Semantic Investigation of Canonical Gödel Hypersequent Systems

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To Arnon, for being a great teacher.

Abstract

We define a general family of hypersequent systems with well-behaved logical rules, of which the known hypersequent calculus for (propositional) Gödel logic, is a particular instance. We present a method to obtain (possibly, non-deterministic) many-valued semantics for every system of this family. The detailed semantic analysis provides simple characterizations of cut-admissibility and axiom-expansion for the systems of this family.

1 Introduction

Gödel logic, known also as Gödel-Dummett logic, is perhaps the most prominent intermediate logic, and one of the three fundamental fuzzy logics [14]. It was introduced in [10] both semantically, by an infinite-valued matrix, and syntactically, with a simple axiomatization, namely the extension of (an axiomatization of) intuitionistic logic with the axiom scheme $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ of linearity. The quest for a (cut-free) Gentzen-type formulation for (propositional) Gödel logic began later, and several calculi were proposed (see e.g. [18, 9, 1, 11, 7, 12]). One of the most important cut-free calculi for Gödel logic is the calculus **HG**, introduced in [2] (see also [6] and [17]). **HG** is relatively simple, especially due to the fact that its logical rules are practically the same rules as in **LJ**, the well-known sequent calculus for intuitionistic logic. This was obtained by working in the slightly richer framework of *hypersequents*, that provides a natural generalization of Gentzen's original sequents framework. The structural part of **HG** consists of all the usual structural rules, both on the sequent level (internal) and on the hypersequent level (external). In addition, it includes the *communication* rule that allows a certain sort of interplay between hypersequents, and, needless to say, the *identity rules*, i.e. the identity axiom and the (admissible) cut rule.

Well-behaved logical rules, as those of **HG**, have a great philosophical benefit. Indeed, according to a guiding principle in the philosophy of logic, attributed to Gentzen, the meanings of the connectives are determined by their derivation rules. To achieve this in (cut-free) sequent or hypersequent calculi, one should have "ideal" logical rules, each of which introduces a unique connective and does not involve other connectives. The notion of *canonical sequent rules*, introduced in [5], provides a precise formulation of the structure of these "ideal" rules in the framework of multiple-conclusion sequent calculi. *Canonical sequent systems* were in turn defined as sequent calculi that include all standard structural rules, the two identity rules, and an arbitrary set of canonical sequent rules. Clearly, **LK**, the well-known calculus for classical logic, is the most important example of a canonical sequent system. However, infinitely many new calculi with various new connectives can be defined in this framework. In [4] the *single-conclusion* counterparts of canonical rules and canonical systems were introduced, and provided a proof-theoretical approach to define constructive connectives.

In the current paper, we define *canonical Gödel systems*. First, we adopt the notion of (single-conclusion) canonical sequent rule to the hypersequents framework, and define the family of canonical *hypersequent* rules. Canonical Gödel systems are defined as (single-conclusion) hypersequent calculi that include all standard structural rules, the identity rules, the communication rule, and an arbitrary set of canonical hypersequent rules. Here **HG** is the prototype example, but again, a variety of new connectives can be defined, and be added to (or replace) the usual connectives of Gödel logic. Then, we study canonical Gödel systems from a *semantic* point of view. First and foremost, this includes a general method to obtain a (strongly) sound and complete many-valued semantics for *every* canonical Gödel system. As in [5] and [4], a major key here is the use of *non-deterministic* semantics, which intuitively occurs whenever the right-introduction rules and the left-introduction rules for a certain connective do not match, leaving some options undetermined. In addition, we also consider the semantic effect of the identity rules and provide finer semantics for canonical Gödel systems in which the identity rules are restricted to apply only on some given set of formulas. This semantics is then used to identify the "good" canonical Gödel systems, namely those that enjoy (strong) cut-admissibility. In fact, we show that the simple *coherence* criterion of [4] characterizes cut-admissibility in canonical Gödel systems as well.

2 Canonical Gödel Systems

In what follows $\mathcal{L}$ denotes a propositional language, and $Frm_{\mathcal{L}}$ stands for its set of well-formed formulas. Without loss of generality, we assume that $p_1, p_2, \ldots$ are the atomic formulas of any language $\mathcal{L}$. For our purposes, we find it most convenient to define sequents and hypersequents using *sets*, so that internal and external exchange, contraction and expansion (the converse of contraction) are built-in.

Definition 2.1. A *single-conclusion $\mathcal{L}$-sequent* is an ordered pair of finite

external weakening	$\dfrac{H}{H \mid \Gamma \Rightarrow E}$
left internal weakening	$\dfrac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \Delta \Rightarrow E}$
right internal weakening	$\dfrac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi}$
(com)	$\dfrac{H \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \quad H \mid \Gamma_1, \Gamma_2 \Rightarrow E_2}{H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2}$
(cut)	$\dfrac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma \Rightarrow E}$
(id)	$\dfrac{}{\varphi \Rightarrow \varphi}$

Table 1: Schemes of the Structural Rules

sets of $\mathcal{L}$-formulas $\langle \Gamma, E \rangle$, where E is either a singleton or empty. A *single-conclusion $\mathcal{L}$-hypersequent* is a finite set of single-conclusion $\mathcal{L}$-sequents (called *components*).

Working only in the single-conclusion framework, henceforth we omit the prefix "single-conclusion". We shall use the usual sequent notation $\Gamma \Rightarrow E$ (for $\langle \Gamma, E \rangle$) and the usual hypersequent notation $s_1 \mid \ldots \mid s_n$ (for $\{s_1, \ldots, s_n\}$). We also employ the standard abbreviations, e.g. $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $H \mid s$ instead of $H \cup \{s\}$.

As defined below, canonical Gödel systems all include the external and internal weakening rules, and the (structural) rules (com), (id) and (cut). The schemes of these rules are given in Table 1. In these schemes H is a metavariable for hypersequents, $\Gamma, \Delta, \Gamma_1, \Gamma_2$ are metavariables for finite sets of formulas, E, E_1, E_2 are metavariables for singletons or empty sets of formulas, and φ is a metavariable for formulas. Note that there are no further restrictions on applications of these rules. For example, we do not require that $\Gamma_1 \cap \Gamma_2 = \emptyset$ in applications of (com). Unlike the structural rules, the logical rules of canonical Gödel systems are not predefined (and they depend on the concrete language of the system). Next we define the general form of the allowed logical rules:

Definition 2.2. Given $n \geq 0$, an *n-clause* is a sequent s consisting solely of formulas from $\{p_1, \ldots, p_n\}$. An n-clause of the form $\Pi \Rightarrow$ is called *negative.*

Definition 2.3. An *$\mathcal{L}$-substitution* is a function $\sigma : Frm_{\mathcal{L}} \to Frm_{\mathcal{L}}$, such that $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. $\mathcal{L}$-substitutions are extended to sequents, sets of sequents, etc. in the obvious way.

Definition 2.4.

1. A *canonical right (single-conclusion hypersequent) $\mathcal{L}$-rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form: $\mathcal{S}/ \Rightarrow \diamond(p_1, \ldots, p_n)$,

where $\mathcal{S}$ is a finite set of n-clauses. An $\mathcal{L}$-*application* of the right rule $\{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}/ \Rightarrow \diamond(p_1, \ldots, p_n)$ is any inference step inferring an $\mathcal{L}$-hypersequent of the form $H \mid \Gamma \Rightarrow \sigma(\diamond(p_1, \ldots, p_n))$ from the set of $\mathcal{L}$-hypersequents $\{H \mid \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$, where H is an arbitrary $\mathcal{L}$-hypersequent, Γ is an arbitrary finite set of $\mathcal{L}$-formulas, and σ is an $\mathcal{L}$-substitution.

2. A *canonical left (single-conclusion hypersequent) $\mathcal{L}$-rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form: $\mathcal{S}_1, \mathcal{S}_2 / \diamond(p_1, \ldots, p_n) \Rightarrow$, where $\mathcal{S}_1$ is a finite set of n-clauses, and $\mathcal{S}_2$ is a finite set of negative n-clauses. An $\mathcal{L}$-*application* of the left rule $\{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}, \{\Sigma_i \Rightarrow \ : 1 \leq i \leq m\} / \diamond(p_1, \ldots, p_n) \Rightarrow$ is any inference step inferring an $\mathcal{L}$-hypersequent of the form $H \mid \Gamma, \sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow E$ from the sets of $\mathcal{L}$-hypersequents $\{H \mid \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$ and $\{H \mid \Gamma, \sigma(\Sigma_i) \Rightarrow E : 1 \leq i \leq m\}$, where H is an arbitrary $\mathcal{L}$-hypersequent, Γ is an arbitrary finite set of $\mathcal{L}$-formulas, E is an empty or singleton set of $\mathcal{L}$-formulas, and σ is an $\mathcal{L}$-substitution.

As usual, if the language is clear from the context we may omit the prefix $\mathcal{L}$ when referring to the above mentioned concepts.

Remark 2.5. Note that the premises of left canonical rules are divided into *two* sets, $\mathcal{S}_1$ and $\mathcal{S}_2$, where non-negative clauses are allowed only in $\mathcal{S}_1$. The difference between the two sets is in the application of the rule: while the premises of $\mathcal{S}_2$ allow the addition of a right context formula, this is forbidden for the premises of $\mathcal{S}_1$ (see [4] for some examples of this difference in the context of *sequent* calculi).

Remark 2.6. Recall that since we define sequents using sets, contraction is implicit. Thus, in applications of left rules it might happen that the introduced formula $\sigma(\diamond(p_1, \ldots, p_n))$ already appears in the left context Γ (the conclusion of such an application has the form $H \mid \Gamma \Rightarrow E$).

In Table 2, we present all logical rules of the hypersequent system **HG** for the standard propositional Gödel logic (see [6]) as canonical rules. It is also possible to introduce new connectives using canonical rules:

Example 2.7. A primal-implication connective (see [13]) can be introduced with the following two rules:

$$\{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \rightsquigarrow p_2 \qquad \{\Rightarrow p_1\}, \{p_2 \Rightarrow\} \ / \ p_1 \rightsquigarrow p_2 \Rightarrow$$

Applications of the left rule are like those of the left rule of implication in **HG**, while applications of the right rule allow us to infer a hypersequent of the form $H \mid \Gamma \Rightarrow \varphi_1 \rightsquigarrow \varphi_2$ from a hypersequent of the form $H \mid \Gamma \Rightarrow \varphi_2$.

Example 2.8. It is possible to combine the usual right rule for conjunction, and the usual left rule for disjunction, and introduce a new binary "asterisk connective" with the following two rules:

$$\{ \Rightarrow p_1 \ , \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 * p_2 \qquad \emptyset, \{p_1 \Rightarrow \ , \ p_2 \Rightarrow \ \} \ / \ p_1 * p_2 \Rightarrow$$

Canonical Rule	Application scheme
$\emptyset, \emptyset \ / \ \perp\Rightarrow$	$\dfrac{}{H \mid \Gamma, \perp\Rightarrow E}$
$\{\Rightarrow p_1 \ , \ \Rightarrow p_2\} \ / \ \Rightarrow p_1 \wedge p_2$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \wedge \varphi_2}$
$\emptyset, \{p_1, p_2 \Rightarrow \ \} \ / \ p_1 \wedge p_2 \Rightarrow$	$\dfrac{H \mid \Gamma, \varphi_1, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E}$
$\{\Rightarrow p_1\} \ / \ \Rightarrow p_1 \vee p_2$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_1}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}$
$\{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \vee p_2$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}$
$\emptyset, \{p_1 \Rightarrow \ , \ p_2 \Rightarrow \ \} \ / \ p_1 \vee p_2 \Rightarrow$	$\dfrac{H \mid \Gamma, \varphi_1 \Rightarrow E \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \vee \varphi_2 \Rightarrow E}$
$\{p_1 \Rightarrow p_2\} \ / \ \Rightarrow p_1 \supset p_2$	$\dfrac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2}$
$\{\Rightarrow p_1\}, \{ p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow$	$\dfrac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \supset \varphi_2 \Rightarrow E}$

Table 2: The Logical Rules of **HG**

Applications of the right rule are like those of the right rule of conjunction in **HG**, while applications of the left rule are like those of the left rule of disjunction in **HG** (see Table 2).

We can now define *canonical Gödel systems*:

Definition 2.9. A *canonical Gödel system* for $\mathcal{L}$ is a (single-conclusion) hypersequent calculus that includes all structural rules from Table 1, and an arbitrary finite set of canonical $\mathcal{L}$-rules (either right canonical rules or left ones). Given a canonical Gödel system $\mathbf{G}$, we write $\mathcal{H} \vdash_{\mathbf{G}} H$ if there exists a proof of the hypersequent H from the set $\mathcal{H}$ of hypersequents (called *assumptions*). As usual, a proof of H from $\mathcal{H}$ in $\mathbf{G}$ is a sequence of hypersequents ending with H, each of which is an assumption from $\mathcal{H}$ or inferred from previous hypersequents by applying one of the rules of $\mathbf{G}$.

