Effective Finite-valued Semantics for Labelled Calculi

Matthias Baaz   Anna Zamansky   Ori Lahav

Vienna University of Technology   Tel Aviv University

IJCAR 2012

The Big Picture

- Goals:
  - Characterization of important proof-theoretic properties of calculi: *cut-admissibility, the subformula property, invertibility of rules,...*
  - Understanding the dependencies between them
- Tool: Non-deterministic semantics
  - Goes back to [Schütte 1960], [Tait 1966]
  - Formalized and studied in [Avron,Lev 2001]
- Framework: Canonical labelled sequent calculi
  - Labelled = many-sided

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{\blacksquare, \blacksquare, \blacksquare, \blacksquare, \ldots\}$
- Labelled formula:= $\square : A$ $\qquad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent:= a finite set of labelled formulas

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \ldots\}$
- Labelled formula$:= \square : A$ $\qquad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent$:=$ a finite set of labelled formulas

$$\mathcal{C} = \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}\} \qquad \{{\color{red}\blacksquare} : p_1, {\color{green}\blacksquare} : \neg p_1\}$$

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\qquad \mathcal{C} \subseteq \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \ldots\}$
- Labelled formula:= $\Box : A$ $\qquad$ where $A \in Frm_{\mathcal{L}}$ and $\Box \in \mathcal{C}$
- Sequent:= a finite set of labelled formulas

$$\mathcal{C} = \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}\} \qquad \{{\color{red}\blacksquare} : p_1, {\color{green}\blacksquare} : \neg p_1\}$$

$$\dfrac{\dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{red}\blacksquare} : \neg p_1, {\color{blue}\blacksquare} : \neg p_1\}} \qquad \dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{blue}\blacksquare} : \neg p_1, {\color{orange}\blacksquare} : \neg p_1\}}}{\{{\color{blue}\blacksquare} : \neg p_1\}}$$

Labelled Sequent Calculi

- A propositional language $\mathcal{L}$
- A finite set of labels $\mathcal{C}$ $\quad \mathcal{C} \subseteq \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}, \ldots\}$
- Labelled formula$:= \square : A$ $\quad$ where $A \in Frm_{\mathcal{L}}$ and $\square \in \mathcal{C}$
- Sequent$:=$ a finite set of labelled formulas

$$\mathcal{C} = \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{orange}\blacksquare}, {\color{green}\blacksquare}\} \qquad \{{\color{red}\blacksquare} : p_1, {\color{green}\blacksquare} : \neg p_1\}$$

$$\dfrac{\dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{red}\blacksquare} : \neg p_1, {\color{blue}\blacksquare} : \neg p_1\}} \qquad \dfrac{\{{\color{red}\blacksquare} : p_1\}}{\{{\color{blue}\blacksquare} : \neg p_1, {\color{orange}\blacksquare} : \neg p_1\}}}{\{{\color{blue}\blacksquare} : \neg p_1\}}$$

$$p_1, p_1 \supset p_2 \Rightarrow p_2 \quad \leftrightsquigarrow \quad \{{\color{red}\blacksquare} : p_1, {\color{red}\blacksquare} : p_1 \supset p_2, {\color{blue}\blacksquare} : p_2\}$$

Canonical Labelled Calculi

1. All standard structural rules
   (exchange, contraction, weakening)
2. A finite set of primitive rules
3. A finite set of canonical logical rules

Primitive Rules

Manipulate labels. Have the form ($\Box$'s are replaced by labels)

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{\{\Box : A, \ldots, \Box : A\} \cup s}$$

Primitive Rules

Manipulate labels. Have the form ($\square$'s are replaced by labels)

$$\frac{\{\square : A, \ldots, \square : A\} \cup s \quad \ldots \quad \{\square : A, \ldots, \square : A\} \cup s}{\{\square : A, \ldots, \square : A\} \cup s}$$

Examples:

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : A\} \cup s}{\{\textcolor{orange}{\blacksquare} : A, \textcolor{green}{\blacksquare} : A\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{blue}{\blacksquare} : A\} \cup s}{s}$$

$$\frac{}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s}$$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

Canonical Rules

- “Ideal” logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

- In Labelled Calculi [Avron, Zamansky 2009]:

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

Canonical Rules

- "Ideal" logical introduction rules [Avron, Lev 2001]:
  - Introduce *exactly one connective*.
  - The active formulas are *immediate subformulas* of the principal formula.
  - The application is *context-independent*.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

- In Labelled Calculi [Avron, Zamansky 2009]:

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

  - May introduce a connective with more than one label.

