A Semantic Proof of Strong Cut-admissibility for First-Order Gödel Logic

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Abstract

We provide a constructive direct semantic proof of the completeness of the cut-free part of the hypersequent calculus **HIF** for the standard first-order Gödel logic (thereby proving both completeness of the calculus for its standard semantics, and the admissibility of the cut rule in the full calculus). The results also apply to derivations from assumptions (or "non-logical axioms"), showing in particular that when the set of assumptions is closed under substitutions, then cuts can be confined to formulas occurring in the assumptions. The methods and results are then extended to handle the (Baaz) Delta connective as well.

1 Introduction

In [17] Gödel introduced a sequence $\{G_n\}$ ($n \geq 2$) of n-valued matrices in the language of propositional intuitionistic logic. He used these matrices to show some important properties of intuitionistic logic. An infinite-valued matrix G_ω in which all the G_ns can be embedded was later introduced by Dummett in [14]. G_ω, in turn, can naturally be embedded in a matrix $G_{[0,1]}$, the truth-values of which are the real numbers between 0 and 1 (inclusive). It has not been difficult to show that the sets of valid formulas in G_ω and in $G_{[0,1]}$ are identical, and both are known today as "Gödel logic".[1] Later it has been shown that this logic is also characterized as the logic of *linearly ordered* intuitionistic Kripke frames (see [15],[16]). Gödel logic is probably the most important intermediate logic, i.e. a logic between intuitionistic logic and classical logic. In the last fifteen years it has again attracted a lot of attention because of its recognition as one of the three most basic fuzzy logics [18].

Gödel logic can be naturally extended to the first-order framework. In particular, the standard first-order Gödel logic (the logic based on $[0, 1]$ as the set of truth-values) has been introduced and investigated in [24] (where it was called "intuitionistic fuzzy logic"). The Kripke-style semantics of this logic is provided by the class of all linearly ordered intuitionistic Kripke frames with *constant domains*.

A cut-free Gentzen-type formulation for (propositional) Gödel logic was first given by Sonobe in [21]. Since then several other such calculi which employ ordinary sequents have been proposed (see [11, 1, 12, 5, 13]). All these calculi have the drawback of using some ad-hoc rules of a nonstandard form, in which several occurrences of connectives are involved. In contrast, in [3] a cut-free Gentzen-type proof system **HG** for

[1] It is also called Gödel-Dummett logic, because it was first introduced and axiomatized in [14]. The name Dummett himself has used is LC.

propositional Gödel logic was introduced, which does not have this drawback. **HG** uses (single-conclusion) *hypersequents* (a natural generalization of Gentzen's original (single-conclusion) sequents), and it has exactly the same logical rules as the usual Gentzen-type system for propositional intuitionistic logic. **HG** was furthermore extended by Baaz, Ciabattoni, Fermüller, and Zach to provide appropriate proof systems for extensions of propositional Gödel logic with quantifiers of various types and modalities (see [7] for a survey). In particular, an extension of **HG** for the standard first-order Gödel logic (called **HIF**) was introduced in [8]. Following the work that started in [3], the framework of hypersequents was used by Metcalfe, Ciabattoni, and others for other fuzzy logics (like Łukasiewicz infinite-valued logic), and nowadays it is the major framework for the proof theory of fuzzy logics (see [20]).

Until recently, in all the works about **HG** and its extensions the proofs of completeness (either for the Gödel's many-valued semantics or for the Kripke semantics) and the proofs of cut-elimination have completely been separated. Completeness has been shown for the full calculus (including cut), while cut-elimination has been proved syntactically by some type of induction on complexity of proofs. It is well-known that the syntactic methods are notoriously prone to errors, especially (but certainly not only) in the case of hypersequent systems.[2] In contrast, the recent [4] provided for the first time a constructive, direct, and simple proof of the completeness of the cut-free part of **HG** for its intended semantics (thereby proving both completeness of the calculus and the admissibility of the cut rule in it).[3] However, [4] did not deal with the first-order extension of **HG**, and it was not clear how to adapt its completeness proof to this case.

In this paper we present a purely *semantic*, easy to verify, direct proof of cut-admissibility for **HIF** (the first-order extension of **HG**). As usual, this proof is actually a completeness proof of the cut-free part of our system for its intended semantics. To overcome the difficulties encountered in adapting the proof of [4] to the first-order case, we introduce *extended sequents* and *extended hypersequents* (see Definitions 39 and 40).

Another difference between this work and previous works (e.g. [7]) is that our results apply also to derivations from assumptions (or non-logical axioms), while previous works concern only proofs without assumptions. Thus we actually prove *strong* cut-admissibility, showing that in case that the set of assumptions is closed under substitutions, derivations can be confined to those in which cuts are made only on formulas which occur in the assumptions. In addition, we identify an additional semantic condition, whose satisfaction ensures the existence of a derivation of this kind (in which cuts are only applied on formulas which occur in the assumptions) also when the set of assumptions is not closed under substitutions.

We end the introduction with an important note: this paper is basically an expanded and improved version of the conference paper [6]. However, its main result, strong cut-admissibility in **HIF**, was proved in [6] only in a roundabout way: the semantic proof of strong cut-admissibility was first given there for a new *multiple*-conclusion hypersequent system for the standard first-order Gödel logic. Then this result was used to derive the analogous result for **HIF** (which is single-conclusion). In contrast, in this paper we directly treat **HIF**, without any detour through another calculus (for this we need several new propositions concerning **HIF**; see Propositions 22, 23, 24). This, in turn, makes it easy to extend the met1hod to some natural extensions of **HIF** with rules for new connectives. We do it in Subsection 4.2 for $\mathbf{HIF}_{\Box}$, the extension of the standard first-order Gödel logic with a globalization connective (also known as the (Baaz) Delta connective). For this we had to further add some new concepts and results (see Definition 54, Proposition 55, and Corollary 56). Finally, for the sake of completeness, in the last section we present also the multiple-conclusion hypersequent system, and reproduce the results of [6].

[2]In fact, the first proof (in [8]) of cut-elimination for **HIF** was erroneous. There has also been a gap in the proof given in [3] in its handling of the case of disjunction. Many other examples, also for ordinary sequent calculi, can be given.

[3]A semantic proof of cut-admissibility for **HG** has been given in [10]. However, a complicated algebraic phase semantics was used there, and the proof is not constructive.

2 Standard First-Order Gödel Logic

Let $\mathcal{L}$ be a first-order language with $\wedge, \vee, \supset$ as binary connectives, $\perp$ as a propositional constant, and $\forall$ and $\exists$ as unary quantifiers. We assume that the set of free variables and the set of bounded variables are disjoint (thus in a well-formed formula, the use of the bound variables is always in the scope of a quantification of the same variables). We use the metavariables a, b to range over the free variables, x to range over the bounded variables, p to range over the predicate symbols of $\mathcal{L}$, c to range over its constant symbols, and f to range over its function symbols. The sets of $\mathcal{L}$-terms and $\mathcal{L}$-formulas are defined as usual, and are denoted by $trm_{\mathcal{L}}$ and $frm_{\mathcal{L}}$, respectively. We mainly use t as a metavariable standing for $\mathcal{L}$-terms, φ, ψ for $\mathcal{L}$-formulas, Γ, Δ for sets of $\mathcal{L}$-formulas, and E, F for sets of $\mathcal{L}$-formulas which are either singletons or empty.

Given an $\mathcal{L}$-term t, a free variable a, and another $\mathcal{L}$-term t', we denote by $t\{t'/a\}$ the $\mathcal{L}$-term obtained from t by replacing all occurrences of a by t'. This notation is extended to formulas, set of formulas, etc. in the obvious way.

Notation 1 To improve readability we use square parentheses in the meta-language, and reserve round parentheses to the first-order language.

As usual, there are two main approaches to define the *standard first-order Gödel logic*:

Proof-theoretically The logic is defined using an Hilbert-style calculus. Such a calculus is obtained by adding the following axioms to any Hilbert-style calculus for first-order intuitionistic logic (see e.g. [7]):

- The "linearity" axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$.
- The "quantifier-shifting" axiom $(\forall x(\varphi\{x/a\} \vee \psi)) \supset (\forall x(\varphi\{x/a\}) \vee \psi)$.

We shall denote such an Hilbert-style calculus by $\mathbf{sG}$, and $\vdash_{\mathbf{sG}}$ will stand for the consequence relation which is naturally associated with it (note that this is a relation between sets of formulas and formulas).

Model-theoretically Here there are two options. First, the standard first-order Gödel logic can be defined as a first-order fuzzy logic based on the real interval $[0, 1]$. Second, it can be seen as an intermediate logic, defined in terms of Kripke-style semantics. In the next two subsections, we briefly review the many-valued semantics and the Kripke-style semantics of this logic.

2.1 Many-Valued Semantics

Definition 2 An $\mathcal{L}$-*algebra* is a pair $\langle D, I\rangle$ where D is a non-empty domain and I is an interpretation of constants and function symbols of $\mathcal{L}$ such that $I[c] \in D$ for every constant symbol c of $\mathcal{L}$, and $I[f] \in D^n \to D$ for every n-ary function symbol f of $\mathcal{L}$.

Definition 3 An $\langle\mathcal{L}, D\rangle$-*predicate interpretation* is a function J from the set of $\mathcal{L}$-predicate symbols such that $J[p] \in D^n \to [0, 1]$ for every n-ary predicate symbol p of $\mathcal{L}$.

Definition 4 Let $M = \langle D, I\rangle$ be an $\mathcal{L}$-algebra. An $\langle\mathcal{L}, M\rangle$-evaluation is a function assigning an element in D to every free variable of $\mathcal{L}$. An $\langle\mathcal{L}, M\rangle$-evaluation e is naturally extended to $trm_{\mathcal{L}}$ as follows: $e[c] = I[c]$ for every constant symbol c; and $e[f(t_1, \ldots, t_n)] = I[f][e[t_1], \ldots, e[t_n]]$ for every $f(t_1, \ldots, t_n) \in trm_{\mathcal{L}}$.

Notation 5 Given an $\langle\mathcal{L}, M\rangle$-evaluation e, a free variable a, and $d \in D$, we denote by $e_{[a:=d]}$ the $\langle\mathcal{L}, M\rangle$-evaluation which is identical to e except that $e_{[a:=d]}[a] = d$.

Definition 6 Let $M = \langle D, I \rangle$ be an $\mathcal{L}$-algebra, J be an $\langle \mathcal{L}, D \rangle$-predicate interpretation, and e be an $\langle \mathcal{L}, M \rangle$-evaluation. The function $\| \bullet \|_{\langle M,J,e \rangle}$ from $\mathit{frm}_{\mathcal{L}}$ to $[0, 1]$ is recursively defined as follows:

1. $\|p(t_1, \dots, t_n)\|_{\langle M,J,e \rangle} = J[p][e[t_1], \dots, e[t_n]]$.
2. $\|\bot\|_{\langle M,J,e \rangle} = 0$.
3. $\|\varphi_1 \supset \varphi_2\|_{\langle M,J,e \rangle} = 1$ if $\|\varphi_1\|_{\langle M,J,e \rangle} \leq \|\varphi_2\|_{\langle M,J,e \rangle}$, and $\|\varphi_1 \supset \varphi_2\|_{\langle M,J,e \rangle} = \|\varphi_2\|_{\langle M,J,e \rangle}$ otherwise.
4. $\|\varphi_1 \vee \varphi_2\|_{\langle M,J,e \rangle} = \max\{\|\varphi_1\|_{\langle M,J,e \rangle}, \|\varphi_2\|_{\langle M,J,e \rangle}\}$.
5. $\|\varphi_1 \wedge \varphi_2\|_{\langle M,J,e \rangle} = \min\{\|\varphi_1\|_{\langle M,J,e \rangle}, \|\varphi_2\|_{\langle M,J,e \rangle}\}$.
6. $\|\forall x(\varphi\{x/a\})\|_{\langle M,J,e \rangle} = \inf\{\|\varphi\|_{\langle M,J,e_{[a:=d]} \rangle} \mid d \in D\}$.
7. $\|\exists x(\varphi\{x/a\})\|_{\langle M,J,e \rangle} = \sup\{\|\varphi\|_{\langle M,J,e_{[a:=d]} \rangle} \mid d \in D\}$.

It is easy to see that $\| \bullet \|_{\langle M,J,e \rangle}$ is well-defined, and in particular in 6 and 7, the exact choice of the free variable a is immaterial.

Definition 7 Let $M = \langle D, I \rangle$ be an $\mathcal{L}$-algebra, J be an $\langle \mathcal{L}, D \rangle$-predicate interpretation. We say that $\langle M, J \rangle$ is a *model* of an $\mathcal{L}$-formula φ, if $\|\varphi\|_{\langle M,J,e \rangle} = 1$ for every $\langle \mathcal{L}, M \rangle$-evaluation e. $\langle M, J \rangle$ is a *model* of a set $\mathcal{T}$ of $\mathcal{L}$-formulas if it is a model of every $\varphi \in \mathcal{T}$.

We now define the consequence relation of Gödel logic in terms of the many-valued semantics:

Definition 8 ($\vdash_{[0,1]}$) Let $\mathcal{T} \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. $\mathcal{T} \vdash_{[0,1]} \varphi$ if whenever $\langle M, J \rangle$ is a model of $\mathcal{T}$, for some $\mathcal{L}$-algebra $M = \langle D, I \rangle$, and $\langle \mathcal{L}, D \rangle$-predicate interpretation J, then $\langle M, J \rangle$ is also a model of φ.

Remark 9 As usual in first-order logics, actually there are two natural consequence relations that can be defined here. In the terminology of [2], in this paper we concentrate on the *validity* consequence relation, rather than the *truth* consequence relation.

Fact 10 ([18],[19]) The Hilbert-style calculus $\mathbf{sG}$ is strongly sound and complete with respect to the many-valued semantics of the standard first-order Gödel logic, i.e. $\vdash_{[0,1]} = \vdash_{\mathbf{sG}}$.

2.2 Kripke-Style Semantics

While the many-valued semantics is perhaps more intuitive, in this paper the Kripke-style semantics is essential to prove our main results. There are two differences between this semantics and the Kripke-style semantics of first-order intuitionistic logic. First, *linearly* ordered Kripke frames are considered. Second, for first-order Gödel logic we use a constant domain, i.e. the same domain in each world, rather than the expanding domains used for intuitionistic logic.

Definition 11 A Kripke $\langle \mathcal{L}, D \rangle$-predicate interpretation is a function assigning a subset of D^n to every n-ary predicate symbol of $\mathcal{L}$.

Definition 12 An $\mathcal{L}$-*frame* is a tuple $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ where:

1. W is a set linearly ordered by $\leq$.
2. $M = \langle D, I \rangle$ is an $\mathcal{L}$-algebra (Definition 2).

3. $\mathcal{I} = \{I_w\}_{w \in W}$, where for every $w \in W$, I_w is a Kripke $\langle \mathcal{L}, D \rangle$-predicate interpretation such that for every predicate symbol p: if $u \leq w$ then $I_u[p] \subseteq I_w[p]$.

Definition 13 Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ be an $\mathcal{L}$-frame, where $M = \langle D, I \rangle$ and $\mathcal{I} = \{I_w\}_{w \in W}$. Let e be an $\langle \mathcal{L}, M \rangle$-evaluation (Definition 4). The satisfaction relation $\vDash$ is recursively defined as follows:

1. $\mathcal{W}, w, e \vDash p(t_1, \ldots, t_n)$ iff $\langle e[t_1], \ldots, e[t_n] \rangle \in I_w[p]$.
2. $\mathcal{W}, w, e \nvDash \bot$.
3. $\mathcal{W}, w, e \vDash \varphi_1 \supset \varphi_2$ iff $\mathcal{W}, u, e \nvDash \varphi_1$ or $\mathcal{W}, u, e \vDash \varphi_2$ for every element $u \geq w$.
4. $\mathcal{W}, w, e \vDash \varphi_1 \vee \varphi_2$ iff $\mathcal{W}, w, e \vDash \varphi_1$ or $\mathcal{W}, w, e \vDash \varphi_2$.
5. $\mathcal{W}, w, e \vDash \varphi_1 \wedge \varphi_2$ iff $\mathcal{W}, w, e \vDash \varphi_1$ and $\mathcal{W}, w, e \vDash \varphi_2$.
6. $\mathcal{W}, w, e \vDash \forall x(\varphi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \vDash \varphi$ for every $d \in D$.[4]
7. $\mathcal{W}, w, e \vDash \exists x(\varphi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \vDash \varphi$ for some $d \in D$.