Remark 2.10. Since the weakening rules are present in every canonical Gödel system, it is always possible to incorporate external weakenings and left internal weakenings in the applications of the rules. Thus for example, we could have defined an application of a canonical right rule as an inference step deriving $H \mid \Gamma \Rightarrow \sigma(\diamond(p_1, \dots, p_n))$ from the set $\{H_i \mid \Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$ of hypersequents, where $H, H_1, \dots, H_k$ are hypersequents such that $H_1, \dots, H_k \subseteq H$, $\Gamma, \Gamma_1, \dots, \Gamma_k$ are finite set of formulas such that $\Gamma_1, \dots, \Gamma_k \subseteq \Gamma$, and σ is a substitution. A similar definition is possible for the left canonical rules and for (*com*) and (*cut*). (In the case of (*com*), the equivalent definition allows us to derive $H \mid \Gamma'_1 \Rightarrow E_1 \mid \Gamma'_2 \Rightarrow E_2$

from $H_1 \mid \Gamma_1, \Delta_1 \Rightarrow E_1$ and $H_2 \mid \Gamma_2, \Delta_2 \Rightarrow E_2$ where $H_1, H_2 \subseteq H$, $\Gamma_1, \Delta_2 \subseteq \Gamma'_1$ and $\Gamma_2, \Delta_1 \subseteq \Gamma'_2$.) Henceforth, we may use freely this kind of applications (which formally might involve additional applications of the weakening rules).

By definition, proofs in canonical Gödel systems admit arbitrary applications of (*cut*) and (*id*). Moreover, any formula of the language can appear in proofs, regardless of the set of assumptions and the proven hypersequent. However, proof theoretic properties often involve restricted proofs, in which some formulas are not allowed to appear (as in the subformula property), or some applications of (*cut*) and (*id*) are forbidden (as in cut-admissibility and axiom-expansion). To uniformly handle these kinds of restricted proofs, we introduce the notion of a *proof-specification*:

Definition 2.11. Let $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$ be an ordered pair of sets of formulas. An $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequent is a sequent $\Gamma \Rightarrow E$ for which $\Gamma \subseteq \mathcal{E}_l$ and $E \subseteq \mathcal{E}_r$. An $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent is a hypersequent consisting solely of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents.

Definition 2.12. A *proof-specification* is a quadruple of sets of formulas $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$, such that $\mathcal{C} \cup \mathcal{A} \subseteq \mathcal{E}_l \cap \mathcal{E}_r$. Given a proof-specification $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$, a proof P in a canonical Gödel system $\mathbf{G}$ is called a *ρ-proof* if the following conditions hold:

1. Only $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequents occur in P.
2. For every application of (*cut*) in P inferring $H \mid \Gamma \Rightarrow E$ from $H \mid \Gamma, \varphi \Rightarrow E$ and $H \mid \Gamma \Rightarrow \varphi$, we have that φ, the *cut-formula*, is in $\mathcal{C}$.
3. For every application of (*id*) in P inferring $\varphi \Rightarrow \varphi$, we have that $\varphi \in \mathcal{A}$.

We write $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H$ if there exists a ρ-proof in $\mathbf{G}$ of H from $\mathcal{H}$.

Clearly, $\mathcal{H} \vdash_{\mathbf{G}} H$ iff $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H$ for $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}} \rangle$. We end this section with two propositions that will turn out to be useful in connection with the proof of Theorem 4.10 below.

Proposition 2.13 (Generalized Communication)**.** Let $\mathbf{G}$ be a canonical Gödel system, and let $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$ be a proof-specification. For all $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequents H_1, H_2, numbers $n, m \geq 0$, $n + m$ singleton or empty sets of formulas $E_1, \ldots, E_n, F_1, \ldots, F_m \subseteq \mathcal{E}_r$, and finite subsets $\Gamma_1, \Delta_1, \Gamma_2, \Delta_2$ of $\mathcal{E}_l$,

$$H_1 \mid H_2 \mid \Gamma_1, \Delta_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Delta_2 \Rightarrow E_n \mid \Gamma_2, \Delta_1 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \Delta_1 \Rightarrow F_m$$

has a ρ-proof in $\mathbf{G}$ from the following two hypersequents:

$$H_1 \mid \Gamma_1, \Delta_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Delta_1 \Rightarrow E_n,$$

$$H_2 \mid \Gamma_2, \Delta_2 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \Delta_2 \Rightarrow F_m.$$

Proof. See Proposition 22 in [16]. Note that only weakenings and (*com*) are used. □

Proposition 2.14. Let $\mathbf{G}$ be a canonical Gödel system, and let $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$ be a proof-specification. Let

$$r = \{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}, \{\Sigma_i \Rightarrow \ : 1 \leq i \leq m\} / \diamond(p_1, \ldots, p_n) \Rightarrow$$

be a left rule of $\mathbf{G}$, with $m > 0$. Let σ be a substitution such that $\psi = \sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{E}_l$, $\sigma(\Pi_i \Rightarrow E_i)$ is an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequent for every $1 \leq i \leq k$, and $\sigma(\Sigma_i) \subseteq \mathcal{E}_l$ for every $1 \leq i \leq m$. Let Γ be a finite subset of $\mathcal{E}_l$, and H be an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent. Denote by $\mathcal{H}_1$ the set $\{H \mid \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$. Then, for every $n_1, \ldots, n_m \geq 0$, $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent H' such that $H \subseteq H'$, and $n_1 + \ldots + n_m$ singleton or empty sets $F_1^1, \ldots, F_{n_1}^1, \ldots, F_1^m, \ldots, F_{n_m}^m \subseteq \mathcal{E}_r$,

$$H' \mid \Gamma, \psi \Rightarrow F_1^1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_{n_1}^1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_1^m \mid \ldots \mid \Gamma, \psi \Rightarrow F_{n_m}^m$$

has a ρ-proof in $\mathbf{G}$ from

$$\mathcal{H}_1 \cup \{H' \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_1^i \mid \ldots \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_{n_i}^i : 1 \leq i \leq m\}.$$

Proof. First, if $n_i = 0$ for some $1 \leq i \leq m$, the claim follows by applying external weakening on H'. Next, we prove the claim for the case that $n_1 = n_2 = \ldots = n_m = 1$, by induction on the size S of the set $\{F_1^1, \ldots, F_1^m\}$. If $S = 1$, then one application of r suffices. Now let $S \geq 2$, and assume that the claim holds for sets of size $S - 1$. Let $F_1^1, \ldots, F_1^m \subseteq \mathcal{E}_r$ be singleton or empty sets such that $|\{F_1^1, \ldots, F_1^m\}| = S$, and let H' be some $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent such that $H \subseteq H'$. Denote:

$$G_0 = H' \mid \Gamma, \psi \Rightarrow F_1^1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_1^m,$$

and for every $1 \leq i \leq m$,

$$G_i = H' \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_1^i.$$

Let i_1, i_2 be two indices such that $F_1^{i_1} \neq F_1^{i_2}$, and let $I_1 = \{1 \leq i \leq m : F_1^i = F_1^{i_1}\}$ and $I_2 = \{1 \leq i \leq m : F_1^i = F_1^{i_2}\}$. For every $j_1 \in I_1$ and $j_2 \in I_2$, we have that (using one application of (com)):

$$\{G_{j_1}, G_{j_2}\} \vdash_{\mathbf{G}}^{\rho} H' \mid \Gamma, \sigma(\Sigma_{j_2}) \Rightarrow F_1^{i_1} \mid \Gamma, \sigma(\Sigma_{j_1}) \Rightarrow F_1^{i_2}.$$

For every $j_1 \in I_1$, the induction hypothesis and the availability of external weakening entail that $G_0 \mid \Gamma, \sigma(\Sigma_{j_1}) \Rightarrow F_1^{i_2}$ has a ρ-proof in $\mathbf{G}$ from

$$\mathcal{H}_1 \cup \{G_j : j \notin I_2\} \cup \{H' \mid \Gamma, \sigma(\Sigma_{j_2}) \Rightarrow F_1^{i_1} \mid \Gamma, \sigma(\Sigma_{j_1}) \Rightarrow F_1^{i_2} : j_2 \in I_2\}.$$

The induction hypothesis and the availability of external weakening again imply that G_0 has a ρ-proof in $\mathbf{G}$ from

$$\mathcal{H}_1 \cup \{G_j : j \notin I_1\} \cup \{G_0 \mid \Gamma, \sigma(\Sigma_{j_1}) \Rightarrow F_1^{i_2} : j_1 \in I_1\}.$$

Together, we have

$$\mathcal{H}_1 \cup \{G_i : 1 \leq i \leq m\} \vdash_{\mathbf{G}}^{\rho} G_0.$$

Next we prove the claim for any $n_1, \ldots, n_m \geq 1$ by induction on $n_1 + \ldots + n_m$. Assume that $n_1 + \ldots + n_m = l$ and that the claim holds for every $n_1, \ldots, n_m$ such that $n_1 + \ldots + n_m < l$. Let H' be some $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent such that $H \subseteq H'$, and $F_1^1, \ldots, F_{n_1}^1, \ldots, F_1^m, \ldots, F_{n_m}^m \subseteq \mathcal{E}_r$ be singleton or empty sets. Denote:

$$G_0 = H' \mid \Gamma, \psi \Rightarrow F_1^1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_{n_1}^1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_1^m \mid \ldots \mid \Gamma, \psi \Rightarrow F_{n_m}^m,$$

$$\mathcal{H} = \{H' \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_1^i \mid \ldots \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_{n_i}^i : 1 \leq i \leq m\}.$$

For every $1 \leq i \leq m$, the induction hypothesis and the availability of external weakening entail that

$$\mathcal{H}_1 \cup \mathcal{H} \vdash_{\mathbf{G}}^{\rho} G_0 \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_1^i.$$

By the proof for the case $n_1 = n_2 = \ldots = n_m = 1$, we have that

$$\mathcal{H}_1 \cup \{G_0 \mid \Gamma, \sigma(\Sigma_i) \Rightarrow F_1^i : 1 \leq i \leq m\} \vdash_{\mathbf{G}}^{\rho} G_0. \qquad \square$$

Example 2.15. Suppose that $\mathbf{G}$ includes the left rule $\emptyset, \{p_1 \Rightarrow \ , \ p_2 \Rightarrow \} \ / \ p_1 * p_2 \Rightarrow$ (see Example 2.8). By Proposition 2.14, the following rule (given by a scheme) is cut-free derivable in $\mathbf{G}$:

$$\frac{H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_{n_1} \quad H \mid \Gamma, \psi \Rightarrow F_1 \mid \ldots \mid \Gamma, \psi \Rightarrow F_{n_2}}{H \mid \Gamma, \varphi * \psi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi * \psi \Rightarrow E_{n_1} \mid \Gamma, \varphi * \psi \Rightarrow F_1 \mid \ldots \mid \Gamma, \varphi * \psi \Rightarrow F_{n_2}}$$

3 Many-Valued Semantics

In this section we provide a method to obtain sound and complete many-valued semantics for any given canonical Gödel system.

Definition 3.1. A *(Gödel) valuation* U (for $\mathcal{L}$) is a function from $Frm_{\mathcal{L}}$ to some non-empty set V_U (of *truth values*) linearly ordered by $\leq_U$, with a maximal element 1_U and a minimal element 0_U.

Notation 3.2. Let U be a valuation. Given a finite set $V \subseteq V_U$, we denote by $\min_U(V)$ the minimum of V with respect to $\leq_U$, where $\min_U(\emptyset)$ is defined to be 1_U. Similarly, $\max_U(V)$ stands for the maximum, where $\max_U(\emptyset)$ is defined to be 0_U. For every two elements $u_1, u_2 \in V_U$, $u_1 \rightarrow_U u_2$ is defined to be 1_U if $u_1 \leq_u u_2$, and $u_1 \rightarrow_U u_2 = u_2$ otherwise.

The canonical rules of a given canonical Gödel system induce a "truth table" for each connective, or more generally, they enforce some restrictions on values of compound formulas. The right rules for a connective $\diamond$ are used to compute a lower bound on the value assigned to $\diamond$-formulas. Symmetrically, the left rules for $\diamond$ are used to compute an upper bound on this value. This is formulated in the next definition.

Definition 3.3. Let U be a valuation.