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s \quad \{{\color{orange}\blacksquare} : B, {\color{red}\blacksquare} : C, {\color{blue}\blacksquare} : C\} \cup s}{\{{\color{red}\blacksquare} : \heartsuit(A, B, C), {\color{blue}\blacksquare} : \heartsuit(A, B, C)\} \cup s}$$

Canonical Labelled Calculi

1. All standard structural rules
   (exchange, contraction, weakening)
2. A finite set of primitive rules
3. A finite set of canonical logical rules

Semantics

Intuition

- The value of A determines which of the labelled formulas ■ : A, ■ : A, ■ : $A, \ldots$ is true.
- In general, there are $2^{|\mathcal{C}|}$ possible options.
- Primitive rules forbid some of them.
- Logical rules are used to determine the values of compound formulas.

Semantics

Intuition

- The value of A determines which of the labelled formulas $\blacksquare : A, \blacksquare : A, \blacksquare : A, \ldots$ is true.
- In general, there are $2^{|\mathcal{C}|}$ possible options.
- Primitive rules forbid some of them.
- Logical rules are used to determine the values of compound formulas.

Formalization

- The set of truth-values $\mathcal{T}_{\mathbf{G}} \subseteq P(\mathcal{C})$ is determined according to the primitive rules of **G**.
- A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}_{\mathbf{G}}$ is a model of $\square : A$ if $\square \in v(A)$.
- A valuation is a model of a sequent s if it is a model of some labelled formula in s.

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\ r_1$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s} \; r_1$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \xcancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$$

$$\frac{\{\textcolor{orange}{\blacksquare} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s}\ r_1 \qquad\qquad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \xcancel{\{\textcolor{orange}{\blacksquare}\}}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\ r_1 \qquad\qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \xcancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$$

$$\frac{\{{\color{orange}\blacksquare} : A\} \cup s}{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : A\} \cup s}\ r_1 \qquad\qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \xcancel{\{{\color{orange}\blacksquare}\}}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{orange}\blacksquare}\}}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}, \xcancel{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$$

Example: Semantic Effect of Primitive Rules

$$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$$

$$\frac{\{\textcolor{orange}{\blacksquare} : A\} \cup s}{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : A\} \cup s}\ r_1 \qquad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : A\} \cup s}{s}\ r_2$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \xcancel{\{\textcolor{orange}{\blacksquare}\}}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \xcancel{\{\textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}\}}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}, \xcancel{\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}}\}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\ \}, \{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$$

Theorem

Given a canonical calculus **G** *without logical rules,* $\Omega \vdash_{\mathbf{G}} s$ *iff every valuation* $v : Frm_{\mathcal{L}} \to \mathcal{T}_{\mathbf{G}}$ *which is a model of every sequent in* Ω *is also a model of* s*.*

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\frac{\{\color{blue}\blacksquare\color{black} : A\} \cup s \quad \{\color{red}\blacksquare\color{black} : B\} \cup s}{\{\color{red}\blacksquare\color{black} : A \supset B\} \cup s}$$

$$\mathcal{T}_{\mathbf{G}} = \{\{\color{red}\blacksquare\color{black}\}, \{\color{blue}\blacksquare\color{black}\}\}$$

$$\frac{\{\color{red}\blacksquare\color{black} : A, \color{blue}\blacksquare\color{black} : B\} \cup s}{\{\color{blue}\blacksquare\color{black} : A \supset B\} \cup s}$$

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$		
$\{{\color{blue}\blacksquare}\}$		

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$	
$\{{\color{blue}\blacksquare}\}$		

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\color{red}\blacksquare\}, \{\color{blue}\blacksquare\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\color{red}\blacksquare\}$	$\{\color{blue}\blacksquare\}$
$\{\color{red}\blacksquare\}$	$\{\color{blue}\blacksquare\}$	
$\{\color{blue}\blacksquare\}$	$\{\color{red}\blacksquare\}$	

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$
$\{\textcolor{blue}{\blacksquare}\}$	$\{\textcolor{red}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}\}$

The Truth-Tables

The table for a connective is algorithmically extracted from its logical rules.

For example:

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

$\tilde{\supset}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{blue}\blacksquare}\}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$

A legal valuation should respect the table:
$$v(\diamond(A_1, \ldots, A_n)) = \tilde{\diamond}(v(A_1), \ldots, v(A_n))$$

What Can Go Wrong?

What Can Go Wrong?

- Non truth-functional connectives,
  e.g. primal implication [Gurevich, Neeman 2009]:

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{\textcolor{red}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}\})$?

What Can Go Wrong?

- More than one option satisfies the conclusion, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$$

$$\frac{\{{\color{blue}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{{\color{red}\blacksquare} : A, {\color{blue}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

What Can Go Wrong?