It is easy to see that $\vDash$ is well-defined, and in particular in 6 and 7, the exact choice of the free variable a is immaterial.

It is a routine matter to prove the following proposition:

Proposition 14 (Persistence) Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ be an $\mathcal{L}$-frame, and e be an $\langle \mathcal{L}, M \rangle$-evaluation. Let φ be an $\mathcal{L}$-formula, and u be an element of W such that $\mathcal{W}, u, e \vDash \varphi$. Then, $\mathcal{W}, w, e \vDash \varphi$ for every element w of W such that $u \leq w$.

Definition 15 Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ be an $\mathcal{L}$-frame. $\mathcal{W}$ is a *model* of an $\mathcal{L}$-formula φ if $\mathcal{W}, w, e \vDash \varphi$ for every $\langle \mathcal{L}, M \rangle$-evaluation e and $w \in W$. $\mathcal{W}$ is a model of a set $\mathcal{T}$ of $\mathcal{L}$-formulas if it is a model of every $\varphi \in \mathcal{T}$.

We now define the consequence relation of Gödel logic in terms of Kripke-style semantics:

Definition 16 ($\vdash_{Kr}$) Let $\mathcal{T} \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. $\mathcal{T} \vdash_{Kr} \varphi$ if every $\mathcal{L}$-frame $\mathcal{W}$ which is a model of $\mathcal{T}$ is also a model of φ.

Fact 17 ([15]) The Hilbert-style calculus $\mathbf{sG}$ is strongly sound and complete with respect to the Kripke-style semantics of the standard first-order Gödel logic, i.e. $\vdash_{Kr} = \vdash_{\mathbf{sG}}$.

3 The System HIF

As indicated in the introduction, the main tool to obtain well-behaved proof systems for Gödel logic is *single-conclusion hypersequents*.

Definition 18 A *single-conclusion sequent* is an ordered pair of finite sets of $\mathcal{L}$-formulas $\langle \Gamma, E \rangle$, where E is either a singleton or empty. A *single-conclusion hypersequent* is a finite set of single-conclusion sequents.

[4] Note that unlike in Kripke-style semantics of intuitionistic logic, the condition here is "local".

Henceforth, we simply write *sequent* instead of *single-conclusion sequent*, and *hypersequent* instead of *single-conclusion hypersequent*. We shall use the usual sequent notation $\Gamma \Rightarrow E$ (for $\langle \Gamma, E \rangle$) and the usual hypersequent notation $s_1 \mid \ldots \mid s_n$ (for $\{s_1, \ldots, s_n\}$). We also employ the standard abbreviations, e.g. $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $H \mid s$ instead of $H \cup \{s\}$.

Notation 19 Given a set $\mathcal{H}$ of hypersequents, we denote by $frm[\mathcal{H}]$ the set of $\mathcal{L}$-formulas that appear in $\mathcal{H}$.

Next, we review the hypersequent system **HIF** from [8].

Definition 20 **HIF** is the (single-conclusion) hypersequent system containing the following rules:[5]

Axioms:

$$\varphi \Rightarrow \varphi \qquad \bot \Rightarrow$$

Structural Rules:

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \varphi \Rightarrow E} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com)\ \frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \quad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \Gamma_2, \Gamma_1' \Rightarrow E_2}$$

$$(cut)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi \quad H_2 \mid \Gamma_2, \varphi \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E}$$

Logical Rules:

$$(\supset\Rightarrow)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi_1 \quad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow E} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2}$$

$$(\vee\Rightarrow)\ \frac{H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E \quad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow E}$$

$$(\Rightarrow\vee_1)\ \frac{H \mid \Gamma \Rightarrow \varphi_1}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2} \qquad (\Rightarrow\vee_2)\ \frac{H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}$$

$$(\wedge\Rightarrow_1)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E} \qquad (\wedge\Rightarrow_2)\ \frac{H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow E}$$

$$(\Rightarrow\wedge)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi_1 \quad H_2 \mid \Gamma_2 \Rightarrow \varphi_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \varphi_1 \wedge \varphi_2}$$

$$(\forall\Rightarrow)\ \frac{H \mid \Gamma, \varphi\{t/a\} \Rightarrow E}{H \mid \Gamma, \forall x(\varphi\{x/a\}) \Rightarrow E} \qquad (\Rightarrow\forall)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \forall x(\varphi\{x/a\})}$$

$$(\exists\Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E} \qquad (\Rightarrow\exists)\ \frac{H \mid \Gamma \Rightarrow \varphi\{t/a\}}{H \mid \Gamma \Rightarrow \exists x(\varphi\{x/a\})}$$

The rules $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ must obey the eigenvariable condition: a must not occur in the lower hypersequent. Note that the sets of formulas denoted by $\Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2'$ need not to be disjoint. Similarly, the hypersequents denoted by H_1 and H_2 not to be disjoint.

Definition 21 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents.

[5]What we present is actually an equivalent version of the system presented in [8]. Thus $\neg\varphi$ is defined here as $\varphi \supset \bot$, and the density rule is not present (see Subsection 4.3). Other insignificant differences are due to the facts that we define hypersequents as *sets* of sequents rather than as *multisets*, and that we use multiplicative versions of the binary rules rather than additive ones.

1. We write $\mathcal{H} \vdash H$ if there exists a derivation of H from $\mathcal{H}$ in **HIF**.

2. Let $\mathcal{E}$ be a set of $\mathcal{L}$-formulas. We write $\mathcal{H} \vdash^{\mathcal{E}} H$ if there exists a derivation of H from $\mathcal{H}$ in **HIF** in which the cut-formula of every application of the cut rule is in $\mathcal{E}$.

As usual, we shall write $\vdash H$ instead of $\emptyset \vdash H$. Using this notation and the one introduced in the last definition, cut-admissibility means that $\vdash H$ iff $\vdash^{\emptyset} H$, while *strong* cut-admissibility means that $\mathcal{H} \vdash H$ iff $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$.

Next, we provide two derivability results for **HIF**, that will be used in the sequel.

Proposition 22 (Generalized Communication) For every $n, m \geq 0$, hypersequents H_1 and H_2, $m + n$ singleton or empty sets of $\mathcal{L}$-formulas $E_1, \dots, E_n, F_1, \dots, F_m$, and sets $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2'$ of $\mathcal{L}$-formulas:

$$H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2' \Rightarrow E_n \mid \Gamma_2, \Gamma_1' \Rightarrow F_1 \mid \dots \mid \Gamma_2, \Gamma_1' \Rightarrow F_m$$

is cut-free derivable in **HIF** from the hypersequents

$$H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_1' \Rightarrow E_n \quad \text{and} \quad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow F_1 \mid \dots \mid \Gamma_2, \Gamma_2' \Rightarrow F_m$$

Proof We prove this by induction on $n + m$. First, when $n = 0$ or $m = 0$, the claim follows by applying an external weakening. Assume that $n, m > 0$, $n + m = l$ and that the claim holds for every n, m such that $n + m < l$. By the induction hypothesis, the following two hypersequents are cut-free derivable in **HIF** from $H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_1' \Rightarrow E_n$ and $H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow F_1 \mid \dots \mid \Gamma_2, \Gamma_2' \Rightarrow F_m$:

$$H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_n \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2' \Rightarrow E_{n-1} \mid \Gamma_2, \Gamma_1' \Rightarrow F_1 \mid \dots \mid \Gamma_2, \Gamma_1' \Rightarrow F_m$$

$$H_1 \mid H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow F_m \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2' \Rightarrow E_n \mid \Gamma_2, \Gamma_1' \Rightarrow F_1 \mid \dots \mid \Gamma_2, \Gamma_1' \Rightarrow F_{m-1}$$

An application of (com) on these two hypersequents provides the desired result. ∎

Proposition 23 (Generalized $(\vee\Rightarrow)$) For every $n, m \geq 0$, hypersequents H_1 and H_2, $m + n$ singleton or empty sets of $\mathcal{L}$-formulas $E_1, \dots, E_n, F_1, \dots, F_m$, sets Γ_1, Γ_2 of $\mathcal{L}$-formulas, and two $\mathcal{L}$-formulas φ_1, φ_2:

$$H_1 \mid H_2 \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_n \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_m$$

is cut-free derivable in **HIF** from the hypersequents

$$H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \Rightarrow E_n \quad \text{and} \quad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_2 \Rightarrow F_m$$

Proof We prove this by induction on $n + m$. First, for $n = 0$ or $m = 0$, the claim follows by applying an external weakening. Assume that $n, m > 0$, $n + m = l$ and that the claim holds for every n, m such that $n + m < l$. By the induction hypothesis, the following two hypersequents are cut-free derivable in **HIF** from $H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \Rightarrow E_n$ and $H_2 \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_2 \Rightarrow F_m$:

$$G_1 = H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_n \mid H_2 \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_{n-1} \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_m$$

$$G_2 = H_1 \mid H_2 \mid \Gamma_2, \varphi_2 \Rightarrow F_m \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_n \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_{m-1}$$

An application of (com) on these two hypersequents yields:

$$H_1 \mid H_2 \mid \Gamma_1, \varphi_2 \Rightarrow E_n \mid \Gamma_2, \varphi_1 \Rightarrow F_m \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_n \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_m$$

By applying $(\vee\Rightarrow)$ on this hypersequent and on G_1 we get:

$$H_1 \mid H_2 \mid \Gamma_2, \varphi_1 \Rightarrow F_m \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \vee \varphi_2 \Rightarrow E_n \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow F_m$$

Another application of $(\vee\Rightarrow)$ on this hypersequent and on G_2 provides the desired hypersequent. ∎

Finally, the next proposition provides a generalization of the rule ($\exists\Rightarrow$), to be used in the sequel. The main idea is taken from the proof of Lemma 30 from [7] (used there in the syntactic proof of cut-elimination for **HIF**). However, here we also consider proofs from non-empty sets of assumptions.

Proposition 24 (Generalized ($\exists\Rightarrow$)) Let $\mathcal{H}$ be a set of hypersequents, and $\mathcal{E}$ be a set of $\mathcal{L}$-formulas. Suppose that both $\mathcal{H}$ and $\mathcal{E}$ are closed under substitutions. Then, for every $n \geq 0$, hypersequent H, n singleton or empty sets of $\mathcal{L}$-formulas $E_1, \dots, E_n$, set Γ of $\mathcal{L}$-formulas, $\mathcal{L}$-formula φ, and free variable a which does not occur in $H, E_1, \dots, E_n$ and Γ: $\mathcal{H} \vdash^{\mathcal{E}} H \mid \Gamma, \varphi \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi \Rightarrow E_n$ implies that $\mathcal{H} \vdash^{\mathcal{E}} H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E_1 \mid \dots \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E_n$.

The following standard lemma is required in the proof:

Lemma 25 Let $\mathcal{H}$ be a set of hypersequents, and $\mathcal{E}$ be a set of $\mathcal{L}$-formulas. Suppose that both $\mathcal{H}$ and $\mathcal{E}$ are closed under substitutions. For every hypersequent H and two free variables a, b, such that b does not occur in H, if $\mathcal{H} \vdash^{\mathcal{E}} H$ then $\mathcal{H} \vdash^{\mathcal{E}} H\{b/a\}$.

Proof (of Proposition 24) We use induction on n. The claim is trivial for $n = 0$. Now assume that the claim holds for $n - 1$, we prove it for n. Let H be a hypersequent, $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n$ be sequents, φ be an $\mathcal{L}$-formula, and let a be a free variable which does not occur in $H, \Gamma, E_1, \dots, E_n$. Let $G_0 = H \mid \Gamma, \varphi \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi \Rightarrow E_n$. Let b be a free variable which does not occur in G_0. By Lemma 25, $\mathcal{H} \vdash^{\mathcal{E}} G_0\{b/a\}$. Note that since a does not occur in $H, \Gamma, E_1, \dots, E_n$, $G_0\{b/a\} = H \mid \Gamma, \varphi\{b/a\} \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi\{b/a\} \Rightarrow E_n$. By Proposition 22, the following hypersequent is cut-free derivable from G_0 and $G_0\{b/a\}$:

$$H \mid \Gamma, \varphi \Rightarrow E_n \mid \Gamma, \varphi\{b/a\} \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi\{b/a\} \Rightarrow E_{n-1}$$

(to see this, take $H_1 = H \mid \Gamma, \varphi \Rightarrow E_n$ and $H_2 = H \mid \Gamma, \varphi\{b/a\} \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi\{b/a\} \Rightarrow E_{n-1}$). By an application of ($\exists\Rightarrow$) on the last hypersequent, we obtain:

$$H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E_n \mid \Gamma, \varphi\{b/a\} \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi\{b/a\} \Rightarrow E_{n-1}$$

The induction hypothesis now entails that $\mathcal{H} \vdash^{\mathcal{E}} H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E_1 \mid \dots \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E_n$. ∎

3.1 Interpretation of Hypersequents

A better understanding of hypersequents is obtained using the following interpretation ([8]).

Definition 26 (Interpretation of Hypersequents)

1. For a non-empty finite set of $\mathcal{L}$-formulas Γ, $\bigwedge \Gamma$ is the conjunction of the formulas in Γ, and $\bigvee \Gamma$ is their disjunction. $\bigwedge \emptyset$ is defined to be $\bot \supset \bot$, and $\bigvee \emptyset$ is defined to be $\bot$.

2. For a sequent $\Gamma \Rightarrow E$, $Int[\Gamma \Rightarrow E]$ is the $\mathcal{L}$-formula $\bigwedge \Gamma \supset \bigvee E$.

3. For a hypersequent H, $Int[H]$ is the $\mathcal{L}$-formula $\bigvee \{Int[s] \mid s \in H\}$.

Theorem 27 H and $Int[H]$ are interderivable in **HIF** for every non-empty hypersequent H (i.e. $H \vdash \Rightarrow Int[H]$ and $\Rightarrow Int[H] \vdash H$).

To prove this theorem, we use the following lemmas:

Lemma 28 $\vdash \Gamma \Rightarrow \bigwedge \Gamma$, for every finite set Γ of $\mathcal{L}$-formulas.

Proof Repeatedly apply identity axioms and $(\Rightarrow\wedge)$. In case Γ is empty, $\bot\Rightarrow\bot$ and $(\Rightarrow\supset)$ are needed. ∎

Lemma 29 Let $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ be a non-empty finite set of $\mathcal{L}$-formulas. Then, $\vdash \bigvee\Gamma \Rightarrow \varphi_1 \mid \ldots \mid \bigvee\Gamma \Rightarrow \varphi_n$.

Proof Repeatedly apply identity axioms and Proposition 23. ∎

Lemma 30 The following hold for every hypersequent H, and sequent $\Gamma \Rightarrow E$:

1. $H \mid \Gamma \Rightarrow E \vdash H \mid \bigwedge\Gamma \Rightarrow \bigvee E$.
2. $H \mid \bigwedge\Gamma \Rightarrow \bigvee E \vdash H \mid \Gamma \Rightarrow E$.
3. $H \mid \Gamma \Rightarrow E \vdash H \mid \Rightarrow Int[\Gamma \Rightarrow E]$.
4. $H \mid \Rightarrow Int[\Gamma \Rightarrow E] \vdash H \mid \Gamma \Rightarrow E$.

