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# A Cut-Free Calculus for Second-Order Gödel Logic

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**Abstract**

We prove that the extension of the known hypersequent calculus for standard first-order Gödel logic with usual rules for second-order quantifiers is sound and (cut-free) complete for Henkin-style semantics for second-order Gödel logic. The proof is semantic, and it is similar in nature to Schütte and Tait's proof of Takeuti's conjecture.

*Keywords:* Proof theory, Cut-admissibility, Second-order logic, Non-classical logics, Fuzzy logics, Gödel logic, Non-deterministic semantics

## 1. Introduction

Fuzzy logics form a natural generalization of classical logic, in which truth values consist of some linearly ordered set, usually taken to be the real interval $[0, 1]$. They have a wide variety of applications, as they provide a reasonable model of certain very common vagueness phenomena. Both their propositional and first-order versions are well-studied by now (see, e.g., [1]). Clearly, for many interesting applications (see, e.g., [2] and Section 5.5.2 in Chapter I of [3]), propositional and first-order fuzzy logics do not suffice, and one has to use higher-order versions. These are much less developed (see, e.g., [4] and [3]), especially from the proof-theoretic perspective. Evidently, higher-order fuzzy logics deserve a proof-theoretic study, with the aim of providing a basis for automated deduction methods, as well as a complementary point of view in the investigation of these logics.

The proof-theory of propositional fuzzy logics is the main subject of [5]. There, an essential tool to develop well-behaved proof calculi for fuzzy logics is the transition from (Gentzen-style) sequents, to *hypersequents*. The latter, that are usually nothing more than disjunctions of sequents, turn to be an adequate proof-theoretic framework for the fundamental fuzzy logics. In particular, propositional Gödel logic (the logic interpreting conjunction as minimum, and disjunction as maximum) is easily captured by a cut-free hypersequent calculus called **HG** (introduced in [6]). The derivation rules of **HG** are the standard hypersequent versions of the sequent rules of Gentzen's **LJ** for intuitionistic logic, and they are augmented by the *communication rule* that allows "exchange of information" between two hypersequents [7]. In [8], it was shown that **HIF**, the extension of **HG** with the natural hypersequent versions of **LJ**'s sequent rules for the first-order quantifiers, is sound and (cut-free) complete for standard first-order Gödel logic.[1] As a corollary, one obtains Herbrand theorem for the prenex fragment of this logic (see [5, 9]).

In this paper, we study the extension of **HIF** with usual rules for *second-order* quantifiers. These consist of the single-conclusion hypersequent version of the rules for introducing second-order quantifiers in the

[1]Note that Gödel logic is the only fundamental fuzzy logic whose first-order version is recursively axiomatizable [5].

ordinary sequent calculus for classical logic (see, e.g., [10, 11]). We denote by $\mathbf{HIF}^2$ the extension of (the cut-free fragment of) **HIF** with these rules. Our main result is that $\mathbf{HIF}^2$ is sound and complete for second-order Gödel logic. Since we do not include the cut rule in $\mathbf{HIF}^2$, this automatically implies the admissibility of cut, which makes this calculus a suitable possible basis for automated theorem proving. It should be noted that like in the case of second-order classical logic, the obtained calculus characterizes *Henkin-style* second-order Gödel logic. Thus second-order quantifiers range over a domain that is directly specified in the second-order structure, and it admits full comprehension (this is a domain of *fuzzy sets* in the case of fuzzy logics). This is in contrast to what is called the *standard semantics*, where second-order quantifiers range over *all* subsets of the universe. Hence $\mathbf{HIF}^2$ is practically a system for two-sorted first-order Gödel logic together with the comprehension axioms (see also [12]).

While the soundness of $\mathbf{HIF}^2$ is straightforward, proving its (cut-free) completeness turns out to be relatively involved. This is similar to the case of second-order classical logic, where the completeness of the *cut-free* sequent calculus was open for several years, and known as *Takeuti's conjecture* [13].[2] While usual syntactic arguments for cut-elimination dramatically fail for the rules of second-order quantifiers, Takeuti's conjecture was initially verified by a semantic proof. This was accomplished in two steps. First, the completeness was proved with respect to three-valued non-deterministic semantics (this was done by Schütte in [14]). Then, it was left to show that from every three-valued non-deterministic counter-model, one can extract a usual (two-valued) counter-model, without losing comprehension (this was done first by Tait in [15]). Our proof has basically a similar general structure. Thus, in Section 5, we present a non-deterministic semantics for $\mathbf{HIF}^2$ with generalized truth values. In Section 6, we use this semantics to derive completeness with respect to the ordinary semantics. Finally, full proofs of the more technical propositions are given in Appendices Appendix A and Appendix B.

## 2. Preliminaries

In what follows, $\mathcal{L}$ denotes an arbitrary second-order language. For the simplicity of the presentation, we follow [10] and restrict ourselves to simplified second-order languages, in which the second-order part of the signature consists only of one predicate symbol $\varepsilon$ (with the intuitive meaning of set membership). This is formulated in the next definition.

**Definition 2.1.** A *simple second-order language* consists of the following:

1. Infinitely many individual variables $\nu_1, \nu_2, ...$ and set variables $\chi_1, \chi_2, ....$ We use the metavariables $x, y, z$ and $X, Y, Z$ (with or without subscripts) for individual and set variables (respectively).
2. A propositional constant $\perp$.
3. Binary connectives $\wedge, \vee, \supset$. We use $\diamond$ as a metavariable for the binary connectives.
4. Individual quantifiers $\forall^i$ and $\exists^i$, and set quantifiers $\forall^s$ and $\exists^s$. We use $Q^i$ and $Q^s$ as metavariables for the individual and set quantifiers (respectively).
5. An arbitrary set of individual constant symbols, and an arbitrary set of set constant symbols. The metavariables $c$ and $C$ are used to range over individual and set constant symbols (respectively).
6. An arbitrary set of function symbols (that take individuals as arguments). The metavariable $f$ is used to range over them.
7. An arbitrary set of predicate symbols (that take individuals as arguments). The metavariable $p$ is used to range over them.
8. A predicate symbol $\varepsilon$, with two places, the first for individuals and the second for sets.
9. Parentheses $'('$ and $')'$.

**Definition 2.2.** The set of $\mathcal{L}$*-terms* consists of *first-order* $\mathcal{L}$*-terms* and *second-order* $\mathcal{L}$*-terms*. These are defined as follows:

[2] More precisely, Takeuti's conjecture concerned full type-theory, namely, the completeness of the cut-free sequent calculus that includes rules for quantifiers of any finite order. However, the proof for second-order fragment was the main breakthrough.

1. *First-order $\mathcal{L}$-terms* are all individual variables of $\mathcal{L}$; all individual constant symbols of $\mathcal{L}$; if $f$ is an $n$-ary function symbol of $\mathcal{L}$ and $t_1, \dots, t_n$ are first-order $\mathcal{L}$-terms then $f(t_1, \dots, t_n)$ is a first-order $\mathcal{L}$-term. We use $t$ (with or without subscripts) as a metavariable for first-order $\mathcal{L}$-terms.
2. *Second-order $\mathcal{L}$-terms* are all set variables of $\mathcal{L}$ and all set constant symbols of $\mathcal{L}$. We use $T$ (with or without subscripts) as a metavariable for second-order $\mathcal{L}$-terms.

The set of (individual) variables occurring in a first-order $\mathcal{L}$-term $t$ is defined as usual, and denoted by $fv[t]$. Similarly, the set of (set) variables occurring in a second-order $\mathcal{L}$-term $T$ is denoted by $fv[T]$.

Following the convention in [10], we define a formula as an *equivalence class* of what we call *concrete* formulas, so that two formulas that differ only by the names of their bound variables are considered as the same.[3] This is convenient for handling the bureaucracy of free and bound variables. Moreover, it simplifies the non-deterministic semantics below (see Remark 5.7).

**Definition 2.3.** *Concrete $\mathcal{L}$-formulas* are inductively defined as follows:

1. $p(t_1, \dots, t_n)$ is a concrete $\mathcal{L}$-formula for $n$-ary every predicate symbol $p$ of $\mathcal{L}$ and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$.
2. $(t \varepsilon T)$ is a concrete $\mathcal{L}$-formula for every first-order $\mathcal{L}$-term $t$ and second-order $\mathcal{L}$-term $T$.
3. $\bot$ is a concrete $\mathcal{L}$-formula.
4. If $\varphi_1^\bullet$ and $\varphi_2^\bullet$ are concrete $\mathcal{L}$-formulas, so are $(\varphi_1^\bullet \wedge \varphi_2^\bullet)$, $(\varphi_1^\bullet \vee \varphi_2^\bullet)$, and $(\varphi_1^\bullet \supset \varphi_2^\bullet)$.
5. If $\varphi^\bullet$ is a concrete $\mathcal{L}$-formula, and $x$ is an individual variable of $\mathcal{L}$, then $(\forall^i x \varphi^\bullet)$ and $(\exists^i x \varphi^\bullet)$ are concrete $\mathcal{L}$-formulas.
6. If $\varphi^\bullet$ is a concrete $\mathcal{L}$-formula, and $X$ is a set variable of $\mathcal{L}$, then $(\forall^s X \varphi^\bullet)$ and $(\exists^s X \varphi^\bullet)$ are concrete $\mathcal{L}$-formulas.

We use $\varphi^\bullet$ (with or without subscripts) as a metavariable for concrete $\mathcal{L}$-formulas. *Free and bound variables* in concrete $\mathcal{L}$-formulas are defined as usual. We denote by $fv[\varphi^\bullet]$, the set of (individual and set) variables occurring free in a concrete $\mathcal{L}$-formula $\varphi^\bullet$. *Alpha-equivalence* between concrete $\mathcal{L}$-formulas is defined as usual (renaming of bound variables). We denote by $[\varphi^\bullet]_\alpha$ the set of all concrete $\mathcal{L}$-formulas which are alpha-equivalent to $\varphi^\bullet$ (i.e. the equivalence class of $\varphi^\bullet$ under alpha-equivalence).

**Definition 2.4.** $cp[\varphi^\bullet]$, the *complexity of a concrete $\mathcal{L}$-formula* $\varphi^\bullet$ is the number of occurrences of quantifiers, connectives (including $\bot$), and atomic concrete formulas (formulas of the form $p(t_1, \dots, t_n)$ or $(t \varepsilon T)$) in $\varphi^\bullet$.

**Definition 2.5.** An *$\mathcal{L}$-formula* is an equivalence class of concrete $\mathcal{L}$-formulas under alpha-equivalence. We mainly use $\varphi, \psi$ (with or without subscripts) as metavariables for $\mathcal{L}$-formulas. The set of free variables and the complexity of an $\mathcal{L}$-formula are defined using representatives, i.e. for an $\mathcal{L}$-formula $\varphi$, $fv[\varphi] = fv[\varphi^\bullet]$ and $cp[\varphi] = cp[\varphi^\bullet]$ for some $\varphi^\bullet \in \varphi$.

In the last definition and henceforth, it is easy to verify that all notions defined using representatives are well-defined.

**Definition 2.6.** We define several operations on $\mathcal{L}$-formulas:

- For $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$: $(\varphi_1 \diamond \varphi_2) = [(\varphi_1^\bullet \diamond \varphi_2^\bullet)]_\alpha$ for some $\varphi_1^\bullet \in \varphi_1$ and $\varphi_2^\bullet \in \varphi_2$.
- For $Q^i \in \{\forall^i, \exists^i\}$, an individual variable $x$ of $\mathcal{L}$, and an $\mathcal{L}$-formula $\varphi$: $(Q^i x \varphi) = [(Q^i x \varphi^\bullet)]_\alpha$ for some $\varphi^\bullet \in \varphi$.
- For $Q^s \in \{\forall^s, \exists^s\}$, a set variable $X$ of $\mathcal{L}$, and an $\mathcal{L}$-formula $\varphi$: $(Q^s X \varphi) = [(Q^s X \varphi^\bullet)]_\alpha$ for some $\varphi^\bullet \in \varphi$.