- More than one option satisfies the conclusion, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{\textcolor{red}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}\}$$

$$\frac{\{\textcolor{blue}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \supset B\} \cup s}$$

$$\frac{\{\textcolor{red}{\blacksquare} : A, \textcolor{blue}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \supset B\} \cup s}$$

How to determine $\tilde{\supset}(\{\textcolor{blue}{\blacksquare}\}, \{\textcolor{red}{\blacksquare}\})$?

Solution: Non-deterministic Truth-Tables [Avron, Lev 2001]

A table of an n-ary connective $\diamond$ is a function $\tilde{\diamond} : \mathcal{T}^n \to P^+(\mathcal{T})$.
A legal valuation satisfies: $v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n))$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\quad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\quad$ $\circ$ is a binary connective

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\quad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\quad$ $\circ$ is a binary connective

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B\} \cup s}$$

$\tilde{\circ}$	$\emptyset$	$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \cancel{\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\quad$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\quad$ $\circ$ is a binary connective

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B, \textcolor{blue}{\blacksquare} : A \circ B\} \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \cancel{\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{red}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{\textcolor{red}{\blacksquare} : A\} \cup s \quad \{\textcolor{orange}{\blacksquare} : B\} \cup s}{\{\textcolor{red}{\blacksquare} : A \circ B, \textcolor{blue}{\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{\textcolor{blue}{\blacksquare} : A, \textcolor{orange}{\blacksquare} : B\} \cup s}{\{\textcolor{blue}{\blacksquare} : A \circ B \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \cancel{\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$
$\{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$	$\{\emptyset, \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}\}, \{\textcolor{blue}{\blacksquare}, \textcolor{orange}{\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\qquad \mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\qquad \circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A, {\color{orange}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \circ B \cup s}$$

$\widetilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

Example: Construction of a Non-deterministic Truth-Table

$\mathcal{C} = \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$ $\mathcal{T}_{\mathbf{G}} = \{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$ $\circ$ is a binary connective

$$\frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{red}\blacksquare} : A\} \cup s \quad \{{\color{orange}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \circ B, {\color{blue}\blacksquare} : A \circ B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A, {\color{orange}\blacksquare} : B\} \cup s}{\{{\color{blue}\blacksquare} : A \circ B \cup s}$$

$\tilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \cancel{\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\cancel{\emptyset}, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

$\tilde{\circ}$	$\emptyset$	$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$
$\emptyset$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\emptyset, \{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}\}, \{{\color{blue}\blacksquare}, {\color{orange}\blacksquare}\}\}$

What Else Can Go Wrong?

What Else Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{\color{red}\blacksquare\color{black}\}, \{\color{blue}\blacksquare\color{black}\}\} \qquad \frac{\{\color{red}\blacksquare\color{black} : B\} \cup s}{\{\color{red}\blacksquare\color{black} : A \diamond B\} \cup s} \qquad \frac{\{\color{blue}\blacksquare\color{black} : A\} \cup s}{\{\color{blue}\blacksquare\color{black} : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{\color{blue}\blacksquare\color{black}\}, \{\color{red}\blacksquare\color{black}\})$?

What Else Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\} \qquad \frac{\{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \diamond B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

$\tilde{\diamond}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}\}$	$\emptyset$	$\{\{{\color{blue}\blacksquare}\}\}$

$\{{\color{blue}\blacksquare}\}$ and $\{{\color{red}\blacksquare}\}$ cannot be used by the same valuation.

What Else Can Go Wrong?

- Contradictions between rules, e.g.

$$\mathcal{T}_{\mathbf{G}} = \{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\} \qquad \frac{\{{\color{red}\blacksquare} : B\} \cup s}{\{{\color{red}\blacksquare} : A \diamond B\} \cup s} \qquad \frac{\{{\color{blue}\blacksquare} : A\} \cup s}{\{{\color{blue}\blacksquare} : A \diamond B\} \cup s}$$

How to determine $\tilde{\diamond}(\{{\color{blue}\blacksquare}\}, \{{\color{red}\blacksquare}\})$?

$\tilde{\diamond}$	$\{{\color{red}\blacksquare}\}$	$\{{\color{blue}\blacksquare}\}$
$\{{\color{red}\blacksquare}\}$	$\{\{{\color{red}\blacksquare}\}\}$	$\{\{{\color{red}\blacksquare}\}, \{{\color{blue}\blacksquare}\}\}$
$\{{\color{blue}\blacksquare}\}$	$\emptyset$	$\{\{{\color{blue}\blacksquare}\}\}$

$\{{\color{blue}\blacksquare}\}$ and $\{{\color{red}\blacksquare}\}$ cannot be used by the same valuation.

Solution: Partial Truth-Tables

Allow empty entries: $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$.

