Effective Finite-valued Semantics for Labelled Calculi

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Abstract. We provide a systematic and modular method to define non-deterministic finite-valued semantics for a natural and very general family of *canonical labelled calculi*, of which many previously studied sequent and labelled calculi are particular instances. This semantics is *effective*, in the sense that it naturally leads to a decision procedure for these calculi. It is then applied to provide simple *decidable* semantic criteria for crucial syntactic properties of these calculi, namely (strong) analyticity and cut-admissibility.

1 Introduction

There are two contrary aims in logic: the first is to find calculi that characterize a given semantics, the second is to find semantics for a logic that is only given as a formal calculus. Roughly speaking, the former aim has been reached for all (ordinary) finite-valued logics (including, of course, classical logic), as well as for non-deterministic finite-valued logics ([2, 3]). As for the latter, there is no known systematic method of constructing for a given general calculus, a corresponding "well-behaved" semantics. By "well-behaved" here we mean that it is *effective* in the sense of naturally inducing a decision procedure for its underlying logic. Moreover, it is desirable that such semantics can be applied to provide simple semantic characterization of important syntactic properties of the corresponding calculi, which are hard to establish by other means. Analyticity and cut-admissibility are just a few cases in point.

In [6] and [4] two families of *labelled sequent calculi* have been studied in this context.[3] [6] considers labelled calculi with generalized forms of cuts and identity

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[3] A remark is in order here on the relationship between the labelled calculi studied here and the general framework of labelled deductive systems (LDS) from [8]. Both frameworks consider consequence relations between labelled formulas. Methodologically, however, they have different aims: [8] constructs a system for a given logic defined in semantic terms, while we define a semantics for a given labelled system. Moreover, in LDS *anything* is allowed to serve as labels, while we assume a finite set of labels. In this sense, our labelled calculi are a particular instance of LDS.

axioms and a restricted form of logical rules, and provides some necessary and sufficient conditions for such calculi to have a characteristic finite-valued matrix. In [4] labelled calculi with a less restrictive form of logical rules (but a more restrictive form of cuts and axioms) are considered. The calculi of [4], satisfying a certain coherence condition, have a semantic characterization using a natural generalization of the usual finite-valued matrix called *non-deterministic matrices* ([2, 3]). The semantics provided in [6, 4] for these families of labelled calculi is well-behaved in the sense defined above, that is the question of whether a sequent s follows in some (non-deterministic) matrix from a finite set of sequents $\mathcal{S}$, can be reduced to considering legal *partial* valuations, defined on the subformulas of $\mathcal{S} \cup \{s\}$. This naturally induces a decision procedure for such logics.

In this paper we show that the class of labelled calculi that have a finite-valued well-behaved semantics is substantially larger than all the families of calculi considered in the literature in this context. We start by defining a general class of fully-structural and propositional labelled calculi, called *canonical labelled calculi*, of which the labelled calculi of [6, 4] are particular examples. In addition to the weakening rule, canonical labelled calculi have rules of two forms: primitive rules and introduction rules. The former operate on labels and do not mention any connective. The generalized cuts and axioms of [6] are specific instances of such rules. As for the latter, each such rule introduces one logical connective of the language. To provide semantics for such calculi in a systematic and modular way, we generalize the notion of non-deterministic matrices to *partial non-deterministic matrices* (PNmatrices), by allowing empty sets of options in the truth tables of logical connectives. Although applicable to a much wider range of calculi, the semantic framework of finite PNmatrices shares a crucial property with both standard and non-deterministic matrices: any calculus that has a characteristic PNmatrix is decidable. Moreover, as opposed to the results in [6, 4], *no* conditions are required for a canonical labelled calculi to have a characteristic PNmatrix: *all* such calculi have one, and so *all* of them are decidable. We then apply PNmatrices to provide simple *decidable* characterizations of two crucial syntactic properties: strong analyticity and strong cut-admissibility.

Due to lack of space, most proofs are omitted, and will appear in an extended version.

2 Preliminaries

In what follows $\mathcal{L}$ is a propositional language, and $\pounds$ is a finite non-empty set of labels. We assume that $p_1, p_2, \ldots$ are the atomic formulas of $\mathcal{L}$. We denote by $Frm_{\mathcal{L}}$ the set of all wffs of $\mathcal{L}$. We usually use φ, ψ as metavariables for formulas, Γ, Δ for finite sets of formulas, l for labels, and L for sets of labels.

Definition 1. *A* labelled formula *is an expression of the form $l : \psi$, where $l \in \pounds$ and $\psi \in Frm_{\mathcal{L}}$. A* sequent *is a finite set of labelled formulas. An n-*clause *is a sequent consisting of atomic formulas from $\{p_1, \ldots, p_n\}$. Given a set $L \subseteq \pounds$, we write $(L : \psi)$ instead of (the sequent) $\{l : \psi \mid l \in L\}$.*

Given a labelled formula γ, we denote by $frm[\gamma]$ the (ordinary) formula appearing in γ, and by $sub[\gamma]$ the set of subformulas of the formula $frm[\gamma]$. *frm* and *sub* are extended to sets of labelled formulas and to sets of sets of labelled formulas in the obvious way.

Remark 1. The usual (two-sided) sequent notation $\psi_1, \ldots, \psi_n \Rightarrow \varphi_1, \ldots, \varphi_m$ can be interpreted as $\{f : \psi_1, \ldots, f : \psi_n, t : \varphi_1, \ldots, t : \varphi_m\}$, i.e. a sequent in the sense of Definition 1 for $\pounds = \{t, f\}$.

Definition 2. *An* $\mathcal{L}$-substitution *is a function* $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$*, which satisfies* $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ *for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$*. A substitution is extended to labelled formulas, sequents, etc. in the obvious way.*

3 Canonical Labelled Calculi

In this section we define the family of *canonical labelled calculi*. This is a general family of labelled calculi, which includes many natural subclasses of previously studied systems. These include the system **LK** for classical logic, the canonical sequent calculi of [2], the signed calculi of [4] and the labelled calculi of [6].[4]

All canonical labelled calculi have in common the *weakening* rule. In addition, they include rules of two types: *primitive rules* and *introduction rules*. Each rule of the latter type introduces exactly one logical connective, while rules of the former type operate on labels and do no mention any logical connectives. Next we provide precise definitions.