Proof

1. Repeatedly apply $(\wedge\Rightarrow_1)$ and $(\wedge\Rightarrow_2)$. Internal weakenings are required in case Γ or E are empty.
2. Obtained by Lemma 28 using a cut. In case E is empty, $(\bot\Rightarrow)$ and another cut are needed.
3. Follows from 1 by applying $(\Rightarrow\supset)$.
4. Using identity axioms and $(\supset\Rightarrow)$, we can prove $\bigwedge\Gamma, Int[\Gamma \Rightarrow E] \Rightarrow \bigvee E$. A cut on $Int[\Gamma \Rightarrow E]$ of this hypersequent and $H \mid \Rightarrow Int[\Gamma \Rightarrow E]$ yields $H \mid \bigwedge\Gamma \Rightarrow \bigvee E$. By 2, one can derive $H \mid \Gamma \Rightarrow E$. ∎

We can now prove Theorem 27.

Proof (of Theorem 27) $H \vdash \Rightarrow Int[H]$ is obtained by repeatedly applying Lemma 30 (3), $(\Rightarrow\vee_1)$ and $(\Rightarrow\vee_2)$. We show that $\Rightarrow Int[H] \vdash H$. Assume that $H = s_1 \mid \ldots \mid s_n$. By Lemma 29, $\vdash Int[H] \Rightarrow Int[s_1] \mid \ldots \mid Int[H] \Rightarrow Int[s_n]$. By n cuts we get $\Rightarrow Int[s_1] \mid \ldots \mid \Rightarrow Int[s_n]$. H is now obtained by repeatedly applying Lemma 30 (4). ∎

Using interpretations of hypersequents, soundness and completeness for **HIF** with respect to the the many-valued semantics were formulated and proved in [8]. Soundness means here that $\vdash H$ implies that $\vdash_{[0,1]} Int[H]$. To have this, one proves that $\vdash_{[0,1]} \varphi$ whenever φ is an interpretation of an instance of **HIF**'s axiom, and that $\varphi_1, \ldots, \varphi_n \vdash_{[0,1]} \varphi$ whenever φ is an interpretation of a conclusion of an instance of **HIF**'s rule, and $\varphi_1, \ldots, \varphi_n$ are the interpretations of the premises of this instance. In turn, completeness means that $\vdash_{[0,1]} Int[H]$ implies $\vdash H$. This was done in [8] relatively to **sG**. Using Fact 10 and Theorem 27, it suffices to prove that $\vdash_{\mathbf{sG}} Int[H]$ implies $\vdash \Rightarrow Int[H]$. To have this, one proves that $\vdash \Rightarrow \varphi$ for every axiom instance φ of **sG**, and that whenever $\varphi_1, \ldots, \varphi_n \rightarrow \varphi$ is an instance of a derivation rule in **sG**, we have $\Rightarrow \varphi_1, \ldots, \Rightarrow \varphi_n \vdash \Rightarrow \varphi$. The existence of the cut rule in **HIF** turns out to be crucial, to handle the Modus Ponens rule, as well as in the proof of Theorem 27. Thus, aiming to obtain a completeness proof of the *cut-free* fragment of **HIF**, we will not follow this approach. In fact, we only need interpretations of hypersequents (and Theorem 27) to obtain the following proposition:

Proposition 31 (Derivability of Substitution) $H \vdash H\{t/a\}$ for every hypersequent H, $\mathcal{L}$-term t, and free variable a.

Proof The claim trivially holds if H is empty. Assume otherwise. By Lemma 30, there exists a derivation of $\Rightarrow Int[H]$ from H in **HIF**. To have a derivation of $H\{t/a\}$ one can proceed as follows. First, apply $(\Rightarrow \forall)$ to obtain $\Rightarrow \forall x(Int[H]\{x/a\})$. Then, Apply $(\forall \Rightarrow)$ on the axiom $Int[H]\{t/a\} \Rightarrow Int[H]\{t/a\}$ to obtain $\forall x(Int[H]\{x/a\}) \Rightarrow Int[H]\{t/a\}$. A cut on $\forall x(Int[H]\{x/a\})$ now gives $\Rightarrow Int[H]\{t/a\}$. From the definition of Int, $Int[H]\{t/a\} = Int[H\{t/a\}]$. By Lemma 30, from $\Rightarrow Int[H\{t/a\}]$ it is possible to derive $H\{t/a\}$ in **HIF**. ∎

Corollary 32 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. Let $\mathcal{H}_*$ be a set of hypersequents, consisting only of substitution instances of the hypersequents in $\mathcal{H}$. If $\mathcal{H}_* \vdash H$, then $\mathcal{H} \vdash H$.

4 Soundness, Completeness and Strong Cut-Admissibility

In this section we obtain the main result of this paper, which is a semantic proof of strong cut-admissibility for **HIF**. First, we define a semantic consequence relation between sets of hypersequents and hypersequents. This relation is based on the Kripke-style semantics presented in Section 2.

Notation 33 Given an $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, M, \mathcal{I}\rangle$, an $\langle \mathcal{L}, M\rangle$-evaluation e, $w \in W$, and a sequent $\Gamma \Rightarrow E$, we write $\mathcal{W}, w, e \vDash \Gamma \Rightarrow E$ if either $\mathcal{W}, w, e \nvDash \varphi$ for some $\varphi \in \Gamma$, or $\mathcal{W}, w, e \vDash \varphi$ for some $\varphi \in E$.

Definition 34 Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I}\rangle$ be an $\mathcal{L}$-frame. $\mathcal{W}$ is a *model* of a hypersequent H if for every $\langle \mathcal{L}, M\rangle$-evaluation e, there exists a component $s \in H$ such that $\mathcal{W}, w, e \vDash s$ for every $w \in W$. $\mathcal{W}$ is a model of a set $\mathcal{H}$ of hypersequents if it is a model of every $H \in \mathcal{H}$.

Definition 35 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. $\mathcal{H} \vdash^{hs}_{Kr} H$ iff every $\mathcal{L}$-frame which is a model of $\mathcal{H}$ is also a model of H.

Next, we prove strong soundness of **HIF** with respect to $\vdash^{hs}_{Kr}$.

Theorem 36 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. If $\mathcal{H} \vdash H$ then $\mathcal{H} \vdash^{hs}_{Kr} H$.

Proof Assume that $\mathcal{H} \vdash H$. Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I}\rangle$ be an $\mathcal{L}$-frame which is a model of $\mathcal{H}$, where $M = \langle D, I\rangle$ and $\mathcal{I} = \{I_w\}_{w \in W}$. We show that for every $\langle \mathcal{L}, M\rangle$-evaluation e, there exists a component $s \in H$ such that $\mathcal{W}, w, e \vDash s$ for every $w \in W$. Since the axioms of **HIF** and the assumptions of $\mathcal{H}$ trivially have this property, it suffices to show that this property is preserved also by applications of the rules of **HIF**. This is a routine matter. We do here only two cases:

(com) Suppose that $H = H_1 \mid H_2 \mid \Gamma_1, \Gamma'_2 \Rightarrow E_1 \mid \Gamma_2, \Gamma'_1 \Rightarrow E_2$ is derived from the hypersequents $H_1 \mid \Gamma_1, \Gamma'_1 \Rightarrow E_1$ and $H_2 \mid \Gamma_2, \Gamma'_2 \Rightarrow E_2$ using (com). Assume for contradiction that $\mathcal{W}$ is not a model of H. Thus there exists an $\langle \mathcal{L}, M\rangle$-evaluation e, such that for every $s \in H$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In particular, for every $s \in H_1 \cup H_2$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In addition, there exist $w_1 \in W$ such that $\mathcal{W}, w_1, e \nvDash \Gamma_1, \Gamma'_2 \Rightarrow E_1$, and $w_2 \in W$ such that $\mathcal{W}, w_2, e \nvDash \Gamma_2, \Gamma'_1 \Rightarrow E_2$. By definition, $\mathcal{W}, w_1, e \vDash \varphi$ for every $\varphi \in \Gamma_1 \cup \Gamma'_2$, while $\mathcal{W}, w_1, e \nvDash \varphi$ for every $\varphi \in E_1$. Analogously, $\mathcal{W}, w_2, e \vDash \varphi$ for every $\varphi \in \Gamma_2 \cup \Gamma'_1$, while $\mathcal{W}, w_2, e \nvDash \varphi$ for every $\varphi \in E_2$. Since $\leq$ is linear, either $w_1 \leq w_2$ or $w_2 \leq w_1$. Assume w.l.o.g that $w_1 \leq w_2$. Then by Proposition 14, $\mathcal{W}, w_2, e \vDash \varphi$ for every $\varphi \in \Gamma'_2$. It follows that $\mathcal{W}, w_2, e \nvDash \Gamma_2, \Gamma'_2 \Rightarrow E_2$. But this implies that $\mathcal{W}$ is not a model of $H_2 \mid \Gamma_2, \Gamma'_2 \Rightarrow E_2$.

($\Rightarrow\forall$) Suppose that $H = H' \mid \Gamma \Rightarrow \forall x(\varphi\{x/a\})$ is derived from the hypersequent $H' \mid \Gamma \Rightarrow \varphi$ using ($\Rightarrow \forall$) (where a is a free variable which does not occur in H). Assume for contradiction that $\mathcal{W}$ is not a model of H. Thus there exists an $\langle \mathcal{L}, M\rangle$-evaluation e, such that for every $s \in H$, there exists $w \in W$ such that $\mathcal{W}, w, e \not\models s$. In particular, for every $s \in H'$, there exists $w \in W$ such that $\mathcal{W}, w, e \not\models s$. In addition, there exists $w_1 \in W$ such that $\mathcal{W}, w_1, e \not\models \Gamma \Rightarrow \forall x(\varphi\{x/a\})$. By definition, $\mathcal{W}, w_1, e \models \varphi$ for every $\varphi \in \Gamma$, and $\mathcal{W}, w_1, e \not\models \forall x(\varphi\{x/a\})$. This implies that there exists some $d \in D$ such that $\mathcal{W}, w_1, e^* \not\models \varphi$ where $e^* = e_{[a:=d]}$. Since a does not occur in Γ, we obtain that $\mathcal{W}, w_1, e^* \models \varphi$ for every $\varphi \in \Gamma$. It follows that $\mathcal{W}, w_1, e^* \not\models \Gamma \Rightarrow \varphi$. Moreover, since a does not occur in H', for every $s \in H'$, there exists $w \in W$ such that $\mathcal{W}, w, e^* \not\models s$. Hence, $\mathcal{W}$ is not a model of $H' \mid \Gamma \Rightarrow \varphi$. ∎

For the completeness proof, we introduce *extended sequents* and *extended hypersequents*, defined as follows:

Definition 37 An *extended sequent* is an ordered pair of (possibly infinite) sets of $\mathcal{L}$-formulas. Given two extended sequents $\mu_1 = \langle \mathcal{T}_1, \mathcal{U}_1\rangle$ and $\mu_2 = \langle \mathcal{T}_2, \mathcal{U}_2\rangle$, we write $\mu_1 \sqsubseteq \mu_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$. An extended sequent is called *finite* if it consists of finite sets of formulas.

Definition 38 An *extended hypersequent* is a (possibly infinite) set of extended sequents. Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every extended sequent $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$. An extended hypersequent is called *finite* if it consists of finite number of finite extended sequents.

We shall use the same notations as above for extended sequents and extended hypersequents. For example, we write $\mathcal{T} \Rightarrow \mathcal{U}$ instead of $\langle \mathcal{T}, \mathcal{U}\rangle$, and $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}$ instead of $\Omega \cup \{\langle \mathcal{T}, \mathcal{U}\rangle\}$.

Definition 39 An extended sequent $\mathcal{T} \Rightarrow \mathcal{U}$ admits *the witness property* if the following hold:

1. If $\forall x(\varphi\{x/a\}) \in \mathcal{U}$ then there exists a free variable b such that $\varphi\{b/a\} \in \mathcal{U}$.
2. If $\exists x(\varphi\{x/a\}) \in \mathcal{T}$ then there exists a free variable b such that $\varphi\{b/a\} \in \mathcal{T}$.

Definition 40 Let Ω be an extended hypersequent, and $\mathcal{H}$ be a set of (ordinary) hypersequents.

1. Ω is called *$\mathcal{H}$-consistent* if $\mathcal{H} \not\vdash^{frm[\mathcal{H}]} H$ for every (ordinary) hypersequent $H \sqsubseteq \Omega$ (see Definition 21 and Notation 19).
2. Let φ be an $\mathcal{L}$-formula. Ω is called *internally $\mathcal{H}$-maximal with respect to* φ if for every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:
   (a) If $\varphi \notin \mathcal{T}$ then $\Omega \mid \mathcal{T}, \varphi \Rightarrow \mathcal{U}$ is not $\mathcal{H}$-consistent.
   (b) If $\varphi \notin \mathcal{U}$ then $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}, \varphi$ is not $\mathcal{H}$-consistent.
3. Ω is called *internally $\mathcal{H}$-maximal* if it is internally $\mathcal{H}$-maximal with respect to any $\mathcal{L}$-formula.
4. Let s be a sequent. Ω is called *externally $\mathcal{H}$-maximal with respect to s* if either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is not $\mathcal{H}$-consistent.
5. Ω is called *externally $\mathcal{H}$-maximal* if it is externally $\mathcal{H}$-maximal with respect to any sequent of the form $E_1 \Rightarrow E_2$ (E_1 and E_2 denote sets of formulas containing at most one formula).
6. Ω admits *the witness property* if every $\mu \in \Omega$ admits the witness property.
7. Ω is called *$\mathcal{H}$-maximal* if it is $\mathcal{H}$-consistent, internally $\mathcal{H}$-maximal, externally $\mathcal{H}$-maximal, and it admits the witness property.

Less formally, an extended hypersequent Ω is internally $\mathcal{H}$-maximal if every formula added on some side of some component of Ω would make it $\mathcal{H}$-inconsistent. Similarly, Ω is externally $\mathcal{H}$-maximal if every sequent of the form $E_1 \Rightarrow E_2$ added to Ω would make it $\mathcal{H}$-inconsistent.

Obviously, every hypersequent is an extended hypersequent, and so all of these properties apply to (ordinary) hypersequents as well.

Proposition 41 Let Ω be an extended hypersequent, which is internally $\mathcal{H}$-maximal with respect to an $\mathcal{L}$-formula φ. For every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:

1. If $\varphi \notin \mathcal{T}$, then $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$ for some hypersequent $H \sqsubseteq \Omega$ and sequents $\Gamma \Rightarrow E_1, \ldots, \Gamma \Rightarrow E_n \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.
2. If $\varphi \notin \mathcal{U}$, then $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma \Rightarrow \varphi$ for some hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \mathcal{T}$.

Proof

1. Let $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\varphi \notin \mathcal{T}$. By internal maximality, $\Omega \mid \mathcal{T}, \varphi \Rightarrow \mathcal{U}$ is not $\mathcal{H}$-consistent, and so there exists a hypersequent $H' \sqsubseteq \Omega \mid \mathcal{T}, \varphi \Rightarrow \mathcal{U}$, such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H'$. Let $H = \{s \in H' \mid \{s\} \sqsubseteq \Omega\}$. Note that for every sequent $\Gamma \Rightarrow E \in H'$ which does not occur in H, we have $\varphi \in \Gamma$, $\Gamma \setminus \{\varphi\} \subseteq \mathcal{T}$, and $E \subseteq \mathcal{U}$. Let $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_n \Rightarrow E_n$ be an enumeration of these sequents, and let $\Gamma = \bigcup \Gamma_i \setminus \{\varphi\}$. By applying internal weakenings on H', we obtain $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$. Clearly, $H \sqsubseteq \Omega$ and $\Gamma \Rightarrow E_1, \ldots, \Gamma \Rightarrow E_n \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.
2. Let $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\varphi \notin \mathcal{U}$. By internal maximality, $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}, \varphi$ is not $\mathcal{H}$-consistent, and so there exists a hypersequent $H' \sqsubseteq \Omega \mid \mathcal{T} \Rightarrow \mathcal{U}, \varphi$, such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H'$. Let $H = \{s \in H' \mid \{s\} \sqsubseteq \Omega\}$. Note that for every sequent $\Gamma \Rightarrow E \in H'$ which does not occur in H, we have $E = \{\varphi\}$ and $\Gamma \subseteq \mathcal{T}$. Let $\Gamma_1 \Rightarrow \varphi, \ldots, \Gamma_n \Rightarrow \varphi$ be an enumeration of the sequents of H' which does not appear in H. Let $\Gamma = \bigcup \Gamma_i$. By applying internal weakenings on H', we obtain $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma \Rightarrow \varphi$. Clearly, $H \sqsubseteq \Omega$ and $\Gamma \subseteq \mathcal{T}$. ∎

Proposition 42 Let Ω be an extended hypersequent, which is externally $\mathcal{H}$-maximal with respect to a sequent s. If $\{s\} \not\sqsubseteq \Omega$, then there exists a hypersequent $H \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid s$.