The next proposition allows us to use inductive definitions and to prove claims by induction on complexity of formulas:

**Proposition 2.7.** Exactly one of the following holds for every $\mathcal{L}$-formula $\varphi$:

[3] Since [10] does not provide all technical details for this convention, we do it here.

- $cp[\varphi] = 1$ and exactly one of the following holds:
  - $\varphi = \{p(t_1, \dots, t_n)\}$ for unique $n$-ary predicate symbol $p$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$.
  - $\varphi = \{(t \varepsilon T)\}$ for unique first-order $\mathcal{L}$-term $t$, and second-order $\mathcal{L}$-term $T$.
  - $\varphi = \{\bot\}$.
- $\varphi = (\varphi_1 \diamond \varphi_2)$ for unique $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ such that $cp[\varphi_1] < cp[\varphi]$ and $cp[\varphi_2] < cp[\varphi]$.
- For every individual variable $x \notin fv[\varphi]$, $\varphi = (Q^i x \psi)$ for unique $Q^i \in \{\forall^i, \exists^i\}$, and $\mathcal{L}$-formula $\psi$ such that $cp[\psi] < cp[\varphi]$.
- For every set variable $X \notin fv[\varphi]$, $\varphi = (Q^s X \psi)$ for unique $Q^s \in \{\forall^s, \exists^s\}$, and $\mathcal{L}$-formula $\psi$ such that $cp[\psi] < cp[\varphi]$.

Substitutions are defined as follows:

**Definition 2.8.** Let $t$ be a first-order $\mathcal{L}$-term, and $x$ be an individual variable of $\mathcal{L}$.

1. Given a first-order $\mathcal{L}$-term $t'$, $t'\{{}^t/_x\}$ is inductively defined by:

$$t'\{{}^t/_x\} = \begin{cases} t & t' = x \\ t' & t' = y \text{ for } y \neq x, \text{ or } t' = c \\ f(t_1\{{}^t/_x\}, \dots, t_n\{{}^t/_x\}) & t' = f(t_1, \dots, t_n) \end{cases}$$

2. Given an $\mathcal{L}$-formula $\varphi$, $\varphi\{{}^t/_x\}$ is inductively defined by:

$$\varphi\{{}^t/_x\} = \begin{cases} \{p(t_1\{{}^t/_x\}, \dots, t_n\{{}^t/_x\})\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \{(t'\{{}^t/_x\} \varepsilon T)\} & \varphi = \{(t' \varepsilon T)\} \\ \varphi & \varphi = \{\bot\} \\ (\varphi_1\{{}^t/_x\} \diamond \varphi_2\{{}^t/_x\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{{}^t/_x\}) & \varphi = (Q^i y \psi) \text{ for } y \notin fv[t] \cup \{x\} \\ (Q^s Y \psi\{{}^t/_x\}) & \varphi = (Q^s Y \psi) \end{cases}$$

**Definition 2.9.** Let $T$ be a second-order $\mathcal{L}$-term, and $X$ be a set variable of $\mathcal{L}$. Given an $\mathcal{L}$-formula $\varphi$, $\varphi\{{}^T/_X\}$ is inductively defined by:

$$\varphi\{{}^T/_X\} = \begin{cases} \varphi & \varphi = \{p(t_1, \dots, t_n)\} \\ \varphi & \varphi = \{(t \varepsilon T')\} \text{ for } T' \neq X \\ \{(t \varepsilon T)\} & \varphi = \{(t \varepsilon X)\} \\ \varphi & \varphi = \{\bot\} \\ (\varphi_1\{{}^T/_X\} \diamond \varphi_2\{{}^T/_X\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{{}^T/_X\}) & \varphi = (Q^i y \psi) \\ (Q^s Y \psi\{{}^T/_X\}) & \varphi = (Q^s Y \psi) \text{ for } Y \notin fv[T] \cup \{X\} \end{cases}$$

Note that the above substitution operations are well-defined. In particular, the choice of the variables $y$ and $Y$ is immaterial.

### *2.1. Henkin-style Second-Order Gödel Logic*

In this section we precisely define Henkin-style second-order Gödel logic, via a semantic presentation. These definitions naturally extend the usual definitions of Henkin-style second-order classical logic, by replacing the usual two truth values *True* and *False* by any bounded complete linearly ordered set of truth values (e.g., the standard real interval $[0, 1]$). From a different angle, these definitions naturally extend the usual definitions of (standard) first-order Gödel logic, by extending the first-order domains with an additional collection of fuzzy sets. The first component of the semantics is the set of truth values. These should form a Gödel set, defined as follows:

**Definition 2.10.** A *(standard) Gödel set* is a bounded complete linearly ordered set $\mathcal{V} = \langle V, \leq \rangle$. We denote by $0_\mathcal{V}$ and $1_\mathcal{V}$ the maximal and minimal elements (respectively) of $V$ with respect to $\leq$. The operations $\min_\mathcal{V}, \max_\mathcal{V}, \inf_\mathcal{V}$ and $\sup_\mathcal{V}$ are defined as usual (where $\min_\mathcal{V} \emptyset = 1_\mathcal{V}$ and $\max_\mathcal{V} \emptyset = 0_\mathcal{V}$). For every pair of elements $u_1, u_2 \in V$, $u_1 \to_\mathcal{V} u_2$ is defined to be $1_\mathcal{V}$ if $u_1 \leq u_2$, and $u_2$ otherwise. The relations $\geq$, $<$, and $>$ are also defined in the obvious way. We omit the subscript $\mathcal{V}$ when it is clear from the context, and sometimes we identify $\mathcal{V}$ with the set $V$ (e.g., when referring to the elements of $V$ as elements of $\mathcal{V}$).

Next, we define the domain of the semantic structures. As done in Henkin-style second order logics, in addition to the non-empty domain of individuals, we also have a domain of sets. In our case the elements of this domain are *fuzzy* sets.

**Definition 2.11.** A function from a set $\mathcal{D}$ to a Gödel set $\mathcal{V}$ is called a *fuzzy subset* of $\mathcal{D}$ over $\mathcal{V}$.

**Definition 2.12.** Let $\mathcal{V}$ be a Gödel set. A *domain* for $\mathcal{L}$ and $\mathcal{V}$ is an ordered triplet $\langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$, where $\mathcal{D}_i$ is some non-empty set, $\mathcal{D}_s$ is a non-empty collection of fuzzy subsets of $\mathcal{D}_i$ over $\mathcal{V}$, and $I$ is a function assigning: an element in $\mathcal{D}_i$ to every individual constant symbol of $\mathcal{L}$, a fuzzy subset in $\mathcal{D}_s$ to every set constant symbol of $\mathcal{L}$, and a function in ${\mathcal{D}_i}^n \to \mathcal{D}_i$ to every $n$-ary function symbol of $\mathcal{L}$.

Note that we include the interpretation of constants and function symbols in the domain itself. Thus, a domain is defined relatively to the language, as well as to the set of truth values (which is used in the definition of a fuzzy subset). Next, we define *$\mathcal{L}$-structures*, that include, in addition to a Gödel set of truth values and a corresponding domain, an interpretation of the predicate symbols. Similarly to the set constant symbols, unary predicate symbols are naturally interpreted as *fuzzy* subsets of the individuals domain, and fuzzy subsets of tuples of individuals are used for predicates of larger arities.

**Definition 2.13.** An *$\mathcal{L}$-structure* is a tuple $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ where:

1. $\mathcal{V}$ is a Gödel set.
2. $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ is a domain for $\mathcal{L}$ and $\mathcal{V}$.
3. $P$ is a function assigning a fuzzy subset of ${\mathcal{D}_i}^n$ over $\mathcal{V}$ to every $n$-ary predicate symbol of $\mathcal{L}$.

As usual, an additional function is used for interpreting the *free* variables.

**Definition 2.14.** Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ be a domain for $\mathcal{L}$ and $\mathcal{V}$.

1. An *$\langle \mathcal{L}, \mathcal{D} \rangle$-assignment* is a function assigning an element of $\mathcal{D}_i$ to every individual variable of $\mathcal{L}$, and an element of $\mathcal{D}_s$ to every set variable of $\mathcal{L}$. An $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$ is extended to all $\mathcal{L}$-terms by: $\sigma[c] = I[c]$ for every individual constant $c$ of $\mathcal{L}$, $\sigma[C] = I[C]$ for every set individual constant $C$ of $\mathcal{L}$, and $\sigma[f(t_1, \dots, t_n)] = I[f][\sigma[t_1], \dots, \sigma[t_n]]$ for every $n$-ary function symbol $f$ of $\mathcal{L}$ and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$.
2. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. Given an individual variable $x$ of $\mathcal{L}$ and $d \in \mathcal{D}_i$, we denote by $\sigma_{x:=d}$ the $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment which is identical to $\sigma$ except for $\sigma_{x:=d}[x] = d$. Similarly, given a set variable $X$ of $\mathcal{L}$, and $D \in \mathcal{D}_s$, we denote by $\sigma_{X:=D}$ the $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment which is identical to $\sigma$ except for $\sigma_{X:=D}[X] = D$. These notations are naturally extended to several distinct variables (e.g. $\sigma_{\nu_1:=d_1, \nu_2:=d_2, \chi_1:=D}$).

We can now define the truth value assigned by a given structure to an arbitrary formula with respect to some assignment. This definition generalizes in a natural way the usual recursive definition used in classical higher-order logics, where instead of the usual truth tables we use their counterparts of Gödel logic: $\wedge$ corresponds to min, $\vee$ to max, and the implication $\supset$ is interpreted as the $\to$ operation. For the quantifiers, we take inf and sup. Since the set of truth values is, by definition, complete, inf and sup are always defined.

**Definition 2.15.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P\rangle$ be an $\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I\rangle$. For every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment $\sigma$, $\mathcal{U}[\varphi, \sigma]$ is the element of $\mathcal{V}$ inductively defined as follows:

$$\mathcal{U}[\varphi,\sigma] = \begin{cases} P[p][\sigma[t_1],\dots,\sigma[t_n]] & \varphi = \{p(t_1,\dots,t_n)\} \\ \sigma[T][\sigma[t]] & \varphi = \{(t\varepsilon T)\} \\ 0 & \varphi = \{\bot\} \\ \min\{\mathcal{U}[\varphi_1,\sigma],\mathcal{U}[\varphi_2,\sigma]\} & \varphi = (\varphi_1 \wedge \varphi_2) \\ \max\{\mathcal{U}[\varphi_1,\sigma],\mathcal{U}[\varphi_2,\sigma]\} & \varphi = (\varphi_1 \vee \varphi_2) \\ \mathcal{U}[\varphi_1,\sigma] \to \mathcal{U}[\varphi_2,\sigma] & \varphi = (\varphi_1 \supset \varphi_2) \\ \inf_{d\in\mathcal{D}_i} \mathcal{U}[\psi,\sigma_{x:=d}] & \varphi = (\forall^i x\psi) \\ \sup_{d\in\mathcal{D}_i} \mathcal{U}[\psi,\sigma_{x:=d}] & \varphi = (\exists^i x\psi) \\ \inf_{D\in\mathcal{D}_s} \mathcal{U}[\psi,\sigma_{X:=D}] & \varphi = (\forall^s X\psi) \\ \sup_{D\in\mathcal{D}_s} \mathcal{U}[\psi,\sigma_{X:=D}] & \varphi = (\exists^s X\psi) \end{cases}$$

It can be verified that the choice of $x$ and $X$ in the last definition is immaterial, and $\mathcal{U}[\varphi, \sigma]$ is well-defined.

The last definition establishes the connection between the predicate symbol $\varepsilon$, and the (fuzzy) set membership. The truth value assigned to a formula of the form $\{(t\varepsilon T)\}$ with respect to an assignment $\sigma$ is equal to the membership degree of $\sigma[t]$ in the fuzzy subset $\sigma[T]$.

Next, we define Henkin-style second-order Gödel logic. This amounts to the set of tautologies induced by the structures defined above with the additional restriction of *comprehension*. Thus, as done in Henkin-style classical second-order logic, we require that all (fuzzy) subsets of the universe that can be captured by some formula, are indeed included in the domain of (fuzzy) subsets. Structures satisfying this property (namely, admit the comprehension axiom) are called *comprehensive*.