Definition 3 (Weakening). *The* weakening *rule allows to infer* $s \cup s'$ *from* s *for every two sequents* s *and* s'*.*

Definition 4 (Primitive Rules). *A* primitive rule *for* $\pounds$ *is an expression of the form* $\{L_1, \ldots, L_n\}/L$ *where* $n \geq 0$ *and* $L_1, \ldots, L_n, L \subseteq \pounds$*. An* application of a primitive rule $\{L_1, \ldots, L_n\}/L$ *is any inference step of the following form:*

$$\frac{(L_1 : \psi) \cup s_1 \quad \ldots \quad (L_n : \psi) \cup s_n}{(L : \psi) \cup s_1 \cup \ldots \cup s_n}$$

where ψ *is a formula, and* s_i *is a sequent for every* $1 \leq i \leq n$*.*

Example 1. Suppose $\pounds = \{a, b, c\}$ and consider the primitive rule $\{\{a\}, \{b\}\}/\{b, c\}$. This rule allows to infer $(\{b, c\} : \psi) \cup s_1 \cup s_2$ from $\{a : \psi\} \cup s_1$ and $\{b : \psi\} \cup s_2$ for every two sequents s_1, s_2 and a formula ψ.

[4] The family of canonical labelled calculi also includes the systems dealt with in [9]. [9] extends the results of [2] by considering also "semi-canonical calculi", which are obtained from (two-sided) canonical calculi by discarding either the cut rule, the identity axioms or both of them. Clearly, these systems are particular instances of canonical labelled calculi, defined in this paper.

Definition 5. *A primitive rule for £ of the form* $\emptyset/L$ *is called a* canonical axiom. *Its applications provide all axioms of the form* $(L : \psi)$.

Example 2. Axiom schemas of two-sided sequent calculi usually have the form $\psi \Rightarrow \psi$. Using the notation from Remark 1, it can be presented as the canonical axiom $\emptyset/\{t, f\}$.

Definition 6. *A primitive rule for £ of the form* $\{L_1, \ldots, L_n\}/\emptyset$ *is called a* canonical cut. *Its applications allow to infer* $s_1 \cup \ldots \cup s_n$ *from the sequents* $(L_i : \psi) \cup s_i$ *for every* $1 \leq i \leq n$ *(the formula* ψ *is called the* cut-formula*).*

Example 3. Applications of the cut rule for two-sided sequent calculi are usually presented by the following schema:

$$\frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Using the notation from Remark 1, the corresponding canonical cut has the form $\{\{t\}, \{f\}\}/\emptyset$.

Definition 7 (Introduction Rules). *A* canonical introduction rule *for an* n*-ary connective* $\diamond$ *of* $\mathcal{L}$ *and £ is an expression of the form* $\mathcal{S}/L : \diamond(p_1, \ldots, p_n)$*, where* $\mathcal{S}$ *is a finite set of* n*-clauses (see Definition 1) (called* premises*), and* L *is a non-empty subset of £. An* application *of a canonical introduction rule* $\{c_1, \ldots, c_m\}/L : \diamond(p_1, \ldots, p_n)$ *is any inference step of the following form:*[5]

$$\frac{\sigma(c_1) \cup s_1 \quad \ldots \quad \sigma(c_m) \cup s_m}{(L : \sigma(\diamond(p_1, \ldots, p_n))) \cup s_1 \cup \ldots \cup s_m}$$

where σ *is an* $\mathcal{L}$*-substitution, and* s_i *is a sequent for every* $1 \leq i \leq m$.

Example 4. The introduction rules for the classical conjunction in **LK** are usually presented as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2 \Rightarrow \Delta_2, \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \psi \wedge \varphi}$$

Using the notation from Remark 1, the corresponding canonical rules are:

$$r_1 = \{\{f : p_1, f : p_2\}\}/\{f\} : p_1 \wedge p_2 \qquad r_2 = \{\{t : p_1\}, \{t : p_2\}\}/\{t\} : p_1 \wedge p_2$$

Their applications have the forms:

$$\frac{\{f : \psi, f : \varphi\} \cup s}{\{f : \psi \wedge \varphi\} \cup s} \qquad \frac{\{t : \psi\} \cup s_1 \quad \{t : \varphi\} \cup s_2}{\{t : \psi \wedge \varphi\} \cup s_1 \cup s_2}$$

[5] Note the full separation between a rule and its application: $p_1, \ldots, p_n$ appearing in the rule serve as schematic variables, which are replaced by actual formulas of the language in the application.

Definition 8 (Canonical Labelled Calculi). *A* canonical labelled calculus **G** *for* $\mathcal{L}$ *and* $\pounds$ *includes the weakening rule, a finite set of primitive rules for* $\pounds$*, and a finite set of introduction rules for the connectives of* $\mathcal{L}$ *and* $\pounds$*. We say that a sequent s* is derivable *in a canonical labelled calculus* **G** *from a set of sequents* $\mathcal{S}$ *(and denote it by* $\mathcal{S} \vdash_{\mathbf{G}} s$*) if there exists a derivation in* **G** *of s from* $\mathcal{S}$*.*

Notation: Given a canonical labelled calculus **G** for $\mathcal{L}$ and $\pounds$, we denote by $\mathsf{P}_{\mathbf{G}}$ the set of primitive rules of **G**. In addition, for every connective $\diamond$ of $\mathcal{L}$, we denote by $\mathsf{R}^{\diamond}_{\mathbf{G}}$ the set of canonical introduction rules for $\diamond$ of **G**.

Example 5. The standard sequent system **LK** can be represented as a canonical labelled calculus for the language of classical logic and $\{t, f\}$ (see Remark 1 and Examples 2 to 4).

Henceforth, to improve readability, we usually omit the parentheses from the set appearing before the "/" symbol in primitive rules and canonical introduction rules.