1. Given a finite set $\Gamma \subseteq Frm_{\mathcal{L}}$:

$$U^l(\Gamma) = \min_U\{U(\psi) : \psi \in \Gamma\} \text{ and } U^r(\Gamma) = \max_U\{U(\psi) : \psi \in \Gamma\}.$$

2. Given a sequent $\Gamma \Rightarrow E$: $U(\Gamma \Rightarrow E) = U^l(\Gamma) \to_U U^r(E)$.

3. U *respects* a canonical right rule $\mathcal{S}/ \Rightarrow \diamond(p_1, \ldots, p_n)$ iff for every substitution σ:

$$\min_U\{U(s) : s \in \sigma(\mathcal{S})\} \leq_U U(\sigma(\diamond(p_1, \ldots, p_n)))$$

4. U *respects* a canonical left rule $\mathcal{S}_1, \mathcal{S}_2/ \diamond(p_1, \ldots, p_n) \Rightarrow$ iff for every substitution σ:

$$U(\sigma(\diamond(p_1, \ldots, p_n))) \leq_U$$
$$\min_U\{U(s) : s \in \sigma(\mathcal{S}_1)\} \to_U \max_U\{U^l(\Gamma) : \Gamma \Rightarrow \in \sigma(\mathcal{S}_2)\}$$

5. U is called **G**-*legal* iff it respects all canonical rules of a canonical Gödel system **G**.

Note that we may obtain *non-deterministic semantics*, where the values assigned to a formula ψ by a legal valuation are not uniquely determined by the values assigned to the subformulas of ψ. A deterministic semantics is obtained only when the lower bound determined by the right rules is equal to the upper bound determined by the left rules.

Example 3.4 (Conjunction)**.** Let U be a valuation for a language $\mathcal{L}$ that includes a binary connective $\wedge$. Consider the usual right rule for $\wedge$: $\{\Rightarrow p_1\ ,\ \Rightarrow p_2\}\ /\ \Rightarrow p_1 \wedge p_2$. For every substitution σ,

$$U(\sigma(\Rightarrow p_1)) = U^l(\emptyset) \to_U U^r(\{\sigma(p_1)\}) = 1_U \to_U U(\sigma(p_1)) = U(\sigma(p_1)),$$

and similarly $U(\sigma(\Rightarrow p_2)) = U(\sigma(p_2))$. It follows that U respects this rule iff for every φ_1 and φ_2: $\min_U\{U(\varphi_1), U(\varphi_2)\} \leq_U U(\varphi_1 \wedge \varphi_2)$. Now, consider the usual left rule for $\wedge$: $\emptyset, \{p_1, p_2 \Rightarrow\ \}\ /\ p_1 \wedge p_2 \Rightarrow$. For every substitution σ,

$$\begin{aligned}\min_U \emptyset \to_U \max_U\{U^l(\{\sigma(p_1), \sigma(p_2)\})\} &= 1_U \to_U \min_U\{U(\sigma(p_1)), U(\sigma(p_2))\} \\ &= \min_U\{U(\sigma(p_1)), U(\sigma(p_2))\}.\end{aligned}$$

It follows that U respects this rule iff for every φ_1 and φ_2: $U(\varphi_1 \wedge \varphi_2) \leq_U \min_U\{U(\varphi_1), U(\varphi_2)\}$. Therefore, it respects both rules for $\wedge$ iff $U(\varphi_1 \wedge \varphi_2) = \min_U\{U(\varphi_1), U(\varphi_2)\}$ for every φ_1 and φ_2. Thus we obtain the usual semantics of $\wedge$ as a particular instance (this is the case for each of the usual connectives).

Example 3.5 (Implication)**.** Let U be a valuation for a language $\mathcal{L}$ that includes a binary connective $\supset$. Consider the usual right rule for $\supset$: $\{p_1 \Rightarrow p_2\}$ / $\Rightarrow p_1 \supset p_2$. For every substitution σ,

$$U(\sigma(p_1) \Rightarrow \sigma(p_2)) = U^l(\{\sigma(p_1)\}) \to_U U^r(\{\sigma(p_2)\}) = U(\sigma(p_1)) \to_U U(\sigma(p_2)).$$

It follows that U respects this rule iff for every φ_1 and φ_2: $U(\varphi_1) \to_U U(\varphi_2) \leq_U U(\varphi_1 \supset \varphi_2)$. Now, consider the usual left rule for $\supset$: $\{\Rightarrow p_1\}, \{ p_2 \Rightarrow\}$ / $p_1 \supset p_2 \Rightarrow$. For every substitution σ, $U(\sigma(\Rightarrow p_1)) = U(\sigma(p_1))$, and $U^l(\{\sigma(p_2)\}) = U(\sigma(p_2))$. It follows that U respects this rule iff for every φ_1 and φ_2: $U(\varphi_1 \supset \varphi_2) \leq_U U(\varphi_1) \to_U U(\varphi_2)$.

Example 3.6 (Primal-Implication)**.** Let U be a valuation for a language $\mathcal{L}$ that includes a binary connective $\rightsquigarrow$. Consider the right rule for $\rightsquigarrow$: $\{\Rightarrow p_2\}$ / $\Rightarrow p_1 \rightsquigarrow p_2$ (see Example 2.7). U respects this rule iff for every φ_1 and φ_2: $U(\varphi_2) \leq_U U(\varphi_1 \rightsquigarrow \varphi_2)$. Now, the left rule is the same as the rule for $\supset$, and thus it enforces the same condition as in the previous example. Therefore, it respects both rules of $\rightsquigarrow$ iff $U(\varphi_2) \leq_U U(\varphi_1 \rightsquigarrow \varphi_2) \leq_U U(\varphi_1) \to_U U(\varphi_2)$ for every φ_1 and φ_2. Note that $\rightsquigarrow$ has a non-deterministic semantics.

Example 3.7 (Asterisk)**.** Let U be a valuation for a language $\mathcal{L}$ that includes a binary connective $*$. Consider the left rule for $*$: $\emptyset, \{p_1 \Rightarrow\ , \ p_2 \Rightarrow\}$ / $p_1 * p_2 \Rightarrow$ (see Example 2.8). U respects this rule iff for every φ_1 and φ_2: $U(\varphi_1 * \varphi_2) \leq_U \max_U\{U(\varphi_1), U(\varphi_2)\}$. Now, the left rule is the same as the rule for $\wedge$, and thus it enforces the same condition as in Example 3.4. Therefore, it respects both rules of $*$ iff $\min_U\{U(\varphi_1), U(\varphi_2)\} \leq_U U(\varphi_1 * \varphi_2) \leq_U \max_U\{U(\varphi_1), U(\varphi_2)\}$ for every φ_1 and φ_2. This connective provides another example of a *non-deterministic* connective.

Our first main result is the following general soundness and completeness theorem. Its proof is omitted since we prove a more general result in the next section.

Definition 3.8. A valuation U is a *model* of a sequent $\Gamma \Rightarrow E$ if $U^l(\Gamma) \leq_U U^r(E)$. U is a model of a hypersequent H if it is a model of some component $s \in H$.

Theorem 3.9. For every canonical Gödel system $\mathbf{G}$, set $\mathcal{H}$ of hypersequents, and a hypersequent H: $\mathcal{H} \vdash_{\mathbf{G}} H$ iff every $\mathbf{G}$-legal valuation which is a model of every hypersequent in $\mathcal{H}$ is also a model of H.

4 Many-Valued Semantics for Proof-Specifications

In this section we generalize the method of the previous section in order to obtain sound and complete many-valued semantics for any given canonical Gödel system and *proof-specification.* This will be helpful to obtain semantic characterizations of proof-theoretical properties in Section 5. The semantics is based on the notion of *Gödel ρ-valuations* defined as follows:

Definition 4.1. Let $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$ be a proof-specification. A *(Gödel) ρ-valuation* U (for $\mathcal{L}$) consists of:

1. A non-empty set V_U (of *truth values*) linearly ordered by $\leq_U$, with a maximal element 1_U and a minimal element 0_U.
2. Two functions: $U^l : \mathcal{E}_l \to V_U$ and $U^r : \mathcal{E}_r \to V_U$ satisfying the following conditions:
   (*a*) $U^r(\psi) \leq_U U^l(\psi)$ for every $\psi \in \mathcal{C}$.
   (*b*) $U^l(\psi) \leq_U U^r(\psi)$ for every $\psi \in \mathcal{A}$.

Notation 4.2. Given a ρ-valuation U, $\geq_U$, $<_U$, and $>_U$ are defined in the obvious way. $\min_U, \max_U$ and $\to_U$ are defined as for valuations (Notation 3.2).

Intuitively, U^l is used to assign values for the formulas occurring on the left sides of components of hypersequents, and U^r for those occurring on the right side. The given proof-specification determines the domain of U^l and U^r. Note that it can happen that $U^l(\psi) \neq U^r(\psi)$ for some formula ψ. This is the key for providing sound and complete semantics for proof-specifications, in which (*cut*) and (*id*) are restricted. Intuitively, (*cut*) and (*id*) are dual – while (*cut*) forces the value on the left to be greater than or equal to the value on the right, (*id*) forces the value on the left to be lower than or equal to the value on the right. When proofs are not restricted at all (so both (*cut*) and (*id*) can be used for all formulas), U^l and U^r are full functions, and $U^l(\psi) = U^r(\psi)$ for every $\psi \in Frm_{\mathcal{L}}$. Thus Gödel valuations (Definition 3.1) are clearly identified with ρ-valuations for $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}} \rangle$. In this case, we write $U(\psi)$ instead of $U^l(\psi)$ (equivalently, $U^r(\psi)$).

Next, we turn to the semantic effect of the canonical rules included in the given canonical Gödel system. We adapt Definition 3.3 for ρ-valuations:

Definition 4.3. Let U be a ρ-valuation for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$.

1. Given a finite set $\Gamma \subseteq \mathcal{E}_l$: $U^l(\Gamma) = \min_U \{U^l(\psi) : \psi \in \Gamma\}$.
2. Given a finite set $\Gamma \subseteq \mathcal{E}_r$: $U^r(\Gamma) = \max_U \{U^r(\psi) : \psi \in \Gamma\}$.
3. Given an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequent $\Gamma \Rightarrow E$: $U(\Gamma \Rightarrow E) = U^l(\Gamma) \to_U U^r(E)$.
4. Given a finite set $\mathcal{S}$ of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents: $U(\mathcal{S}) = \min_U \{U(s) : s \in \mathcal{S}\}$, and $U^N(\mathcal{S}) = \max_U \{U^l(\Gamma) : \Gamma \Rightarrow E \in \mathcal{S}\}$.

Definition 4.4. Let U be a ρ-valuation for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$.

1. U *respects* a canonical right rule $\mathcal{S}/ \Rightarrow \diamond(p_1, \ldots, p_n)$ iff for every substitution σ for which $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{E}_r$ and $\sigma(\mathcal{S})$ is a set of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents, we have $U(\sigma(\mathcal{S})) \leq_U U^r(\sigma(\diamond(p_1, \ldots, p_n)))$.
2. U *respects* a canonical left rule $\mathcal{S}_1, \mathcal{S}_2 / \diamond(p_1, \ldots, p_n) \Rightarrow$ iff for every substitution σ for which $\sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{E}_l$, and $\sigma(\mathcal{S}_1)$ and $\sigma(\mathcal{S}_2)$ are sets of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents, we have $U^l(\sigma(\diamond(p_1, \ldots, p_n))) \leq_U U(\sigma(\mathcal{S}_1)) \to_U U^N(\sigma(\mathcal{S}_2))$.

3. U is called **G**-*legal* iff it respects all canonical rules of a canonical Gödel system **G**.

Note that when (*cut*) and/or (*id*) are not allowed to be applied on some formula $\diamond(\psi_1, \ldots, \psi_n)$, we obtain *non-deterministic semantics* by definition, since either $U^l(\diamond(\psi_1, \ldots, \psi_n))$ or $U^r(\diamond(\psi_1, \ldots, \psi_n))$ is restricted only from one side.

Example 4.5 (Conjunction)**.** Let U be a ρ-valuation for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$. Consider the usual right rule for $\wedge$: $\{ \Rightarrow p_1 \ , \ \ \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \wedge p_2$. Let σ be a substitution, assigning φ_1 to p_1 and φ_2 to p_2. We have

$$U(\sigma(\Rightarrow p_1)) = U(\Rightarrow \varphi_1) = U^l(\emptyset) \to_U U^r(\{\varphi_1\}) = 1_U \to_U U^r(\varphi_1) = U^r(\varphi_1).$$

Similarly, $U(\sigma(\Rightarrow p_2)) = U^r(\varphi_2)$. Thus

$$U(\sigma(\{ \Rightarrow \varphi_1 \ , \ \ \Rightarrow \varphi_2\})) = \min_U \{U^r(\varphi_1), U^r(\varphi_2)\}.$$

It follows that U respects this rule iff for every substitution σ: if we have $\{\sigma(p_1 \wedge p_2), \sigma(p_1), \sigma(p_2)\} \subseteq \mathcal{E}_r$, then we have $\min_U\{U^r(\sigma(p_1)), U^r(\sigma(p_2))\} \leq_U U^r(\sigma(p_1 \wedge p_2))$. Now, consider the usual left rule for $\wedge$: $\emptyset, \{p_1, p_2 \Rightarrow \ \} \ / \ p_1 \wedge p_2 \Rightarrow$. Again, let σ be a substitution, assigning φ_1 to p_1 and φ_2 to p_2. We have $U(\sigma(\emptyset)) = U(\emptyset) = 1_U$, and

$$\begin{aligned} U^N(\sigma(\{p_1, p_2 \Rightarrow \ \})) &= U^N(\{\varphi_1, \varphi_2 \Rightarrow \ \}) \\ &= U^l(\{\varphi_1, \varphi_2\}) = \min_U \{U^l(\varphi_1), U^l(\varphi_2)\}. \end{aligned}$$

Thus,

$$\begin{aligned} U(\sigma(\mathcal{S}_1)) \to_U U^N(\sigma(\mathcal{S}_2)) &= 1_U \to_U \min_U \{U^l(\varphi_1), U^l(\varphi_2)\} \\ &= \min_U \{U^l(\varphi_1), U^l(\varphi_2)\}. \end{aligned}$$

It follows that U respects this rule iff for every substitution σ: if we have $\{\sigma(p_1 \wedge p_2), \sigma(p_1), \sigma(p_2)\} \subseteq \mathcal{E}_l$, then we have $U^l(\sigma(p_1 \wedge p_2)) \leq_U \min_U\{U^l(\sigma(p_1)), U^l(\sigma(p_2))\}$.