**Definition 2.16.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P\rangle$ be an $\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I\rangle$. Given an $\mathcal{L}$-formula $\varphi$, an individual variable $x$ of $\mathcal{L}$, and an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment $\sigma$, we denote by $\mathcal{U}[\varphi, \sigma, x]$ the fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ defined by $\lambda d \in \mathcal{D}_i.\, \mathcal{U}[\varphi, \sigma_{x:=d}]$. $\mathcal{U}$ is called *comprehensive* if $\mathcal{U}[\varphi, \sigma, x] \in \mathcal{D}_s$ for every $\varphi$, $x$, and $\sigma$.

**Definition 2.17.** For an $\mathcal{L}$-formula $\varphi$, we write $\vdash^{\mathbf{G^2_\mathcal{L}}} \varphi$ if $\mathcal{U}[\varphi, \sigma] = 1$ for every comprehensive $\mathcal{L}$-structure $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P\rangle$ and $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment $\sigma$. $\mathbf{G^2_\mathcal{L}}$ is the logic consisting of all formulas $\varphi$ such that $\vdash^{\mathbf{G^2_\mathcal{L}}} \varphi$.

*Remark* 2.18. For simplicity, we identify a logic with its set of theorems, and do not consider consequence relations.

**Example 2.19.** It is easy to see that the comprehension axiom scheme is valid in $\mathbf{G^2_\mathcal{L}}$, i.e.

$$\vdash^{\mathbf{G^2_\mathcal{L}}} (\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi))))$$

for every $\mathcal{L}$-formula $\varphi$, set variable $X \notin fv[\varphi]$, and individual variable $x$. Indeed, let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P\rangle$ be a comprehensive $\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I\rangle$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment. By definition, $\mathcal{U}[\varphi, \sigma, x] \in \mathcal{D}_s$. Thus

$$\mathcal{U}[(\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\})\wedge(\{(x\varepsilon X)\} \supset \varphi)))), \sigma] \geq \mathcal{U}[(\forall^i x((\varphi \supset \{(x\varepsilon X)\})\wedge(\{(x\varepsilon X)\} \supset \varphi))), \sigma_{X:=\mathcal{U}[\varphi,\sigma,x]}].$$

By definition, for every $d \in \mathcal{D}_i$ we have:

$$\mathcal{U}[\{(x\varepsilon X)\}, \sigma_{X:=\mathcal{U}[\varphi,\sigma,x],x:=d}] = \mathcal{U}[\varphi, \sigma, x][d] = \mathcal{U}[\varphi, \sigma_{x:=d}].$$

Since $X \notin fv[\varphi]$ (using Lemma Appendix A.1, see Appendix Appendix A),

$$\mathcal{U}[(\varphi \supset \{(x\varepsilon X)\}), \sigma_{X:=\mathcal{U}[\varphi,\sigma,x],x:=d}] = \mathcal{U}[\varphi, \sigma_{x:=d}] \to \mathcal{U}[\varphi, \sigma_{x:=d}] = 1,$$

and similarly,

$$\mathcal{U}[(\{(x\varepsilon X)\} \supset \varphi), \sigma_{X:=\mathcal{U}[\varphi,\sigma,x],x:=d}] = 1.$$

It follows that

$$\mathcal{U}[((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)), \sigma_{X:=\mathcal{U}[\varphi,\sigma,x],x:=d}] = \min\{1,1\} = 1.$$

Since this holds for every $d \in \mathcal{D}_i$, we have:

$$\inf_{d\in\mathcal{D}_i} \mathcal{U}[((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)), \sigma_{X:=\mathcal{U}[\varphi,\sigma,x],x:=d}] = 1.$$

Consequently,

$$\mathcal{U}[(\exists^s X(\forall^i x((\varphi \supset \{(x\varepsilon X)\}) \wedge (\{(x\varepsilon X)\} \supset \varphi)))), \sigma] = 1.$$

## 3. Hypersequent Calculus

In this section we present a cut-free hypersequent calculus for $\mathbf{G}^{\mathbf{2}}_{\mathcal{L}}$. This calculus is obtained by augmenting the hypersequent calculus **HIF** for standard first-order Gödel logic (introduced in [8]) with rules for second-order quantifiers. We begin by presenting **HIF** itself. We follow its formulation from [16] (adapted to our definitions, where formulas are equivalence classes of concrete formulas).

**Definition 3.1.** A *single-conclusion $\mathcal{L}$-sequent* is an ordered pair $\langle \Gamma, E\rangle$ of finite sets of $\mathcal{L}$-formulas, where $E$ is either a singleton or empty. A *single-conclusion $\mathcal{L}$-hypersequent* is a finite set of single-conclusion $\mathcal{L}$-sequents.

Henceforth, we simply write *$\mathcal{L}$-sequent* instead of *single-conclusion $\mathcal{L}$-sequent*, and *$\mathcal{L}$-hypersequent* instead of *single-conclusion $\mathcal{L}$-hypersequent*. The set of free variables and substitution operations are defined as expected for sets of $\mathcal{L}$-formulas, $\mathcal{L}$-sequents, and $\mathcal{L}$-hypersequents. We usually employ the standard sequent notation $\Gamma \Rightarrow E$ (for $\langle \Gamma, E\rangle$) and the hypersequent notation $s_1 \mid \ldots \mid s_n$ (for $\{s_1, \ldots, s_n\}$). We also employ the standard abbreviations, e.g. $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $H \mid s$ instead of $H \cup \{s\}$.

**Definition 3.2.** **HIF** is the hypersequent calculus consisting of the following derivation rules:

**Structural Rules:**

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \varphi \Rightarrow E} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com)\ \frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \quad H \mid \Gamma_1, \Gamma_2 \Rightarrow E_2}{H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2} \qquad (id)\ \frac{}{H \mid \Gamma, \varphi \Rightarrow \varphi}$$

**Logical Rules:**

$$(\bot\Rightarrow)\ \frac{}{H \mid \Gamma, \{\bot\} \Rightarrow E}$$

$$(\supset\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, (\varphi_1 \supset \varphi_2) \Rightarrow E} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow (\varphi_1 \supset \varphi_2)}$$

$$(\vee\Rightarrow)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow E \quad H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, (\varphi_1 \vee \varphi_2) \Rightarrow E}$$

$$(\Rightarrow\vee_1)\ \frac{H \mid \Gamma \Rightarrow \varphi_1}{H \mid \Gamma \Rightarrow (\varphi_1 \vee \varphi_2)} \qquad (\Rightarrow\vee_2)\ \frac{H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow (\varphi_1 \vee \varphi_2)}$$

$$(\wedge\Rightarrow_1)\ \frac{H \mid \Gamma, \varphi_1 \Rightarrow E}{H \mid \Gamma, (\varphi_1 \wedge \varphi_2) \Rightarrow E} \qquad (\wedge\Rightarrow_2)\ \frac{H \mid \Gamma, \varphi_2 \Rightarrow E}{H \mid \Gamma, (\varphi_1 \wedge \varphi_2) \Rightarrow E}$$

$$(\Rightarrow\wedge)\ \frac{H \mid \Gamma \Rightarrow \varphi_1 \quad H \mid \Gamma \Rightarrow \varphi_2}{H \mid \Gamma \Rightarrow (\varphi_1 \wedge \varphi_2)}$$

$$(\forall^i \Rightarrow)\ \frac{H \mid \Gamma, \varphi\{^t/_x\} \Rightarrow E}{H \mid \Gamma, (\forall^i x\varphi) \Rightarrow E} \qquad (\Rightarrow \forall^i)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow (\forall^i x\varphi)}$$

$$(\exists^i \Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, (\exists^i x\varphi) \Rightarrow E} \qquad (\Rightarrow \exists^i)\ \frac{H \mid \Gamma \Rightarrow \varphi\{^t/_x\}}{H \mid \Gamma \Rightarrow (\exists^i x\varphi)}$$

Applications of the rules $(\Rightarrow \forall^i)$ and $(\exists^i \Rightarrow)$ must obey the eigenvariable condition: $x$ is not a free variable in the lower hypersequent.

Several clarifications should be noted:

1. As usual, rules are formulated by schemes using metavariables. In addition to the metavariables declared above, we use: $H$ for $\mathcal{L}$-hypersequents; $\Gamma$ for finite sets of $\mathcal{L}$-formulas; $E$ for singleton or empty sets of $\mathcal{L}$-formulas. For example, an $\mathcal{L}$-hypersequent $H_1$ can be derived from an $\mathcal{L}$-hypersequent $H_2$ by applying the rule $(\forall^i \Rightarrow)$ iff $H_1 = H \cup \{\langle \Gamma, (\forall^i x\varphi)\rangle\}$ and $H_2 = H \cup \{\langle \Gamma \cup \{\varphi\{^t/_x\}\}, E\rangle\}$ for some $\mathcal{L}$-hypersequent $H$, finite set $\Gamma$ of $\mathcal{L}$-formulas, individual variable $x$ of $\mathcal{L}$, $\mathcal{L}$-formula $\varphi$, first-order $\mathcal{L}$-term $t$, and singleton or empty set $E$ of $\mathcal{L}$-formulas.
2. Since formulas are equivalence classes, the rules $(\Rightarrow \forall^i)$, $(\exists^i \Rightarrow)$ could be written also as:

$$(\Rightarrow \forall^i)\ \frac{H \mid \Gamma \Rightarrow \varphi\{^y/_x\}}{H \mid \Gamma \Rightarrow (\forall^i x\varphi)} \qquad (\exists^i \Rightarrow)\ \frac{H \mid \Gamma, \varphi\{^y/_x\} \Rightarrow E}{H \mid \Gamma, (\exists^i x\varphi) \Rightarrow E}$$

   where $y$ is not a free variable in the lower hypersequent.
3. In the presence of the weakening rules, it is always possible to incorporate external weakenings and left internal weakenings in the applications of the rules. Thus for example, we could have defined an application of $(\supset \Rightarrow)$ as an inference step deriving $H \mid \Gamma, \varphi_1 \supset \varphi_2 \Rightarrow E$ from $H_1 \mid \Gamma_1 \Rightarrow \varphi_1$ and $H_2 \mid \Gamma_2, \varphi_2 \Rightarrow E$ with the requirement that $H_1 \cup H_2 \subseteq H$ and $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$. Note that in the case of $(com)$, the equivalent definition allows us to derive $H \mid \Gamma'_1 \Rightarrow E_1 \mid \Gamma'_2 \Rightarrow E_2$ from $H_1 \mid \Gamma_1, \Delta_1 \Rightarrow E_1$ and $H_2 \mid \Gamma_2, \Delta_2 \Rightarrow E_2$ where $H_1 \cup H_2 \subseteq H$, $\Gamma_1 \cup \Delta_2 \subseteq \Gamma'_1$ and $\Gamma_2 \cup \Delta_1 \subseteq \Gamma'_2$. Henceforth, we freely use this kind of applications (that formally might involve additional applications of the weakening rules).

Next, we introduce the rules schemes for the set quantifiers. These are the single-conclusion hypersequent versions of the sequent rules used for classical logic (see the calculus $\mathbf{L}^2K$ in [10]), and they have the same structure of the rules for individual quantifiers, where instead of using first-order terms, one uses *abstraction terms* (*abstracts* for short). Abstracts are syntactic objects of the form $\{\!\!\{x \mid \varphi\}\!\!\}$ that intuitively represent sets of individuals. Note that abstracts are just a syntactic tool for formulating the rules of the set quantifiers. Derivations in the calculus still consist solely of hypersequents, and no abstracts are mentioned in them. As we did for formulas, we first define *concrete abstracts*, and abstracts are defined as alpha-equivalence classes of concrete ones.