Example 6. For $\pounds = \{a, b, c\}$, the canonical labelled calculus $\mathbf{G}_{abc}$ includes the primitive rules $\emptyset/\{a, b\}, \emptyset/\{b, c\}, \emptyset/\{a, c\}$, and $\{a, b, c\}/\emptyset$. It also has the following canonical introduction rules for a ternary connective $\circ$:

$$\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}/\{a, c\} : \circ(p_1, p_2, p_3)$$

$$\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}/\{b, c\} : \circ(p_1, p_2, p_3)$$

Their applications are of the forms:

$$\frac{\{a : \psi_1, c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_2\} \cup s_2}{(\{a, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2}$$

$$\frac{\{c : \psi_2\} \cup s_1 \quad \{a : \psi_3, b : \psi_3\} \cup s_2 \quad \{c : \psi_1\} \cup s_3}{(\{b, c\} : \circ(\psi_1, \psi_2, \psi_3)) \cup s_1 \cup s_2 \cup s_3}$$

Note that the canonical labelled calculi studied here are substantially more general than the signed calculi of [4] and the labelled calculi of [6], as the primitive rules of both of these families of calculi include only canonical cuts and axioms. Moreover, in the latter only introduction rules which introduce a singleton are allowed, which is not the case for the calculus in Example 6. In the former, all systems have $\emptyset/\pounds$ as their only axiom, and the set of cuts is always assumed to be $\{\{l_1\}, \{l_2\}/\emptyset \mid l_1 \neq l_2\}$ (again leaving the calculus in Example 6 out of scope).

4 Partial Non-deterministic Matrices

Non-deterministic matrices (Nmatrices) are a natural generalization of the notion of a standard many-valued matrix. These are structures, in which the truth value of a complex formula is chosen non-deterministically out of a *non-empty* set of options (determined by the truth values of its subformulas). For further discussion on Nmatrices we refer the reader to [2, 3]. In this paper we introduce a further generalization of the concept of an Nmatrix, in which this set of options is allowed to be *empty*. Intuitively, empty sets of options in a truth table corresponds to forbidding some combinations of truth values. As we shall see, this will allow us to characterize a wider class of calculi than that obtained by applying usual Nmatrices. However, as shown in the sequel, the property of *effectiveness* is preserved in PNmatrices, and like finite-valued matrices and Nmatrices, (calculi characterized by) finite-valued PNmatrices are decidable.

4.1 Introducing PNmatrices

Definition 9. *A* partial non-deterministic matrix *(*PNmatrix *for short)* $\mathcal{M}$ *for* $\mathcal{L}$ *and* $\pounds$ *consists of: (i) set* $\mathcal{V}_{\mathcal{M}}$ *of truth values, (ii) a function* $\mathcal{D}_{\mathcal{M}} : \pounds \to P(\mathcal{V}_{\mathcal{M}})$ *assigning a set of (designated) truth values to the labels of* $\pounds$*, and (iii) a function* $\diamond_{\mathcal{M}} : {\mathcal{V}_{\mathcal{M}}}^n \to P(\mathcal{V}_{\mathcal{M}})$ *for every n-ary connective* $\diamond$ *of* $\mathcal{L}$*. We say that* $\mathcal{M}$ *is* finite *if so is* $\mathcal{V}_{\mathcal{M}}$*.*

Definition 10. *Let* $\mathcal{M}$ *be a PNmatrix for* $\mathcal{L}$ *and* $\pounds$*.*

1. *An* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation is a function* $v : Frm_{\mathcal{L}} \to \mathcal{V}_{\mathcal{M}}$ *satisfying the condition* $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ *for every compound formula* $\diamond(\psi_1, \ldots, \psi_n) \in Frm_{\mathcal{L}}$*.*
2. *Let* v *be an* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation. A sequent* s *is* true *in* v *for* $\mathcal{M}$ *(denoted by* $v \models_{\mathcal{M}} s$*) if* $v(\psi) \in \mathcal{D}_{\mathcal{M}}(l)$ *for some* $l : \psi \in s$*. A set* $\mathcal{S}$ *of sequents is* true *in* v *for* $\mathcal{M}$ *(denoted by* $v \models_{\mathcal{M}} \mathcal{S}$*) if* $v \models_{\mathcal{M}} s$ *for every* $s \in \mathcal{S}$*.*
3. *Given a set of sequents* $\mathcal{S}$ *and a single sequent* s*,* $\mathcal{S} \vdash_{\mathcal{M}} s$ *if for every* $\mathcal{M}$*-legal* $\mathcal{L}$*-valuation* v*,* $v \models_{\mathcal{M}} s$ *whenever* $v \models_{\mathcal{M}} \mathcal{S}$*.*

We now define a special subclass of PNmatrices, in which no empty sets of truth values are allowed in the truth tables of logical connectives. This corresponds to the case of Nmatrices from [2–4].

Definition 11. *We say that a PNmatrix* $\mathcal{M}$ *for* $\mathcal{L}$ *and* $\pounds$ *is* proper *if* $\mathcal{V}_{\mathcal{M}}$ *is non-empty and* $\diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ *is non-empty for every n-ary connective* $\diamond$ *of* $\mathcal{L}$ *and* $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{M}}$*.*

Remark 2. Nmatrices in their original formulation can be viewed as proper PNmatrices for $\mathcal{L}$ and $\pounds$, where $\pounds$ is a singleton. In this case $\mathcal{D}_{\mathcal{M}}$ is practically a set of designated truth values. This is useful to define consequence relations between sets of formulas and formulas in the following way: $T \vdash_{\mathcal{M}} \psi$ if whenever the formulas of T are "true in v for $\mathcal{M}$" (that is $v(\varphi) \in \mathcal{D}_{\mathcal{M}}$ for every $\varphi \in T$), also ψ is "true in v for $\mathcal{M}$" ($v(\psi) \in \mathcal{D}_{\mathcal{M}}$). However, in this paper we study consequence relations of a different type, namely relations between a set of labelled sequents and a labelled sequent. We need, therefore, a notion of "being true for $\mathcal{M}$" for every $l \in \pounds$. This is achieved by taking $\mathcal{D}_{\mathcal{M}}$ to be a function from $\pounds$ to $P(\mathcal{V}_{\mathcal{M}})$. Finally, note that for simplicity of presentation, unlike in previous works, we allow the set of designated truth values (for every $l \in \pounds$) to be empty or to include all truth values in $\mathcal{V}_{\mathcal{M}}$.