Proof Immediately follows from the definitions using internal and external weakenings. ∎

A certain $\mathcal{H}$-maximal extended hypersequent serves as the set of worlds in the refuting frame constructed in our completeness proof. Lemma 45 below ensures the existence of that extended hypersequent. In turn, for the proof of Lemma 45 we need the next two lemmas.

Lemma 43 Let $\mathcal{H}$ be a set of hypersequents closed under substitutions, and let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent finite extended hypersequent. Then there exists a $\mathcal{H}$-consistent finite extended hypersequent H' of the form $\Gamma'_1 \Rightarrow \Delta'_1 \mid \ldots \mid \Gamma'_n \Rightarrow \Delta'_n$, such that $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$, and H' admits the witness property.

Proof This extension is done in steps. In every step, we take some extended sequent $\Gamma \Rightarrow \Delta \in H$. If Δ contains a formula of the form $\forall x(\varphi\{x/a\})$, we take a fresh free variable b (a free variable that does not occur in the current extended hypersequent), and add the formula $\varphi\{b/a\}$ to Δ. Furthermore, if Γ contains a formula of the form $\exists x(\varphi\{x/a\})$, we again take a fresh free variable b, and add the formula $\varphi\{b/a\}$ to Γ. We continue this procedure until the obtained extended hypersequent admits the witness property. Note that since the number of formulas in H is finite, and the complexity of the formulas which are added is decreasing, this procedure would terminate after a finite number of steps. H' is the finite extended hypersequent obtained from H by this procedure. We show that every such extension keeps the extended hypersequent $\mathcal{H}$-consistent (and thus H' is $\mathcal{H}$-consistent):

- Suppose that a $\mathcal{H}$-consistent extended hypersequent H_1 contains a sequent $\Gamma \Rightarrow \Delta$, where Δ contains a formula of the form $\forall x(\varphi\{x/a\})$. Let H_2 be the extended hypersequent obtained from H_1 by adding $\varphi\{b/a\}$ to Δ, where b is a free variable which does not occur in H_1. Assume for contradiction that H_2 is not $\mathcal{H}$-consistent. By Proposition 41, there exist a hypersequent $G \sqsubseteq H_1$, and a finite set $\Gamma' \subseteq \Gamma$, such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} G \mid \Gamma' \Rightarrow \varphi\{b/a\}$. By applying $(\Rightarrow\forall)$, we obtain $\mathcal{H} \vdash^{frm[\mathcal{H}]} G \mid \Gamma' \Rightarrow \forall x(\varphi\{b/a\}\{x/b\})$. Obviously, $\forall x(\varphi\{b/a\}\{x/b\}) = \forall x(\varphi\{x/a\})$, and so this contradicts the fact the H_1 is $\mathcal{H}$-consistent.

- Suppose that a $\mathcal{H}$-consistent extended hypersequent H_1 contains a sequent $\Gamma \Rightarrow \Delta$, where Γ contains a formula of the form $\exists x(\varphi\{x/a\})$. Let H_2 be the extended hypersequent obtained from H_1 by adding $\varphi\{b/a\}$ to Γ, where b is a free variable which does not occur in H_1. Assume for contradiction that H_2 is not $\mathcal{H}$-consistent. By Proposition 41, there exist a hypersequent $G \sqsubseteq H_1$, and sequents $\Gamma' \Rightarrow E_1, \dots, \Gamma' \Rightarrow E_n \sqsubseteq \Gamma \Rightarrow \Delta$, such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} G \mid \Gamma', \varphi\{b/a\} \Rightarrow E_1 \mid \dots \mid \Gamma', \varphi\{b/a\} \Rightarrow E_n$. Proposition 24 entails that $\mathcal{H} \vdash^{frm[\mathcal{H}]} G \mid \Gamma', \exists x(\varphi\{b/a\}\{x/b\}) \Rightarrow E_1 \mid \dots \mid \Gamma', \exists x(\varphi\{b/a\}\{x/b\}) \Rightarrow E_n$. Obviously, $\exists x(\varphi\{b/a\}\{x/b\}) = \exists x(\varphi\{x/a\})$, and so this contradicts the fact the H_1 is $\mathcal{H}$-consistent. ∎

Lemma 44 Let $\mathcal{H}$ be a set of hypersequents closed under substitutions. and $H = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent finite extended hypersequent. Let φ be an $\mathcal{L}$-formula, and s be a sequent. Then there exists a $\mathcal{H}$-consistent finite extended hypersequent H', such that:

- $H' = \Gamma'_1 \Rightarrow \Delta'_1 \mid \dots \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$, where $n' \in \{n, n+1\}$, $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$.
- H' is internally $\mathcal{H}$-maximal with respect to φ.
- H' is externally $\mathcal{H}$-maximal with respect to s.
- H' admits the witness property.

Proof Suppose $s = \Gamma^* \Rightarrow E$. First, if $H \mid s$ is $\mathcal{H}$-consistent, let $n' = n + 1$ and define $\Gamma_{n+1} = \Gamma^*$ and $\Delta_{n+1} = E$. Otherwise, let $n' = n$. We recursively define a finite sequence of finite extended hypersequents, $H_0 = \Gamma^0_1 \Rightarrow \Delta^0_1 \mid \dots \mid \Gamma^0_{n'} \Rightarrow \Delta^0_{n'}, \dots, H_{n'} = \Gamma^{n'}_1 \Rightarrow \Delta^{n'}_1 \mid \dots \mid \Gamma^{n'}_{n'} \Rightarrow \Delta^{n'}_{n'}$, in which $\Gamma^i_j \subseteq \Gamma^{i+1}_j$ and $\Delta^i_j \subseteq \Delta^{i+1}_j$ for every $1 \leq j \leq n'$ and $0 \leq i \leq n' - 1$.

First, define $\Gamma^0_j = \Gamma_j$, $\Delta^0_j = \Delta_j$ for every $1 \leq j \leq n'$. Let $0 \leq i \leq n' - 1$. Assume that $H_i = \Gamma^i_1 \Rightarrow \Delta^i_1 \mid \dots \mid \Gamma^i_{n'} \Rightarrow \Delta^i_{n'}$ is defined. We show how to construct $H_{i+1} = \Gamma^{i+1}_1 \Rightarrow \Delta^{i+1}_1 \mid \dots \mid \Gamma^{i+1}_{n'} \Rightarrow \Delta^{i+1}_{n'}$:

1. If $\Gamma^i_1 \Rightarrow \Delta^i_1 \mid \dots \mid \Gamma^i_{i+1}, \varphi \Rightarrow \Delta^i_{i+1} \mid \dots \mid \Gamma^i_{n'} \Rightarrow \Delta^i_{n'}$ is $\mathcal{H}$-consistent, then $\Gamma^{i+1}_{i+1} = \Gamma^i_{i+1} \cup \{\varphi\}$, $\Delta^{i+1}_{i+1} = \Delta^i_{i+1}$, and $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \Delta^i_j$ for every $j \neq i + 1$.
2. Otherwise, if $\Gamma^i_1 \Rightarrow \mathcal{U}^i_1 \mid \dots \mid \Gamma^i_{i+1} \Rightarrow \Delta^i_{i+1}, \varphi \mid \dots \mid \Gamma^i_{n'} \Rightarrow \Delta^i_{n'}$ is $\mathcal{H}$-consistent, then $\Gamma^{i+1}_{i+1} = \Gamma^i_{i+1}$, $\Delta^{i+1}_{i+1} = \Delta^i_{i+1} \cup \{\varphi\}$, and $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \Delta^i_j$ for every $j \neq i + 1$.
3. If both do not hold, then $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \mathcal{U}^i_j$ for every $1 \leq j \leq n'$.

It is easy to verify that $H_{n'} = \Gamma^{n'}_1 \Rightarrow \Delta^{n'}_1 \mid \dots \mid \Gamma^{n'}_{n'} \Rightarrow \Delta^{n'}_{n'}$ is a $\mathcal{H}$-consistent finite extended hypersequent. By Lemma 43, there exists a $\mathcal{H}$-consistent finite extended hypersequent, H' of the form $\Gamma'_1 \Rightarrow \Delta'_1 \mid \dots \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$, such that $\Gamma^{n'}_j \subseteq \Gamma'_j$ and $\Delta^{n'}_j \subseteq \Delta'_j$ for every $1 \leq j \leq n'$, and H' admits the witness property. It is again easy to see that H' has all the required properties. For example, we show that H' is internally $\mathcal{H}$-maximal with respect to φ. Let $\Gamma \Rightarrow \Delta \in H'$. Suppose that $\Gamma = \Gamma'_j$ and $\Delta = \Delta'_j$.

- Assume that $\varphi \notin \Gamma$. Then, since $\Gamma^j_j \subseteq \Gamma$, this implies that $\Gamma^{j-1}_1 \Rightarrow \Delta^{j-1}_1 \mid \dots \mid \Gamma^{j-1}_j, \varphi \Rightarrow \Delta^{j-1}_j \mid \dots \mid \Gamma^{j-1}_{n'} \Rightarrow \Delta^{j-1}_{n'}$ is not $\mathcal{H}$-consistent. It easily follows that $H' \mid \Gamma, \varphi \Rightarrow \Delta$ (which extends this finite extended hypersequent) is not $\mathcal{H}$-consistent.

- Now, assume that $\varphi \notin \Delta$. If $\varphi \in \Gamma$, then since $\varphi \Rightarrow \varphi$ is an axiom of **HIF**, $H' \mid \Gamma \Rightarrow \Delta, \varphi$ is not $\mathcal{H}$-consistent. Otherwise, $\varphi \notin \Gamma$, and since $\Gamma^j_j \subseteq \Gamma$ and $\Delta^j_j \subseteq \Delta$, our construction ensures that $\Gamma^{j-1}_1 \Rightarrow \Delta^{j-1}_1 \mid \ldots \mid \Gamma^{j-1}_j \Rightarrow \Delta^{j-1}_j, \varphi \mid \ldots \mid \Gamma^{j-1}_{n'} \Rightarrow \Delta^{j-1}_{n'}$ is not $\mathcal{H}$-consistent. It easily follows that $H' \mid \Gamma \Rightarrow \Delta, \varphi$ (which extends this finite extended hypersequent) is not $\mathcal{H}$-consistent. ∎

Lemma 45 Let $\mathcal{H}$ be a set of hypersequents closed under substitutions. Every $\mathcal{H}$-consistent hypersequent H can be extended to a $\mathcal{H}$-maximal extended hypersequent Ω.

Proof Suppose that $H = \Gamma_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_n \Rightarrow E_n$. Let $\varphi_0, \varphi_1 \ldots$ be an enumeration of all $\mathcal{L}$-formulas, in which every formula appears an infinite number of times. Let $s_0, s_1 \ldots$ be an enumeration of all sequents of the form $E'_1 \Rightarrow E'_2$.

We recursively define an infinite sequence of $\mathcal{H}$-consistent finite extended hypersequents, $H_0 = \Gamma^0_1 \Rightarrow \Delta^0_1 \mid \ldots \mid \Gamma^0_{n_0} \Rightarrow \Delta^0_{n_0}, H_1 = \Gamma^1_1 \Rightarrow \Delta^1_1 \mid \ldots \mid \Gamma^1_{n_1} \Rightarrow \Delta^1_{n_1}, \ldots$, in which: (a) $n_0 \leq n_1 \leq \ldots$ and (b) $\Gamma^i_j \subseteq \Gamma^{i+1}_j$ and $\Delta^i_j \subseteq \Delta^{i+1}_j$ for every $i \geq 0$ and $1 \leq j \leq n_i$.

First, let $n_0 = n$ and let $\Gamma^0_j = \Gamma_j$, $\Delta^0_j = E_j$ for every $1 \leq j \leq n_0$. Let $i \geq 0$. Assume $H_i = \Gamma^i_1 \Rightarrow \Delta^i_1 \mid \ldots \mid \Gamma^i_{n_i} \Rightarrow \Delta^i_{n_i}$ is defined. By Lemma 44, there exists a $\mathcal{H}$-consistent hypersequent H' such that:

- $H' = \Gamma'_1 \Rightarrow \Delta'_1 \mid \ldots \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$ where $n' \in \{n_i, n_i + 1\}$, and $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n_i$.
- H' is internally $\mathcal{H}$-maximal with respect to φ_i.
- H' is externally $\mathcal{H}$-maximal with respect to s_i.
- H' admits the witness property.

Let $n_{i+1} = n'$, and $\Gamma^{i+1}_j = \Gamma'_j$, $\Delta^{i+1}_j = \Delta'_j$ for every $1 \leq j \leq n_{i+1}$.

Note that after every step we have a $\mathcal{H}$-consistent finite extended hypersequent, so Lemma 44 can be applied. Finally, let N be $\max\{n_0, n_1, \ldots\} + 1$, if such a maximum exists, and infinity otherwise. Let $n(j) = \min\{i \mid j \leq n_i\}$ for every $1 \leq j < N$. Define $\mathcal{T}_j = \cup_{i \geq n(j)} \Gamma^i_j$ and $\mathcal{U}_j = \cup_{i \geq n(j)} \Delta^i_j$ for every $1 \leq j < N$. Let Ω be the extended hypersequent $\mathcal{T}_1 \Rightarrow \mathcal{U}_1 \mid \mathcal{T}_2 \Rightarrow \mathcal{U}_2 \mid \ldots$. Obviously, Ω extends H. We prove that Ω is $\mathcal{H}$-maximal:

Consistency Suppose by way of contradiction that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$ for some hypersequent $H \sqsubseteq \Omega$. Assume that $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$. The construction of Ω ensures that for every $1 \leq i \leq n$, there exists $k_i \geq 1$ such that $\Gamma_i \subseteq \mathcal{T}_{k_i}$ and $\Delta_i \subseteq \mathcal{U}_{k_i}$. This entails that for every $1 \leq i \leq n$, there exists $m_i \geq 0$ such that $\Gamma_i \subseteq \Gamma^{m_i}_{k_i}$ and $\Delta_i \subseteq \Delta^{m_i}_{k_i}$. By the construction of the Γ^i_j's and Δ^i_j's, we have that for every $1 \leq i \leq n$ and $l \geq m_i$, $\Gamma_i \subseteq \Gamma^l_{k_i}$ and $\Delta_i \subseteq \Delta^l_{k_i}$. Let $m = \max\{m_1, \ldots, m_n\}$. Then, by definition $H \sqsubseteq H_m$. Since $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$, it follows that H_m is not $\mathcal{H}$-consistent. But, this contradicts the fact that H_0 is consistent, and that each application of Lemma 44 yields a $\mathcal{H}$-consistent extended hypersequent.

Internal Maximality We show that Ω is internally $\mathcal{H}$-maximal with respect to every $\mathcal{L}$-formula. Let φ be an $\mathcal{L}$-formula, and let $\mathcal{T}_j \Rightarrow \mathcal{U}_j \in \Omega$. Since we included φ infinite number of times in the enumeration of the formulas, there exists some $i \geq n(j)$ such that $\varphi_i = \varphi$. Our construction ensures that H_{i+1} is internally $\mathcal{H}$-maximal with respect to φ, and so if $\varphi \notin \Gamma^{i+1}_j$ then $H_{i+1} \mid \Gamma^{i+1}_j, \varphi \Rightarrow \Delta^{i+1}_j$ is not $\mathcal{H}$-consistent, and if $\varphi \notin \Delta^{i+1}_j$ then $H_{i+1} \mid \Gamma^{i+1}_j \Rightarrow \Delta^{i+1}_j, \varphi$ is not $\mathcal{H}$-consistent. Since $H_{i+1} \sqsubseteq \Omega$, it follows that if $\varphi \notin \mathcal{T}_j$ then $\Omega \mid \mathcal{T}_j, \varphi \Rightarrow \mathcal{U}_j$ is not $\mathcal{H}$-consistent, and if $\varphi \notin \mathcal{U}_j$ then $\Omega \mid \mathcal{T}_j \Rightarrow \mathcal{U}_j, \varphi$ is not $\mathcal{H}$-consistent.