**Definition 3.3.** A *concrete $\mathcal{L}$-abstract* is an expression of the form $\{\!\!\{x \mid \varphi^\bullet\}\!\!\}$, where $x$ is an individual variable of $\mathcal{L}$, and $\varphi^\bullet$ is a concrete $\mathcal{L}$-formula. Alpha-equivalence between concrete $\mathcal{L}$-abstracts is defined as usual (where $x$ is considered bound in $\{\!\!\{x \mid \varphi^\bullet\}\!\!\}$), and $[\{\!\!\{x \mid \varphi^\bullet\}\!\!\}]_\alpha$ is standing for the set of all concrete $\mathcal{L}$-abstracts which are alpha-equivalent to $\{\!\!\{x \mid \varphi^\bullet\}\!\!\}$. An *$\mathcal{L}$-abstract* is an equivalence class of concrete $\mathcal{L}$-abstracts under alpha-equivalence. We mainly use $\tau$ as a metavariable for $\mathcal{L}$-abstracts. The *set of free variables* of an $\mathcal{L}$-abstract is defined using representatives, i.e. for an $\mathcal{L}$-abstract $\tau$, $fv[\tau] = fv[\{\!\!\{x \mid \varphi^\bullet\}\!\!\}]$ for some $\{\!\!\{x \mid \varphi^\bullet\}\!\!\} \in \tau$.

**Definition 3.4.** Given an individual variable $x$ of $\mathcal{L}$ and an $\mathcal{L}$-formula $\varphi$, $\{\!\!\{x \mid \varphi\}\!\!\}$ is the $\mathcal{L}$-abstract $[\{\!\!\{x \mid \varphi^\bullet\}\!\!\}]_\alpha$ for some $\varphi^\bullet \in \varphi$.

**Proposition 3.5.** For every $\mathcal{L}$-abstract $\tau$ and individual variable $x \notin fv[\tau]$, there exists a unique $\mathcal{L}$-formula $\varphi$, such that $\tau = \{\!\!\{x \mid \varphi\}\!\!\}$.

**Definition 3.6.** Let $\tau$ be an $\mathcal{L}$-abstract, and $t$ be a first-order $\mathcal{L}$-term. $\tau[t]$ is defined to be the $\mathcal{L}$-formula $\varphi\{^t/_x\}$ for some individual variable $x$ and $\mathcal{L}$-formula $\varphi$, such that $\tau = \{\!\!\{x \mid \varphi\}\!\!\}$.

It is easy to see that $\tau[t]$ is well-defined. First, Proposition 3.5 ensures the existence of $x$ and $\varphi$ such that $\tau = \lbrace\!\!\{ x \mid \varphi \rbrace\!\!\}$. Additionally, it is easy to see that the result does not depend on the choice of $x$. Next, we define substitution of an abstract for a set variable in an arbitrary formula $\varphi$.

**Definition 3.7.** Let $\tau$ be an $\mathcal{L}$-abstract, and $X$ be a set variable of $\mathcal{L}$. Given an $\mathcal{L}$-formula $\varphi$, $\varphi\{^{\tau}/_{X}\}$ is inductively defined by:

$$\varphi\{^{\tau}/_{X}\} = \begin{cases} \varphi & \varphi = \{p(t_1, \dots, t_n)\}, \varphi = \{(t \varepsilon T)\} \text{ for } T \neq X, \text{ or } \varphi = \{\bot\} \\ \tau[t] & \varphi = \{(t \varepsilon X)\} \\ (\varphi_1\{^{\tau}/_{X}\} \diamond \varphi_2\{^{\tau}/_{X}\}) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i y \psi\{^{\tau}/_{X}\}) & \varphi = (Q^i y \psi) \text{ for } y \notin fv[\tau] \\ (Q^s Y \psi\{^{\tau}/_{X}\}) & \varphi = (Q^s Y \psi) \text{ for } Y \notin fv[\tau] \cup \{X\} \end{cases}$$

Note that the this substitution operation is well-defined. In particular, the choice of the variables $y$ and $Y$ is immaterial.

**Example 3.8.** For $\varphi = (\forall^i \nu_1(\{(\nu_1 \varepsilon \chi_1)\} \supset (\exists^s \chi_2\{(\nu_1 \varepsilon \chi_2)\})))$, and $\tau = \lbrace\!\!\{ \nu_2 \mid \{p(\nu_2, \nu_2)\} \rbrace\!\!\}$, we have

$$\varphi\{^{\tau}/_{\chi_1}\} = (\forall^i \nu_1(\{p(\nu_1, \nu_1)\} \supset (\exists^s \chi_2\{(\nu_1 \varepsilon \chi_2)\}))).$$

The following lemma will be useful in the sequel.

*Notation* 3.9. Given a second-order $\mathcal{L}$-term $T$, $T_{abs}$ denotes the $\mathcal{L}$-abstract $\lbrace\!\!\{ \nu_1 \mid \{(\nu_1 \varepsilon T)\} \rbrace\!\!\}$.

**Lemma 3.10.** For every second-order $\mathcal{L}$-term $T$, $\mathcal{L}$-formula $\varphi$, and set variable $X$ of $\mathcal{L}$, $\varphi\{^{T_{abs}}/_{X}\} = \varphi\{^{T}/_{X}\}$.

Proof. By usual induction on the complexity of $\varphi$. □

Using abstracts, the rules for the second-order quantifiers are formulated as follows.

**Definition 3.11.** $\mathbf{HIF}^2$ is the hypersequent calculus obtained by augmenting **HIF** with the following derivation rules:

$$(\forall^s \Rightarrow) \quad \frac{H \mid \Gamma, \varphi\{^{\tau}/_{X}\} \Rightarrow E}{H \mid \Gamma, (\forall^s X \varphi) \Rightarrow E} \qquad (\Rightarrow \forall^s) \quad \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow (\forall^s X \varphi)}$$

$$(\exists^s \Rightarrow) \quad \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, (\exists^s X \varphi) \Rightarrow E} \qquad (\Rightarrow \exists^s) \quad \frac{H \mid \Gamma \Rightarrow \varphi\{^{\tau}/_{X}\}}{H \mid \Gamma \Rightarrow (\exists^s X \varphi)}$$

where $X$ is not a free variable in the lower hypersequent in applications of the rules $(\Rightarrow \forall^s)$ and $(\exists^s \Rightarrow)$. Below, we write $\vdash H$ to denote that a hypersequent $H$ is provable in $\mathbf{HIF}^2$.

As before, since formulas are equivalence classes, the rules $(\Rightarrow \forall^s)$, and $(\exists^s \Rightarrow)$ could be written as:

$$(\Rightarrow \forall^s) \quad \frac{H \mid \Gamma \Rightarrow \varphi\{^{Y}/_{X}\}}{H \mid \Gamma \Rightarrow (\forall^s X \varphi)} \qquad (\exists^s \Rightarrow) \quad \frac{H \mid \Gamma, \varphi\{^{Y}/_{X}\} \Rightarrow E}{H \mid \Gamma, (\exists^s X \varphi) \Rightarrow E}$$

where $Y$ is not a free variable in the lower hypersequent.

*Remark* 3.12. Note that rules given by the schemes

$$\frac{H \mid \Gamma, \varphi\{^{T}/_{X}\} \Rightarrow E}{H \mid \Gamma, (\forall^s X \varphi) \Rightarrow E} \qquad \frac{H \mid \Gamma \Rightarrow \varphi\{^{T}/_{X}\}}{H \mid \Gamma \Rightarrow (\exists^s X \varphi)}$$

where $T$ is a second-order $\mathcal{L}$-term, are particular instances of $(\forall^s \Rightarrow)$ and $(\Rightarrow \exists^s)$, obtained by choosing $\tau = T_{abs}$ (see Lemma 3.10).

**Example 3.13.** Let $\varphi$ and $\psi$ be $\mathcal{L}$-formulas, and $X$ be a set variable such that $X \notin fv[\varphi] \cup fv[\psi]$. We show that $\vdash \Rightarrow ((\forall^s X(\varphi \vee \psi)) \supset (\varphi \vee (\forall^s X\psi)))$:

$$\dfrac{\dfrac{\overline{\varphi \Rightarrow \varphi}\ (id) \qquad \overline{\psi \Rightarrow \psi}\ (id)}{\psi \Rightarrow \varphi \mid \varphi \Rightarrow \psi}\ (com) \qquad \overline{\psi \Rightarrow \psi}\ (id)}{\psi \Rightarrow \varphi \mid (\varphi \vee \psi) \Rightarrow \psi}\ (\vee\Rightarrow)$$

$$\dfrac{\overline{\varphi \Rightarrow \varphi}\ (id) \qquad \psi \Rightarrow \varphi \mid (\varphi \vee \psi) \Rightarrow \psi}{(\varphi \vee \psi) \Rightarrow \varphi \mid (\varphi \vee \psi) \Rightarrow \psi}\ (\vee\Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\varphi \vee \psi) \Rightarrow \varphi \mid (\varphi \vee \psi) \Rightarrow \psi}{(\forall^s X(\varphi \vee \psi)) \Rightarrow \varphi \mid (\varphi \vee \psi) \Rightarrow \psi}\ (\forall^s\Rightarrow)}{(\forall^s X(\varphi \vee \psi)) \Rightarrow \varphi \mid (\forall^s X(\varphi \vee \psi)) \Rightarrow \psi}\ (\forall^s\Rightarrow)}{(\forall^s X(\varphi \vee \psi)) \Rightarrow \varphi \mid (\forall^s X(\varphi \vee \psi)) \Rightarrow (\forall^s X\psi)}\ (\Rightarrow\forall^s)}{(\forall^s X(\varphi \vee \psi)) \Rightarrow (\varphi \vee (\forall^s X\psi)) \mid (\forall^s X(\varphi \vee \psi)) \Rightarrow (\forall^s X\psi)}\ (\Rightarrow\vee_1)$$

$$\dfrac{\dfrac{(\forall^s X(\varphi \vee \psi)) \Rightarrow (\varphi \vee (\forall^s X\psi)) \mid (\forall^s X(\varphi \vee \psi)) \Rightarrow (\forall^s X\psi)}{(\forall^s X(\varphi \vee \psi)) \Rightarrow (\varphi \vee (\forall^s X\psi))}\ (\Rightarrow\vee_2)}{\Rightarrow ((\forall^s X(\varphi \vee \psi)) \supset (\varphi \vee (\forall^s X\psi)))}\ (\Rightarrow\supset)$$

### *3.1. Some Admissible and Derivable Rules*

In this section, we prove some properties of $\mathbf{HIF}^2$, to be used below for proving its completeness. First, the following standard lemma establishes the admissibility of substitution:

**Lemma 3.14.** Let $H$ be an $\mathcal{L}$-hypersequent.

- For every individual variables $x, y$ of $\mathcal{L}$, such that $y \notin fv[H]$, if $\vdash H$ then $\vdash H\{y/x\}$.
- For every set variables $X, Y$ of $\mathcal{L}$, such that $Y \notin fv[H]$, if $\vdash H$ then $\vdash H\{Y/X\}$.

Next, the rules $(com)$, $(\vee\Rightarrow)$, $(\exists^i\Rightarrow)$, and $(\exists^s\Rightarrow)$ can be generalized as follows.

**Proposition 3.15** (Generalized $(com)$)**.** The following rule is derivable in $\mathbf{HIF}^2$:

$$\frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_1' \Rightarrow E_n \qquad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \Gamma_2' \Rightarrow F_m}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \Gamma_2' \Rightarrow E_n \mid \Gamma_2, \Gamma_1' \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \Gamma_1' \Rightarrow F_m}$$

Proof. The proof is similar to the proof of Proposition 22 in [16]. □

**Proposition 3.16** (Generalized $(\vee\Rightarrow)$)**.** The following rule is derivable in $\mathbf{HIF}^2$:

$$\frac{H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \ldots \mid \Gamma_1, \varphi_1 \Rightarrow E_n \qquad H_2 \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \ldots \mid \Gamma_2, \varphi_2 \Rightarrow F_m}{H_1 \mid H_2 \mid \Gamma_1, (\varphi_1 \vee \varphi_2) \Rightarrow E_1 \mid \ldots \mid \Gamma_1, (\varphi_1 \vee \varphi_2) \Rightarrow E_n \mid \Gamma_2, (\varphi_1 \vee \varphi_2) \Rightarrow F_1 \mid \ldots \mid \Gamma_2, (\varphi_1 \vee \varphi_2) \Rightarrow F_m}$$

Proof. The proof is similar to the proof of Proposition 23 in [16]. □

**Proposition 3.17** (Generalized $(\exists^i\Rightarrow)$)**.** For every $\mathcal{L}$-hypersequent $H$, singleton or empty sets of $\mathcal{L}$-formulas $E_1, \ldots, E_n$, finite set $\Gamma$ of $\mathcal{L}$-formulas, $\mathcal{L}$-formula $\varphi$, and individual variable $x \notin fv[H] \cup fv[\Gamma \cup E_1 \cup \ldots \cup E_n]$: if $\vdash H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$, then $\vdash H \mid \Gamma, (\exists^i x\varphi) \Rightarrow E_1 \mid \ldots \mid \Gamma, (\exists^i x\varphi) \Rightarrow E_n$.