Example 7. Let $\pounds = \{a, b\}$ and suppose that $\mathcal{L}$ contains one unary connective $\star$. The PNmatrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as follows: $\mathcal{V}_{\mathcal{M}_1} = \mathcal{V}_{\mathcal{M}_2} = \{t, f\}$, $\mathcal{D}_{\mathcal{M}_1}(a) = \mathcal{D}_{\mathcal{M}_2}(a) = \{t\}$ and $\mathcal{D}_{\mathcal{M}_1}(b) = \mathcal{D}_{\mathcal{M}_2}(b) = \{f\}$. The respective truth tables for $\star$ are defined as follows:

x	$\star_{\mathcal{M}_1}(x)$
t	$\{f\}$
f	$\{t, f\}$

x	$\star_{\mathcal{M}_2}(x)$
t	$\emptyset$
f	$\{t, f\}$

While both $\mathcal{M}_1$ and $\mathcal{M}_2$ are (finite) PNmatrices, only $\mathcal{M}_1$ is proper. Note that in this case we have $\{a : p_1\} \vdash_{\mathcal{M}_2} \emptyset$, simply because there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that assigns t to p_1.

Finally, we extend the notion of *simple refinements* of Nmatrices ([3]) to the context of PNmatrices:

Definition 12. *Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$. We say that $\mathcal{N}$ is a* simple refinement *of $\mathcal{M}$, denoted by $\mathcal{N} \subseteq \mathcal{M}$, if $\mathcal{V}_{\mathcal{N}} \subseteq \mathcal{V}_{\mathcal{M}}$, $\mathcal{D}_{\mathcal{N}}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{\mathcal{N}}$ for every $l \in \pounds$, and $\diamond_{\mathcal{N}}(x_1, \ldots, x_n) \subseteq \diamond_{\mathcal{M}}(x_1, \ldots, x_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{\mathcal{N}}$.*

Proposition 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be PNmatrices for $\mathcal{L}$ and $\pounds$, such that $\mathcal{N} \subseteq \mathcal{M}$. Then:* (1) *Every $\mathcal{N}$-legal $\mathcal{L}$-valuation is also $\mathcal{M}$-legal; and* (2) $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{N}}$.

4.2 Decidability

For a denotational semantics to be useful, it should be *effective*: the question of whether some conclusion follows from a finite set of assumptions, should be decidable by considering some *computable* set of partial valuations defined on some *finite* set of "relevant" formulas. Usually, the "relevant" formulas are taken as all subformulas occurring in the conclusion and the assumptions. Next, we show that the semantics induced by PNmatrices is effective in this sense.

Definition 13. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation is a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$ satisfying $v(\diamond(\psi_1, \ldots, \psi_n)) \in \diamond_{\mathcal{M}}(v(\psi_1), \ldots, v(\psi_n))$ for every formula $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$. $\models_{\mathcal{M}}$ is defined for $\mathcal{F}$-valuations exactly as for $\mathcal{L}$-valuations. We say that an $\mathcal{M}$-legal $\mathcal{F}$-valuation is* extendable in $\mathcal{M}$ *if it can be extended to an $\mathcal{M}$-legal $\mathcal{L}$-valuation.*

In proper PNmatrices, all partial valuations are extendable:

Proposition 2. *Let $\mathcal{M}$ be a* proper *PNmatrix for $\mathcal{L}$ and $\pounds$, and let $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. Then any $\mathcal{M}$-legal $\mathcal{F}$-valuation is* extendable in $\mathcal{M}$.

Proof. The proof goes exactly like the one for Nmatrices in [1]. Note that the non-emptiness of $\mathcal{V}_{\mathcal{M}}$ is needed in order to extend the empty valuation. Clearly, the different definition of $\mathcal{D}_{\mathcal{M}}$ is immaterial here. $\square$

However, this is not the case for arbitrary PNmatrices:

Example 8. Consider the PNmatrix $\mathcal{M}_2$ from Example 7. Let v be the $\mathcal{M}_2$-legal $\{p_1\}$-valuation defined by $v(p_1) = t$. Obviously, there is no $\mathcal{M}_2$-legal $\mathcal{L}$-valuation that extends v (as there is no way to assign a truth value to $\star p_1$). Thus v is not extendable in $\mathcal{M}_2$.

Theorem 1. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$ and $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. An $\mathcal{M}$-legal $\mathcal{F}$-valuation v is extendable in $\mathcal{M}$ iff v is $\mathcal{N}$-legal for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.*

Corollary 1. *Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$, a finite $\mathcal{F} \subseteq Frm_{\mathcal{L}}$ closed under subformulas. and a function $v : \mathcal{F} \to \mathcal{V}_{\mathcal{M}}$, it is decidable whether v is an $\mathcal{M}$-legal $\mathcal{F}$-valuation which is extendable in $\mathcal{M}$.*

Proof. Checking whether v is $\mathcal{M}$-legal is straightforward. To verify that it is extendable in $\mathcal{M}$, we go over all (finite) proper PNmatrices $\mathcal{N} \subseteq \mathcal{M}$ (there is a finite number of them since $\mathcal{M}$ is finite), and check whether v is $\mathcal{N}$-legal for some such $\mathcal{N}$. We return a positive answer iff we have found some $\mathcal{N} \subseteq \mathcal{M}$ such that v is $\mathcal{N}$-legal. The correctness is guaranteed by Theorem 1. □

Corollary 2. *Given a finite PNmatrix $\mathcal{M}$ for $\mathcal{L}$ and $\pounds$, a finite set $\mathcal{S}$ of sequents, and a sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathcal{M}} s$ or not.*

In the literature of Nmatrices (see e.g. [1]) effectiveness is usually identified with the property given in Proposition 2.[6] In this case Corollary 1 trivially holds: to check that v is an extendable $\mathcal{M}$-legal $\mathcal{F}$-valuation, it suffices to check that it is $\mathcal{M}$-legal, as extendability is a priori guaranteed. However, the results above show that this property is not a necessary condition for decidability. To guarantee the latter, instead of requiring that *all* partial valuations are extendable, it is sufficient to have an algorithm that establishes which of them are.