External Maximality We show that Ω is externally $\mathcal{H}$-maximal with respect to every sequent of the form $E'_1 \Rightarrow E'_2$. Let s be such a sequent. Assume that $s = s_i$ ($i \geq 0$), our construction ensures that H_{i+1} is externally $\mathcal{H}$-maximal with respect to s. Hence, either $\{s\} \sqsubseteq H_{i+1}$, or $H_{i+1} \mid s$ is not $\mathcal{H}$-consistent. Since $H_{i+1} \sqsubseteq \Omega$, either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is not $\mathcal{H}$-consistent.

The Witness Property Let $1 \leq j < N$. We show that $\mathcal{T}_j \Rightarrow \mathcal{U}_j$ admits the witness property. Assume $\forall x(\varphi\{x/a\}) \in \mathcal{U}_j$. Then $\forall x(\varphi\{x/a\}) \in \Delta^i_j$ for some $i \geq n(j)$. We can assume that $i > 0$ (if it holds for $i = 0$ then it holds for $i = 1$ as well). Our construction ensures that H_i admits the witness property, and so there exists a free variable b such that $\varphi\{b/a\} \in \Delta^i_j$. Since $\Delta^i_j \subseteq \mathcal{U}_j$. It follows that there exists a free variable b such that $\varphi\{b/a\} \in \mathcal{U}_j$. The case in which $\exists x(\varphi\{x/a\}) \in \mathcal{T}_j$ is analogous. ∎

Next we define the $\mathcal{L}$-algebra used in the completeness proof.

Definition 46 The *Herbrand $\mathcal{L}$-algebra* is an $\mathcal{L}$-algebra, $\langle D, I\rangle$, such that $D = trm_{\mathcal{L}}$ (the set of *all* $\mathcal{L}$-terms), $I[c] = c$ for every constant c, and $I[f][t_1, \ldots, t_n] = f(t_1, \ldots, t_n)$ for every n-ary function symbol f and $t_1, \ldots, t_n \in D$.

Note that the domain of the Herbrand $\mathcal{L}$-algebra contains also non-closed terms. However, recall that we assume that the set of free variables and the set of bounded variables are disjoint, and so an $\mathcal{L}$-term cannot contain a bounded variable. The following technical proposition is proved by a standard structural induction:

Proposition 47 Let $M = \langle D, I\rangle$ be the Herbrand $\mathcal{L}$-algebra, and e be the identity $\langle \mathcal{L}, M\rangle$-evaluation (defined by $e[a] = a$ for every free variable a). For every $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, M, \mathcal{I}\rangle$, $t_1, \ldots, t_n \in D$, distinct free variables $a_1, \ldots, a_n$, and $w \in W$:

- For every $\mathcal{L}$-term t: $e_{[a_1:=t_1,\ldots,a_n:=t_n]}[t] = t\{t_1/a_1, \ldots, t_n/a_n\}$. In particular, $e[t] = t$.
- For every $\mathcal{L}$-formula φ: $\mathcal{W}, w, e_{[a_1:=t_1,\ldots,a_n:=t_n]} \models \varphi$ iff $\mathcal{W}, w, e \models \varphi\{t_1/a_1, \ldots, t_n/a_n\}$.
- For every sequent s: $\mathcal{W}, w, e_{[a_1:=t_1,\ldots,a_n:=t_n]} \models s$ iff $\mathcal{W}, w, e \models s\{t_1/a_1, \ldots, t_n/a_n\}$.

We are now ready to establish the main completeness theorem.

Theorem 48 Let $\mathcal{H}_0$ be a set of hypersequents closed under substitutions, and H_0 be a hypersequent. If $\mathcal{H}_0 \vdash^{hs}_{Kr} H_0$ then $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_0$.

Proof Assume that $\mathcal{H}_0 \not\vdash^{frm[\mathcal{H}_0]} H_0$. We construct an $\mathcal{L}$-frame $\mathcal{W}$ which is a model of $\mathcal{H}_0$ but not of H_0. The availability of external and internal weakenings ensures that H_0 is $\mathcal{H}_0$-consistent. Thus by Lemma 45, there exists a $\mathcal{H}_0$-maximal extended hypersequent Ω such that $H_0 \sqsubseteq \Omega$. Using Ω, $\mathcal{W} = \langle W, \leq, M, \mathcal{I}\rangle$ is defined as follows:

- $W = \Omega$.
- For every $\mathcal{T}_1 \Rightarrow \mathcal{U}_1, \mathcal{T}_2 \Rightarrow \mathcal{U}_2 \in W$, $\mathcal{T}_1 \Rightarrow \mathcal{U}_1 \leq \mathcal{T}_2 \Rightarrow \mathcal{U}_2$ iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- $M = \langle D, I\rangle$ is the Herbrand $\mathcal{L}$-algebra.
- $\mathcal{I} = \{I_w\}_{w \in W}$ where $\langle t_1, \ldots, t_n\rangle \in I_{\mathcal{T} \Rightarrow \mathcal{U}}[p]$ iff $p(t_1, \ldots, t_n) \in \mathcal{T}$.

We first prove that $\langle W, \leq\rangle$ is linearly ordered:

Partial Order Obviously $\leq$ is reflexive and transitive. To see that it is also anti-symmetric, let w_1, w_2 be elements of W such that $w_1 \leq w_2$ and $w_2 \leq w_1$. Assume that $w_1 = \mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $w_2 = \mathcal{T}_2 \Rightarrow \mathcal{U}_2$. By definition, $\mathcal{T}_1 = \mathcal{T}_2$ in this case. Assume for contradiction that $\mathcal{U}_1 \neq \mathcal{U}_2$, and let $\varphi \in \mathcal{U}_1 \setminus \mathcal{U}_2$ (w.l.o.g.). Since Ω is internally $\mathcal{H}$-maximal, by Proposition 41, $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \varphi$ for some hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \mathcal{T}_2$. But, $\Gamma \Rightarrow \varphi \sqsubseteq w_1$, and this contradicts Ω's $\mathcal{H}_0$-consistency. Hence $\mathcal{U}_1 = \mathcal{U}_2$, and so $w_1 = w_2$.

Linearity Let $\mathcal{T}_1 \Rightarrow \mathcal{U}_1, \mathcal{T}_2 \Rightarrow \mathcal{U}_2 \in W$. Assume for contradiction that $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$. Let $\varphi_1 \in \mathcal{T}_1 \setminus \mathcal{T}_2$ and $\varphi_2 \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Since Ω is internally $\mathcal{H}_0$-maximal, by Proposition 41, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$ and sequents $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $\Gamma_2 \Rightarrow F_1, \ldots, \Gamma_2 \Rightarrow F_m \sqsubseteq \mathcal{T}_2 \Rightarrow \mathcal{U}_2$ such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid \Gamma_1, \varphi_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \varphi_2 \Rightarrow E_n$ and $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_2 \mid \Gamma_2, \varphi_1 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \varphi_1 \Rightarrow F_m$. By Proposition 22, $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid H_2 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \varphi_1 \Rightarrow E_n \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \varphi_2 \Rightarrow F_m$. But, note that Ω extends this hypersequent, and so this contradicts Ω's $\mathcal{H}_0$-consistency.

Now, let e be the identity $\langle \mathcal{L}, M \rangle$-evaluation (defined by $e[a] = a$ for every free variable a). Next we prove that the following hold for every $w = \mathcal{T} \Rightarrow \mathcal{U} \in W$:

(a) If $\psi \in \mathcal{T}$ then $\mathcal{W}, w, e \models \psi$.

(b) If $\psi \in \mathcal{U}$ then $\mathcal{W}, w, e \not\models \psi$.

(a) and **(b)** are proved together using a simultaneous induction on the complexity of ψ. Let $w = \mathcal{T} \Rightarrow \mathcal{U} \in W$.

- Suppose ψ is an atomic $\mathcal{L}$-formula of the form $p(t_1, \ldots, t_n)$. By definition, $\mathcal{W}, w, e \models \psi$ iff $\langle e[t_1], \ldots, e[t_n] \rangle \in I_w[p]$. By Proposition 47, $e[t_i] = t_i$ for every $1 \leq i \leq n$. Hence our construction ensures that $\mathcal{W}, w, e \models \psi$ iff $\psi \in \mathcal{T}$. This proves $(\mathbf{a})$. For $(\mathbf{b})$, note that $\varphi \Rightarrow \varphi$ is an axiom (for every $\mathcal{L}$-formula φ), and since Ω is $\mathcal{H}_0$-consistent, $\psi \in \mathcal{U}$ implies $\psi \notin \mathcal{T}$. It follows that $\mathcal{W}, w, e \not\models \psi$.

- Suppose $\psi = \bot$. Then $(\mathbf{b})$ is trivially true. On the other hand, $(\mathbf{a})$ is vacuously true. To see this, assume that $\bot \in \mathcal{T}$. Since $\bot \Rightarrow$ is an axiom, $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} \bot \Rightarrow$. But, in this case $\{\bot \Rightarrow\} \sqsubseteq \Omega$, and this contradicts Ω's $\mathcal{H}_0$-consistency.

- Suppose $\psi = \varphi_1 \supset \varphi_2$.

  1. Assume that $\psi \in \mathcal{T}$. We show that for every element $w' \in W$ such that $w \leq w'$ either $\mathcal{W}, w', e \not\models \varphi_1$ or $\mathcal{W}, w', e \models \varphi_2$.

     Let $w' = \mathcal{T}' \Rightarrow \mathcal{U}'$ be an element of W, such that $w \leq w'$ (and so, $\mathcal{T} \subseteq \mathcal{T}'$). By the induction hypothesis, it suffices to show that either $\varphi_1 \in \mathcal{U}'$ or $\varphi_2 \in \mathcal{T}'$. Assume otherwise. By Proposition 41, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, a finite set $\Gamma_1 \subseteq \mathcal{T}'$, and sequents $\Gamma_2 \Rightarrow E_1, \ldots, \Gamma_2 \Rightarrow E_n \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid \Gamma_1 \Rightarrow \varphi_1$, and $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_2 \mid \Gamma_2, \varphi_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_2, \varphi_2 \Rightarrow E_n$. By n consecutive applications of $(\supset\Rightarrow)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2, \psi \Rightarrow E_n$. But since $\psi \in \mathcal{T}$, $\psi \in \mathcal{T}'$ and so $H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2, \psi \Rightarrow E_n \sqsubseteq \Omega$. This contradicts Ω's $\mathcal{H}_0$-consistency.

  2. Assume that $\psi \in \mathcal{U}$.

     First we claim that $\mathcal{H}_0 \not\vdash^{frm[\mathcal{H}_0]} H \mid \varphi_1 \Rightarrow \varphi_2$ for every hypersequent $H \sqsubseteq \Omega$. To see this, assume for contradiction that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \varphi_1 \Rightarrow \varphi_2$ for some $H \sqsubseteq \Omega$. By applying $(\Rightarrow\supset)$ to this hypersequent we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Rightarrow \psi$. But this contradicts Ω's $\mathcal{H}_0$-consistency (because $H \mid \Rightarrow \psi \sqsubseteq \Omega$).

Therefore, by Proposition 42, Ω's external $\mathcal{H}_0$-maximality entails that $\varphi_1 \Rightarrow \varphi_2 \sqsubseteq \Omega$. Thus there exists an extended sequent $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\varphi_1 \in \mathcal{T}'$ and $\varphi_2 \in \mathcal{U}'$. By the induction hypothesis, $\mathcal{W}, \mathcal{T}' \Rightarrow \mathcal{U}', e \vDash \varphi_1$ and $\mathcal{W}, \mathcal{T}' \Rightarrow \mathcal{U}', e \nvDash \varphi_2$. It follows that if $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{W}, w, e \nvDash \psi$ and we are done.

Assume now that $\mathcal{T} \nsubseteq \mathcal{T}'$. By linearity, $\mathcal{T}' \subseteq \mathcal{T}$, and so $\varphi_1 \in \mathcal{T}$. By the induction hypothesis, $\mathcal{W}, w, e \vDash \varphi_1$. Now notice that $\varphi_2 \in \mathcal{U}$. To see this assume for contradiction that $\varphi_2 \notin \mathcal{U}$. Then by Ω's internal $\mathcal{H}_0$-maximality, there exist a hypersequent $H \sqsubseteq \Omega$, and a finite set $\Gamma \subseteq \mathcal{T}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \varphi_2$. By applying internal weakening we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2$. By $(\Rightarrow\supset)$ we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \psi$. But this contradicts Ω's $\mathcal{H}_0$-consistency (because $H \mid \Gamma \Rightarrow \psi \sqsubseteq \Omega$). Finally, since $\varphi_2 \in \mathcal{U}$, the induction hypothesis entails that $\mathcal{W}, w, e \nvDash \varphi_2$, and this again implies that $\mathcal{W}, w, e \nvDash \psi$.

- Suppose $\psi = \varphi_1 \vee \varphi_2$.

  1. Assume that $\psi \in \mathcal{T}$. Then either $\varphi_1 \in \mathcal{T}$ or $\varphi_2 \in \mathcal{T}$. To see this, suppose $\varphi_1 \notin \mathcal{T}$ and $\varphi_2 \notin \mathcal{T}$. By Proposition 41, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, and sequents $\Gamma_1 \Rightarrow E_1, \dots, \Gamma_1 \Rightarrow E_n, \Gamma_2 \Rightarrow F_1, \dots, \Gamma_2 \Rightarrow F_m \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \Rightarrow E_n$ and $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_2 \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_2 \Rightarrow F_m$. By Proposition 23, it follows that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid H_2 \mid \Gamma_1, \psi \Rightarrow E_1 \mid \dots \mid \Gamma_1, \psi \Rightarrow E_n \mid \Gamma_2, \psi \Rightarrow F_1 \mid \dots \mid \Gamma_2, \psi \Rightarrow F_m$. But this contradicts Ω's $\mathcal{H}_0$-consistency.

     By the induction hypothesis, $\mathcal{W}, w, e \vDash \varphi_1$ or $\mathcal{W}, w, e \vDash \varphi_2$. And so, $\mathcal{W}, w, e \vDash \psi$.

  2. Assume that $\psi \in \mathcal{U}$. Then $\varphi_1 \in \mathcal{U}$ and $\varphi_2 \in \mathcal{U}$. To see this, suppose for example that $\varphi_1 \notin \mathcal{U}$. By Proposition 41, there exist hypersequent $H \sqsubseteq \Omega$, and finite set $\Gamma \subseteq \mathcal{T}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \varphi_1$. By applying $(\Rightarrow \vee_1)$ we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \psi$. But this contradicts Ω's $\mathcal{H}_0$-consistency.

     By the induction hypothesis, $\mathcal{W}, w, e \nvDash \varphi_1$ and $\mathcal{W}, w, e \nvDash \varphi_2$. And so, $\mathcal{W}, w, e \nvDash \psi$.

- Suppose $\psi = \varphi_1 \wedge \varphi_2$.

  1. Assume that $\psi \in \mathcal{T}$. Then $\varphi_1 \in \mathcal{T}$ and $\varphi_2 \in \mathcal{T}$. To see this, suppose for example that $\varphi_1 \notin \mathcal{T}$. By Proposition 41, there exist hypersequent $H \sqsubseteq \Omega$, and sequents $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi_1 \Rightarrow E_n$. By n consecutive applications of $(\wedge \Rightarrow_1)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \psi \Rightarrow E_1 \mid \dots \mid \Gamma, \psi \Rightarrow E_n$. But this contradicts Ω's $\mathcal{H}_0$-consistency.

     By the induction hypothesis, $\mathcal{W}, w, e \vDash \varphi_1$ and $\mathcal{W}, w, e \vDash \varphi_2$, and so, $\mathcal{W}, w, e \vDash \psi$.

