Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices

Ori Lahav, Tel Aviv University

Yoni Zohar, Bar Ilan University

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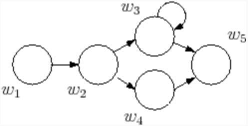
The Modal Logic $\mathtt{K}$

Axiomatic System

- Axiom $\mathtt{K}$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- Necessitation Rule: $\dfrac{\psi}{\Box\psi}$ when ψ is a theorem
- extensions: $\mathtt{T}$: $\Box\varphi \to \varphi$ / 4: $\Box\varphi \to \Box\Box\varphi$

Standard Semantics – Kripke Models (W, R, V)

- W is a set (of worlds)
- R is an accessibility relation
- $v : W \to P$ is a valuation
- restrictions: reflexivity / transitivity



Kearns Semantics [Kearns 1981]

"For I do not think there are such things as possible worlds..."

4-valued Semantics for KT

- T, t, F, f
- "not functional"

A	$\sim A$
T	F
t	f
f	t
F	T

A	$\Box A$
T	T, t
t	f, F
f	f, F
F	f, F

A	B	$[A \vee B]$
f	T	T
f	t	T, t
f	f	f
f	F	f

But...

$v(p) = \mathsf{f},\ v(\neg p) = \mathsf{t},\ v(p \vee \neg p) = \mathsf{t},\ v(\Box(p \vee \neg p)) = \mathsf{f}$

Levels

A more satisfactory definition follows.

A 0*th-level T-valuation of* L_0 is a function which assigns one of **T, t, f, F** to each sentence of L_0 in a manner consistent with the matrices.

Let $\mathscr{V}$ be an *m*th-level T-valuation of L_0. $\mathscr{V}$ is an $m + 1$*st level T-valuation of* L_0 iff $\mathscr{V}$ assigns T to every sentence A which is true (which has value T or t) for every *m*th-level T-valuation.

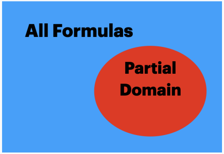
Can be seen as being based on NMatrices [Avron & Lev 2005]

Related Work

- Simplification, extension, connection to NMatrices [Omori & Skurt 2016]
- More Extensions [Coniglio, del Cerro, Peron 2015]
- Missing ingredient: effectiveness
- Recent: S4 and KT with 3 values, effective [Grätz 2022]

Our Contribution: Semantics for K and KT

- Effective
- Basic truth tables are more intuitive
- KT emerges by deleting a truth value
- Connection to sequent calculi



Intuition

	Holds in acc. worlds	Doesn't hold in acc. worlds
Holds here	T	t
Doesn't hold here	F	f

$$\mathcal{V}_4 \overset{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \overset{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

Truth Tables: Classical Connectives

$$t \mapsto \{\mathsf{t}, \mathsf{T}\} \qquad f \mapsto \{\mathsf{f}, \mathsf{F}\}$$

$x \tilde{\supset} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
t	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
F	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
f	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

$x \tilde{\wedge} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
t	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
F	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

$x \tilde{\vee} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
F	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

x	$\tilde{\neg} x$
T	$\overline{\mathcal{D}}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\mathcal{D}$

Four Values

Intuition

	Holds in acc. worlds	Doesn't hold in acc. worlds
Holds here	T	t
Doesn't hold here	F	f

$$\mathcal{V}_4 \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

Truth Tables: $\Box$

x	$\tilde{\Box} x$
T	$\mathcal{D}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\overline{\mathcal{D}}$

Intuition:

$v(\varphi) \in \{\mathsf{T}, \mathsf{F}\}$ iff $v(\Box\varphi) \in \mathcal{D}$

Levels

NMatrices Are Not Enough

A formula that is entailed from a set of formulas that hold in all acc. worlds should hold in all acc. worlds

$$(*)\ v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}$$

Example

$v(p) = \mathsf{T}, v(q) = \mathsf{T} \implies v(p \land q) = \mathsf{T}$

Levels To The Rescue

$\mathbb{V}^{0}_{\mathrm{K}} \overset{\text{def}}{=} \{v \mid v \text{ respects } \mathrm{M}_{\mathrm{K}}\}$