4.3 Minimality

In the next section, we show that the framework of PNmatrices provides a semantic way of characterizing canonical labelled calculi. A natural question in this context is how one can obtain *minimal* such characterizations. Next we provide lower bounds on the number of truth values that are needed to characterize $\vdash_{\mathcal{M}}$ of some PNmatrix $\mathcal{M}$ satisfying a separability condition defined below. Moreover, we provide a method to extract from a given (separable) PNmatrix an equivalent PNmatrix with the *minimal* number of truth values.

Definition 14. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$.*

1. *A truth value $x \in \mathcal{V}_{\mathcal{M}}$ is called* useful in $\mathcal{M}$ *if $x \in \mathcal{V}_{\mathcal{N}}$ for some proper PNmatrix $\mathcal{N} \subseteq \mathcal{M}$.*

[6] This property is sometimes called (semantic) *analyticity*. Note that in this paper the term 'analyticity' refers to a *proof-theoretic* property (see Definition 20).

2. *The PNmatrix $R[\mathcal{M}]$ is the simple refinement of $\mathcal{M}$, defined as follows: $\mathcal{V}_{R[\mathcal{M}]}$ consists of all truth values in $\mathcal{V}_{\mathcal{M}}$ which are useful in $\mathcal{M}$; for every $l \in \pounds$, $\mathcal{D}_{R[\mathcal{M}]}(l) = \mathcal{D}_{\mathcal{M}}(l) \cap \mathcal{V}_{R[\mathcal{M}]}$; and for every n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \mathcal{V}_{R[\mathcal{M}]}$, $\diamond_{R[\mathcal{M}]}(x_1, \ldots, x_n) = \diamond_{\mathcal{M}}(x_1, \ldots, x_n) \cap \mathcal{V}_{R[\mathcal{M}]}$.*

Proposition 3. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$, and let v be an $\mathcal{M}$-legal $\mathcal{L}$-valuation. Then:* (1) *For every formula ψ, $v(\psi)$ is useful in $\mathcal{M}$; and* (2) *Every $\mathcal{M}$-legal $\mathcal{L}$-valuation is also $R[\mathcal{M}]$-legal.*

Corollary 3. $\vdash_{\mathcal{M}} = \vdash_{R[\mathcal{M}]}$ *for every PNmatrix $\mathcal{M}$.*

Proof. One direction follows from Proposition 1, simply because $R[\mathcal{M}]$ is a simple refinement of $\mathcal{M}$ by definition. The converse is easily established using Proposition 3. We leave the details to the reader. □

Definition 15. *Let $\mathcal{M}$ be a PNmatrix for $\mathcal{L}$ and $\pounds$. We say that two truth values $x_1, x_2 \in \mathcal{V}_{\mathcal{M}}$ are* separable *in $\mathcal{M}$ for $l \in \pounds$ if $x_1 \in \mathcal{D}_{\mathcal{M}}(l) \Leftrightarrow x_2 \notin \mathcal{D}_{\mathcal{M}}(l)$ holds. $\mathcal{M}$ is called* separable *if every pair of truth values in $\mathcal{V}_{\mathcal{M}}$ are separable in $\mathcal{M}$ for some $l \in \pounds$.*

We are now ready to obtain a *lower bound* on the number of truth values needed to characterize $\vdash_{\mathcal{M}}$ for a given separable PNmatrix $\mathcal{M}$:

Theorem 2. *Let $\mathcal{M}$ be a separable PNmatrix for $\mathcal{L}$ and $\pounds$. If $\vdash_{\mathcal{M}} = \vdash_{\mathcal{N}}$ for some PNmatrix $\mathcal{N}$ for $\mathcal{L}$ and $\pounds$, then $\mathcal{N}$ contains at least $|\mathcal{V}_{R[\mathcal{M}]}|$ truth values.*

Remark 3. As done for usual matrices, it is also possible to define $\vdash_F$, the consequence relation induced by *a family of proper PNmatrices* to be $\bigcap_{\mathcal{N} \in F} \vdash_{\mathcal{N}}$. A PNmatrix can then be thought of as a succinct presentation of a family of proper PNmatrices in the following sense. The consequence relation induced by a PNmatrix $\mathcal{M}$ can be shown to be equivalent to the relation induced by the family of all the proper PNmatrices $\mathcal{N}$, such that $\mathcal{N} \subseteq \mathcal{M}$. Conversely, for every family of proper PNmatrices it is possible to construct an equivalent PNmatrix.

5 Finite PNmatrices for Canonical Labelled Systems

Definition 16. *We say that a PNmatrix $\mathcal{M}$ (for $\mathcal{L}$ and $\pounds$) is* characteristic *for a canonical labelled calculus $\mathbf{G}$ (for $\mathcal{L}$ and $\pounds$) if $\vdash_{\mathcal{M}} = \vdash_{\mathbf{G}}$.*

Next we provide a systematic way to obtain a characteristic PNmatrix $\mathcal{M}_{\mathbf{G}}$ for every canonical labelled calculus $\mathbf{G}$. The intuitive idea is as follows: the primitive rules of $\mathbf{G}$ determine the set of the truth values of $\mathcal{M}_{\mathbf{G}}$, while the introduction rules for the logical connectives dictate their corresponding truth tables. The semantics based on PNmatrices is thus *modular*: each such rule corresponds to a certain semantic condition, and the semantics of a system is obtained by joining the semantic effects of each of its derivation rules.

Definition 17. *Let $r = \{L_1, \ldots, L_n\}/L_0$ be a primitive rule for $\pounds$. Define:*

$$r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \textit{ for some } 1 \leq i \leq n \textit{ or } L_0 \cap L \neq \emptyset\}$$

Example 9. For an axiom $r = \emptyset/L_0$, we have $r^* = \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\}$. For a cut $r = \{L_1, \ldots, L_n\}/\emptyset$, $r^* = \{L \subseteq \pounds \mid L_i \cap L = \emptyset \textit{ for some } 1 \leq i \leq n\}$. In particular, continuing Examples 2 and 3 (for $\pounds = \{t, f\}$), $r^* = \{\{t\}, \{f\}, \{t, f\}\}$ for the classical axiom, and $r^* = \{\emptyset, \{t\}, \{f\}\}$ for the classical cut.