  2. Assume that $\psi \in \mathcal{U}$. Then $\varphi_1 \in \mathcal{U}$ or $\varphi_2 \in \mathcal{U}$. To see this, suppose that $\varphi_1 \notin \mathcal{U}$ and $\varphi_2 \notin \mathcal{U}$. Then by Proposition 41, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, and finite sets $\Gamma_1, \Gamma_2 \subseteq \mathcal{T}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid \Gamma_1 \Rightarrow \varphi_1$ and $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_2 \mid \Gamma_2 \Rightarrow \varphi_2$. By applying $(\Rightarrow \wedge)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \psi$. But this contradicts Ω's $\mathcal{H}_0$-consistency.

     By the induction hypothesis, $\mathcal{W}, w, e \nvDash \varphi_1$ or $\mathcal{W}, w, e \nvDash \varphi_2$, and so, $\mathcal{W}, w, e \nvDash \psi$.

- Suppose $\psi = \forall x(\varphi\{x/a\})$.

  1. Assume that $\mathcal{W}, w, e \nvDash \psi$. We show that $\psi \notin \mathcal{T}$. By definition, there exists some $t \in D$ such that $\mathcal{W}, w, e_{[a:=t]} \nvDash \varphi$. By Proposition 47, $\mathcal{W}, w, e \nvDash \varphi\{t/a\}$. By the induction hypothesis, $\varphi\{t/a\} \notin \mathcal{T}$. By Proposition 41, there exist hypersequent $H \sqsubseteq \Omega$, and sequents $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \varphi\{t/a\} \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi\{t/a\} \Rightarrow E_n$. By n consecutive applications of $(\forall\Rightarrow)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \psi \Rightarrow E_1 \mid \dots \mid \Gamma, \psi \Rightarrow E_n$. Since Ω is $\mathcal{H}_0$-consistent, $\psi \notin \mathcal{T}$.

2. Assume that $\psi \in \mathcal{U}$. By Ω's witness property, there exists a free variable b such that $\varphi\{b/a\} \in \mathcal{U}$. From the induction hypothesis it follows that $\mathcal{W}, w, e \not\models \varphi\{b/a\}$. By Proposition 47, it follows that $\mathcal{W}, w, e_{[a:=b]} \not\models \varphi$, and so $\mathcal{W}, w, e \not\models \psi$ (since $b \in D$).

- Suppose $\psi = \exists x(\varphi\{x/a\})$.

1. Assume that $\psi \in \mathcal{T}$. By Ω's witness property, there exists a free variable b such that $\varphi\{b/a\} \in \mathcal{T}$. By the induction hypothesis, $\mathcal{W}, w, e \models \varphi\{b/a\}$. By Proposition 47, it follows that $\mathcal{W}, w, e_{[a:=b]} \models \varphi$, and so $\mathcal{W}, w, e \models \psi$ (since $b \in D$).
2. Assume that $\mathcal{W}, w, e \models \psi$. We show that $\psi \notin \mathcal{U}$. By definition, there exists some $t \in D$ such that $\mathcal{W}, w, e_{[a:=t]} \models \varphi$. By Proposition 47, it follows that $\mathcal{W}, w, e \models \varphi\{t/a\}$. By the induction hypothesis, $\varphi\{t/a\} \notin \mathcal{U}$. By Proposition 41, there exist hypersequent $H \sqsubseteq \Omega$, and finite set $\Gamma \subseteq \mathcal{T}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \varphi\{t/a\}$. By applying $(\Rightarrow \exists)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma \Rightarrow \psi$. Since Ω is $\mathcal{H}_0$-consistent, $\psi \notin \mathcal{U}$.

It remains to show that $\mathcal{W}$ is a model of $\mathcal{H}_0$ but not of H_0. First, note that for every $\varphi \in frm[\mathcal{H}_0]$ and $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$, either $\varphi \in \mathcal{T}$ or $\varphi \in \mathcal{U}$. Otherwise, by Proposition 41, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, sequents $\Gamma_1 \Rightarrow E_1, \ldots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, and a finite set $\Gamma_2 \subseteq \mathcal{T}$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid \Gamma_1, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \varphi \Rightarrow E_n$, and $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_2 \mid \Gamma_2 \Rightarrow \varphi$. Now using n (legal) applications of the cut rule, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2 \Rightarrow E_n$, but this contradicts Ω's $\mathcal{H}_0$-consistency.

Second, we prove that for every hypersequent $H \not\sqsubseteq \Omega$ such that $frm[H] \subseteq frm[\mathcal{H}_0]$, there exists some $s \in H$ such that $\mathcal{W}, w, e \models s$ for every $w \in W$. Indeed, if $H \not\sqsubseteq \Omega$, then there exists a sequent $s = \Gamma \Rightarrow E$, such that $s \in H$ and $s \not\sqsubseteq w$ for every $w \in W$. Let $w \in W$. Assume that $w = \mathcal{T} \Rightarrow \mathcal{U}$. Since $s \not\sqsubseteq w$, there either exists $\varphi \in \Gamma$ such that $\varphi \notin \mathcal{T}$, or $\varphi \in E$ such that $\varphi \notin \mathcal{U}$. By the previous claim (since $frm[H] \subseteq frm[\mathcal{H}_0]$), this implies that there either exists $\varphi \in \Gamma$ such that $\varphi \in \mathcal{U}$, or $\varphi \in E$ such that $\varphi \in \mathcal{T}$. By **(a)** and **(b)**, either there exists $\varphi \in \Gamma$ such that $\mathcal{W}, w, e \not\models \varphi$, or there exists $\varphi \in E$ such that $\mathcal{W}, w, e \models \varphi$. Therefore, $\mathcal{W}, w, e \models s$.

Now let $H \in \mathcal{H}_0$, and let e^* be an arbitrary $\langle \mathcal{L}, M \rangle$-evaluation. We show that there exists some $s \in H$ such that $\mathcal{W}, w, e^* \models s$ for every $w \in W$. Let $a_1, \ldots, a_k$ be the free variables appearing in H, and let $H' = H\{e^*[a_1]/a_1, \ldots, e^*[a_k]/a_k\}$. Since $\mathcal{H}_0$ is closed under substitutions $H' \in \mathcal{H}_0$, and obviously $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H'$. Since Ω is $\mathcal{H}_0$-consistent, $H' \not\sqsubseteq \Omega$. By the previous claim, there exists some $s' \in H'$, such that $\mathcal{W}, w, e \models s'$ for every $w \in W$. Let s be a sequent in H such that $s' = s\{e^*[a_1]/a_1, \ldots, e^*[a_k]/a_k\}$. By Proposition 47, it follows that $\mathcal{W}, w, e_{[a_1:=e^*[a_1], \ldots, a_k:=e^*[a_k]]} \models s$ for every $w \in W$. Since $e_{[a_1:=e^*[a_1], \ldots, a_k:=e^*[a_k]]}$ and e^* agree on $a_1, \ldots, a_k$, we have that $\mathcal{W}, w, e^* \models s$ for every $w \in W$.

We end the proof by showing that $\mathcal{W}$ is not a model of H_0. Indeed, we show that for every $s \in H_0$, there exists some $w \in W$ such that $\mathcal{W}, w, e \not\models s$. Let $s \in H_0$. Assume that $s = \Gamma \Rightarrow E$. Since $H_0 \sqsubseteq \Omega$, there exists an extended sequent $w = \mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $s \sqsubseteq w$. By **(a)**, for every $\varphi \in \Gamma$, $\mathcal{W}, w, e \models \varphi$. By **(b)**, for every $\varphi \in E$, $\mathcal{W}, w, e \not\models \varphi$. Thus, $\mathcal{W}, w, e \not\models \Gamma \Rightarrow E$. ∎

Corollary 49 (Strong Soundness and Completeness) **HIF** is strongly sound and complete with respect to the Kripke semantics of the standard first-order Gödel logic, i.e. $\mathcal{H} \vdash H$ iff $\mathcal{H} \vdash^{hs}_{Kr} H$.

Proof One direction follows from Theorem 36. For the converse, suppose that $\mathcal{H} \vdash^{hs}_{Kr} H$. Let $\mathcal{H}_*$ be the closure of $\mathcal{H}$ under substitutions. Obviously, $\mathcal{H}_* \vdash^{hs}_{Kr} H$. By Theorem 48, $\mathcal{H}_* \vdash^{frm[\mathcal{H}_*]} H$, and so $\mathcal{H}_* \vdash H$. By Corollary 32, $\mathcal{H} \vdash H$. ∎

We can now provide semantic basis for the interpretation of hypersequents (see Subsection 3.1).

Corollary 50 An $\mathcal{L}$-frame $\mathcal{W}$ is a model of a hypersequent H, iff it is a model of (the $\mathcal{L}$-formula) $Int[H]$.

Proof By Theorem 27, $H \vdash\, \Rightarrow Int[H]$. Hence, $H \vdash^{hs}_{Kr} \Rightarrow Int[H]$. It follows that every $\mathcal{L}$-frame which is a model of H is also a model of $\Rightarrow Int[H]$. By definition, an $\mathcal{L}$-frame is a model of a formula φ iff it is a model of $\Rightarrow \varphi$. The converse is similar. ∎

Recall that hypersequents are just a tool to obtain a well-behaved deduction system for the standard first-order Gödel logic. The following corollary reestablishes this link:

Corollary 51 $\mathcal{T} \vdash_{Kr} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash\, \Rightarrow \varphi$.

Proof From the definition of $\vdash^{hs}_{Kr}$, it easily follows that a $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash^{hs}_{Kr} \Rightarrow \varphi$ iff $\mathcal{T} \vdash_{Kr} \varphi$. Thus the claim follows from Corollary 49. ∎

Obviously, following facts 10 and 17, $\vdash_{Kr}$ in the last corollary can be replaced with $\vdash_{[0,1]}$ or $\vdash_{\mathbf{sG}}$.

4.1 Strong Cut-Admissibility

Taken together, Theorems 36 and 48 naturally entail the following strong cut-admissibility result.

Corollary 52 $\mathcal{H} \vdash H$ implies $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$, for every set $\mathcal{H}$ of hypersequents closed under substitutions, and a hypersequent H. In particular, for every hypersequent H, $\vdash H$ implies that there exists a proof of H in **HIF** $- (cut)$.

Proof By Theorem 36, $\mathcal{H} \vdash H$ entails that $\mathcal{H} \vdash^{hs}_{Kr} H$. Since $\mathcal{H}$ is closed under substitutions, Theorem 48 implies that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$. ∎

Note that in the last corollary, it is necessary to require that the set of assumptions $\mathcal{H}$ is closed under substitutions. Indeed, $\Rightarrow p(a_1) \vdash\, \Rightarrow p(a_2)$, but there is no derivation of $\Rightarrow p(a_2)$ from $\Rightarrow p(a_1)$ in **HIF** with cuts only on $p(a_1)$.

Remark 53 Equivalently, one can consider the substitution rule (allowing to infer $H\{t/a\}$ from H for every hypersequent H, $\mathcal{L}$-term t and free variable a), and prove that for every set $\mathcal{H} \cup \{H\}$ of hypersequents, if $\mathcal{H} \vdash H$, then there exists a proof of H from $\mathcal{H}$ in **HIF** $+ (substitution)$, in which the substitution rule is applied only to hypersequents of $\mathcal{H}$ and all cuts are on *substitution instances* of formulas which occur in $\mathcal{H}$. Note that cuts on formulas which occur in $\mathcal{H}$ do not suffice. Indeed, $\Rightarrow p(a,c)\ ,\ p(c,a) \Rightarrow \vdash\, \Rightarrow$, but there is no derivation of the empty sequent from $\Rightarrow p(a,c)$ and $p(c,a) \Rightarrow$ in **HIF** $+ (substitution)$, in which cuts are only on $p(a,c)$ and $p(c,a)$.

In the rest of this subsection, we identify a natural, more restrictive, semantic consequence relation $\vdash^{hs-t}_{Kr}$ between hypersequents. Unlike $\vdash^{hs}_{Kr}$, $\mathcal{H} \vdash^{hs-t}_{Kr} H$ ensures that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$ also when $\mathcal{H}$ is not closed under substitutions.

Definition 54

1. Let $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ be an $\mathcal{L}$-frame, and e be an $\langle \mathcal{L}, M \rangle$-evaluation. Given a hypersequent H, we write $\mathcal{W}, e \vDash H$ if there exists a component $s \in H$ such that $\mathcal{W}, w, e \vDash s$ for every $w \in W$. Given a set $\mathcal{H}$ of hypersequents, we write $\mathcal{W}, e \vDash \mathcal{H}$ if $\mathcal{W}, e \vDash H$ for every $H \in \mathcal{H}$.

2. Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. $\mathcal{H} \vdash^{hs-t}_{Kr} H$ iff $\mathcal{W}, e \vDash \mathcal{H}$ implies $\mathcal{W}, e \vDash H$ for every $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ and $\langle \mathcal{L}, M \rangle$-evaluation e.

Clearly, $\mathcal{H} \vdash^{hs-t}_{Kr} H$ implies that $\mathcal{H} \vdash^{hs}_{Kr} H$ (compare the last definition to Definition 35). In addition, the following reduction easily follows from the definitions:

Proposition 55 Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents, and let $\mathcal{H}^* \cup \{H^*\}$ be the set obtained from $\mathcal{H} \cup \{H\}$ by simultaneous substitutions of new constant symbols for the free variables in $\mathcal{H} \cup \{H\}$. Then, $\mathcal{H} \vdash^{hs-t}_{Kr} H$ iff $\mathcal{H}^* \vdash^{hs}_{Kr} H^*$.

Corollary 52 and the last proposition easily entail the following strong cut-admissibility result:

Corollary 56 If $\mathcal{H} \vdash^{hs-t}_{Kr} H$ then $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$, for every set $\mathcal{H}$ of hypersequents and a hypersequent H.

Proof Suppose that $\mathcal{H} \vdash^{hs-t}_{Kr} H$. By Proposition 55, $\mathcal{H}^* \vdash^{hs}_{Kr} H^*$, where $\mathcal{H}^*$ and H^* are, respectively, the set obtained from $\mathcal{H}$ and the hypersequent obtained from H by simultaneous substitutions of new constant symbols for the free variables (if necessary, one can augment the language with infinitely many new constant symbols). Now, since $\mathcal{H}^*$ is closed under substitutions, Corollary 52 entails that there exists a derivation P in **HIF** of H^* from $\mathcal{H}^*$ in which cuts are only applied on formulas from $frm[\mathcal{H}^*]$. Evidently, without a loss in generality we can further assume that the set of free variables occurring in P and the set of free variables occurring in $\mathcal{H} \cup \{H\}$ are disjoint. Thus, by back-substituting the new constant symbols in P with the former free variables of $\mathcal{H}$ and H, we obtain a derivation in **HIF** of H from $\mathcal{H}$ in which cuts are only applied on formulas from $frm[\mathcal{H}]$. ∎

4.2 Globalization Connective

The extension of Gödel logic with a globalization connective $\Box$ (also known as the (Baaz) Delta connective) was widely studied (see e.g. [25], [7] and [9]). Intuitively, $\Box\varphi$ means that "φ is completely true". Next we show that our results and methods are easily extended to the extension of the standard first-order Gödel logic with $\Box$.

As before, there are two approaches to formally define the extension of Gödel logic with $\Box$. Proof-theoretically, the logic is defined by augmenting the Hilbert-style calculus **sG** with appropriate axioms and the necessitation rules (see e.g. [9]). Model-theoretically, one adds the following interpretation of $\Box$ to the definition of $\| \bullet \|_{\langle M,J,e\rangle}$ (Definition 6):

$$\|\Box\varphi\|_{\langle M,J,e\rangle} = 1 \text{ if } \|\varphi\|_{\langle M,J,e\rangle} = 1, \text{ and } \|\Box\varphi\|_{\langle M,J,e\rangle} = 0 \text{ otherwise.}$$

Alternatively, the following interpretation of $\Box$ is added to the Kripke-style semantics (Definition 13):

$$\mathcal{W}, w, e \vDash \Box\varphi \text{ iff } \mathcal{W}, u, e \vDash \varphi \text{ for every element } u \in W.$$

We denote by $\vdash^{hs}_{Kr_\Box}$ and $\vdash^{hs-t}_{Kr_\Box}$ the semantic consequence relations between hypersequents obtained by the last addition (see Definitions 35 and 54). It is very easy to see that the reduction of $\vdash^{hs-t}_{Kr}$ to $\vdash^{hs}_{Kr}$ given in Proposition 55 holds between $\vdash^{hs}_{Kr_\Box}$ and $\vdash^{hs-t}_{Kr_\Box}$ as well.