We are now ready to define the semantic consequence relation induced by a canonical Gödel system and a proof-specification.

Definition 4.6. Let U be a ρ-valuation for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$. U is a *model* of:

- an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-*sequent* $\Gamma \Rightarrow E$ if $U^l(\Gamma) \leq_U U^r(E)$ (equivalently, $U(\Gamma \Rightarrow E) = 1_U$).
- a *hypersequent* H if H is an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent and U is a model of some component $s \in H$.
- a *set* $\mathcal{H}$ *of hypersequents* if it is a model of every $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent $H \in \mathcal{H}$.

We write $U \models X$ to denote that U is a model of X (here X is either a sequent, a hypersequent, or a set of hypersequents).[1]

Definition 4.7. Let $\mathbf{G}$ be a canonical Gödel system, and $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$ be a proof-specification. Given a set $\mathcal{H}$ of hypersequents and a hypersequent H, we write $\mathcal{H} \Vdash^{\rho}_{\mathbf{G}} H$ iff H is an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent, and every $\mathbf{G}$-legal ρ-valuation which is a model of $\mathcal{H}$ is also a model of H.

Our next goal is to show that the semantic consequence relation induced by a canonical Gödel system and a proof-specification indeed coincides with the corresponding derivability relation.

Theorem 4.8 (Strong Soundness)**.** For every canonical Gödel system $\mathbf{G}$ and proof-specification ρ, $\vdash^{\rho}_{\mathbf{G}} \subseteq \Vdash^{\rho}_{\mathbf{G}}$.

Proof. Let $\mathcal{H}$ be a set of hypersequents and H_0 be a hypersequent. Assume that there exists a ρ-proof P in $\mathbf{G}$ of H_0 from $\mathcal{H}$, where $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$. Obviously, H_0 is an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent. Now let U be a $\mathbf{G}$-legal ρ-valuation which is a model of $\mathcal{H}$. Using induction on the length of P, we show that $U \models H$ for every hypersequent H appearing in P. It then follows that $U \models H_0$. Note that since all hypersequents in P are $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequents, it suffices to prove that U is a model of some component $s \in H$ for every hypersequent H appearing in P. This trivially holds for the hypersequents of $\mathcal{H}$ that appear in P. We show that this property is also preserved by applications in P of the rules of $\mathbf{G}$. Consider such an application, and assume that U is a model of some component of every premise of this application. We show that U is also a model of some component of the conclusion:

Weakenings For applications of the weakening rules, this is obvious.

(**id**) Suppose that $\psi \Rightarrow \psi$ is derived using (id). Thus $\psi \in \mathcal{A}$, and so $U^l(\psi) \leq_U U^r(\psi)$. Consequently, $U \models \psi \Rightarrow \psi$.

(**com**) Suppose that $H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2$ is derived from $H \mid \Gamma_1, \Gamma_2 \Rightarrow E_1$ and $H \mid \Gamma_1, \Gamma_2 \Rightarrow E_2$ using (com). If $U \models s$ for some component $s \in H$, then we are done. Otherwise, $U \models \Gamma_1, \Gamma_2 \Rightarrow E_1$ and $U \models \Gamma_1, \Gamma_2 \Rightarrow E_2$. Thus $U^l(\Gamma_1 \cup \Gamma_2) \leq_U U^r(E_1)$ and $U^l(\Gamma_1 \cup \Gamma_2) \leq U^r(E_2)$. By definition, either $U^l(\Gamma_1 \cup \Gamma_2) = U^l(\Gamma_2)$ or $U^l(\Gamma_1 \cup \Gamma_2) = U^l(\Gamma_1)$. It follows that either $U^l(\Gamma_1) \leq_U U^r(E_1)$ or $U^l(\Gamma_2) \leq U^r(E_2)$. Therefore, $U \models \Gamma_1 \Rightarrow E_1$ or $U \models \Gamma_2 \Rightarrow E_2$. In both cases, U is also a model of some component of $H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2$.

(**cut**) Suppose that $H \mid \Gamma \Rightarrow E$ is derived from $H \mid \Gamma, \psi \Rightarrow E$ and $H \mid \Gamma \Rightarrow \psi$ using (cut). Thus $\psi \in \mathcal{C}$. If $U \models s$ for some component $s \in H$, then we are done. Otherwise, $U \models \Gamma, \psi \Rightarrow E$ and $U \models \Gamma \Rightarrow \psi$. We show that $U \models \Gamma \Rightarrow E$. By definition, we have $U^l(\Gamma \cup \{\psi\}) \leq_U U^r(E)$ and $U^l(\Gamma) \leq_U U^r(\psi)$. Since $\psi \in \mathcal{C}$, $U^r(\psi) \leq_U U^l(\psi)$, and so $U^l(\Gamma \cup \{\psi\}) = U^l(\Gamma)$. It follows that $U^l(\Gamma) \leq_U U^r(E)$, and so $U \models \Gamma \Rightarrow E$.

[1] Note that $U \models \emptyset$ is ambiguous: it holds for the empty set of hypersequents, and does not hold for the empty hypersequent.

Right rules Suppose that $H \mid \Gamma \Rightarrow \sigma(\diamond(p_1, \ldots, p_n))$ is derived from the set $\{H \mid \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$, using the right rule $r = \{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\} / \ \Rightarrow \diamond(p_1, \ldots, p_n)$. If $U \models s$ for some component $s \in H$, then we are done. Otherwise, $U \models \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)$ for every $1 \leq i \leq k$. Let $\psi = \sigma(\diamond(p_1, \ldots, p_n))$, and suppose for a contradiction that $U \not\models \Gamma \Rightarrow \psi$. This implies that $U^l(\Gamma) >_U U^r(\psi)$. Since U is **G**-legal, U respects r. Thus since $\psi \in \mathcal{E}_r$ and $\sigma(\{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\})$ is a set of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents, we have $U^r(\psi) \geq_U U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\})$, and so $U^l(\Gamma) >_U U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\})$. By definition, there exists some $1 \leq i \leq k$ such that $U^l(\Gamma) >_U U(\sigma(\Pi_i) \Rightarrow \sigma(E_i))$. Now $U(\sigma(\Pi_i) \Rightarrow \sigma(E_i)) = U^l(\sigma(\Pi_i)) \rightarrow_U U^r(\sigma(E_i))$, and it follows that $U^l(\sigma(\Pi_i)) >_U U^r(\sigma(E_i))$ and $U^l(\Gamma) >_U U^r(\sigma(E_i))$. But then we have $U^l(\Gamma \cup \sigma(\Pi_i)) >_U U^r(\sigma(E_i))$, in contradiction to the fact that $U \models \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)$.

Left rules Suppose that $H \mid \Gamma, \sigma(\diamond(p_1, \ldots, p_n)) \Rightarrow E$ is derived from the sets $\{H \mid \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}$ and $\{H \mid \Gamma, \sigma(\Sigma_i) \Rightarrow E : 1 \leq i \leq m\}$, using the left rule

$$r = \{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}, \{\Sigma_i \Rightarrow \quad : 1 \leq i \leq m\} / \diamond(p_1, \ldots, p_n) \Rightarrow .$$

If $U \models s$ for some component s of H, then we are done. Otherwise, $U \models \Gamma, \sigma(\Pi_i) \Rightarrow \sigma(E_i)$ for every $1 \leq i \leq k$, and $U \models \Gamma, \sigma(\Sigma_i) \Rightarrow E$ for every $1 \leq i \leq m$. Hence: (1) for every $1 \leq i \leq k$, either $U^l(\Gamma) \leq_U U^r(\sigma(E_i))$ or $U^l(\sigma(\Pi_i)) \leq_U U^r(\sigma(E_i))$; and (2) either $U^l(\Gamma) \leq_U U^r(E)$, or $U^l(\sigma(\Sigma_i)) \leq_U U^r(E)$ for every $1 \leq i \leq m$. Let $\psi = \sigma(\diamond(p_1, \ldots, p_n))$, and suppose for a contradiction that $U \not\models \Gamma, \psi \Rightarrow E$. Then, $U^l(\Gamma \cup \{\psi\}) >_U U^r(E)$. Thus we have: (3) $U^r(E) <_U U^l(\Gamma)$; and (4) $U^r(E) <_U U^l(\psi)$. From (2) and (3) we obtain (5): for every $1 \leq i \leq m$, $U^l(\sigma(\Sigma_i)) \leq_U U^r(E)$. Since U is **G**-legal, U respects r. Let

$$x = U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}) \rightarrow_U U^N(\{\sigma(\Sigma_i) \Rightarrow \quad : 1 \leq i \leq m\}).$$

Thus since $\psi \in \mathcal{E}_l$ and $\sigma(\{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\})$ and $\sigma(\{\Sigma_i \Rightarrow \quad : 1 \leq i \leq m\})$ are sets of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents, we have $U^l(\psi) \leq_U x$. Together with (4), we have that $U^r(E) <_U x$. By (5), we obtain that $U^l(\sigma(\Sigma_i)) <_U x$ for every $1 \leq i \leq m$. Let $1 \leq i_0 \leq m$ be an index for which $U^l(\sigma(\Sigma_i))$ obtains a maximal value (i.e. $U^l(\sigma(\Sigma_{i_0})) = U^N(\{\sigma(\Sigma_i) \Rightarrow \quad : 1 \leq i \leq m\})$). In particular, we have that

$$U^l(\sigma(\Sigma_{i_0})) <_U U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\})) \rightarrow_U U^l(\sigma(\Sigma_{i_0})).$$

This entails that

$$U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\}) \leq_U U^l(\sigma(\Sigma_{i_0})).$$

Now, (3) and (5) imply that $U^l(\sigma(\Sigma_{i_0})) <_U U^l(\Gamma)$. It then follows that

$$U(\{\sigma(\Pi_i) \Rightarrow \sigma(E_i) : 1 \leq i \leq k\})) <_U U^l(\Gamma).$$

Hence $U(\sigma(\Pi_{i_1} \Rightarrow E_{i_1})) <_U U^l(\Gamma)$ for some $1 \leq i_1 \leq k$. Equivalently, $U^l(\sigma(\Pi_{i_1})) \rightarrow_U U^r(\sigma(E_{i_1})) <_U U^l(\Gamma)$. This implies that $U^r(\sigma(E_{i_1})) <_U U^l(\sigma(\Pi_{i_1}))$, and $U^r(\sigma(E_{i_1})) <_U U^l(\Gamma)$. But this contradicts (1) above. □

The general soundness theorem can be used to prove that certain identity axioms or cuts are unavoidable, or perhaps to show that in all derivations of some hypersequent H, a certain formula ψ appears on a left (right) side of some component.

Example 4.9. Let H be the hypersequent $\Rightarrow (p_1 \supset p_2) \vee (p_2 \supset p_1)$, and let A be the set of all subformulas of H. We show that all proofs of H in **HG** that consist solely of formulas from A include the identity axiom $p_1 \Rightarrow p_1$. Let $\rho = \langle A, A, A, A \setminus \{p_1\}\rangle$. By Theorem 4.8, it suffices to provide a **HG**-legal ρ-valuation which is not a model of H. For that, we can choose $V_U = [0, 1]$, $\leq_U = \leq$, and:

- $U^l(p_1) = 1, U^r(p_1) = 0, U^l(p_2) = U^r(p_2) = 0.5$,
- $U^l(p_1 \supset p_2) = U^r(p_1 \supset p_2) = 0.5$,
- $U^l(p_2 \supset p_1) = U^r(p_2 \supset p_1) = 0$,
- $U^l((p_1 \supset p_2) \vee (p_2 \supset p_1)) = U^r((p_1 \supset p_2) \vee (p_2 \supset p_1)) = 0.5$.

It is straightforward to verify that U is a **HG**-legal ρ-valuation which is not a model of H.

Theorem 4.10 (Strong Completeness)**.** For every canonical Gödel system **G** and proof-specification ρ, $\Vdash^\rho_{\mathbf{G}} \subseteq \vdash^\rho_{\mathbf{G}}$.