Definition 18. *Let $\diamond$ be an n-ary connective, and let $r = \mathcal{S}/L_0 : \diamond(p_1, \ldots, p_n)$ be a canonical introduction rule for $\diamond$ and $\pounds$. For every $L_1, \ldots, L_n \subseteq \pounds$, define:*

$$r^*[L_1, \ldots, L_n] = \begin{cases} \{L \subseteq \pounds \mid L_0 \cap L \neq \emptyset\} & \forall s \in \mathcal{S}.((L_1 : p_1) \cup \ldots \cup (L_n : p_n)) \cap s \neq \emptyset \\ P(\pounds) & \textit{otherwise} \end{cases}$$

Example 10. Let $\pounds = \{t, f\}$. Recall the usual introduction rules for conjunction from Example 4. By Definition 18:

$$r_1^*[L_1, L_2] = \begin{cases} \{\{f\}, \{t, f\}\} & f \in L_1 \cup L_2 \\ P(\{t, f\}) & \textit{otherwise} \end{cases}$$

$$r_2^*[L_1, L_2] = \begin{cases} \{\{t\}, \{t, f\}\} & t \in L_1 \cap L_2 \\ P(\{t, f\}) & \textit{otherwise} \end{cases}$$

Definition 19 (The PNmatrix $\mathcal{M}_{\mathbf{G}}$). *Let $\mathbf{G}$ be a canonical labelled calculus for $\mathcal{L}$ and $\pounds$. The PNmatrix $\mathcal{M}_{\mathbf{G}}$ (for $\mathcal{L}$ and $\pounds$) is defined by:*

1. $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \{L \subseteq \pounds \mid L \in r^* \textit{ for every } r \in \mathsf{P}_{\mathbf{G}}\}$.
2. *For every $l \in \pounds$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) = \{L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \mid l \in L\}$.*
3. *For every n-ary connective $\diamond$ of $\mathcal{L}$ and $L_1, \ldots, L_n \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}}$:*

$$\diamond_{\mathcal{M}_{\mathbf{G}}}(L_1, \ldots, L_n) = \{L \in \mathcal{V}_{\mathcal{M}_{\mathbf{G}}} \mid L \in r^*[L_1, \ldots, L_n] \textit{ for every } r \in \mathsf{R}_{\mathbf{G}}^{\diamond}\}$$

Example 11. Let $\pounds = \{t, f\}$ and consider the calculus $\mathbf{G}_\wedge$ whose primitive rules include only the classical axiom, and the classical cut (see Examples 2 and 3), and whose only introduction rules are the two usual rules for conjunction (see Example 4). By Example 9 and the construction above, $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_\wedge}} = \{\{t\}, \{f\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(t) = \{t\}$, and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}_\wedge}}(f) = \{f\}$. Using Example 10, we obtain the following interpretation of $\wedge$:

$\wedge_{\mathcal{M}_{\mathbf{G}_\wedge}}$	$\{t\}$	$\{f\}$
$\{t\}$	$\{t\}$	$\{f\}$
$\{f\}$	$\{f\}$	$\{f\}$

Example 12. Let $\pounds = \{a, b, c\}$, and assume that $\mathcal{L}$ contains only a unary connective $\star$. Let us start with the calculus $\mathbf{G}_0$, the primitive rules of which include the canonical axiom $\emptyset/\{a, b, c\}$ and the canonical cuts $\{a\}, \{c\}/\emptyset$ and $\{a\}, \{b\}/\emptyset$, while $\mathbf{G}_0$ has no introduction rules. Here we have $\mathcal{V}_{\mathcal{M}_{\mathbf{G}_0}} = \{\{a\}, \{b\}, \{c\}, \{b, c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(a) = \{\{a\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(b) = \{\{b\}, \{b, c\}\}$ and $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(c) = \{\{c\}, \{b, c\}\}$. $\star_{\mathcal{M}_{\mathbf{G}_0}}$ is

given in the table below (it is completely non-deterministic). One can now obtain a calculus $\mathbf{G}_1$ by adding the rule $\{a : p_1\}/\{b, c\} : \star p_1$. This leads to a refinement of the truth table, described below. Finally, one can obtain the calculus $\mathbf{G}_2$ by adding $\{b : p_1\}/\{a\} : \star p_1$, resulting in another refinement of truth table, also described below.

x	$\star_{\mathcal{M}_{\mathbf{G}_0}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_1}}(x)$	$\star_{\mathcal{M}_{\mathbf{G}_2}}(x)$
$\{a\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{b\},\{b,c\}\}$	$\{\{b\},\{b,c\}\}$
$\{b\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$
$\{c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$
$\{b,c\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\},\{b,c\}\}$	$\{\{a\}\}$

Theorem 3 (Soundness and completeness). *For every canonical labelled calculus* $\mathbf{G}$*,* $\mathcal{M}_{\mathbf{G}}$ *is a* characteristic PNmatrix *for* $\mathbf{G}$.

Corollary 4 (Decidability). *Given a canonical labelled calculus* $\mathbf{G}$*, a finite set* $\mathcal{S}$ *of sequents, and a sequent* s*, it is decidable whether* $\mathcal{S} \vdash_{\mathbf{G}} s$ *or not.*

Corollary 5. *The question whether a given canonical labelled calculus* $\mathbf{G}$ *is consistent (i.e.* $\not\vdash_{\mathbf{G}} \emptyset$*) is decidable.*

$\mathcal{M}_{\mathbf{G}}$ provides a semantic characterization for $\mathbf{G}$, however it may not be a minimal one (in terms of the number of truth values). For a minimal semantic representation, we should consider the equivalent PNmatrix $R[\mathcal{M}_{\mathbf{G}}]$:

Corollary 6 (Minimality). *For every canonical labelled calculus* $\mathbf{G}$*,* $R[\mathcal{M}_{\mathbf{G}}]$ *is a minimal (in terms of number of truth values) characteristic PNmatrix for* $\mathbf{G}$.