Now, let $\mathbf{HIF}_\Box$ be the system obtained by augmenting **HIF** with the following two rules:

$$(\Box\Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Box\varphi \Rightarrow \mid \Gamma \Rightarrow E} \qquad (\Rightarrow\Box)\ \frac{H \mid \Rightarrow \varphi}{H \mid \Rightarrow \Box\varphi}$$

All of our results are easily extended to $\mathbf{HIF}_\Box$. First, it is straightforward to prove the soundness of $\mathbf{HIF}_\Box$ with respect to $\vdash^{hs}_{Kr_\Box}$. For this, one simply extends the proof of Theorem 36 with two more cases, one for each additional rule. We do here the case of $(\Box\Rightarrow)$, leaving the other rule for the reader:

> Suppose that $H = H' \mid \Box\varphi \Rightarrow \mid \Gamma \Rightarrow E$ is derived from the hypersequent $H \mid \Gamma, \varphi \Rightarrow E$ using $(\Box\Rightarrow)$. Assume that $\mathcal{W}$ is not a model of H. Thus there exists an $\langle \mathcal{L}, M\rangle$-evaluation e, such that for every $s \in H$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In particular, for every $s \in H'$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In addition, there exist $w_1 \in W$ such that

$\mathcal{W}, w_1, e \not\models \Box\varphi \Rightarrow$, and $w_2 \in W$ such that $\mathcal{W}, w_2, e \not\models \Gamma \Rightarrow E$. By definition, $\mathcal{W}, w_1, e \models \Box\varphi$. Therefore, the semantics of $\Box$ implies that $\mathcal{W}, w_2, e \models \varphi$. Thus we have $\mathcal{W}, w_2, e \not\models \Gamma, \varphi \Rightarrow E$. But this implies that $\mathcal{W}$ is not a model of $H' \mid \Gamma, \varphi \Rightarrow E$.

Next, Theorem 48 is extended to $\mathbf{HIF}_\Box$ by adding one case to the proof of **(a)** and **(b)**:

- Suppose $\psi = \Box\varphi$.
  1. Assume that $\psi \in \mathcal{T}$. We show that for every element $w' \in W$ we have $\mathcal{W}, w', e \models \varphi$. Let $w' = \mathcal{T}' \Rightarrow \mathcal{U}'$ be an element of W. By the induction hypothesis, it suffices to show that $\varphi \in \mathcal{T}'$. Assume otherwise. By Proposition 41, there exist a hypersequent $H \sqsubseteq \Omega$, and sequents $\Gamma \Rightarrow E_1, \ldots, \Gamma \Rightarrow E_n \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$, such that $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$. By n consecutive applications of $(\Box\Rightarrow)$, we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \psi \Rightarrow \mid \Gamma \Rightarrow E_1 \mid \ldots \mid \Gamma \Rightarrow E_n$. But since $\psi \in \mathcal{T}$, $H \mid \psi \Rightarrow \mid \Gamma \Rightarrow E_1 \mid \ldots \mid \Gamma \Rightarrow E_n \sqsubseteq \Omega$. This contradicts Ω's $\mathcal{H}_0$-consistency.
  2. Assume that $\psi \in \mathcal{U}$. First note that $\mathcal{H}_0 \not\vdash^{frm[\mathcal{H}_0]} H \mid \Rightarrow \varphi$ for every hypersequent $H \sqsubseteq \Omega$ (otherwise By applying $(\Rightarrow \Box)$ we obtain $\mathcal{H}_0 \vdash^{frm[\mathcal{H}_0]} H \mid \Rightarrow \psi$, but $H \mid \Rightarrow \psi \sqsubseteq \Omega$). Therefore, by Proposition 42, Ω's external $\mathcal{H}_0$-maximality entails that $\Rightarrow \varphi \sqsubseteq \Omega$. Thus there exists an extended sequent $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\varphi \in \mathcal{U}'$. By the induction hypothesis, $\mathcal{W}, \mathcal{T}' \Rightarrow \mathcal{U}', e \not\models \varphi$. It follows that $\mathcal{W}, w, e \not\models \psi$.

Now Corollaries 49, 52 and 56 can easily be extended to $\mathbf{HIF}_\Box$. Thus we obtain strong soundness and completeness for $\mathbf{HIF}_\Box$ (with respect to $\vdash^{hs}_{Kr_\Box}$), as well as the two strong cut-admissibility results we had for **HIF**.

Remark 57 In [9] a different single-conclusion hypersequent system for the standard first-order Gödel logic with $\Box$ was provided, and it was (syntactically) proved that this system admits cut-elimination. Instead of the two rules above, the system in [9] includes the following three rules:

$$(\Box\Rightarrow^*)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, \Box\varphi \Rightarrow E} \qquad (\Rightarrow\Box^*)\ \frac{H \mid \Box\Gamma \Rightarrow \varphi}{H \mid \Box\Gamma \Rightarrow \Box\varphi} \qquad (cl_\Box)\ \frac{H \mid \Box\Gamma, \Gamma' \Rightarrow E}{H \mid \Box\Gamma \Rightarrow \mid \Gamma' \Rightarrow E}$$

where $\Box\Gamma$ is standing for any set of formulas prefixed by $\Box$. Clearly, our two rules for $\Box$ are derivable from these three rules without using the cut rule. Hence our strong cut-admissibility result applies also to the system in [9]. On the other hand, it is easy to verify that each of these rules describes a valid inference in $\vdash^{hs-t}_{Kr_\Box}$, and consequently, our results imply that each of these rules is derivable in $\mathbf{HIF}_\Box$ using only permissible cuts. For example, for every $\Box\Gamma, \Gamma'$ and E, we have $\Box\Gamma, \Gamma' \Rightarrow E \vdash^{hs-t}_{Kr_\Box} \Box\Gamma \Rightarrow \mid \Gamma' \Rightarrow E$ (this is easy to verify from the semantic definitions). Thus Corollary 56 (adapted to $\mathbf{HIF}_\Box$) entails that there exists a derivation of $\Box\Gamma \Rightarrow \mid \Gamma' \Rightarrow E$ from $\Box\Gamma, \Gamma' \Rightarrow E$ using only cuts on formulas from $\Gamma \cup \Gamma' \cup E$. The hypersequential context (denoted by H in the formulation of the rules) can obviously be added, and hence $(cl_\Box)$ is derivable in $\mathbf{HIF}_\Box$ using only permissible cuts. Here is an example of such a derivation (where $\Box\Gamma$ consists of one formula):

$$\frac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \mid \Rightarrow \varphi}\,(\Box\Rightarrow)}{\Box\varphi \Rightarrow \mid \Rightarrow \Box\varphi}\,(\Rightarrow\Box) \qquad H \mid \Box\varphi, \Gamma' \Rightarrow E}{H \mid \Box\varphi \Rightarrow \mid \Gamma' \Rightarrow E}\,cut$$

4.3 Density Rule

In [24] the following density rule was introduced and used to axiomatize standard first-order Gödel logic:

$$\frac{\Gamma \Rightarrow \varphi \vee (\psi_1 \supset p) \vee (p \supset \psi_2)}{\Gamma \Rightarrow \varphi \vee (\psi_1 \supset \psi_2)}$$

where p (a metavariable for a nullary predicate symbol) does not occur in the conclusion. In [22] this rule was proved to be admissible (using a semantic proof). The (single-conclusion) hypersequent version of this rule has the form (see [7]):

$$(density) \quad \frac{H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E}{H \mid \Gamma_1, \Gamma_2 \Rightarrow E}$$

where, again, p does not occur in the conclusion. Note that if we allow also derivations from a non-empty set of assumptions, then this rule is not valid (for example, $\Rightarrow p \vdash \Rightarrow$ using external weakening and this rule). To solve this, we should require that p does not occur in the set of assumptions as well. We next prove the strong soundness of $(density)$ under this condition.

Proposition 58 If $\mathcal{H} \vdash^{hs}_{Kr} H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$ then $\mathcal{H} \vdash^{hs}_{Kr} H \mid \Gamma_1, \Gamma_2 \Rightarrow E$, provided that the nullary predicate symbol p does not occur in $\mathcal{H}, H, \Gamma_1, \Gamma_2$ and E.

Proof Assume that $\mathcal{H} \vdash^{hs}_{Kr} H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$. Let $\mathcal{W} = \langle W, \leq, M, \{I_w\}_{w \in W}\rangle$ be an $\mathcal{L}$-frame, which is a model of $\mathcal{H}$. We show that it is a model of $H \mid \Gamma_1, \Gamma_2 \Rightarrow E$. Since $\mathcal{H} \vdash^{hs}_{Kr} H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$, $\mathcal{W}$ is a model of $H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$. If $\mathcal{W}$ is a model of H then we are done. Otherwise, $\mathcal{W}$ is a model of $\Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$. We prove that $\mathcal{W}$ is a model of $\Gamma_1, \Gamma_2 \Rightarrow E$. Assume otherwise. Thus there exists an $\langle \mathcal{L}, M\rangle$-evaluation e_0 and $w_0 \in W$, such that $\mathcal{W}, w_0, e_0 \not\models \Gamma_1, \Gamma_2 \Rightarrow E$. Let $\mathcal{W}' = \langle W', \leq', M, \{I'_w\}_{w \in W'}\rangle$, where:

- $W' = W \cup \{w'_0\}$ where w'_0 is a new world.
- $\leq' = \leq \cup \{\langle w, w'_0\rangle \mid w < w_0\} \cup \{\langle w'_0, w\rangle \mid w_0 \leq w\} \cup \{\langle w'_0, w'_0\rangle\}$
- For every $w \in W$, $I'_w = I_w$, except (possibly) for $I'_w[p]$ which is $\emptyset$ for $w < w_0$, and $\{\langle\rangle\}$ for $w \geq w_0$. In addition, $I'_{w'_0} = I_{w_0}$, except (possibly) for $I'_{w'_0}[p]$ which is $\emptyset$.

It is easy to see that for every sequent s in which p does not occur, $w \in W$, and $\langle \mathcal{L}, M\rangle$-evaluation e, $\mathcal{W}, w, e \not\models s$ iff $\mathcal{W}', w, e \not\models s$, and $\mathcal{W}, w_0, e \not\models s$ iff $\mathcal{W}', w'_0, e \not\models s$. Hence, $\mathcal{W}', w_0, e_0 \not\models \Gamma_1, \Gamma_2 \Rightarrow E$, and $\mathcal{W}', w'_0, e_0 \not\models \Gamma_1, \Gamma_2 \Rightarrow E$. Now, note that $\mathcal{W}', w_0, e_0 \not\models p \Rightarrow$ and $\mathcal{W}', w'_0, e_0 \not\models \Rightarrow p$. It follows that $\mathcal{W}'$ is not a model of $\Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$. Moreover, since $\mathcal{W}$ is not a model of H (and p does not occur in H), so does W', and so $\mathcal{W}'$ is not a model of $H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$. Since $\mathcal{H} \vdash^{hs}_{Kr} H \mid \Gamma_1 \Rightarrow p \mid \Gamma_2, p \Rightarrow E$, $\mathcal{W}'$ is not a model of $\mathcal{H}$. But, p does not occur in $\mathcal{H}$, and so this contradicts the fact that $\mathcal{W}$ is a model of $\mathcal{H}$. ∎

Finally, note that by Corollary 49, it follows that $(density)$ is admissible in **HIF** (also with non-empty set of assumptions).

5 Multiple-Conclusion Version

In this section we present the multiple-conclusion version from [6] of the system **HIF**, and prove its strong soundness and completeness, as well as strong cut-admissibility. The proposed system, which we call **MCG**, can be seen as a combination of **HIF** and the well-known multiple-conclusion sequent system for intuitionistic logic (called **LJ′** in [23]).

Definition 59 A *multiple-conclusion sequent* is an ordered pair of finite sets of $\mathcal{L}$-formulas $\langle\Gamma, \Delta\rangle$. A *multiple-conclusion hypersequent* is a finite set of multiple-conclusion sequents.

We shall again use the usual sequent notation $\Gamma \Rightarrow \Delta$ (for $\langle\Gamma, \Delta\rangle$) and the usual hypersequent notation $s_1 \mid \ldots \mid s_n$ (for $\{s_1, \ldots, s_n\}$). $frm[\mathcal{H}]$ for a set of multiple-conclusion hypersequents $\mathcal{H}$ is defined as for sets of single-conclusion hypersequents (see Notation 19).

Definition 60 MCG is the multiple-conclusion hypersequent system containing the following rules:[6]

Axioms:

$$\varphi \Rightarrow \varphi \qquad \bot \Rightarrow$$

Structural Rules:

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, \varphi \Rightarrow \Delta} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, \varphi} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow \Delta}$$

$$(com)\ \frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow \Delta_1 \quad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow \Delta_1 \mid \Gamma_2, \Gamma_1' \Rightarrow \Delta_2}$$

$$(cut)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad H_2 \mid \Gamma_2, \varphi \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$(\supset\Rightarrow)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi_1 \quad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \varphi_1 \supset \varphi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow \varphi_1 \supset \varphi_2}$$

$$(\vee\Rightarrow)\ \frac{H_1 \mid \Gamma_1, \varphi_1 \Rightarrow \Delta_1 \quad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \varphi_1 \vee \varphi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\vee)\ \frac{H \mid \Gamma \Rightarrow \Delta, \varphi_1, \varphi_2}{H \mid \Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}$$

$$(\wedge\Rightarrow)\ \frac{H \mid \Gamma, \varphi_1, \varphi_2 \Rightarrow \Delta}{H \mid \Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi_1 \quad H_2 \mid \Gamma_2 \Rightarrow \Delta_2, \varphi_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi_1 \wedge \varphi_2}$$

$$(\forall\Rightarrow)\ \frac{H \mid \Gamma, \varphi\{t/a\} \Rightarrow \Delta}{H \mid \Gamma, \forall x(\varphi\{x/a\}) \Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{H \mid \Gamma \Rightarrow \Delta, \varphi}{H \mid \Gamma \Rightarrow \Delta, \forall x(\varphi\{x/a\})}$$

$$(\exists\Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{H \mid \Gamma \Rightarrow \Delta, \varphi\{t/a\}}{H \mid \Gamma \Rightarrow \Delta, \exists x(\varphi\{x/a\})}$$

The rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ must obey the eigenvariable condition: a must not occur in the lower hypersequent. Note that right context-formulas are *not* allowed in $(\Rightarrow\supset)$, and so this rule looks exactly like its single-conclusion counterpart.

We denote by $\vdash_{\mathbf{MCG}}$ the provability relation (between sets of multiple-conclusion hypersequents and multiple-conclusion hypersequents) induced by $\mathbf{MCG}$. We denote by $\vdash^{\mathcal{E}}_{\mathbf{MCG}}$ the provability relation induced by $\mathbf{MCG}$, in which the cut-formulas are confined to the formulas of $\mathcal{E}$ (see Definition 21 for similar definitions for $\mathbf{HIF}$). To avoid confusion, we henceforth use $\vdash_{\mathbf{HIF}}$ instead of $\vdash$.

Proposition 61 $H \mid \Gamma \Rightarrow \Delta_1, \Delta_2 \vdash^{frm[\Delta_1 \cup \Delta_2]}_{\mathbf{MCG}} H \mid \Gamma \Rightarrow \Delta_1 \mid \Gamma \Rightarrow \Delta_2$, for every multiple-conclusion hypersequent H, and finite sets $\Gamma, \Delta_1, \Delta_2$.

[6]The system described in [6] includes an extra $(split)$ rule. The results of this paper show that $(split)$ is redundant.