The proof of Theorem 4.10 is based on a construction of a counter-model whenever $\mathcal{H} \nvdash^\rho_{\mathbf{G}} H$. For that we use a similar structure to the one used in [16] (here, however, we only deal with a propositional language). Next, we reproduce and adapt the required definitions and propositions from [16].

Definition 4.11. An *extended sequent* is an ordered pair of (possibly infinite) sets of formulas. Given two extended sequents $\mu_1 = \langle \mathtt{L}_1, \mathtt{R}_1\rangle$ and $\mu_2 = \langle \mathtt{L}_2, \mathtt{R}_2\rangle$, we write $\mu_1 \sqsubseteq \mu_2$ if $\mathtt{L}_1 \subseteq \mathtt{L}_2$ and $\mathtt{R}_1 \subseteq \mathtt{R}_2$. An *extended hypersequent* is a (possibly infinite) set of extended sequents. Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every extended sequent $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$.

We shall use the same notations as above for extended sequents and extended hypersequents. For example, we write $\mathtt{L} \Rightarrow \mathtt{R}$ instead of $\langle \mathtt{L}, \mathtt{R}\rangle$, and $\Omega \mid \mathtt{L} \Rightarrow \mathtt{R}$ instead of $\Omega \cup \{\mathtt{L} \Rightarrow \mathtt{R}\}$. An extended $\langle \mathcal{E}_l, \mathcal{E}_r\rangle$-hypersequent is also defined as expected (see Definition 2.11).

Definition 4.12. Let Ω be an extended hypersequent, **G** be a canonical Gödel system, $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A}\rangle$ be a proof-specification and $\mathcal{H}$ be a set of (ordinary) hypersequents. Ω is called:

1. $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-consistent* if $\mathcal{H} \nvdash^{\rho}_{\mathbf{G}} H$ for every (ordinary) hypersequent $H \sqsubseteq \Omega$.
2. *left internally* $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal with respect to a formula* φ if for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$, if $\varphi \notin \mathtt{L}$ then $\Omega \mid \mathtt{L}, \varphi \Rightarrow \mathtt{R}$ is not $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-consistent.
3. *right internally* $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal with respect to a formula* φ if for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$, if $\varphi \notin \mathtt{R}$ then $\Omega \mid \mathtt{L} \Rightarrow \mathtt{R}, \varphi$ is not $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-consistent.
4. *internally* $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal* if it is left internally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to any formula in $\mathcal{E}_l$, and right internally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to any formula in $\mathcal{E}_r$.
5. *externally* $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal with respect to a sequent* s if either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is not $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-consistent.
6. Ω is called *externally* $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal* if it is externally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to any $\langle \mathcal{E}_l, \mathcal{E}_r\rangle$-sequent.
7. $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$*-maximal* if it is an extended $\langle \mathcal{E}_l, \mathcal{E}_r\rangle$-hypersequent, $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-consistent, internally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal, and externally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal.

Obviously, every hypersequent is an extended hypersequent, and so all of these properties apply to (ordinary) hypersequents as well. The following proposition and lemma are proved similarly to their counterparts in [16].[2]

Proposition 4.13. Let Ω be an extended hypersequent.

1. Assume that Ω is left internally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to a formula φ. Then, for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$: if $\varphi \notin \mathtt{L}$, then $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$ for some hypersequent $H \sqsubseteq \Omega$ and sequents $\Gamma \Rightarrow E_1, \ldots, \Gamma \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$.
2. Assume that Ω is right internally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to a formula φ. Then, for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$: if $\varphi \notin \mathtt{R}$, then $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H \mid \Gamma \Rightarrow \varphi$ for some hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \mathtt{L}$.
3. Assume that Ω is externally $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal with respect to a sequent s. Then, if $\{s\} \not\sqsubseteq \Omega$, then there exists a hypersequent $H \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H \mid s$.

Lemma 4.14. Let Ω be an extended hypersequent, $\mathbf{G}$ be a canonical Gödel system, $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A}\rangle$ be a proof-specification and $\mathcal{H}$ be a set of (ordinary) hypersequents. Every $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-consistent $\langle \mathcal{E}_l, \mathcal{E}_r\rangle$-hypersequent H can be extended to a $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal extended hypersequent Ω.

We are now ready to prove the completeness theorem. We assume that $\mathcal{H} \nvdash^{\rho}_{\mathbf{G}} H$, construct a $\langle \mathbf{G}, \rho, \mathcal{H}\rangle$-maximal extended hypersequent Ω that extends H, and use it to provide the required counter-model. Its set of truth values consists of subsets of Ω (which are also extended hypersequents), and it is linearly ordered by $\subseteq$.

[2] The proofs in [16] are for a specific (first-order) hypersequent system and a less general notion of a proof-specification. However, they are easily adapted to the current case.

Proof of Theorem 4.10. Let $\mathbf{G}$ be a canonical Gödel system, and $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, A \rangle$ be a proof-specification. Assume that $\mathcal{H} \not\vdash^{\rho}_{\mathbf{G}} H$. If H is not an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent, then $\mathcal{H} \not\Vdash^{\rho}_{\mathbf{G}} H$ by definition. Otherwise, we construct a $\mathbf{G}$-legal ρ-valuation U which is a model of $\mathcal{H}$, but not of H. By Lemma 4.14 there exists a $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-maximal extended hypersequent Ω such that $H \sqsubseteq \Omega$. Next, define the following sets for every formula ψ:

$$L(\psi) = \{\mathtt{L} \Rightarrow \mathtt{R} \in \Omega : \psi \in \mathtt{L}\} \qquad \text{and} \qquad R(\psi) = \{\mathtt{L} \Rightarrow \mathtt{R} \in \Omega : \psi \notin \mathtt{R}\}.$$

The ρ-valuation U is defined by:

1. $V_U = \{L(\psi) : \psi \in \mathcal{E}_l\} \cup \{R(\psi) : \psi \in \mathcal{E}_r\} \cup \{\Omega, \emptyset\}$.
2. $\leq_U = \subseteq$ (so, $1_U = \Omega$ and $0_U = \emptyset$).
3. $U^l(\psi) = L(\psi)$ for every $\psi \in \mathcal{E}_l$.
4. $U^r(\psi) = R(\psi)$ for every $\psi \in \mathcal{E}_r$.

Clearly, $\subseteq$ is a partial order relation on V_U, Ω is a maximal element, and $\emptyset$ is a minimal element. To see that V_U is linearly ordered by $\subseteq$, it suffices to prove the following:

1. $L(\psi_1) \subseteq L(\psi_2)$ or $L(\psi_2) \subseteq L(\psi_1)$ for every $\psi_1, \psi_2 \in \mathcal{E}_l$. To see this, suppose for a contradiction that $L(\psi_1) \not\subseteq L(\psi_2)$ and $L(\psi_2) \not\subseteq L(\psi_1)$ for some $\psi_1, \psi_2 \in \mathcal{E}_l$. Thus there exist extended sequents $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in L(\psi_1) \setminus L(\psi_2)$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in L(\psi_2) \setminus L(\psi_1)$. Hence, $\psi_1 \in \mathtt{L}_1$, $\psi_1 \notin \mathtt{L}_2$, $\psi_2 \in \mathtt{L}_2$ and $\psi_2 \notin \mathtt{L}_1$. Since Ω is internally $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-maximal, by Proposition 4.13, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$ and sequents $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathtt{L}_1 \Rightarrow \mathtt{R}_1$ and $\Gamma_2 \Rightarrow F_1, \ldots, \Gamma_2 \Rightarrow F_m \sqsubseteq \mathtt{L}_2 \Rightarrow \mathtt{R}_2$ such that the following hold:

   $$\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid \Gamma_1, \psi_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \psi_2 \Rightarrow E_n,$$
   $$\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_2 \mid \Gamma_2, \psi_1 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \psi_1 \Rightarrow F_m.$$

   By Proposition 2.13, $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid H_2 \mid H'$, where:

   $$H' = \Gamma_1, \psi_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \psi_1 \Rightarrow E_n \mid \Gamma_2, \psi_2 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \psi_2 \Rightarrow F_m.$$

   But, Ω extends this hypersequent, and this contradicts Ω's $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency.

2. $R(\psi_1) \subseteq R(\psi_2)$ or $R(\psi_2) \subseteq R(\psi_1)$ for every $\psi_1, \psi_2 \in \mathcal{E}_r$. To see this, suppose for a contradiction that $R(\psi_1) \not\subseteq R(\psi_2)$ and $R(\psi_2) \not\subseteq R(\psi_1)$ for some $\psi_1, \psi_2 \in \mathcal{E}_r$. Thus there exist extended sequents $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in R(\psi_1) \setminus R(\psi_2)$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in R(\psi_2) \setminus R(\psi_1)$. Hence, $\psi_1 \notin \mathtt{R}_1$, $\psi_1 \in \mathtt{R}_2$, $\psi_2 \in \mathtt{R}_1$ and $\psi_2 \notin \mathtt{R}_2$. Since Ω is internally $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-maximal, by Proposition 4.13, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$ and finite sets $\Gamma_1 \subseteq \mathtt{L}_1$ and $\Gamma_2 \subseteq \mathtt{L}_2$ such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid \Gamma_1 \Rightarrow \psi_1$ and $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_2 \mid \Gamma_2 \Rightarrow \psi_2$. By applying (*com*), we obtain $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid H_2 \mid \Gamma_2 \Rightarrow \psi_1 \mid \Gamma_1 \Rightarrow \psi_2$. Again, since Ω extends this hypersequent, this contradicts Ω's $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency.

3. $L(\psi_1) \subseteq R(\psi_2)$ or $R(\psi_2) \subseteq L(\psi_1)$ for every $\psi_1 \in \mathcal{E}_l$ and $\psi_2 \in \mathcal{E}_r$. To see this, suppose for a contradiction that $L(\psi_1) \not\subseteq R(\psi_2)$ and $R(\psi_2) \not\subseteq L(\psi_1)$ for some $\psi_1 \in \mathcal{E}_l$ and $\psi_2 \in \mathcal{E}_r$. Thus there exist extended sequents $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in L(\psi_1) \setminus R(\psi_2)$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in R(\psi_2) \setminus L(\psi_1)$. Hence, $\psi_1 \in \mathtt{L}_1$, $\psi_1 \notin \mathtt{L}_2$, $\psi_2 \in \mathtt{R}_1$ and $\psi_2 \notin \mathtt{R}_2$. Since Ω is internally $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-maximal, by Proposition 4.13, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, sequents $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathtt{L}_2 \Rightarrow \mathtt{R}_2$ and a finite set $\Gamma_2 \subseteq \mathtt{L}_2$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid \Gamma_1, \psi_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \psi_1 \Rightarrow E_n$ and $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_2 \mid \Gamma_2 \Rightarrow \psi_2$. By Proposition 2.13, it follows that

$$\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2 \Rightarrow E_n \mid \psi_1 \Rightarrow \psi_2.$$

Again, this contradicts Ω's $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency.

Now, to show that U is indeed a ρ-valuation, it remains to prove conditions (a) and (b) from Definition 4.1:

(a) Let $\psi \in \mathcal{C}$. Assume for a contradiction that $U^r(\psi) \not\subseteq U^l(\psi)$, and thus there exists some extended sequent $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ such that $\mathtt{L} \Rightarrow \mathtt{R} \in R(\psi)$ but $\mathtt{L} \Rightarrow \mathtt{R} \notin L(\psi)$. Thus $\psi \notin \mathtt{R}$ and $\psi \notin \mathtt{L}$. Since Ω is internally $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-maximal, by Proposition 4.13, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, sequents $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$ and a finite set $\Gamma_2 \subseteq \mathtt{L}$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid \Gamma_1, \psi_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \psi_1 \Rightarrow E_n$ and $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_2 \mid \Gamma_2 \Rightarrow \psi_1$. By n consecutive applications of (cut) (on ψ, which is an element of $\mathcal{C}$), we obtain $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2 \Rightarrow E_n$, but this contradicts Ω's $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency.

(b) Let $\psi \in \mathcal{A}$. Thus $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} \psi \Rightarrow \psi$ (by applying (id) with ψ, which is an element of $\mathcal{A}$). Since Ω is $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistent, either $\psi \notin \mathtt{L}$ or $\psi \notin \mathtt{R}$ for every element $\mathtt{L} \Rightarrow \mathtt{R}$ of Ω. Equivalently, for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$, either $\mathtt{L} \Rightarrow \mathtt{R} \notin L(\psi)$ or $\mathtt{L} \Rightarrow \mathtt{R} \in R(\psi)$. It follows that $L(\psi) \subseteq R(\psi)$.

To show that U is $\mathbf{G}$-legal, we first prove that the following hold for every $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequent $\Gamma \Rightarrow E$:

$*_1$ If $U^l(\Gamma) \subseteq U^r(E)$ then there exists a hypersequent $H' \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H' \mid \Gamma \Rightarrow E$.