Proof. The claim follows by Theorem 2 from the fact that $\mathcal{M}_{\mathbf{G}}$ is separable for every system $\mathbf{G}$. □

6 Proof-Theoretic Applications

In this section we apply the semantic framework of PNmatrices to provide *decidable* semantic criteria for syntactic properties of canonical labelled calculi that are usually hard to generally characterize by other means. We focus on the notions of *analyticity* and *cut-admissibility*, extended to the context of reasoning with assumptions.

6.1 Strong Analyticity

Strong analyticity is a crucial property of a useful (propositional) calculus, as it implies its consistency and decidability. Intuitively, a calculus is *strongly analytic* if whenever a sequent s is provable in it from a set of assumptions $\mathcal{S}$, then s can be proven using only the formulas available within $\mathcal{S}$ and s.

Definition 20. *A canonical labelled calculus* $\mathbf{G}$ *is* strongly analytic *if whenever* $\mathcal{S} \vdash_{\mathbf{G}} s$*, there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *consisting solely of (sequents consisting of) formulas from* $sub[\mathcal{S} \cup \{s\}]$.

Below we provide a *decidable* semantic characterization of strong analyticity of canonical labelled calculi:

Theorem 4 (Characterization of Strong Analyticity). *Let* $\mathbf{G}$ *be a canonical labelled calculus for* $\mathcal{L}$ *and* $\pounds$*. Suppose that* $\mathbf{G}$ *does not include the (trivial) primitive rule* $\emptyset/\emptyset$*. Then,* $\mathbf{G}$ *is strongly analytic iff* $\mathcal{M}_{\mathbf{G}}$ *is proper.*

Corollary 7. *The question whether a given canonical labelled calculus is strongly analytic is decidable.*

6.2 Strong Cut-Admissibility

As the property of strong analyticity is sometimes difficult to establish, it is traditional in proof theory to investigate the property of *cut-admissibility*, which means that whenever s is provable in $\mathbf{G}$, it has a cut-free derivation in $\mathbf{G}$. In this paper we investigate a stronger notion of this property, defined as follows for labelled calculi:

Definition 21. *A labelled calculus* $\mathbf{G}$ *enjoys* strong cut-admissibility *if whenever* $\mathcal{S} \vdash_{\mathbf{G}} s$*, there exists a derivation in* $\mathbf{G}$ *of* s *from* $\mathcal{S}$ *in which only formulas from* $frm[\mathcal{S}]$ *serve as cut-formulas.*

Due to the special form of primitive and introduction rules of canonical calculi (which, except for canonical cuts, enjoy the subformula property), the above property guarantees strong analyticity:

Proposition 4. *Let* $\mathbf{G}$ *be a canonical labelled calculus. If* $\mathbf{G}$ *enjoys strong cut-admissibility, then* $\mathbf{G}$ *is strongly analytic.*

Although for two-sided canonical sequent calculi the notions of strong analyticity and strong cut-admissibility coincide (see [3]), this is not the case for general labelled calculi, for which the converse of Proposition 4 does not necessarily hold, as shown by the following example:

Example 13. Let $\pounds = \{a, b, c\}$, and assume that $\mathcal{L}$ contains only a unary connective $\star$. Let $\mathbf{G}$ be the canonical labelled calculus $\mathbf{G}$ for $\mathcal{L}$ and $\pounds$, the primitive rules of which include only the canonical cuts $\{a\}, \{b\}/\emptyset$, $\{a\}, \{c\}/\emptyset$, and $\{b\}, \{c\}/\emptyset$, and its only introduction rules are $\{a : p_1\}/\{a, b\} : \star p_1$ and $\{a : p_1\}/\{b, c\} : \star p_1$. To see that this system is strongly analytic, by Theorem 4, it suffices to construct $\mathcal{M}_{\mathbf{G}}$ and check that it is proper. The construction proceeds as follows: $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $\mathcal{D}_{\mathcal{M}_{\mathbf{G}}}(l) = \{l\}$ for $l \in \{a, b, c\}$, and the truth table for $\star$ is the following:

x	$\star_{\mathcal{M}_{\mathbf{G}}}(x)$
$\emptyset$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{a\}$	$\{\{b\}\}$
$\{b\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$
$\{c\}$	$\{\emptyset, \{a\}, \{b\}, \{c\}\}$

This is a proper PNmatrix, and so **G** is strongly analytic. However, it impossible to derive the sequent $\{b : \star p_1\}$ from the singleton set $\{\{a : p_1\}\}$ using only p_1 as a cut-formula. This is possible by applying the two introduction rules of **G** and then using the cut $\{a\}, \{c\}/\emptyset$ (with $\star p_1$ as the cut-formula). Thus although this system is strongly analytic, it does not enjoy strong cut-admissibility.

The intuitive explanation is that non-eliminable applications of canonical cuts (like the one in the above example) are not harmful for strong analyticity because they enjoy the subformula property. Thus, the equivalence between strong analyticity and cut-admissibility can be restored if we enforce the following condition:

Definition 22. *A canonical labelled calculus* $\mathbf{G}$ *for* $\mathcal{L}$ *and* $\pounds$ *is* cut-saturated *if for every canonical cut* $\{L_1, \dots, L_n\}/\emptyset$ *of* $\mathbf{G}$ *and* $l \in \pounds$*,* $\mathbf{G}$ *contains the primitive rule* $\{L_1, \dots, L_n\}/\{l\}$*.*

Proposition 5. *For every canonical labelled calculus* $\mathbf{G}$*, there is an equivalent cut-saturated canonical labelled calculus* $\mathbf{G}'$ *(i.e.* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{G}'}$*).*

Example 14. Revisiting the system from Example 13, we observe that **G** is not cut-saturated. To obtain a cut-saturated equivalent system $\mathbf{G}'$, we add (among others) the three primitive rules: $r_1 = \{\{a\}, \{b\}\}/\{c\}$, $r_2 = \{\{a\}, \{c\}\}/\{b\}$, and $r_3 = \{\{b\}, \{c\}\}/\{a\}$. Note that the addition of these rules does not affect the set of truth values, i.e., $\mathcal{V}_{\mathcal{M}_{\mathbf{G}}} = \mathcal{V}_{\mathcal{M}_{\mathbf{G}'}}$. However, we can now derive $\{b : \star p_1\}$ from $\{\{a : p_1\}\}$ without any cuts by the two introduction rules and the new rule r_2. Moreover, by Theorem 5 below, $\mathbf{G}'$ *does* enjoy strong cut-admissibility.