Proof We prove the claim by induction on $|\Delta_1|+|\Delta_2|$. First, if Δ_1 or Δ_2 is empty, then the claim follows by applying external weakening. Assume that $|\Delta_1|, |\Delta_2| > 0$, $|\Delta_1|+|\Delta_2| = l$ and that the claim holds for every Δ_1, Δ_2 such that $|\Delta_1|+|\Delta_2| < l$. Let $\varphi_1 \in \Delta_1$ and $\varphi_2 \in \Delta_2$, and let $\Delta_1' = \Delta_1 \setminus \{\varphi_1\}$ and $\Delta_2' = \Delta_2 \setminus \{\varphi_2\}$. Using (com) on the identity axioms $\varphi_1 \Rightarrow \varphi_1$ and $\varphi_2 \Rightarrow \varphi_2$, we derive $\varphi_1 \Rightarrow \varphi_2 \mid \varphi_2 \Rightarrow \varphi_1$. By a cut on φ_1 of this hypersequent and the assumption $H \mid \Gamma \Rightarrow \Delta_1, \Delta_2$, we obtain $H \mid \Gamma \Rightarrow \Delta_1', \Delta_2 \mid \varphi_2 \Rightarrow \varphi_1$. Another cut on φ_2 of this hypersequent and the same assumption now yields $H \mid \Gamma \Rightarrow \Delta_1', \Delta_2 \mid \Gamma \Rightarrow \Delta_1, \Delta_2'$. By applying (twice) the induction hypothesis, we obtain a derivation of $H \mid \Gamma \Rightarrow \Delta_1' \mid \Gamma \Rightarrow \Delta_2 \mid \Gamma \Rightarrow \Delta_1 \mid \Gamma \Rightarrow \Delta_2'$ in which cuts are only made on formulas from $\Delta_1 \cup \Delta_2$. The claim follows by applying internal weakenings. ∎

Next, we study the relation between **MCG** and **HIF**.

Definition 62 Given a multiple-conclusion sequent $s = (\Gamma \Rightarrow \varphi_1, \dots, \varphi_n)$, $s^{\leq 1}$ is the (single-conclusion) hypersequent $\Gamma \Rightarrow \varphi_1 \mid \dots \mid \Gamma \Rightarrow \varphi_n \mid \Gamma \Rightarrow$. Given a multiple-conclusion hypersequent H, $H^{\leq 1}$ is the (single-conclusion) hypersequent $\bigcup_{s \in H} s^{\leq 1}$. For a set of multiple-conclusion hypersequents $\mathcal{H}$, we denote the set $\{H^{\leq 1} \mid H \in \mathcal{H}\}$ by $\mathcal{H}^{\leq 1}$.

For example, $(\Rightarrow p_1(a_1) \mid p_2(a_1) \Rightarrow p_3(a_1), p_4(a_1))^{\leq 1}$ is the hypersequent: $\Rightarrow p_1(a_1) \mid \Rightarrow \mid p_2(a_1) \Rightarrow p_3(a_1) \mid p_2(a_1) \Rightarrow p_4(a_1) \mid p_2(a_1) \Rightarrow$.

Theorem 63 Let $\mathcal{H} \cup \{H\}$ be a set of multiple-conclusion hypersequents, and $\mathcal{E}$ be a set of $\mathcal{L}$-formulas. If $\mathcal{H}^{\leq 1} \vdash^{\mathcal{E}}_{\mathbf{HIF}} H^{\leq 1}$ then $\mathcal{H} \vdash^{\mathcal{E} \cup frm[\mathcal{H}]}_{\mathbf{MCG}} H$.

Proof Suppose that $\mathcal{H}^{\leq 1} \vdash^{\mathcal{E}}_{\mathbf{HIF}} H^{\leq 1}$. We show that H is derivable from $\mathcal{H}$ in **MCG**, using only cuts on formulas from $\mathcal{E}$ and $frm[\mathcal{H}]$. First, by Proposition 61, $G \vdash^{frm[G]}_{\mathbf{MCG}} G^{\leq 1}$ for every $G \in \mathcal{H}$. In addition, every proof in **HIF** can obviously be modified into a proof in **MCG** with the same set of cut-formulas (simple internal weakenings should be used before applying the rules $(\wedge\Rightarrow)$ and $(\Rightarrow\vee)$). Thus, we can now proceed according to the derivation in **HIF** of $H^{\leq 1}$ from $\mathcal{H}^{\leq 1}$. Then, by applying internal weakenings we can obtain H from $H^{\leq 1}$. ∎

Theorem 64 Let $\mathcal{H} \cup \{H\}$ be a set of multiple-conclusion hypersequents, and $\mathcal{E}$ be a set of $\mathcal{L}$-formulas. Suppose that both $\mathcal{H}$ and $\mathcal{E}$ are closed under substitutions. Then, $\mathcal{H} \vdash^{\mathcal{E}}_{\mathbf{MCG}} H$ implies $\mathcal{H}^{\leq 1} \vdash^{\mathcal{E}}_{\mathbf{HIF}} H^{\leq 1}$.

Proof (Outline) Proved by a standard induction on the length of proofs in **MCG**. The base case is easily established, since **MCG** and **HIF** have the same set of axioms. We include here only some cases of the induction step. The other cases are left to the reader.

1. Consider an application of (com) that derives the multiple-conclusion hypersequent $G = H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow \varphi_1, \dots, \varphi_n \mid \Gamma_2, \Gamma_1' \Rightarrow \psi_1, \dots, \psi_m$ from $G_1 = H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow \varphi_1, \dots, \varphi_n$ and $G_2 = H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \psi_1, \dots, \psi_m$. We prove that $G_1^{\leq 1}, G_2^{\leq 1} \vdash^{\emptyset}_{\mathbf{HIF}} G^{\leq 1}$. For this, it suffices to prove that for every $n, m \geq 0$, hypersequents H_1 and H_2, $\mathcal{L}$-formulas $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m$, and sets of $\mathcal{L}$-formulas $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2'$, the hypersequent $H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow \mid \Gamma_1, \Gamma_2' \Rightarrow \varphi_1 \mid \dots \mid \Gamma_1, \Gamma_2' \Rightarrow \varphi_n \mid \Gamma_2, \Gamma_1' \Rightarrow \mid \Gamma_2, \Gamma_1' \Rightarrow \psi_1 \mid \dots \mid \Gamma_2, \Gamma_1' \Rightarrow \psi_m$ is cut-free derivable in **HIF** from the hypersequents $H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow \mid \Gamma_1, \Gamma_1' \Rightarrow \varphi_1 \mid \dots \mid \Gamma_1, \Gamma_1' \Rightarrow \varphi_n$ and $H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \mid \Gamma_2, \Gamma_2' \Rightarrow \psi_1 \mid \dots \mid \Gamma_2, \Gamma_2' \Rightarrow \psi_m$. This follows directly from Proposition 22.

2. Applications of $(\vee\Rightarrow)$ and $(\exists\Rightarrow)$ are treated similarly using Propositions 23 and 24.

3. Consider an application of $(\Rightarrow \forall)$ that derives the multiple-conclusion hypersequent $G = H_0 \mid \Gamma \Rightarrow \psi_1, \dots, \psi_n, \forall x(\varphi\{x/a\})$ from $G_1 = H_0 \mid \Gamma \Rightarrow \psi_1, \dots, \psi_n, \varphi$ (a does not occur in G). Here $G_1^{\leq 1} = H_0^{\leq 1} \mid \Gamma \Rightarrow \mid \Gamma \Rightarrow \psi_1 \mid \dots \mid \Gamma \Rightarrow \psi_n \mid \Gamma \Rightarrow \varphi$, and $G^{\leq 1} = H_0^{\leq 1} \mid \Gamma \Rightarrow \mid \Gamma \Rightarrow \psi_1 \mid \dots \mid \Gamma \Rightarrow \psi_n \mid \Gamma \Rightarrow \forall x(\varphi\{x/a\})$. Now, $G^{\leq 1}$ is obtained from $G_1^{\leq 1}$ by one application of $(\Rightarrow\forall)$ in **HIF**. ∎

Corollary 65 Let $\mathcal{H} \cup \{H\}$ be a set of multiple-conclusion hypersequents. If $\mathcal{H} \vdash_{\mathbf{MCG}} H$ then $\mathcal{H}^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$.

Proof Suppose that $\mathcal{H} \vdash_{\mathbf{MCG}} H$. Let H_* be the set of multiple-conclusion hypersequents obtained from $\mathcal{H}$ by closing it under substitutions. Obviously, $\mathcal{H}_* \vdash_{\mathbf{MCG}} H$. By Theorem 64, $\mathcal{H}_*^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$ (take $\mathcal{E} = \mathit{frm}_{\mathcal{L}}$). Since $\mathcal{H}_*^{\leq 1}$ consists of substitution instances of $\mathcal{H}^{\leq 1}$, Corollary 32 implies that $\mathcal{H}^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$. ∎

Minor modifications are needed in the semantic consequence relation between (single-conclusion) hypersequents to support multiple-conclusion hypersequents. First, Notation 33 is extended to multiple-conclusion sequents in the obvious way ($\mathcal{W}, w, e \vDash \Gamma \Rightarrow \Delta$ if either $\mathcal{W}, w, e \not\vDash \varphi$ for some $\varphi \in \Gamma$, or $\mathcal{W}, w, e \vDash \varphi$ for some $\varphi \in \Delta$). Models of multiple-conclusion hypersequents, and the relation $\vdash^{hs}_{\mathit{Kr}}$ between sets of multiple-conclusion hypersequents and multiple-conclusion hypersequents are defined exactly as for single-conclusion hypersequents (see Definitions 34 and 35). Next, we prove the following semantic relation:

Theorem 66 Let $\mathcal{H} \cup \{H\}$ be a set of multiple-conclusion hypersequents. $\mathcal{H} \vdash^{hs}_{\mathit{Kr}} H$ iff $\mathcal{H}^{\leq 1} \vdash^{hs}_{\mathit{Kr}} H^{\leq 1}$.

Proof We prove that an $\mathcal{L}$-frame is a model of a multiple-conclusion hypersequent H iff it is a model of $H^{\leq 1}$. The claim then easily follows. Suppose first that $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ is a model of $H^{\leq 1}$. Then, for every $\langle \mathcal{L}, M \rangle$-evaluation e, there exists a component $\Gamma \Rightarrow E \in H^{\leq 1}$ such that $\mathcal{W}, w, e \vDash \Gamma \Rightarrow E$ for every $w \in W$. Now, for every $\Gamma \Rightarrow E \in H^{\leq 1}$, there exists a component $\Gamma \Rightarrow \Delta \in H$ such that $E \subseteq \Delta$. In this case, by definition, if $\mathcal{W}, w, e \vDash \Gamma \Rightarrow E$ then $\mathcal{W}, w, e \vDash \Gamma \Rightarrow \Delta$. It follows that for every $\langle \mathcal{L}, M \rangle$-evaluation e, there exists a component $\Gamma \Rightarrow \Delta \in H$ such that $\mathcal{W}, w, e \vDash \Gamma \Rightarrow \Delta$ for every $w \in W$. Therefore $\mathcal{W}$ is a model of H.

Next, suppose that $\mathcal{W} = \langle W, \leq, M, \mathcal{I} \rangle$ is not a model of $H^{\leq 1}$. Thus, there exists an $\langle \mathcal{L}, M \rangle$-evaluation e, such that for every component $s \in H^{\leq 1}$, $\mathcal{W}, w, e \not\vDash s$ for some $w \in W$. We show that for every component $s \in H$, $\mathcal{W}, w, e \not\vDash s$ for some $w \in W$. It follows that $\mathcal{W}$ is not a model of H. Let $\Gamma \Rightarrow \{\varphi_1, \ldots, \varphi_n\}$ be a component of H. For every $1 \leq i \leq n$, let w_i be an element of W such that $\mathcal{W}, w_i, e \not\vDash \Gamma \Rightarrow \varphi_i$, and let w_0 be a element of W such that $\mathcal{W}, w_i, e \not\vDash \Gamma \Rightarrow$. Then, by definition, for every $1 \leq i \leq n$, $\mathcal{W}, w_i, e \vDash \varphi$ for every $\varphi \in \Gamma$, and $\mathcal{W}, w_i, e \not\vDash \varphi_i$. Similarly, $\mathcal{W}, w_0, e \vDash \varphi$ for every $\varphi \in \Gamma$. Let w be the minimum of $\{w_1, \ldots, w_n\}$. Thus $\mathcal{W}, w, e \vDash \varphi$ for every $\varphi \in \Gamma$, and by Proposition 14, $\mathcal{W}, w, e \not\vDash \varphi_i$ for every $1 \leq i \leq n$. This implies that $\mathcal{W}, w, e \not\vDash \Gamma \Rightarrow \{\varphi_1, \ldots, \varphi_n\}$. ∎

We finally establish strong soundness and completeness, as well as strong cut-admissibility, for $\mathbf{MCG}$.

Corollary 67 (Strong Soundness and Completeness) $\mathbf{MCG}$ is strongly sound and complete with respect to the Kripke semantics of the standard first-order Gödel logic, i.e. $\mathcal{H} \vdash_{\mathbf{MCG}} H$ iff $\mathcal{H} \vdash^{hs}_{\mathit{Kr}} H$.

Proof Theorem 63 and Corollary 65 imply that $\mathcal{H} \vdash_{\mathbf{MCG}} H$ iff $\mathcal{H}^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$. Soundness and completeness of **HIF** (Corollary 49) implies that $\mathcal{H}^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$ iff $\mathcal{H}^{\leq 1} \vdash^{hs}_{\mathit{Kr}} H^{\leq 1}$. Finally, Theorem 66 provides the last link. ∎

Again, we can straightforwardly reduce $\vdash_{\mathit{Kr}}$ to $\vdash_{\mathbf{MCG}}$ (equivalently, $\vdash_{[0,1]}$ or $\vdash_{\mathbf{sG}}$):

Corollary 68 $\mathcal{T} \vdash_{\mathit{Kr}} \varphi$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{MCG}} \Rightarrow \varphi$.

Proof Theorems 63 and 64 imply that $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{MCG}} \Rightarrow \varphi$ iff $\{\Rightarrow \psi \mid \Rightarrow \ \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{HIF}} \Rightarrow \varphi \mid \Rightarrow$. It is easy to see that $\{\Rightarrow \psi \mid \Rightarrow \ \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{HIF}} \Rightarrow \varphi \mid \Rightarrow$ iff $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\} \vdash_{\mathbf{HIF}} \Rightarrow \varphi$. Thus the claim follows from Corollary 51. ∎

Corollary 69 (Strong Cut-Admissibility for $\mathbf{MCG}$) $\mathcal{H} \vdash_{\mathbf{MCG}} H$ implies $\mathcal{H} \vdash^{\mathit{frm}[\mathcal{H}]}_{\mathbf{MCG}} H$, for every set $\mathcal{H}$ of multiple-conclusion hypersequents closed under substitutions, and a multiple-conclusion hypersequent H.

Proof Assume $\mathcal{H} \vdash_{\mathbf{MCG}} H$. By Corollary 65, $\mathcal{H}^{\leq 1} \vdash_{\mathbf{HIF}} H^{\leq 1}$. Obviously, $\mathcal{H}^{\leq 1}$ is closed under substitutions. Strong cut-admissibility of **HIF** (Corollary 52) then implies that $\mathcal{H}^{\leq 1} \vdash^{\mathit{frm}[\mathcal{H}]}_{\mathbf{HIF}} H^{\leq 1}$. By Theorem 63, $\mathcal{H} \vdash^{\mathit{frm}[\mathcal{H}]}_{\mathbf{MCG}} H$. ∎

6 Further Research

Many other (multiple or single-conclusion) hypersequent systems for various propositional and first-order fuzzy logics and intermediate logics have only syntactic proofs of (usual) cut-elimination theorem (see e.g. [20]). It should be interesting to find for them too simpler semantic proofs and derive corresponding strong cut-admissibility theorems. However, for other fuzzy logics, simple Kripke-style semantics do not exist (to the best of our knowledge), and new methods should be developed.

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