Proof. Suppose that there does not exist a hypersequent $H' \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H' \mid \Gamma \Rightarrow E$. Then Proposition 4.13 implies that $\{\Gamma \Rightarrow E\} \sqsubseteq \Omega$. Hence $\Gamma \subseteq \mathtt{L}$ and $E \subseteq \mathtt{R}$ for some $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$. By definition, we have $\mathtt{L} \Rightarrow \mathtt{R} \in L(\psi)$ for every $\psi \in \Gamma$, and $\mathtt{L} \Rightarrow \mathtt{R} \notin R(\psi)$ for every $\psi \in E$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in U^l(\Gamma)$ and $\mathtt{L} \Rightarrow \mathtt{R} \notin U^r(E)$.

$*_2$ If $\mathtt{L} \Rightarrow \mathtt{R} \in U(\Gamma \Rightarrow E)$, then there exist a hypersequent $H' \sqsubseteq \Omega$, and a finite set $\Gamma' \subseteq \mathtt{L}$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H' \mid \Gamma', \Gamma \Rightarrow E$.

Proof. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \in U(\Gamma \Rightarrow E) = U^l(\Gamma) \to_U U^r(E)$. Now, if $U^l(\Gamma) \subseteq U^r(E)$, then the claim follows by $*_1$ (take $\Gamma' = \emptyset$). Otherwise, $\mathtt{L} \Rightarrow \mathtt{R} \in U^r(E)$. Hence there exists some $\psi \in E$ (i.e. $E = \{\psi\}$) such that

$\mathtt{L} \Rightarrow \mathtt{R} \in R(\psi)$. Thus $\psi \notin \mathtt{R}$. By (possibly) using weakening, Proposition 4.13 implies that there exist a hypersequent $H' \sqsubseteq \Omega$, and a finite set $\Gamma' \subseteq \mathtt{L}$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H' \mid \Gamma', \Gamma \Rightarrow \psi$.

$*_3$ If $\mathtt{L} \Rightarrow \mathtt{R} \notin U^l(\Gamma)$ then there exist a hypersequent $H' \sqsubseteq \Omega$ and sequents $\Gamma' \Rightarrow F_1, \ldots, \Gamma' \Rightarrow F_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$ such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H' \mid \Gamma', \Gamma \Rightarrow F_1 \mid \ldots \mid \Gamma', \Gamma \Rightarrow F_n$.

Proof. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin U^l(\Gamma)$. Thus there exists some $\psi \in \Gamma$ such that $\mathtt{L} \Rightarrow \mathtt{R} \notin L(\psi)$, and so $\psi \notin \mathtt{L}$. The claim follows from Proposition 4.13 (possibly by using weakenings).

Next we show that U is $\mathbf{G}$-legal, i.e. that it respects all canonical rules of $\mathbf{G}$:

- Let $r = \mathcal{S}/ \ \Rightarrow \diamond(p_1, \ldots, p_n)$ be a right rule of $\mathbf{G}$, and let σ be a substitution, such that $\psi = \sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{E}_r$, and $\sigma(\mathcal{S})$ is a set of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents. We prove that $U(\sigma(\mathcal{S})) \subseteq R(\psi)$. Let $\mathtt{L} \Rightarrow \mathtt{R} \in U(\sigma(\mathcal{S}))$. Suppose that $\mathcal{S} = \{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}$. Since $\mathtt{L} \Rightarrow \mathtt{R} \in U(\sigma(\mathcal{S}))$, $*_2$ entails that for every $1 \leq i \leq k$, there exist a hypersequent $H_i \sqsubseteq \Omega$ and a finite set $\Gamma_i \subseteq \mathtt{L}$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_i \mid \Gamma_i, \sigma(\Pi_i) \Rightarrow \sigma(E_i)$. By applying the rule r, we obtain that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_1 \mid \ldots \mid H_k \mid \Gamma_1, \ldots, \Gamma_k \Rightarrow \psi$. The $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency of Ω then entails that $\psi \notin \mathtt{R}$, and so $\mathtt{L} \Rightarrow \mathtt{R} \in R(\psi)$.

- Let $r = \mathcal{S}_1, \mathcal{S}_2 / \diamond (p_1, \ldots, p_n) \Rightarrow$ be a left rule of $\mathbf{G}$, and let σ be a substitution, such that $\psi = \sigma(\diamond(p_1, \ldots, p_n)) \in \mathcal{E}_l$, and $\sigma(\mathcal{S}_1)$ and $\sigma(\mathcal{S}_2)$ are sets of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents. We show that $L(\psi) \subseteq U(\sigma(\mathcal{S}_1)) \to_U U^N(\sigma(\mathcal{S}_2))$. Let $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ and suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin U(\sigma(\mathcal{S}_1)) \to_U U^N(\sigma(\mathcal{S}_2))$. Hence, $U(\sigma(\mathcal{S}_1)) \not\subseteq U^N(\sigma(\mathcal{S}_2))$ and $\mathtt{L} \Rightarrow \mathtt{R} \notin U^N(\sigma(\mathcal{S}_2))$. Let $\mathtt{L}' \Rightarrow \mathtt{R}' \in U(\sigma(\mathcal{S}_1))$ such that $\mathtt{L}' \Rightarrow \mathtt{R}' \notin U^N(\sigma(\mathcal{S}_2))$. Suppose that $\mathcal{S}_1 = \{\Pi_i \Rightarrow E_i : 1 \leq i \leq k\}$ and $\mathcal{S}_2 = \{\Sigma_i \Rightarrow \ : 1 \leq i \leq m\}$. We have the following:

  (1) Since $\mathtt{L}' \Rightarrow \mathtt{R}' \in U(\sigma(\mathcal{S}_1))$, we have $\mathtt{L}' \Rightarrow \mathtt{R}' \in U(\sigma(\Pi_i) \Rightarrow \sigma(E_i)))$ for every $1 \leq i \leq k$. $*_2$ entails that for every $1 \leq i \leq k$, there exist a hypersequent $H_i \sqsubseteq \Omega$, and a finite set $\Gamma'_i \subseteq \mathtt{L}'$, such that $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H_i \mid \Gamma'_i, \sigma(\Pi_i) \Rightarrow \sigma(E_i)$.

  (2) Since $\mathtt{L}' \Rightarrow \mathtt{R}' \notin U^N(\sigma(\mathcal{S}_2))$, we have $\mathtt{L}' \Rightarrow \mathtt{R}' \notin U^l(\sigma(\Sigma_i))$ for every $1 \leq i \leq m$. $*_3$ entails that for every $1 \leq i \leq m$, there exist a hypersequent $H'_i \sqsubseteq \Omega$ and sequents $\Gamma^i \Rightarrow F^i_1, \ldots, \Gamma^i \Rightarrow F^i_{n_i} \sqsubseteq \mathtt{L}' \Rightarrow \mathtt{R}'$ such that
  $$\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H'_i \mid \Gamma^i, \sigma(\Sigma_i) \Rightarrow F^i_1 \mid \ldots \mid \Gamma^i, \sigma(\Sigma_i) \Rightarrow F^i_{n_i}.$$

  (3) Similarly, since $\mathtt{L} \Rightarrow \mathtt{R} \notin U^N(\sigma(\mathcal{S}_2))$, for every $1 \leq i \leq m$, there exist a hypersequent $H''_i \sqsubseteq \Omega$ and sequents $\Delta^i \Rightarrow E^i_1, \ldots, \Delta^i \Rightarrow E^i_{n_i} \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$ such that
  $$\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H''_i \mid \Delta^i, \sigma(\Sigma_i) \Rightarrow E^i_1 \mid \ldots \mid \Delta^i, \sigma(\Sigma_i) \Rightarrow E^i_{n'_i}.$$

Let

$$H^* = H_1 \mid \ldots \mid H_k \mid H'_1 \mid \ldots \mid H'_k \mid H''_1 \mid \ldots \mid H''_k$$

and

$$\Gamma' = \Gamma'_1 \cup \ldots \cup \Gamma'_m \cup \Gamma^1 \cup \ldots \cup \Gamma^m.$$

From (1) and (2), by applying weakenings and Proposition 2.14 we obtain that the following hypersequent has a ρ-proof in $\mathbf{G}$ from $\mathcal{H}$:

$$H^* \mid \Gamma', \psi \Rightarrow F_1^1 \mid \ldots \mid \Gamma', \psi \Rightarrow F_{n_1}^1 \mid \ldots \mid \Gamma', \psi \Rightarrow F_1^m \mid \ldots \mid \Gamma', \psi \Rightarrow F_{n_m}^m.$$

Let $\Gamma = \Delta^1 \cup \ldots \cup \Delta^m$. From (1) and (3), again by applying weakenings and Proposition 2.14 we obtain that:

$$\mathcal{H} \vdash_{\mathbf{G}}^{\rho} H^* \mid \Gamma', \Gamma, \psi \Rightarrow E_1^1 \mid \ldots \mid \Gamma', \Gamma, \psi \Rightarrow E_{n'_1}^1 \mid \ldots$$
$$\mid \Gamma', \Gamma, \psi \Rightarrow E_1^m \mid \ldots \mid \Gamma', \Gamma, \psi \Rightarrow E_{n'_m}^m.$$

By Proposition 2.13, it follows that:

$$\mathcal{H} \vdash_{\mathbf{G}}^{\rho} H^* \mid \Gamma' \Rightarrow F_1^1 \mid \ldots \mid \Gamma' \Rightarrow F_{n_1}^1 \mid \ldots \mid \Gamma' \Rightarrow F_1^m \mid \ldots \mid \Gamma' \Rightarrow F_{n_m}^m$$
$$\mid \Gamma, \psi \Rightarrow E_1^1 \mid \ldots \mid \Gamma, \psi \Rightarrow E_{n'_1}^1 \mid \ldots \mid \Gamma, \psi \Rightarrow E_1^m \mid \ldots \mid \Gamma, \psi \Rightarrow E_{n'_m}^m.$$

Now, if $\psi \in \mathtt{L}$, then Ω extends this hypersequent, and this contradicts Ω's $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistency. Therefore, $\psi \notin \mathtt{L}$, and consequently $\mathtt{L} \Rightarrow \mathtt{R} \notin L(\psi)$.

Finally, it remains to show that U is a model of $\mathcal{H}$ but not of H:

- Let H' be an $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequent in $\mathcal{H}$. Since Ω is $\langle \mathbf{G}, \rho, \mathcal{H} \rangle$-consistent, $H' \not\sqsubseteq \Omega$. Thus there exists some sequent $\Gamma \Rightarrow E \in H'$ such that $\Gamma \Rightarrow E \not\sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$ for every extended sequent $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$. This implies that for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$, either $\mathtt{L} \Rightarrow \mathtt{R} \notin L(\psi)$ for some $\psi \in \Gamma$, or $\mathtt{L} \Rightarrow \mathtt{R} \in R(\psi)$ for some $\psi \in E$. It follows that for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$, we have $\mathtt{L} \Rightarrow \mathtt{R} \notin U^l(\Gamma)$ or $\mathtt{L} \Rightarrow \mathtt{R} \in U^r(E)$. Thus $U^l(\Gamma) \subseteq U^r(E)$, and so $U \models \Gamma \Rightarrow E$. Consequently, $U \models H'$.
- Let $\Gamma \Rightarrow E \in H$. Since $H \sqsubseteq \Omega$, there exists an extended sequent $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ such that $\Gamma \Rightarrow E \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$. Hence $\mathtt{L} \Rightarrow \mathtt{R} \in L(\psi)$ for every $\psi \in \Gamma$, and $\mathtt{L} \Rightarrow \mathtt{R} \notin R(\psi)$ for every $\psi \in E$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in U^l(\Gamma)$ and $\mathtt{L} \Rightarrow \mathtt{R} \notin U^r(E)$. Therefore, $U^l(\Gamma) \not\subseteq U^r(E)$, and so $U \not\models \Gamma \Rightarrow E$. □

Remark 4.15. Theorem 3.9 immediately follows from Theorems 4.8 and 4.10, by choosing $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}} \rangle$.

Remark 4.16. The linearly ordered set of truth values used in the completeness proof is countable, and can be easily embedded into the unit interval $[0, 1]$. Thus we can fix $V_U = [0, 1]$ and $\leq_U = \leq$ in the definition of a valuation, and obtain "standard" semantics.

The following is an immediate corollary of the completeness proof.

Corollary 4.17. If $\mathcal{H} \not\vdash_{\mathbf{G}}^{\rho} H$, then there exists a $\mathbf{G}$-legal ρ-valuation U, which is a model of $\mathcal{H}$ but not of H, and $|V_U| \leq |\mathcal{E}_l| + |\mathcal{E}_r| + 2$.