We are now ready to provide a decidable semantic characterization of strong cut-admissibility.

Theorem 5. *Let* $\mathbf{G}$ *be a cut-saturated canonical labelled calculus for* $\mathcal{L}$ *and* $\pounds$*. Suppose that* $\mathbf{G}$ *does not include the (trivial) primitive rule* $\emptyset/\emptyset$*. Then the following statements concerning* $\mathbf{G}$ *are equivalent: (i)* $\mathcal{M}_{\mathbf{G}}$ *is proper, (ii)* $\mathbf{G}$ *is strongly analytic, and (iii)* $\mathbf{G}$ *enjoys strong cut-admissibility.*

7 Conclusions and Further Research

Establishing proof-theoretical properties of syntactic calculi is in many cases a complex and error-prone task. For instance, proving that a calculus admits cut-elimination is often carried out using heavy syntactic arguments and many case-distinctions, leaving room for mistakes and omissions. This leads to the need of *automatizing* the process of reasoning about calculi. However, a faithful formalization is an elusive goal, as such important properties as cut-admissibility, analyticity and decidability, as well as the dependencies between them are little understood for the general case. We believe that the *abstract* view on labelled calculi taken in this paper is a substantial step towards finding the right level of abstraction for reasoning about these properties. Moreover, the simple and decidable semantic characterizations of these properties for canonical labelled calculi are a key to their faithful axiomatization in this context. To provide these characterizations, we have introduced *PNmatrices*, a generalization of Nmatrices, in which empty entries in logical truth tables are allowed, while still preserving the *effectiveness* of the semantics. A characteristic PNmatrix $\mathcal{M}_{\mathbf{G}}$ has been constructed for every canonical labelled calculus, which in turn implies its decidability. If in addition $\mathcal{M}_{\mathbf{G}}$ has no empty entries (i.e, is proper) — which is *decidable*, **G** is strongly analytic. For cut-saturated canonical calculi, the latter is also equivalent to strong cut-admissibility.

The results of this paper extend the theory of canonical sequent calculi of [2], as well as of the labelled calculi of [6] and signed calculi of [4], all of which are particular instances of canonical labelled calculi defined in this paper. Moreover, the semantics obtained for these families of calculi in the above mentioned papers, coincide with the PNmatrices semantics obtained for them here. It is particularly interesting to note that [6] provides a list of conditions, under which a labelled calculus has a characteristic finite-valued logic. These conditions include (i) reducibility of cuts (which can be shown to be equivalent to the criterion of coherence of [4]), which entails that $\mathcal{M}_{\mathbf{G}}$ is proper, and (ii) eliminability of compound axioms,[7] which entails that $\mathcal{M}_{\mathbf{G}}$ is completely deterministic (in other words, it can be identified with an ordinary finite-valued matrix). We conclude that, as shown in this paper, none of the conditions required in any of the mentioned papers [2, 6, 4] from a "well-behaved" calculus are necessary when moving to the more general semantic framework of PNmatrices, where *any* canonical labelled calculus has an effective finite-valued semantics.

An immediate direction for further research is investigating the applications of the theory of canonical labelled calculi developed here. One possibility is exploiting this theory for sequent calculi, whose rules are more complex than the canonical ones, but which can be reformulated in terms of canonical *labelled* calculi. This applies, e.g., to the large family of sequent calculi for paraconsistent logics given in [5]. For example, consider the (two-sided) Gentzen-type system $\mathbf{G_K}$ of [5] over the language $\mathcal{L}_C = \{\wedge, \vee, \supset, \circ, \neg\}$, obtained from **LK** by discarding the left rule for negation and adding the following schemas for the unary connective $\circ$:

$$(\circ \Rightarrow)\ \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \qquad \Gamma_2 \Rightarrow \neg\psi, \Delta_2}{\Gamma_1, \Gamma_2, \circ\psi \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\Rightarrow \circ)\ \frac{\Gamma, \psi, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\psi, \Delta}$$

Clearly, these schemas cannot be formulated as canonical rules in the sense of [2] (since they use $\neg\psi$ as a principal formula). However, we can reformulate $\mathbf{G_K}$ in terms of canonical labelled calculi by using the set of labels $\pounds_4 = \{t^+, t^-, f^+, f^-\}$, where t and f denote the side on which the formula occurs, and $+$ and $-$ determine whether its occurrence is positive or negative (i.e. preceded with negation). Now each (two-sided) rule of $\mathbf{G_K}$ can be translated into

[7] This property intuitively means that compound axioms can be reduced to atomic ones. It is called 'axiom-expansion' in [7].

a labelled *canonical* rule over $\pounds_4$. For instance, $(\circ \Rightarrow)$ and $(\Rightarrow \circ)$ above are translated into $\{t^+ : p_1\}, \{t^- : p_1\}/\{f^+\} : \circ p_1$ and $\{f^+ : p_1, f^- : p_1\}/\{t^+\} : \circ p_1$ respectively. Adding further rules, it can be shown that for each (non-canonical) two-sided calculus **G** from [5], an equivalent labelled *canonical* calculus $\mathbf{G}'$ can be constructed (this automatically implies the decidability of the calculi from [5]). Detailed analysis of such situations is left for future work. Another direction is generalizing the results of this paper to more complex classes of labelled calculi, e.g., like those defined in [10] for inquisitive logic. Extending the results to the first-order case is another future goal. Finally, it would be interesting to explore the relation between the systems studied in this paper and the resolution proof systems of [11].

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