Proof. We use induction on $n$. The claim is trivial for $n = 0$. Now assume that the claim holds for $n - 1$, we prove it for $n$. Let $H$ be an $\mathcal{L}$-hypersequent, $E_1, \ldots, E_n$ be singleton or empty sets of $\mathcal{L}$-formulas, $\Gamma$ be a finite set of $\mathcal{L}$-formulas, $\varphi$ be an $\mathcal{L}$-formula, and $x$ be an individual variable of $\mathcal{L}$ such that $x \notin fv[H] \cup fv[\Gamma \cup E_1 \cup \ldots \cup E_n]$. Let $H_0 = H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$. Suppose that $\vdash H_0$. Let $y$ be an individual variable of $\mathcal{L}$ such that $y \notin fv[H_0]$. By Lemma 3.14, $\vdash H_0\{y/x\}$. By Proposition 3.15, the following $\mathcal{L}$-hypersequent is derivable from $H_0$ and $H_0\{y/x\}$:

$$H \mid \Gamma, \varphi \Rightarrow E_n \mid \Gamma, \varphi\{y/x\} \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi\{y/x\} \Rightarrow E_{n-1}$$

(to see this, take $H_1 = H \mid \Gamma, \varphi \Rightarrow E_n$ and $H_2 = H \mid \Gamma, \varphi\{y/x\} \Rightarrow E_1 \mid ... \mid \Gamma, \varphi\{y/x\} \Rightarrow E_{n-1}$). By an application of $(\exists^i \Rightarrow)$ on the last hypersequent, we obtain:

$$H \mid \Gamma, (\exists^i x\varphi) \Rightarrow E_n \mid \Gamma, \varphi\{y/x\} \Rightarrow E_1 \mid ... \mid \Gamma, \varphi\{y/x\} \Rightarrow E_{n-1}$$

The induction hypothesis now entails that $\vdash H \mid \Gamma, (\exists^i x\varphi) \Rightarrow E_1 \mid ... \mid \Gamma, (\exists^i x\varphi) \Rightarrow E_n$.

Similarly, we have the following:

**Proposition 3.18** (Generalized $(\exists^s \Rightarrow)$)**.** For $\mathcal{L}$-hypersequent $H$, singleton or empty sets of $\mathcal{L}$-formulas $E_1, \dots, E_n$, finite set $\Gamma$ of $\mathcal{L}$-formulas, $\mathcal{L}$-formula $\varphi$, and set variable $X \notin fv[H] \cup fv[\Gamma \cup E_1 \cup ... \cup E_n]$: if $\vdash H \mid \Gamma, \varphi \Rightarrow E_1 \mid ... \mid \Gamma, \varphi \Rightarrow E_n$, then $\vdash H \mid \Gamma, (\exists^s X\varphi) \Rightarrow E_1 \mid ... \mid \Gamma, (\exists^s X\varphi) \Rightarrow E_n$.

## 4. Soundness

In this section we prove the soundness of $\mathbf{HIF}^2$ for $\mathbf{G}^{\mathbf{2}}_{\mathcal{L}}$.

**Definition 4.1.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ be an $\mathcal{L}$-structure, where $\mathcal{V} = \langle V, \leq \rangle$.

1. An $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$ is a *model* (with respect to $\mathcal{U}$) of:
   (a) an $\mathcal{L}$-*sequent* $\Gamma \Rightarrow E$ (denoted by: $\sigma \models^{\mathcal{U}} \Gamma \Rightarrow E$) if $\min_{\varphi \in \Gamma} \mathcal{U}[\varphi, \sigma] \leq \max_{\varphi \in E} \mathcal{U}[\varphi, \sigma]$.[4]
   (b) an $\mathcal{L}$-*hypersequent* $H$ (denoted by: $\sigma \models^{\mathcal{U}} H$) if $\sigma \models^{\mathcal{U}} s$ for some component $s \in H$.
2. $\mathcal{U}$ is a *model* of an $\mathcal{L}$-hypersequent $H$ if $\sigma \models^{\mathcal{U}} H$ for every $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$.

**Theorem 4.2.** Let $H$ be an $\mathcal{L}$-hypersequent. If $\vdash H$, then every comprehensive $\mathcal{L}$-structure is a model of $H$.

Soundness for $\mathbf{G}^{\mathbf{2}}_{\mathcal{L}}$ is an obvious corollary:

**Corollary 4.3.** For every $\mathcal{L}$-formula $\varphi$, if $\vdash \Rightarrow \varphi$, then $\vdash^{\mathbf{G}^{\mathbf{2}}_{\mathcal{L}}} \varphi$.

Proof. Directly follows from Theorem 4.2, using the fact that $\mathcal{U}$ is a model of $\Rightarrow \varphi$ iff $\mathcal{U}[\varphi, \sigma] = 1$ for every comprehensive $\mathcal{L}$-structure $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$. □

Theorem 4.2 is proved in the usual way, by induction on the length of the derivation in $\mathbf{HIF}^2$. Note that the soundness proof for **HIF** in [16], was with respect to *Kripke-style* semantics, where here we prove soundness of $\mathbf{HIF}^2$ for the *many-valued* semantics described above. We use the following technical lemmas (full proofs are given in Appendix Appendix A):

**Lemma 4.4.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ be an $\mathcal{L}$-structure, $t$ be a first-order $\mathcal{L}$-term, and $x$ be an individual variable of $\mathcal{L}$. For every $\mathcal{L}$-formula $\varphi$, and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$: $\mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}] = \mathcal{U}[\varphi\{t/x\}, \sigma]$.

**Lemma 4.5.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ be an $\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$, $\tau$ be an $\mathcal{L}$-abstract, $x \notin fv[\tau]$ be an individual variable, and $X$ be a set variable of $\mathcal{L}$. For every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, if $\mathcal{U}[\tau[x], \sigma, x] \in \mathcal{D}_s$, then $\mathcal{U}[\varphi\{\tau/X\}, \sigma] = \mathcal{U}[\varphi, \sigma_{X:=\mathcal{U}[\tau[x],\sigma,x]}]$.

Proof (Theorem 4.2). Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ be an $\mathcal{L}$-structure, where $\mathcal{V} = \langle V, \leq \rangle$ and $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$. It suffices to prove soundness of each possible application of a rule of $\mathbf{HIF}^2$. We do here several cases, and leave the other cases to the reader:

[4] Dealing with single-conclusion sequences, $E$ is either a singleton $\{\varphi\}$ and then $\max_{\varphi \in E} \mathcal{U}[\varphi, \sigma] = \mathcal{U}[\varphi, \sigma]$, or empty and then $\max_{\varphi \in E} \mathcal{U}[\varphi, \sigma] = 0$.

**(com)** Suppose that $H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2$ is derived from $H \mid \Gamma_1, \Gamma_2 \Rightarrow E_1$ and $H \mid \Gamma_1, \Gamma_2 \Rightarrow E_2$ using $(com)$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. If $\sigma \models^{\mathcal{U}} s$ for some component $s \in H$, then we are done. Otherwise, $\sigma \models^{\mathcal{U}} \Gamma_1, \Gamma_2 \Rightarrow E_1$ and $\sigma \models^{\mathcal{U}} \Gamma_1, \Gamma_2 \Rightarrow E_2$. Thus $\min_{\varphi \in \Gamma_1 \cup \Gamma_2} \mathcal{U}[\varphi, \sigma] \leq \max_{\varphi \in E_1} \mathcal{U}[\varphi, \sigma]$, and $\min_{\varphi \in \Gamma_1 \cup \Gamma_2} \mathcal{U}[\varphi, \sigma] \leq \max_{\varphi \in E_2} \mathcal{U}[\varphi, \sigma]$. Now, either we have $\min_{\varphi \in \Gamma_1 \cup \Gamma_2} \mathcal{U}[\varphi, \sigma] = \min_{\varphi \in \Gamma_1} \mathcal{U}[\varphi, \sigma]$ or $\min_{\varphi \in \Gamma_1 \cup \Gamma_2} \mathcal{U}[\varphi, \sigma] = \min_{\varphi \in \Gamma_2} \mathcal{U}[\varphi, \sigma]$. It follows that either $\min_{\varphi \in \Gamma_1} \mathcal{U}[\varphi, \sigma] \leq_U \max_{\varphi \in E_1} \mathcal{U}[\varphi, \sigma]$ or $\min_{\varphi \in \Gamma_2} \mathcal{U}[\varphi, \sigma] \leq \max_{\varphi \in E_2} \mathcal{U}[\varphi, \sigma]$. Therefore, $\sigma \models^{\mathcal{U}} \Gamma_1 \Rightarrow E_1$ or $\sigma \models^{\mathcal{U}} \Gamma_2 \Rightarrow E_2$. In both cases, $\sigma \models^{\mathcal{U}} H \mid \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2$.

$(\supset\Rightarrow)$ Suppose that $H \mid \Gamma, (\varphi_1 \supset \varphi_2) \Rightarrow E$ is derived from $H \mid \Gamma \Rightarrow \varphi_1$ and $H \mid \Gamma, \varphi_2 \Rightarrow E$ using $(\supset\Rightarrow)$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. If $\sigma \models^{\mathcal{U}} s$ for some $s \in H$, then we are done. Otherwise, $\sigma \models^{\mathcal{U}} \Gamma \Rightarrow \varphi_1$ and $\sigma \models^{\mathcal{U}} \Gamma, \varphi_2 \Rightarrow E$. Let $u_1 = \min_{\psi \in \Gamma} \mathcal{U}[\psi, \sigma]$ and $u_2 = \max_{\psi \in E} \mathcal{U}[\psi, \sigma]$. If $u_1 \leq u_2$, then we have $\sigma \models^{\mathcal{U}} \Gamma, (\varphi_1 \supset \varphi_2) \Rightarrow E$, and we are done. Otherwise, $u_1 \leq \mathcal{U}[\varphi_1, \sigma]$, $\mathcal{U}[\varphi_2, \sigma] \leq u_2$, and so $\mathcal{U}[\varphi_2, \sigma] < \mathcal{U}[\varphi_1, \sigma]$. It follows that $\mathcal{U}[(\varphi_1 \supset \varphi_2), \sigma] = \mathcal{U}[\varphi_1, \sigma] \to \mathcal{U}[\varphi_2, \sigma] = \mathcal{U}[\varphi_2, \sigma] \leq u_2$. Consequently, $\sigma \models^{\mathcal{U}} \Gamma, (\varphi_1 \supset \varphi_2) \Rightarrow E$ in this case as well.

$(\forall^i \Rightarrow)$ Suppose that $H = H' \mid \Gamma, (\forall^i x \varphi) \Rightarrow E$ is derived from the $\mathcal{L}$-hypersequent $H' \mid \Gamma, \varphi\{{}^{t}/{}_{x}\} \Rightarrow E$ using $(\forall^i \Rightarrow)$. Assume that $\sigma \not\models^{\mathcal{U}} H$ for some $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$. Hence, $\sigma \not\models^{\mathcal{U}} s$ for every $s \in H'$, and $\sigma \not\models^{\mathcal{U}} \Gamma, (\forall^i x \varphi) \Rightarrow E$. Let $u = \max_{\psi \in E} \mathcal{U}[\psi, \sigma]$. Since $\sigma \not\models^{\mathcal{U}} \Gamma, (\forall^i x \varphi) \Rightarrow E$, we have $\min_{\psi \in \Gamma} \mathcal{U}[\psi, \sigma] > u$, and $\mathcal{U}[(\forall^i x \varphi), \sigma] > u$. By definition, $\mathcal{U}[(\forall^i x \varphi), \sigma] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\varphi, \sigma_{x:=d}]$. Thus $\mathcal{U}[\varphi, \sigma_{x:=d}] > u$ for every $d \in \mathcal{D}_i$. In particular, $\mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}] > u$. Lemma 4.4 implies that $\mathcal{U}[\varphi\{{}^{t}/{}_{x}\}, \sigma] > u$. It follows that $\sigma \not\models^{\mathcal{U}} \Gamma, \varphi\{{}^{t}/{}_{x}\} \Rightarrow E$. Consequently, $\mathcal{U}$ is not a model of $H' \mid \Gamma, \varphi\{{}^{t}/{}_{x}\} \Rightarrow E$.