It follows that the semantics of $\mathbf{G}$-legal ρ-valuations is effective, in the sense that it naturally induces a procedure to decide whether $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H$ or not for a given canonical Gödel system $\mathbf{G}$, finite proof-specification ρ, finite set $\mathcal{H}$ of hypersequents and hypersequent H. Note that a syntactic decision procedure is trivial, since the number of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequents is bounded by $M = 2^{2^{|\mathcal{E}_l|+|\mathcal{E}_r|}}$. Obviously, one can enumerate all lists of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-hypersequents of size at most M, and return "true" iff one of them is a ρ-proof in $\mathbf{G}$ of H from $\mathcal{H}$. Of-course, the problem is more interesting when ρ is not given, and one has to decide whether $\mathcal{H} \vdash_{\mathbf{G}} H$ or not. We consider this problem in the next section.

5 Proof-Theoretic Consequences

The general soundness and completeness theorems of the previous section are easily applicable to characterize important properties of canonical Gödel systems from a *semantic* point of view. In this section, this is done for cut-admissibility and axiom-expansion.

5.1 Cut-Admissibility

By restricting the cut-formulas in proof-specifications, one automatically obtains a semantic property which is completely equivalent to cut-admissibility. Next, we use this characterization to identify the canonical Gödel systems that enjoy the following strong form of cut-admissibility:

Notation 5.1. Given a sequent s, we denote by $\mathit{frm}[s]$ the set of formulas occurring in s. frm is extended to hypersequents and sets of hypersequents in the obvious way.

Definition 5.2. A canonical Gödel system $\mathbf{G}$ enjoys *strong cut-admissibility* if $\mathcal{H} \vdash^{\rho}_{\mathbf{G}} H$ for $\rho = \langle \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathcal{A} \rangle$ implies that $\mathcal{H} \vdash^{\mu}_{\mathbf{G}} H$ for $\mu = \langle \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathit{frm}[\mathcal{H}], \mathcal{A} \rangle$ (i.e. there exists a proof in $\mathbf{G}$ of H from $\mathcal{H}$, without new applications of (id), in which only formulas from $\mathit{frm}[\mathcal{H}]$ serve as cut-formulas).

Note that usual cut-admissibility (if there exists a proof in $\mathbf{G}$ of H, then there exists a cut-free proof of H) follows from *strong* cut-admissibility by taking $\mathcal{H} = \emptyset$ and $\mathcal{A} = \mathit{Frm}_{\mathcal{L}}$. Obviously, by Theorems 4.8 and 4.10, we have the following semantic characterization of strong cut-admissibility:

Corollary 5.3. A canonical Gödel system $\mathbf{G}$ enjoys strong cut-admissibility if $\mathcal{H} \Vdash^{\rho}_{\mathbf{G}} H$ for $\rho = \langle \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathcal{A} \rangle$ implies that $\mathcal{H} \Vdash^{\mu}_{\mathbf{G}} H$ for $\mu = \langle \mathit{Frm}_{\mathcal{L}}, \mathit{Frm}_{\mathcal{L}}, \mathit{frm}[\mathcal{H}], \mathcal{A} \rangle$.

We use the last corollary to prove that a canonical Gödel system enjoys strong cut-admissibility if it satisfies the following coherence criterion (this is exactly the same criterion used for single-conclusion canonical sequent systems in [4]):

Definition 5.4. A set R of canonical rules for an n-ary connective $\diamond$ is called *coherent* if $\mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is classically inconsistent whenever R contains both $\mathcal{S}/ \ \Rightarrow \diamond(p_1, \dots, p_n)$ and $\mathcal{S}_1, \mathcal{S}_2/ \diamond (p_1, \dots, p_n) \Rightarrow$. A canonical Gödel system $\mathbf{G}$ (for $\mathcal{L}$) is called *coherent* if for each connective $\diamond$ of $\mathcal{L}$, the set of rules in $\mathbf{G}$ for $\diamond$ is coherent.

It is easy to verify that **HG** is a coherent system. Moreover, all sets of rules considered in previous examples are coherent. Coherence is a natural requirement for any canonical Gödel system. Indeed, in non-coherent systems the existence of one provable hypersequent of the form $\Rightarrow \psi$ and another provable hypersequent of the form $\varphi \Rightarrow$ implies that all (non-empty) hypersequents are provable:

Proposition 5.5. If $\mathbf{G}$ is a non-coherent canonical Gödel system then $\Rightarrow p_1\ ,\ p_2 \Rightarrow\ \vdash_{\mathbf{G}} H$ for every non-empty hypersequent H.

Proof. Similar to the proof of Theorem 4.10 in [4] for single conclusion canonical systems. The fact that $\mathbf{G}$ manipulates hypersequents is immaterial here. $\square$

This easily entails that non-coherent systems do not enjoy strong cut-admissibility. To see this, take $H = \{\emptyset\}$, i.e. the hypersequent consisting solely of the empty sequent. Obviously, in any canonical Gödel system there does not exist a proof of H from $\Rightarrow p_1$ and $p_2 \Rightarrow$ using only cuts on p_1 and p_2.

Next we show that coherence is also a sufficient condition for strong cut-admissibility. For that we use the following definition and lemmas.

Definition 5.6. Let $\mathbf{G}$ be a canonical Gödel system, and U be a ρ-valuation, for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$. For every compound formula $\psi = \diamond(\psi_1, \dots, \psi_n)$, σ_ψ is the substitution defined by $\sigma(p_i) = \psi_i$ for every $1 \leq i \leq n$, and $\sigma(p_i) = p_i$ (say) for every $i > n$. $U_r^{\mathbf{G}}(\psi)$ and $U_l^{\mathbf{G}}(\psi)$ are defined as follows:

- $U_r^{\mathbf{G}}(\psi)$ is the maximum (with respect to $\leq_U$) of the values $U(\sigma_\psi(\mathcal{S}))$, where $\mathcal{S}/ \ \Rightarrow \diamond(p_1, \dots, p_n)$ is a rule of $\mathbf{G}$, and $\sigma_\psi(\mathcal{S})$ is a set of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents.
- $U_l^{\mathbf{G}}(\psi)$ is the minimum (with respect to $\leq_U$) of the values $U(\sigma_\psi(\mathcal{S}_1)) \to_U U^N(\sigma_\psi(\mathcal{S}_2))$, where $\mathcal{S}_1, \mathcal{S}_2/ \diamond (p_1, \dots, p_n) \Rightarrow$ is a rule of $\mathbf{G}$, and $\sigma_\psi(\mathcal{S}_1) \cup \sigma_\psi(\mathcal{S}_2)$ is a set of $\langle \mathcal{E}_l, \mathcal{E}_r \rangle$-sequents.

In other words, $U_l^{\mathbf{G}}(\psi)$ is the upper bound on $U^l(\psi)$ induced by the left rules for $\diamond$, and $U_r^{\mathbf{G}}(\psi)$ is the lower bound on $U^r(\psi)$ induced by the rules rules. Note that by definition, we have the following:

Lemma 5.7. Let $\mathbf{G}$ be a canonical Gödel system, and U be a ρ-valuation, for $\rho = \langle \mathcal{E}_l, \mathcal{E}_r, \mathcal{C}, \mathcal{A} \rangle$. Then, U is $\mathbf{G}$-legal iff $U_r^{\mathbf{G}}(\psi) \leq_U U^r(\psi)$ for every compound formula $\psi \in \mathcal{E}_r$, and $U^l(\psi) \leq_U U_l^{\mathbf{G}}(\psi)$ for every compound formula $\psi \in \mathcal{E}_l$.

Lemma 5.8. Let $\mathbf{G}$ be a canonical Gödel system, and U be a ρ-valuation, for $\rho = \langle \mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{A} \rangle$. If $\mathbf{G}$ is coherent, then for every compound formula ψ, $U_r^{\mathbf{G}}(\psi) \leq_U U_l^{\mathbf{G}}(\psi)$.

Proof. Let $\psi = \diamond(\psi_1, \ldots, \psi_n)$, and let $\sigma = \sigma_\psi$ (see Definition 5.6). Suppose that $U_r^{\mathbf{G}}(\psi) >_U U_l^{\mathbf{G}}(\psi)$. This implies that there exist a right rule $\mathcal{S}/ \Rightarrow \diamond(p_1, \ldots, p_n)$ and a left rule $\mathcal{S}_1, \mathcal{S}_2/\diamond(p_1, \ldots, p_n) \Rightarrow$ in $\mathbf{G}$, such that $\sigma(\mathcal{S})$,$\sigma(\mathcal{S}_1)$ and $\sigma(\mathcal{S}_2)$ are all sets of $\langle \mathcal{E}, \mathcal{E} \rangle$-sequents, and $U(\sigma(\mathcal{S})) >_U U(\sigma(\mathcal{S}_1)) \to_U U^N(\sigma(\mathcal{S}_2))$. Hence $U(\sigma(\mathcal{S})) >_U U^N(\sigma(\mathcal{S}_2))$ and $U(\sigma(\mathcal{S}_1)) >_U U^N(\sigma(\mathcal{S}_2))$. Consider the classical valuation on $p_1, \ldots, p_n$ defined by $v(p_i) = t$ iff $U^r(\psi_i) >_U U^N(\sigma(\mathcal{S}_2))$. We prove that v satisfies every n-clause in $\mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$, and so $\mathbf{G}$ is not coherent.

Let $s = \Pi \Rightarrow E$ be an n-clause in $\mathcal{S} \cup \mathcal{S}_1$. Since $U(\sigma(\mathcal{S})) >_U U^N(\sigma(\mathcal{S}_2))$ and $U(\sigma(\mathcal{S}_1)) >_U U^N(\sigma(\mathcal{S}_2))$, $U(\sigma(s)) >_U U^N(\sigma(\mathcal{S}_2))$, and so $U^l(\sigma(\Pi)) \to_U U^r(\sigma(E)) >_U U^N(\sigma(\mathcal{S}_2))$. If $U^r(\sigma(E)) >_U U^N(\sigma(\mathcal{S}_2))$, it follows that $E = \{p_i\}$ (for some $1 \leq i \leq n$) and $U^r(\psi_i) >_U U^N(\sigma(\mathcal{S}_2))$, and so $v(p_i) = t$. Thus v classically satisfies s. Assume now that $U^r(\sigma(E)) \leq_U U^N(\sigma(\mathcal{S}_2))$. This implies that $U^l(\sigma(\Pi)) \leq_U U^r(\sigma(E))$. It follows that $U^l(\sigma(\Pi)) \leq_U U^N(\sigma(\mathcal{S}_2))$, and so there exists some $p_i \in \Pi$ such that $U^l(\psi_i) \leq_U U^N(\sigma(\mathcal{S}_2))$. Since U is a ρ-valuation and $\psi_i \in \mathcal{E}$, $U^r(\psi_i) \leq_U U^l(\psi_i)$, and so $v(p_i) = f$. Thus v classically satisfies s.

Now, let $s = \Pi \Rightarrow$ be an n-clause in $\mathcal{S}_2$. Obviously, $U^l(\sigma(\Pi)) \leq_U U^N(\sigma(\mathcal{S}_2))$. This implies that there exists some $p_i \in \Pi$ such that $U^l(\psi_i) \leq_U U^N(\sigma(\mathcal{S}_2))$. Since U is a ρ-valuation and $\psi_i \in \mathcal{E}$, $U^r(\psi_i) \leq_U U^l(\psi_i)$, and so $v(p_i) = f$. Again v classically satisfies s. $\square$

Theorem 5.9 (Strong Cut-Admissibility)**.** Every coherent canonical Gödel system $\mathbf{G}$ enjoys strong cut-admissibility.

Proof. Let $\mathbf{G}$ be a coherent canonical Gödel system. We use Corollary 5.3 to prove that $\mathbf{G}$ enjoys strong cut-admissibility. Suppose that $\mathcal{H} \nVdash^{\mu}_{\mathbf{G}} H$ for $\mu = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, frm[\mathcal{H}], \mathcal{A} \rangle$. Thus there exists a $\mathbf{G}$-legal μ-valuation W, such that $W \models \mathcal{H}$ but $W \not\models H$. Let $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, \mathcal{A} \rangle$. We construct a $\mathbf{G}$-legal ρ-valuation U, such that $U \models \mathcal{H}$ but $U \not\models H$. It then follows that $\mathcal{H} \nVdash^{\rho}_{\mathbf{G}} H$. First, we define $V_U = V_W$ and $\leq_U = \leq_W$ (thus $1_U = 1_W$, $0_U = 0_W$). We write $\leq$ instead of $\leq_W$ for the rest of this proof. Next, let $\psi \in Frm_{\mathcal{L}}$, $U^l(\psi)$ and $U^r(\psi)$ are determined as follows. If $W^r(\psi) \leq W^l(\psi)$ then $U^l(\psi) = W^l(\psi)$ and $U^r(\psi) = W^r(\psi)$. Otherwise, if ψ is an atomic formula, $U^l(\psi) = U^r(\psi) = W^l(\psi)$. Finally, if ψ is a compound formula and $W^l(\psi) < W^r(\psi)$, we define:

$$U^l(\psi) = U^r(\psi) = \begin{cases} W^l(\psi) & U_l^{\mathbf{G}}(\psi) < W^l(\psi), \\ U_l^{\mathbf{G}}(\psi) & W^l(\psi) \leq U_l^{\mathbf{G}}(\psi) \leq W^r(\psi), \\ W^r(\psi) & W^r(\psi) < U_l^{\mathbf{G}}(\psi). \end{cases}$$

Note that $U_l^{\mathbf{G}}(\psi)$ depends only on the values assigned to proper subformulas of ψ, and hence this construction is well-defined. We first show that U is indeed a ρ-valuation. Obviously, $U^r(\psi) \leq U^l(\psi)$ for every formula ψ. It remains to prove that $U^l(\psi) \leq U^r(\psi)$ for every $\psi \in \mathcal{A}$. To see this, note that the only case in which we have $U^l(\psi) \neq U^r(\psi)$ is when $W^r(\psi) < W^l(\psi)$. Since W is a μ-valuation, this can only happen for $\psi \notin \mathcal{A}$. Next, we show that U is $\mathbf{G}$-legal. For this we use the following properties:

1. $W^l(\psi) \leq U^l(\psi)$ and $U^r(\psi) \leq W^r(\psi)$ for every formula ψ.
2. $W^l(\Gamma) \leq U^l(\Gamma)$ and $U^r(\Gamma) \leq W^r(\Gamma)$ for every finite set Γ of formulas.
3. $U(s) \leq W(s)$ for every sequent s.
4. $U(\mathcal{S}) \leq W(\mathcal{S})$ and $W^N(\mathcal{S}) \leq U^N(\mathcal{S})$ for every finite set $\mathcal{S}$ of sequents.
5. $U_r^{\mathbf{G}}(\psi) \leq W_r^{\mathbf{G}}(\psi)$ and $W_l^{\mathbf{G}}(\psi) \leq U_l^{\mathbf{G}}(\psi)$ for every compound formula ψ.