$\mathbb{V}^{m+1}_{\mathrm{K}} \overset{\text{def}}{=} \{v \in \mathbb{V}^{m}_{\mathrm{K}} \mid \forall\varphi.\, v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash^{\mathbb{V}^{m}_{\mathrm{K}}} \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}\}$

$\bigcap_{m\in\mathbb{N}} \mathbb{V}_{\mathrm{K}}$ is a maximal set satisfying $(*)$

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Example

$v(p) = \mathsf{T}, v(q) = \mathsf{T} \implies v(p \wedge q) = \mathsf{T}$

Levels To The Rescue

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F},0} \stackrel{\text{def}}{=} \{v \mid v \text{ respects } \mathrm{M}_{\mathrm{K}} \text{ with domain } \mathcal{F}\}$

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m+1} \stackrel{\text{def}}{=} \{v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}\}$

$\bigcap_{m \in \mathbb{N}} \mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ is a maximal set satisfying $(*)$

Main Results

Satisfaction And Consequence

- $v \models \varphi$ if $v(\varphi) \in \mathcal{D}$
- $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$: $\forall\ v \in \mathbb{V}_{\mathtt{K}}$, if $v \models \mathcal{T}$ then $v \models \varphi$ $\quad (\mathbb{V}_{\mathtt{K}} \stackrel{\text{def}}{=} \bigcap_{m \geq 0} \mathbb{V}_{\mathtt{K}}^{m})$

Thm. (Soundness and Completeness)
φ follows from $\mathcal{T}$ in $\mathtt{K}$ iff $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$

Thm. (Effectiveness)
For each φ there is a computable $m_\varphi \in \mathbb{N}$ such that:

$$\varphi \text{ is } \mathbb{V}_{\mathtt{K}}\text{-SAT iff it is } \mathbb{V}_{\mathtt{K}}^{sub(\varphi), m_\varphi}\text{-SAT}$$

First Result: Soundness and Completeness

Thm. (Soundness and Completeness)
φ follows from $\mathcal{T}$ in $\mathtt{K}$ iff $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$

Sequent Calculus

- We did not translate Kripke to Kearns (future work)
- We did not work with the Hilbert Calculus
- Instead: went through a sequent calculus for $\mathtt{K}$
- Bonus: connection between levels and applications of rule K

Sequent Calculus

The Sequent Calculus $\mathbb{G}_{\mathbb{K}}$

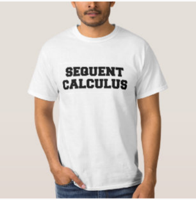
$$
(\textsc{weak})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad
(\textsc{id})\ \frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta} \quad
(\textsc{cut})\ \frac{\begin{array}{c}\Gamma, \varphi \Rightarrow \Delta \\ \Gamma \Rightarrow \varphi, \Delta\end{array}}{\Gamma \Rightarrow \Delta} \quad
(K)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}
$$

$$
(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \quad
(\Rightarrow \neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \quad
(\supset\Rightarrow)\ \frac{\begin{array}{c}\Gamma \Rightarrow \varphi, \Delta \\ \Gamma, \psi \Rightarrow \Delta\end{array}}{\Gamma, \varphi \supset \psi, \Delta} \quad
(\Rightarrow\supset)\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}
$$

$$
(\wedge \Rightarrow)\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \quad
(\Rightarrow \wedge)\ \frac{\begin{array}{c}\Gamma \Rightarrow \varphi, \Delta \\ \Gamma \Rightarrow \psi, \Delta\end{array}}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \quad
(\vee \Rightarrow)\ \frac{\begin{array}{c}\Gamma, \varphi \Rightarrow \Delta \\ \Gamma, \psi \Rightarrow \Delta\end{array}}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \quad
(\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}
$$

Properties

- Sound and complete for $\mathbb{K}$
- Subformula Property



Derivations and Levels

Theorem:
$\Gamma \vdash^{\mathbb{V}_K} \varphi \quad \text{iff} \vdash_{G_K} \Gamma \Rightarrow \varphi$

Definition

- $\vdash_{G_K}^{\mathcal{F},m} \Gamma \Rightarrow \Delta$: The sequent $\Gamma \Rightarrow \Delta$ is deribable:
  - using only $\mathcal{F}$-formulas
  - max number of K applications in each branch is m

Derivations and Levels

Theorem:
$\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}^{m}} \varphi$ iff $\vdash_{\mathsf{G}_{\mathrm{K}}}^{m} \Gamma \Rightarrow \varphi$