$(\forall^s \Rightarrow)$ Suppose that $H = H' \mid \Gamma, (\forall^s X \varphi) \Rightarrow E$ is derived from the $\mathcal{L}$-hypersequent $H' \mid \Gamma, \varphi\{\tau/X\} \Rightarrow E$ using $(\forall^s \Rightarrow)$. Assume that $\sigma \not\models^{\mathcal{U}} H$ for some $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$. Hence, $\sigma \not\models^{\mathcal{U}} s$ for every $s \in H'$, and $\sigma \not\models^{\mathcal{U}} \Gamma, (\forall^s X \varphi) \Rightarrow E$. Let $u = \max_{\psi \in E} \mathcal{U}[\psi, \sigma]$. Since $\sigma \not\models^{\mathcal{U}} \Gamma, (\forall^s X \varphi) \Rightarrow E$, we have $\min_{\psi \in \Gamma} \mathcal{U}[\psi, \sigma] > u$, and $\mathcal{U}[(\forall^s X \varphi), \sigma] > u$. By definition, $\mathcal{U}[(\forall^s X \varphi), \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{U}[\varphi, \sigma_{X:=D}]$. Thus $\mathcal{U}[\varphi, \sigma_{X:=D}] > u$ for every $D \in \mathcal{D}_s$. Let $x$ be an individual variable such that $x \notin fv[\tau]$, and let $D_0 = \mathcal{U}[\tau[x], \sigma, x]$. Since $\mathcal{U}$ is comprehensive, $D_0 \in \mathcal{D}_s$, and in particular, $\mathcal{U}[\varphi, \sigma_{X:=D_0}] > u$. Lemma 4.5 implies that $\mathcal{U}[\varphi\{\tau/X\}, \sigma] > u$. It follows that $\sigma \not\models^{\mathcal{U}} \Gamma, \varphi\{\tau/X\} \Rightarrow E$. Consequently, $\mathcal{U}$ is not a model of $H' \mid \Gamma, \varphi\{\tau/X\} \Rightarrow E$. □

## 5. Complete Non-deterministic Semantics

It remains to prove the completeness of $\mathbf{HIF}^2$ for Henkin-style second-order Gödel logic. This will be established in two stages. First, in this section, we present a non-deterministic semantics for which $\mathbf{HIF}^2$ is complete. In the next section, we prove completeness with respect to the semantics described above, by extracting an ordinary counter-model out of a non-deterministic one. The non-deterministic semantics that we use here extends the semantics that was presented in [17] for the propositional fragment. It is based on *quasi*-$\mathcal{L}$-structures, defined as follows.

**Definition 5.1.** A *quasi-domain* for $\mathcal{L}$ is an ordered triplet $\langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$, where $\mathcal{D}_i$ and $\mathcal{D}_s$ are non-empty sets, and $I$ is a function assigning: an element of $\mathcal{D}_i$ to every individual constant symbol of $\mathcal{L}$, an element of $\mathcal{D}_s$ to every set constant symbol of $\mathcal{L}$, and a function in ${\mathcal{D}_i}^n \to \mathcal{D}_i$ to every $n$-ary function symbol of $\mathcal{L}$.

Note that the elements of $\mathcal{D}_s$ in quasi-domains may not be fuzzy subsets. This allows us to compose $\mathcal{D}_s$ out of abstracts (as done in the completeness proof). $\langle \mathcal{L}, \mathcal{D} \rangle$-assignments are defined for quasi-domains exactly as for domains (see Definition 2.14).

**Definition 5.2.** Let $\mathcal{V} = \langle V, \leq \rangle$ be a Gödel set.

1. A *non-empty closed interval for* $\mathcal{V}$ is a set of elements of the form $\{u \in V : l \leq u \leq r\}$ (denoted by $[l, r]$) where $l, r \in V$ and $l \leq r$. We denote by $\mathsf{Int}_{\mathcal{V}}$ the set of all non-empty closed intervals for $\mathcal{V}$.
2. Given some non-empty set $\mathcal{D}$, a function $D$ from $\mathcal{D}$ to $\mathsf{Int}_{\mathcal{V}}$ is called a *quasi fuzzy subset* of $\mathcal{D}$ over $\mathcal{V}$.

**Definition 5.3.** A *quasi-*$\mathcal{L}$*-structure* is a tuple $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$ where:

1. $\mathcal{V}$ is a Gödel set.
2. $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ is a quasi-domain for $\mathcal{L}$.
3. $P$ is a function assigning a quasi fuzzy subset of ${\mathcal{D}_i}^n$ over $\mathcal{V}$ to every $n$-ary predicate symbol of $\mathcal{L}$, and a quasi fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ to every element of $\mathcal{D}_s$.
4. $v$ is a function assigning an interval in $\mathsf{Int}_{\mathcal{V}}$ to every (pair of) $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, such that the following hold:
   (a) For every pair of individual variables $x, y$ such that $y \notin fv[\varphi]$, and every $d \in \mathcal{D}_i$:
   $$v[\varphi, \sigma_{x:=d}] = v[\varphi\{y/x\}, \sigma_{y:=d}].$$
   (b) For every pair of set variables $X, Y$ such that $Y \notin fv[\varphi]$, and every $D \in \mathcal{D}_s$:
   $$v[\varphi, \sigma_{X:=D}] = v[\varphi\{Y/X\}, \sigma_{Y:=D}].$$

Quasi-$\mathcal{L}$-structures are different from $\mathcal{L}$-structures in several important aspects:

1. The interpretations of the predicate symbols are not fuzzy subsets of tuples of elements of $\mathcal{D}_i$, but *quasi* fuzzy subsets, namely $P[p][d_1, \ldots, d_n]$ is a non-empty closed *interval* of truth values.
2. The interpretation function $P$ of a quasi-$\mathcal{L}$-structure also assigns a quasi fuzzy subset of $\mathcal{D}_i$ over $\mathcal{V}$ to every element of $\mathcal{D}_s$.
3. Quasi-$\mathcal{L}$-structures include also a valuation function $v$ that assigns to *every* formula and assignment some interval of truth values. Intuitively, the left endpoint of $v[\varphi, \sigma]$ should be thought of as the value of $\varphi$ with respect to $\sigma$ when $\varphi$ occurs on left sides of sequents, and the right endpoint is the corresponding value when it occurs on right sides of sequents. This flexibility is the key for proving completeness when the cut rule is absent. Indeed, intuitively speaking, from a semantic point of view, the cut rule and the identity axiom bind the left and right value of each formula. The reason to have $v$ included in the structure lies in the fact that the resulting semantics is *non-deterministic* (see discussion below). Thus, in contrast to (ordinary) structures, in quasi structures the values of the atomic formulas do not uniquely determine the values of all compound formulas. The function $v$ is then used to "store" the values of the compound formulas.

Obviously, in order to be able to extract an ordinary counter-model out of a quasi-structure, further conditions should be imposed:

*Notation* 5.4. For each function $F$ whose range is $\mathsf{Int}_{\mathcal{V}}$ from Definition 5.3 (namely, $P[p]$ for every predicate symbol $p$, $P[D]$ for every $D \in \mathcal{D}_s$, and $v$), we denote by $F^l$ and $F^r$ the functions obtained from $F$ by taking only the left and the right endpoints (respectively). For instance, for every $\varphi$ and $\sigma$, $v^l[\varphi, \sigma]$ is the left endpoint of the interval $v[\varphi, \sigma]$.

**Definition 5.5.** Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$ be a quasi-$\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$. For every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, $\mathcal{Q}[\varphi, \sigma]$ is the interval in $\mathsf{Int}_{\mathcal{V}}$ defined as follows:

$$\mathcal{Q}[\varphi,\sigma] = \begin{cases} P[p][\sigma[t_1], \ldots, \sigma[t_n]] & \varphi = \{p(t_1, \ldots, t_n)\} \\ P[\sigma[T]][\sigma[t]] & \varphi = \{(t \varepsilon T)\} \\ \{0\} & \varphi = \{\bot\} \\ [\min\{v^l[\varphi_1,\sigma], v^l[\varphi_2,\sigma]\}, \min\{v^r[\varphi_1,\sigma], v^r[\varphi_2,\sigma]\}] & \varphi = (\varphi_1 \wedge \varphi_2) \\ [\max\{v^l[\varphi_1,\sigma], v^l[\varphi_2,\sigma]\}, \max\{v^r[\varphi_1,\sigma], v^r[\varphi_2,\sigma]\}] & \varphi = (\varphi_1 \vee \varphi_2) \\ [v^r[\varphi_1,\sigma] \to v^l[\varphi_2,\sigma], v^l[\varphi_1,\sigma] \to v^r[\varphi_2,\sigma]] & \varphi = (\varphi_1 \supset \varphi_2) \\ [\inf_{d \in \mathcal{D}_i} v^l[\psi, \sigma_{x:=d}], \inf_{d \in \mathcal{D}_i} v^r[\psi, \sigma_{x:=d}]] & \varphi = (\forall^i x \psi) \\ [\sup_{d \in \mathcal{D}_i} v^l[\psi, \sigma_{x:=d}], \sup_{d \in \mathcal{D}_i} v^r[\psi, \sigma_{x:=d}]] & \varphi = (\exists^i x \psi) \\ [\inf_{D \in \mathcal{D}_s} v^l[\psi, \sigma_{X:=D}], \inf_{D \in \mathcal{D}_s} v^r[\psi, \sigma_{X:=D}]] & \varphi = (\forall^s X \psi) \\ [\sup_{D \in \mathcal{D}_s} v^l[\psi, \sigma_{X:=D}], \sup_{D \in \mathcal{D}_s} v^r[\psi, \sigma_{X:=D}]] & \varphi = (\exists^s X \psi) \end{cases}$$

Conditions (a) and (b) in Definition 5.3 ensure that $\mathcal{Q}$ is well-defined, namely that the choice of $x$ and $X$ is immaterial. It is straightforward to verify that $\mathcal{Q}[\varphi, \sigma]$ is indeed non-empty for every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$ (for $\supset$, note that if $u_1 \leq u_2$ and $u_3 \leq u_4$ then $u_2 \to u_3 \leq u_1 \to u_4$).

**Definition 5.6.** A quasi-$\mathcal{L}$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$ is called *legal* if $\mathcal{Q}[\varphi, \sigma] \subseteq v[\varphi, \sigma]$ for every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$.

We can now demonstrate the non-deterministic nature of the semantics. Suppose that $v[\varphi_1, \sigma] = [l_1, r_1]$ and $v[\varphi_2, \sigma] = [l_2, r_2]$. All we require from, e.g., $v[(\varphi_1 \wedge \varphi_2), \sigma]$ is $[\min\{l_1, l_2\}, \min\{r_1, r_2\}] \subseteq v[(\varphi_1 \wedge \varphi_2), \sigma]$. In other words, every interval $[l, r]$ such that $l \leq \min\{l_1, l_2\}$ and $\min\{r_1, r_2\} \leq r$ can be chosen as a value for $v[(\varphi_1 \wedge \varphi_2), \sigma]$. In contrast to ordinary structures, here the values of $\langle \varphi_1, \sigma \rangle$ and $\langle \varphi_2, \sigma \rangle$ do not uniquely determine the value of $\langle (\varphi_1 \wedge \varphi_2), \sigma \rangle$. Intuitively speaking, non-determinism is a direct result of the "split" truth values: each logical introduction rule enforces only one-sided bound on values of formulas when they appear on a certain side of the sequent.

