarrow$ omit $\bot$
(cut) $\Longrightarrow$ omit $\top$

What are the Truth-Tables?

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Theorem

Some simple semi-canonical systems (e.g. $LK - (id)$ or $LK - (cut)$) do not have a finite (ordinary) matrix.

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Some simple semi-canonical systems (e.g. $LK - (id)$ or $LK - (cut)$) do not have a finite (ordinary) matrix.

Theorem

Every (consistent) semi-canonical system has a three or four valued non-deterministic matrix.

In fact, we provide a method to obtain such a matrix.

Example: Implication without (id) and (cut)

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

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$\bot$	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$
F	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$
T	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \mathrm{F}, \cancel{\mathrm{T}}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \cancel{\mathrm{T}}, \top\}$
$\top$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \cancel{\mathrm{T}}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \mathrm{T}, \top\}$	$\{\cancel{\bot}, \cancel{\mathrm{F}}, \cancel{\mathrm{T}}, \top\}$

$\supset$	$\bot$	F	T	$\top$
$\bot$	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\mathrm{T}, \top\}$	$\{\mathrm{T}, \top\}$
F	$\{\mathrm{T}, \top\}$	$\{\mathrm{T}, \top\}$	$\{\mathrm{T}, \top\}$	$\{\mathrm{T}, \top\}$
T	$\{\bot, \mathrm{F}, \mathrm{T}, \top\}$	$\{\mathrm{F}, \top\}$	$\{\mathrm{T}, \top\}$	$\{\top\}$
$\top$	$\{\mathrm{T}, \top\}$	$\{\top\}$	$\{\mathrm{T}, \top\}$	$\{\top\}$

Proof-Theoretic Applications

Semantics of (id)-free and/or (cut)-free systems makes it possible to obtain simple semantic proofs of proof-theoretic properties, e.g.:

1. Cut-admissibility

$$\vdash \Gamma \Rightarrow \Delta \qquad \Longrightarrow \qquad \vdash^{cf} \Gamma \Rightarrow \Delta$$

2. Axiom Expansion

$$A \Rightarrow A \, ; \; B \Rightarrow B \;\vdash\; A \diamond B \Rightarrow A \diamond B$$

Related Works

- [Schütte 1960],[Girard 1987] - semantics of $LK - (cut)$.
- [Hösli, Jäger 1994] - semantics of $LK - (id)$.

- Our framework is a unified approach: both logics can be defined by finite valued non-deterministic matrices.

Conclusions and Extensions

- We provided simple and modular semantic characterization for a natural family of sequent calculi.
- Two essential components:

  multiple values   non-determinism
- Application: semantic proofs of proof-theoretic properties

Conclusions and Extensions

- We provided simple and modular semantic characterization for a natural family of sequent calculi.
- Two essential components:

  multiple values non-determinism

- Application: semantic proofs of proof-theoretic properties
- Similar ideas can be applied for:
  - Many-sided systems
  - Sequent systems for intuitionistic logic and modal logics

Thank you!