The proofs of these properties easily follow from our definitions (note that if $a \leq b$ and $c \leq d$ then $b \to c \leq a \to d$). Now, we show that U is **G**-legal using Lemma 5.7. Let ψ be a compound formula.

- We show that $U_r^{\mathbf{G}}(\psi) \leq U^r(\psi)$. Since W is **G**-legal, we have $W_r^{\mathbf{G}}(\psi) \leq W^r(\psi)$. Thus if $U^r(\psi) = W^r(\psi)$, then the claim follows by Item 5. Otherwise, the construction of U ensures that $U^r(\psi) = \max_U\{U_l^{\mathbf{G}}(\psi), W^l(\psi)\}$. Thus $U_l^{\mathbf{G}}(\psi) \leq U^r(\psi)$. Now, by Lemma 5.8, the coherence of **G** entails that $U_r^{\mathbf{G}}(\psi) \leq U_l^{\mathbf{G}}(\psi)$.
- We show that $U^l(\psi) \leq U_l^{\mathbf{G}}(\psi)$. By Lemma 5.7, since W is **G**-legal, we have $W^l(\psi) \leq W_l^{\mathbf{G}}(\psi)$. Thus if $U^l(\psi) = W^l(\psi)$, then the claim follows by Item 5. Otherwise, the construction of U ensures that $U^l(\psi) = \min_U\{U_l^{\mathbf{G}}(\psi), W^r(\psi)\}$, and so $U^l(\psi) \leq U_l^{\mathbf{G}}(\psi)$.

It remains to show that $U \models \mathcal{H}$ but $U \not\models H$. Let $H' \in \mathcal{H}$. Since $W \models \mathcal{H}$, there exists some $\Gamma \Rightarrow E \in H'$ such that $W^l(\Gamma) \leq W^r(E)$. Since W is a μ-valuation and $frm[\Gamma \Rightarrow E] \subseteq frm[\mathcal{H}]$, $W^r(\psi) \leq W^l(\psi)$ for every $\psi \in \Gamma \cup E$. The construction of U ensures that $U^l(\psi) = W^l(\psi)$ for every $\psi \in \Gamma$ and $U^r(\psi) = W^r(\psi)$ for every $\psi \in E$. Hence, $U^l(\Gamma) \leq U^r(E)$, and consequently $U \models H'$. Finally, we show that $U \not\models H$. Let $s \in H$. Since $W \not\models H$, we have $W \not\models s$. Thus $W(s) < 1_W$. Item 3 above entails that $U(s) < 1_U$ as well, and so $U \not\models s$. □

Example 5.10. Since **HG** is coherent, it enjoys strong cut-admissibility. The extension of **HG** with the rules for $\rightsquigarrow$ and $*$ from Examples 2.7 and 2.8 enjoys strong cut-admissibility as well.

Finally, the subformula property and decidability of coherent systems are easy corollaries of strong cut-admissibility.

Corollary 5.11 (Subformula Property)**.** Every coherent canonical Gödel system **G** enjoys the subformula property, i.e. $\mathcal{H} \vdash_{\mathbf{G}}^{\rho} H$ for $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, \mathcal{A}\rangle$ implies that $\mathcal{H} \vdash_{\mathbf{G}}^{\mu} H$ for $\mu = \langle \mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{A} \cap \mathcal{E}\rangle$, where $\mathcal{E}$ is the set of all subformulas occurring in $\mathcal{H} \cup \{H\}$.

Proof. Easily follows from Theorem 5.9 using the fact that (*cut*) is the only rule without the subformula property. □

Corollary 5.12 (Decidability)**.** Given a coherent canonical Gödel system **G**, a finite set $\mathcal{H}$ of hypersequents and a hypersequent H, it is decidable whether $\mathcal{H} \vdash_{\mathbf{G}} H$ or not.

Proof. Follows from Corollaries 4.17 and 5.11 (see also the discussion after Corollary 4.17). □

5.2 Determinism and Axiom-Expansion

In this section we use the soundness and completeness theorems to establish a connection between determinism of the semantics of a certain connective, and the fact that this connective admits axiom-expansion. A similar connection was shown in [8] for canonical single-conclusions *sequent* systems. Roughly speaking, an n-ary connective $\diamond$ is deterministic in a system $\mathbf{G}$ if for every formula $\diamond(\psi_1, \ldots, \psi_n)$, the truth values assigned to $\psi_1, \ldots, \psi_n$ uniquely determine the truth value assigned to $\diamond(\psi_1, \ldots, \psi_n)$. Following Lemma 5.7, it is natural to define this property as follows:

Notation 5.13. Given $n \geq 0$, we denote by $At^n_{\mathcal{L}}$ the set $\{p_1, \ldots, p_n\}$.

Definition 5.14. An *n-atomic (Gödel) valuation* is a ρ-valuation for $\rho = \langle At^n_{\mathcal{L}}, At^n_{\mathcal{L}}, At^n_{\mathcal{L}}, At^n_{\mathcal{L}} \rangle$.

Definition 5.15. Let $\mathbf{G}$ be a canonical Gödel system. An n-ary connective $\diamond$ is *deterministic* in $\mathbf{G}$ if for every $\mathbf{G}$-legal n-atomic valuation U, $U^{\mathbf{G}}_r(\diamond(p_1, \ldots, p_n)) = U^{\mathbf{G}}_l(\diamond(p_1, \ldots, p_n))$.

Indeed, if $\diamond$ is deterministic in $\mathbf{G}$ and U is a $\mathbf{G}$-legal valuation, $U(\diamond(\psi_1, \ldots, \psi_n))$ is forced to be equal to $U^{\mathbf{G}}_r(\diamond(\psi_1, \ldots, \psi_n))$ (equivalently, to $U^{\mathbf{G}}_l(\diamond(\psi_1, \ldots, \psi_n))$). In turn, $U^{\mathbf{G}}_r(\diamond(\psi_1, \ldots, \psi_n))$ is (deterministically) computed from $U(\psi_1), \ldots, U(\psi_n)$.[3]

Axiom-expansion means that derivations can be confined to include only atomic application of (id), namely applications of the form $p_i \Rightarrow p_i$. For a given connective, this property is defined as follows:

Definition 5.16. An n-ary connective $\diamond$ admits *axiom-expansion*[4] in a canonical Gödel system $\mathbf{G}$, if $\vdash^{\rho}_{\mathbf{G}} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ for $\rho = \{Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, At^n_{\mathcal{L}}\}$.

Theorem 5.17. Let $\mathbf{G}$ be a coherent canonical Gödel system. A connective admits axiom-expansion in $\mathbf{G}$ iff it is deterministic in $\mathbf{G}$.

Proof. Let $\diamond$ be an n-ary connective, $\psi = \diamond(p_1, \ldots, p_n)$, $H = \psi \Rightarrow \psi$, and $\rho = \langle Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, Frm_{\mathcal{L}}, At^n_{\mathcal{L}} \rangle$. Suppose that $\diamond$ is deterministic in $\mathbf{G}$. We show that $\vdash^{\rho}_{\mathbf{G}} H$. By Theorem 4.10, it suffices to show that every $\mathbf{G}$-legal ρ-valuation is a model of H. Let U be a $\mathbf{G}$-legal ρ-valuation. By Lemma 5.7, $U^{\mathbf{G}}_r(\psi) \leq_U U^r(\psi)$, and $U^l(\psi) \leq_U U^{\mathbf{G}}_l(\psi)$. Since $\diamond$ is deterministic in $\mathbf{G}$, $U^{\mathbf{G}}_r(\psi) = U^{\mathbf{G}}_l(\psi)$ (this holds by definition to the restriction of U to $At^n_{\mathcal{L}}$, and so it holds for U as well). It follows that $U^l(\psi) \leq_U U^r(\psi)$, and so $U \models H$.

For the converse, suppose that $U^{\mathbf{G}}_r(\psi) \neq U^{\mathbf{G}}_l(\psi)$ for some $\mathbf{G}$-legal n-atomic valuation U. Since $\mathbf{G}$ is coherent, Lemma 5.8 entails that $U^{\mathbf{G}}_r(\psi) <_U U^{\mathbf{G}}_l(\psi)$. Let $\mathcal{E} = At^n_{\mathcal{L}} \cup \{\psi\}$, and let $\mu = \langle \mathcal{E}, \mathcal{E}, \mathcal{E}, At^n_{\mathcal{L}} \rangle$. Define a μ-valuation W as follows. First, $V_W = V_U$ and $\leq_W = \leq_U$, Next, define $W^l(p_i) = W^r(p_i) = U^l(p_i)$ for every $1 \leq i \leq n$, $W^l(\psi) = U^{\mathbf{G}}_l(\psi)$, and $W^r(\psi) = U^{\mathbf{G}}_r(\psi)$. It is easy to see

[3] Recall that in valuations, $U^l(\psi) = U^r(\psi)$ for every formula ψ, and we denote this value by $U(\psi)$.

[4] The term "axiom-expansion" is commonly used, but it is somewhat unfortunate. In fact, this property concerns the *reducibility* of arbitrary axioms to atomic ones.

that W is a $\mathbf{G}$-legal μ-valuation, which is not a model of H. By Theorem 4.8, $\not\vdash^{\mu}_{\mathbf{G}} H$. Corollary 5.11 entails that $\not\vdash^{\rho}_{\mathbf{G}} H$. $\square$

6 Conclusions and Further Work

In this paper we studied the family of canonical Gödel systems, of which **HG**, the hypersequent system for Gödel logic, is a particular important example. We showed that each of these systems is characterized by a (possibly) non-deterministic many-valued semantics. The semantics of each connective $\diamond$ is read off from the introduction rules for $\diamond$ that are included in the given system. Indeed, the right rules and the left rules for $\diamond$ provide two functions, that are used to determine a lower bound and an upper bound (respectively) on the truth values of $\diamond$-formulas. The value assigned to each formula od the form $\diamond(\psi_1, \dots, \psi_n)$ should lie within an interval, whose edges are computed by applying these functions on the values of $\psi_1, \dots, \psi_n$. A deterministic (i.e. truth-functional) semantics is obtained when these two functions coincide, as happens for all usual connectives. Otherwise, if the system is coherent, one always obtains non-degenerate intervals, that lead to a non-deterministic semantics. When the system is not coherent, these intervals might be empty, and consequently the system becomes trivial (see Proposition 5.5). Note also that the functions for the two bounds are built only from "atomic Gödel functions", namely min, max, 0, 1, and Gödel implication.

In Section 4, we provided a generalization of the many-valued semantics, based on the notion of *quasi-valuations*. This semantics characterizes the derivability relation of a canonical Gödel system, when proofs are confined to a given proof-specification. Consequently, we were able to have semantic characterizations of proof-theoretical properties of canonical Gödel systems, and use them, in particular, to show that all coherent canonical Gödel systems enjoy a strong form of cut-admissibility. Note that the proof presented in the current paper generalizes the ideas that can be found already in [3], where the first semantic proof of cut-admissibility for the (coherent) canonical Gödel system **HG** was presented. A minor difference is that while our proof uses the many-valued semantics, the proof in [3] exploits the Kripke-style semantics of Gödel logic.

Finally, an interesting question arises as to whether this semantic proof technique is applicable for other calculi with many-valued semantics. Two particularly important cases are the (sub-structural) hypersequent systems for the other two fundamental fuzzy logics ([17]).

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