*Remark* 5.7. Since formulas are defined to be alpha equivalence classes of concrete formulas, we do not have to enforce in the definition of a quasi-structure that two alpha-equivalent formulas obtain the same value. Previous works on non-deterministic semantics for languages with quantifiers, such as [18], studied structures in which truth values are non-deterministically assigned to concrete formulas. In this case, additional (technically complicated) restrictions are needed.

We adapt the definition of comprehensive structures to quasi-structures, keeping in mind that the function $P$ interprets the elements of $\mathcal{D}_s$ as quasi fuzzy sets.

**Definition 5.8.** A quasi-$\mathcal{L}$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$, is called *comprehensive* if for every $\mathcal{L}$-formula $\varphi$, individual variable $x$, and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, there exist some $D \in \mathcal{D}_s$ such that $P[D] = \lambda d \in \mathcal{D}_i.\ v[\varphi, \sigma_{x:=d}]$.

The notion of *model* for quasi-$\mathcal{L}$-structures is given in Definition 5.9. Note that the definition is "liberal", taking the smallest possible value (the left endpoint of each interval) when a formula occurs on the left side of a sequent, and the greatest one (the right endpoint) when a formula occurs on the right side of a sequent.

**Definition 5.9.** Let $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$ be a quasi-$\mathcal{L}$-structure, where $\mathcal{V} = \langle V, \leq \rangle$.

1. An $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$ is a *model* (with respect to $\mathcal{Q}$) of:
   (a) an *$\mathcal{L}$-sequent* $\Gamma \Rightarrow E$ (denoted by: $\sigma \models^{\mathcal{Q}} \Gamma \Rightarrow E$) if $\min_{\varphi \in \Gamma} v^l[\varphi, \sigma] \leq \max_{\varphi \in E} v^r[\varphi, \sigma]$.
   (b) an *$\mathcal{L}$-hypersequent* $H$ (denoted by: $\sigma \models^{\mathcal{Q}} H$) if $\sigma \models^{\mathcal{Q}} s$ for some component $s \in H$.
2. $\mathcal{Q}$ is a *model* of an $\mathcal{L}$-hypersequent $H$ if $\sigma \models^{\mathcal{Q}} H$ for every $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$.

**Theorem 5.10.** Suppose that $\not\vdash H_0$, for some $\mathcal{L}$-hypersequent $H_0$. Then there exists a legal comprehensive quasi-$\mathcal{L}$-structure which is not a model of $H_0$.

The rest of this section is devoted to prove this theorem. First, we introduce the two main ingredients of this proof: maximal extended hypersequents and Herbrand quasi-domains.

*Remark* 5.11. Every $\mathcal{L}$-structure $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ naturally induces a quasi-$\mathcal{L}$-structure $\mathcal{Q} = \langle \mathcal{V}', \mathcal{D}', P', v \rangle$, by taking $\mathcal{V}' = \mathcal{V}$, $\mathcal{D}' = \mathcal{D}$, $P'[p] = \lambda d_1, \ldots, d_n \in \mathcal{D}_i.\ \{P[p][d_1, \ldots, d_n]\}$ for every $n$-ary predicate symbol, $P'[D] = \lambda d \in \mathcal{D}_i.\ \{D[d]\}$ for every $D \in \mathcal{D}_s$, and $v[\varphi, \sigma] = \{\mathcal{U}[\varphi, \sigma]\}$ for every $\varphi$ and $\sigma$. It is easy to verify that $\mathcal{Q}$ is always legal, as well as comprehensive assuming that $\mathcal{U}$ is comprehensive. In addition, it is straightforward to verify the soundness of $\mathbf{HIF}^2$ with respect to the semantics of legal comprehensive quasi-structures (i.e. $\vdash H$ implies that every legal comprehensive quasi-$\mathcal{L}$-structure is a model of $H$). Soundness would not hold if we added $(cut)$ to $\mathbf{HIF}^2$, since we allow cases in which $v^l[\varphi, \sigma] < v^r[\varphi, \sigma]$ and then $\mathcal{Q}$ might be both a model of $\Rightarrow \varphi$ and $\varphi \Rightarrow$.

### 5.1. Maximal Extended Hypersequents

Maximal extended $\mathcal{L}$-hypersequents will play a crucial role in the completeness proof below. These are straightforward adaptations of the corresponding notions in [16], that were used to prove cut-free completeness of the first-order fragment. The full proofs, that are also adaptations of the corresponding proofs in [16], are given in Appendix Appendix B.

**Definition 5.12.** An *extended $\mathcal{L}$-sequent* is an ordered pair of (possibly infinite) sets of $\mathcal{L}$-formulas. Given two extended $\mathcal{L}$-sequents $\mu_1 = \langle \mathtt{L}_1, \mathtt{R}_1 \rangle$ and $\mu_2 = \langle \mathtt{L}_2, \mathtt{R}_2 \rangle$, we write $\mu_1 \sqsubseteq \mu_2$ if $\mathtt{L}_1 \subseteq \mathtt{L}_2$ and $\mathtt{R}_1 \subseteq \mathtt{R}_2$. An extended $\mathcal{L}$-sequent is called *finite* if it consists of finite sets of formulas.

**Definition 5.13.** An *extended $\mathcal{L}$-hypersequent* is a (possibly infinite) set of extended $\mathcal{L}$-sequents. Given two extended $\mathcal{L}$-hypersequents $\Omega_1, \Omega_2$, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that $\Omega_2$ *extends* $\Omega_1$) if for every extended $\mathcal{L}$-sequent $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$. An extended $\mathcal{L}$-hypersequent is called *finite* if it consists of finite number of finite extended $\mathcal{L}$-sequents.

We shall use the same notations as above for extended $\mathcal{L}$-sequents and extended $\mathcal{L}$-hypersequents. For example, we write $\mathtt{L} \Rightarrow \mathtt{R}$ instead of $\langle \mathtt{L}, \mathtt{R} \rangle$, and $\Omega \mid \mathtt{L}, \varphi \Rightarrow \mathtt{R}$ instead of $\Omega \cup \{\langle \mathtt{L} \cup \{\varphi\}, \mathtt{R} \rangle\}$. Obviously, every $\mathcal{L}$-hypersequent is an extended $\mathcal{L}$-hypersequent, and so the definition above and the properties defined below apply to (ordinary) $\mathcal{L}$-hypersequents as well. Note that finite extended sequents (hypersequents) correspond to *multiple*-conclusion sequents (hypersequents).

**Definition 5.14.** An extended $\mathcal{L}$-sequent $\mathtt{L} \Rightarrow \mathtt{R}$ admits *the witness property* if the following hold for every $\mathcal{L}$-formula $\varphi$, individual variable $x$ of $\mathcal{L}$, and set variable $X$ of $\mathcal{L}$:

1. If $(\forall^i x \varphi) \in \mathtt{R}$, then $\varphi\{y/x\} \in \mathtt{R}$ for some individual variable $y$ of $\mathcal{L}$.
2. If $(\exists^i x \varphi) \in \mathtt{L}$, then $\varphi\{y/x\} \in \mathtt{L}$ for some individual variable $y$ of $\mathcal{L}$.
3. If $(\forall^s X \varphi) \in \mathtt{R}$, then $\varphi\{Y/X\} \in \mathtt{R}$ for some set variable $Y$ of $\mathcal{L}$.
4. If $(\exists^s X \varphi) \in \mathtt{L}$, then $\varphi\{Y/X\} \in \mathtt{L}$ for some set variable $Y$ of $\mathcal{L}$.

**Definition 5.15.** Let $\Omega$ be an extended $\mathcal{L}$-hypersequent.

1. $\Omega$ is called *unprovable* if $\not\vdash H$ for every (ordinary) $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$. Otherwise, $\Omega$ is called *provable*.
2. Let $\varphi$ be an $\mathcal{L}$-formula. $\Omega$ is called *internally maximal with respect to* $\varphi$ if for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$:
   (a) If $\varphi \notin \mathtt{L}$ then $\Omega \mid \mathtt{L}, \varphi \Rightarrow \mathtt{R}$ is provable.
   (b) If $\varphi \notin \mathtt{R}$ then $\Omega \mid \mathtt{L} \Rightarrow \varphi, \mathtt{R}$ is provable.
3. $\Omega$ is called *internally maximal* if it is internally maximal with respect to any $\mathcal{L}$-formula.
4. Let $s$ be an $\mathcal{L}$-sequent. $\Omega$ is called *externally maximal with respect to* $s$ if either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is provable.
5. $\Omega$ is called *externally maximal* if it is externally maximal with respect to any $\mathcal{L}$-sequent.
6. $\Omega$ admits *the witness property* if every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ admits the witness property.
7. $\Omega$ is called *maximal* if it is unprovable, internally maximal, externally maximal, and it admits the witness property.

Less formally, an extended hypersequent $\Omega$ is internally maximal if every formula added on some side of some component of $\Omega$ would make it provable. Similarly, $\Omega$ is externally maximal if every sequent added to $\Omega$ would make it provable. Note that the availability of external and internal weakenings ensures that for ordinary hypersequents the previous notion of provability (denoted by $\vdash$) and the current one are equivalent.

The following propositions easily follow from the definitions using internal and external weakenings.

**Proposition 5.16.** Let $\Omega$ be an extended $\mathcal{L}$-hypersequent, which is internally maximal with respect to an $\mathcal{L}$-formula $\varphi$. For every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$:

1. If $\varphi \notin \mathtt{L}$, then we have $\vdash H \mid \Gamma, \varphi \Rightarrow E_1 \mid \ldots \mid \Gamma, \varphi \Rightarrow E_n$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$ and $\mathcal{L}$-sequents $\Gamma \Rightarrow E_1, \ldots, \Gamma \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$.

2. If $\varphi \notin \mathtt{R}$, then we have $\vdash H \mid \Gamma \Rightarrow \varphi$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$ and finite set $\Gamma \subseteq \mathtt{L}$.

**Proposition 5.17.** Let $\Omega$ be an extended $\mathcal{L}$-hypersequent, which is externally maximal with respect to an $\mathcal{L}$-sequent $s$. If $\{s\} \not\sqsubseteq \Omega$, then there exists an $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$ such that $\vdash H \mid s$.

**Lemma 5.18.** Every unprovable $\mathcal{L}$-hypersequent can be extended to a *maximal* extended $\mathcal{L}$-hypersequent.

Proof. See Appendix Appendix B. □

*5.2. The Herbrand Quasi-Domain*

**Definition 5.19.** The *Herbrand quasi-domain for* $\mathcal{L}$ is the quasi-domain $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ where $\mathcal{D}_i$ is the set of all first-order $\mathcal{L}$-terms, $\mathcal{D}_s$ is the set of all $\mathcal{L}$-abstracts, $I[c] = c$ for every individual constant symbol $c$ of $\mathcal{L}$, $I[C] = C_{abs} = \{\!\{\nu_1 \mid \{(\nu_1 \varepsilon C)\}\}\!\}$ for every set constant symbol $C$ of $\mathcal{L}$ (see Notation 3.9), and $I[f] = \lambda t_1, \dots, t_n \in \mathcal{D}_i.\ f(t_1, \dots, t_n)$ for every $n$-ary function symbol $f$ of $\mathcal{L}$.

In an Herbrand quasi-domain, we can extend $\langle \mathcal{L}, \mathcal{D} \rangle$-assignments to apply on formulas. Roughly speaking, every occurrence of a free variable $x$ or $X$ in a formula $\varphi$ is replaced in $\sigma[\varphi]$ by $\sigma[x]$ or $\sigma[X]$. Formally, this is defined as follows.

**Definition 5.20.** Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ be the Herbrand quasi-domain for $\mathcal{L}$. Let $\varphi$ be an $\mathcal{L}$-formula, and $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. The *set of free variables of the pair* $\langle \varphi, \sigma \rangle$ (denoted by $fv[\langle \varphi, \sigma \rangle]$) consists of the variables of $\sigma[x]$ for every individual variable $x \in fv[\varphi]$, and the free variables of $\sigma[X]$ for every set variable $X \in fv[\varphi]$.