Definition

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Derivations and Levels

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    - using only $\mathcal{F}$-formulas
    - max number of K applications in each branch is m

Second Result: Effectiveness

Thm. (Effectiveness)
For each φ there is a computable $m_\varphi \in \mathbb{N}$ such that:

$$\varphi \text{ is } \mathbb{V}_{\mathtt{K}}\text{-SAT iff it is } \mathbb{V}_{\mathtt{K}}^{sub(\varphi),m_\varphi}\text{-SAT}$$

- Motivation: Decision procedure based on the semantics
- Cannot iterate through all valuations
- Cannot even check if a valuation is in $\mathbb{V}_{\mathtt{K}}$ (or even in $\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}$)

Effectiveness

Algorithm Deciding $\Gamma \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$.

```
for v : 𝓛 → {T, t, F, f} do
    if v ∈ 𝕍_K and v ⊨ Γ and v ⊭ φ then
        return "NO"
return "YES"
```

Justification

Soundness and Completeness

Effectiveness

Algorithm Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

1: $\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
2: **for** $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
3:     **if** $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
4:         **return** "NO"
5: **return** "YES"

Justification

1. $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi \Longrightarrow$ (def.)
2. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F}}} \varphi \Longrightarrow$ (completeness)
3. $\not\vdash_{\mathrm{G_K}}^{\mathcal{F}} \Gamma \Rightarrow \varphi \Longrightarrow$ (subformula property)
4. $\not\vdash_{\mathrm{G_K}} \Gamma \Rightarrow \varphi \Longrightarrow$ (soundness)
5. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$

Effectiveness

Algorithm Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

```
F ← sub(Γ ∪ {φ})
m ← md(F)                                                  ▷ modal depth
for v : F → {T, t, F, f} do
    if v ∈ V_K^{F,m} and v ⊨ Γ and v ⊭ φ then
        return "NO"
return "YES"
```

Justification

1. $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi \Longrightarrow$ (def.)
2. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \Longrightarrow$ (completeness)
3. $\not\vdash^{\mathcal{F},m}_{\mathrm{G_K}} \Gamma \Rightarrow \varphi \Longrightarrow$ (modal depth)
4. $\not\vdash^{\mathcal{F}}_{\mathrm{G_K}} \Gamma \Rightarrow \varphi \Longrightarrow$ (subformula property)
5. $\not\vdash_{\mathrm{G_K}} \Gamma \Rightarrow \varphi \Longrightarrow$ (soundness)
6. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$

Effectiveness

Algorithm Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

$\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
$m \leftarrow md(\mathcal{F})$ ▷ modal depth
for $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
    if $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
        return “NO”
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for $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
    if $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
        return "NO"
return "YES"

Justification

1. $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi \implies$ (def.)
2. $\Gamma \nvdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \implies$ (completeness)
3. $\nvdash_{\mathrm{G_K}}^{\mathcal{F},m} \Gamma \Rightarrow \varphi \implies$ (modal depth)
4. $\nvdash_{\mathrm{G_K}}^{\mathcal{F}} \Gamma \Rightarrow \varphi \implies$ (subformula property)
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$\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
$m \leftarrow md(\mathcal{F})$ ▷ modal depth
for $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
  if $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
    return "NO"
return "YES"

Justification

1. $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi \Longrightarrow$ (def.)
2. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \Longrightarrow$ (completeness)
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5. $\not\vdash_{\mathrm{G_K}} \Gamma \Rightarrow \varphi \Longrightarrow$ (soundness)
6. $\Gamma \not\vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$

Model Generation

- Often YES/NO is not enough
- Model generation
- Is v a "real" model?

Effectiveness

Algorithm Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

$\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
$m \leftarrow 4^{|\mathcal{F}|}$
for $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
  if $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
    return "NO", v
return "YES"

Justification

- $\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m} = \mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$
- Every $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ can be extended to some $v' \in \mathbb{V}_{\mathrm{K}}$

Conclusion

We Have Seen

- 4-valued semantics for K
- Based on NMatrices (RNMatrices [Coniglio & Toledo 2021])
- Effective, model constructing
- Also KT

Future Work

- More modal logics
- Complexity
- Implementation using a SAT solver


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Thank You!