The Semantic Framework

Partial Non-deterministic Matrices

A PNmatrix **M** for $\mathcal{L}$ and $\mathcal{C}$ consists of:

- A set $\mathcal{T}$ of truth-values.
- A function $\mathcal{D} : \mathcal{C} \to P(\mathcal{T})$ assigning a set of designated truth-values for every label.
- A truth-table $\tilde{\diamond} : \mathcal{T}^n \to P(\mathcal{T})$ for every n-ary connective of $\mathcal{L}$.

A valuation $v : Frm_{\mathcal{L}} \to \mathcal{T}$ is:

- a model (in **M**) of a sequent s if $v(A) \in \mathcal{D}(\Box)$ for some $\Box : A$ in s.
- **M**-legal if $v(\diamond(A_1, \dots, A_n)) \in \tilde{\diamond}(v(A_1), \dots, v(A_n))$ for every $\diamond(A_1, \dots, A_n) \in Frm_{\mathcal{L}}$.

Main Result

Theorem

For every canonical labelled calculus $\mathbf{G}$*, there exists a strongly characteristic PNmatrix* $\mathbf{M_G}$ *(i.e.* $\Omega \vdash_{\mathbf{G}} s$ *iff every* $\mathbf{M_G}$*-legal valuation which is a model of every sequent in* Ω *is also a model of* s*).*

Moreover, we provide a uniform algorithm to obtain $\mathbf{M_G}$ *from* $\mathbf{G}$*.*

Main Result

Theorem

For every canonical labelled calculus $\mathbf{G}$*, there exists a strongly characteristic PNmatrix* $\mathbf{M_G}$ *(i.e.* $\Omega \vdash_{\mathbf{G}} s$ *iff every* $\mathbf{M_G}$*-legal valuation which is a model of every sequent in* Ω *is also a model of* s*).*

Moreover, we provide a uniform algorithm to obtain $\mathbf{M_G}$ *from* $\mathbf{G}$*.*

In many cases, the obtained semantics coincides with a known one:

- Propositional fragment of **LK**
- **LK** without cut [Girard 1987]
- **LK** without identity axiom [Hösli,Jäger 1994]
- Two-sided canonical systems [Avron,Lev 2001]
- Labelled calculi studied in [Baaz et al. 1998] and [Avron,Zamansky 2009]

Effectiveness

Theorem

Semantic consequence relations induced by PNmatrices are decidable.

Corollary

All canonical labelled calculi are decidable.

Effectiveness

Theorem

Semantic consequence relations induced by PNmatrices are decidable.

Proof Outline.

- Usual method: To decide whether $\Omega \vdash_{\mathbf{M}} s$, check one-by-one all **M**-legal partial valuations defined $sub[\Omega, s]$.
- Hidden assumption: All **M**-legal partial valuations can be extended to full ones (semantic analyticity).
  But, it does not hold for PNmatrices (recall $\tilde{\diamond}(\{\blacksquare\}, \{\blacksquare\}) = \emptyset$!).
- Lemma: It is decidable whether an **M**-legal partial valuations can be extended to a full one.
- Solution: Check one-by-one all extendable **M**-legal partial valuations defined $sub[\Omega, s]$.

□

Characterization of Cut-Admissibility

A cut is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

A is called the cut-formula, s is called the cut-context

Characterization of Cut-Admissibility

A cut is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

A is called the cut-formula, s is called the cut-context

Strong Cut-Admissibility

$\Omega \vdash_{\mathbf{G}} s \Longrightarrow$ there is a derivation of s from Ω in **G** in which: the cut-formula of each cut occurs either in the cut-context or in Ω.

Characterization of Cut-Admissibility

A cut is a primitive rule of the form:

$$\frac{\{\Box : A, \ldots, \Box : A\} \cup s \quad \ldots \quad \{\Box : A, \ldots, \Box : A\} \cup s}{s}$$

A is called the cut-formula, s is called the cut-context

Strong Cut-Admissibility

$\Omega \vdash_{\mathbf{G}} s \Longrightarrow$ there is a derivation of s from Ω in **G** in which: the cut-formula of each cut occurs either in the cut-context or in Ω.

Theorem

For every canonical labelled calculus **G***:*

G *enjoys strong cut-admissibility iff* $\mathbf{M_G}$ *does not include empty entries.*

Summary

- We provided effective and modular semantic characterization for labelled canonical sequent calculi using partial non-deterministic matrices.
- Application: semantic characterization of proof-theoretic properties.
- Similar ideas can be applied for: single-conclusion canonical calculi, sequent calculi for modal logics, canonical Gödel hypersequent calculi...
- Future research directions:
  - First-order
  - Less restrictive primitive and introduction rules

Thank you!