**Definition 5.21.** Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ be the Herbrand quasi-domain for $\mathcal{L}$. $\langle \mathcal{L}, \mathcal{D} \rangle$-assignments are extended to $\mathcal{L}$-formulas, according to the following inductive definition:

$$\sigma[\varphi] = \begin{cases} \{p(\sigma[t_1], \dots, \sigma[t_n])\} & \varphi = \{p(t_1, \dots, t_n)\} \\ \sigma[T][\sigma[t]] & \varphi = \{(t \varepsilon T)\} \\ \{\bot\} & \varphi = \{\bot\} \\ (\sigma[\varphi_1] \diamond \sigma[\varphi_2]) & \varphi = (\varphi_1 \diamond \varphi_2) \\ (Q^i x \sigma_{x:=x}[\psi]) & \varphi = (Q^i x \psi) \text{ for } x \notin fv[\langle \varphi, \sigma \rangle] \\ (Q^s X \sigma_{X:=X_{abs}}[\psi]) & \varphi = (Q^s X \psi) \text{ for } X \notin fv[\langle \varphi, \sigma \rangle] \end{cases}$$

Note that the choice of $x$ and $X$ in the last definition is immaterial, and thus $\sigma[\varphi]$ is well-defined. The following properties of the Herbrand quasi-domain are needed in the completeness proof. Their proofs are given in Appendix Appendix A.

**Lemma 5.22.** Let $\mathcal{D}$ be the Herbrand quasi-domain for $\mathcal{L}$.

1. Let $t$ be a first-order $\mathcal{L}$-term. For every $\mathcal{L}$-formula $\varphi$, $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, and individual variables $x, y$ such that $y \notin fv[\varphi]$, $\sigma_{x:=t}[\varphi] = \sigma_{y:=t}[\varphi\{^{y}/_{x}\}]$.
2. Let $\tau$ be an $\mathcal{L}$-abstract. For every $\mathcal{L}$-formula $\varphi$, $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, and set variables $X, Y$ such that $Y \notin fv[\varphi]$, $\sigma_{X:=\tau}[\varphi] = \sigma_{Y:=\tau}[\varphi\{^{Y}/_{X}\}]$.

**Lemma 5.23.** Let $\mathcal{D}$ be the Herbrand quasi-domain for $\mathcal{L}$.

1. Let $t$ be a first-order $\mathcal{L}$-term. For every $\mathcal{L}$-formula $\varphi$, $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, and individual variables $x, z$ such that $z \notin fv[\sigma[\varphi]]$, $\sigma_{x:=z}[\varphi]\{^{t}/_{z}\} = \sigma_{x:=t}[\varphi]$.
2. Let $\tau$ be an $\mathcal{L}$-abstract. For every $\mathcal{L}$-formula $\varphi$, $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, and set variable $X \notin fv[\sigma[\varphi]]$, $\sigma_{X:=X_{abs}}[\varphi]\{^{\tau}/_{X}\} = \sigma_{X:=\tau}[\varphi]$.

### 5.3. Proof of Theorem 5.10

Suppose that $\not\vdash H_0$. The availability of external and internal weakenings ensures that $H_0$ is unprovable (seen as an extended hypersequent). By Lemma 5.18, there exists a maximal extended $\mathcal{L}$-hypersequent $\Omega^*$ such that $H_0 \sqsubseteq \Omega^*$. We use $\Omega^*$ to construct a counter-model for $H_0$ in the form of a quasi-$\mathcal{L}$-structure $\mathcal{Q} = \langle \mathcal{V}, \mathcal{D}, P, v \rangle$.

First, we define a bounded linearly ordered set $\mathcal{V}_0$, that will be used to construct (using the Dedekind-MacNeille completion) the Gödel set $\mathcal{V}$. For every $\mathcal{L}$-formula $\varphi$ we define:

$$L[\varphi] = \{\mathtt{L} \Rightarrow \mathtt{R} \in \Omega^* : \varphi \in \mathtt{L}\}, \quad R[\varphi] = \{\mathtt{L} \Rightarrow \mathtt{R} \in \Omega^* : \varphi \notin \mathtt{R}\}.$$

Let $\mathcal{V}_0 = \langle V_0, \subseteq \rangle$, where

$$V_0 = \{L[\varphi] : \varphi \text{ is an } \mathcal{L}\text{-formula}\} \cup \{R[\varphi] : \varphi \text{ is an } \mathcal{L}\text{-formula}\} \cup \{\Omega^*, \emptyset\}.$$

Clearly, $\mathcal{V}_0$ is partially ordered set, bounded by $0 = \emptyset$ and $1 = \Omega^*$. To see that $V_0$ is linearly ordered by $\subseteq$, it suffices to prove the following:

1. $L[\varphi_1] \subseteq L[\varphi_2]$ or $L[\varphi_2] \subseteq L[\varphi_1]$ for every pair of $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$. To see this, suppose that there are $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega^*$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega^*$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in L[\varphi_1] \setminus L[\varphi_2]$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in L[\varphi_2] \setminus L[\varphi_1]$. Hence, we have $\varphi_1 \in \mathtt{L}_1$, $\varphi_1 \notin \mathtt{L}_2$, $\varphi_2 \in \mathtt{L}_2$ and $\varphi_2 \notin \mathtt{L}_1$. Since $\Omega^*$ is internally maximal, by Proposition 5.16, there exist an $\mathcal{L}$-hypersequent $H_1 \sqsubseteq \Omega^*$ and an $\mathcal{L}$-sequent $\Gamma_1 \Rightarrow E_1, \dots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathtt{L}_1 \Rightarrow \mathtt{R}_1$ such that $\vdash H_1 \mid \Gamma_1, \varphi_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_2 \Rightarrow E_n$. Similarly, there exist an $\mathcal{L}$-hypersequent $H_2 \sqsubseteq \Omega^*$ and an $\mathcal{L}$-sequent $\Gamma_2 \Rightarrow F_1, \dots, \Gamma_2 \Rightarrow F_m \sqsubseteq \mathtt{L}_2 \Rightarrow \mathtt{R}_2$ such that $\vdash H_2 \mid \Gamma_2, \varphi_1 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_1 \Rightarrow F_m$. By Proposition 3.15, $\vdash H_1 \mid H_2 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \Rightarrow E_n \mid \Gamma_2, \varphi_2 \Rightarrow F_1 \mid \dots \mid \Gamma_2, \varphi_2 \Rightarrow F_m$. But, $\Omega^*$ extends this hypersequent, and this contradicts $\Omega^*$'s unprovability.
2. $R[\varphi_1] \subseteq R[\varphi_2]$ or $R[\varphi_2] \subseteq R[\varphi_1]$ for every pair of $\mathcal{L}$-formulas $\varphi_1, \varphi_2$. To see this, suppose that there are $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega^*$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega^*$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in R[\varphi_1] \setminus R[\varphi_2]$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in R[\varphi_2] \setminus R[\varphi_1]$. Hence, $\varphi_1 \notin \mathtt{R}_1$, $\varphi_1 \in \mathtt{R}_2$, $\varphi_2 \in \mathtt{R}_1$ and $\varphi_2 \notin \mathtt{R}_2$. Since $\Omega^*$ is internally maximal, by Proposition 5.16, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$ and finite sets $\Gamma_1 \subseteq \mathtt{L}_1$ and $\Gamma_2 \subseteq \mathtt{L}_2$ such that $\vdash H_1 \mid \Gamma_1 \Rightarrow \varphi_1$ and $\vdash H_2 \mid \Gamma_2 \Rightarrow \varphi_2$. By applying $(com)$, we obtain $\vdash H_1 \mid H_2 \mid \Gamma_2 \Rightarrow \varphi_1 \mid \Gamma_1 \Rightarrow \varphi_2$. Again, since $\Omega^*$ extends this hypersequent, this contradicts $\Omega^*$'s unprovability.
3. $L[\varphi_1] \subseteq R[\varphi_2]$ or $R[\varphi_2] \subseteq L[\varphi_1]$ for every pair of $\mathcal{L}$-formulas $\varphi_1, \varphi_2$. To see this, suppose that there are $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in \Omega^*$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in \Omega^*$, such that $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \in L[\varphi_1] \setminus R[\varphi_2]$ and $\mathtt{L}_2 \Rightarrow \mathtt{R}_2 \in R[\varphi_2] \setminus L[\varphi_1]$. Hence, $\varphi_1 \in \mathtt{L}_1$, $\varphi_1 \notin \mathtt{L}_2$, $\varphi_2 \in \mathtt{R}_1$ and $\varphi_2 \notin \mathtt{R}_2$. Since $\Omega^*$ is internally maximal, by Proposition 5.16, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, sequents $\Gamma_1 \Rightarrow E_1, \dots, \Gamma_1 \Rightarrow E_n \sqsubseteq \mathtt{L}_2 \Rightarrow \mathtt{R}_2$ and a finite set $\Gamma_2 \subseteq \mathtt{L}_2$, such that $\vdash H_1 \mid \Gamma_1, \varphi_1 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \varphi_1 \Rightarrow E_n$ and $\vdash H_2 \mid \Gamma_2 \Rightarrow \varphi_2$. By Proposition 3.15, it follows that $\vdash H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2 \Rightarrow E_n \mid \varphi_1 \Rightarrow \varphi_2$. Again, this contradicts $\Omega^*$'s unprovability.

Now, since $\mathcal{V}_0$ might not be complete, we consider its Dedekind-MacNeille completion $\mathcal{V} = \langle V, \subseteq \rangle$ defined by:

$$V = \{\Pi \subseteq V_0 : (\Pi^{\uparrow})^{\downarrow} = \Pi\}$$

where $\Pi^{\uparrow} = \{\Omega \in V_0 : \Omega' \subseteq \Omega \text{ for all } \Omega' \in \Pi\}$ and $\Pi^{\downarrow} = \{\Omega \in V_0 : \Omega \subseteq \Omega' \text{ for all } \Omega' \in \Pi\}$. $\mathcal{V}$ is a bounded complete linearly ordered set (see [5]), and thus it forms a Gödel set. Note that using $\subseteq$ as the order relation, min and max are sets intersection and sets union (respectively). In addition, the function $\eta : V_0 \to V$ defined by $\eta(\Omega) = \{\Omega\}^{\downarrow}$ is injective and it satisfies the following properties:[5]

- $\{\emptyset\} = \eta(\emptyset)$.
- For every $\Omega, \Omega' \in V_0$:
  - $\Omega \subseteq \Omega'$ iff $\eta(\Omega) \subseteq \eta(\Omega')$.
  - $\eta(\Omega) \cap \eta(\Omega') = \eta(\Omega \cap \Omega')$.

[5] All operations notations from Definition 2.10 are adopted to $\mathcal{V}_0$ in the obvious way.

  - $\eta(\Omega) \cup \eta(\Omega') = \eta(\Omega \cup \Omega')$.
  - $\eta(\Omega) \to \eta(\Omega') = \eta(\Omega \to \Omega')$
- For every $\Omega \in V_0$ and $\Pi \subseteq V_0$:
  - If $\Omega \subseteq \bigcap_{\Omega' \in \Pi} \Omega'$, then $\eta(\Omega) \subseteq \inf_{\Omega' \in \Pi} \eta(\Omega')$.
  - If $\bigcap_{\Omega' \in \Pi} \Omega' \subseteq \Omega$, then $\inf_{\Omega' \in \Pi} \eta(\Omega') \subseteq \eta(\Omega)$.
  - If $\Omega \subseteq \bigcup_{\Omega' \in \Pi} \Omega'$, then $\eta(\Omega) \subseteq \sup_{\Omega' \in \Pi} \eta(\Omega')$.
  - If $\bigcup_{\Omega' \in \Pi} \Omega' \subseteq \Omega$, then $\sup_{\Omega' \in \Pi} \eta(\Omega') \subseteq \eta(\Omega)$.

The proofs of these properties are straightforward (note that the linearity of $V_0$ is needed in some of them). *Henceforth, we will identify each element $\Omega$ of $V_0$ with the corresponding element $\{\Omega\}^{\downarrow}$ in $V$, and freely use the properties above.*

Next, for every formula $\varphi$, let $\Omega^*[\varphi]$ be the non-empty closed interval for $\mathcal{V}$ given by: $\Omega^*[\varphi] = [L[\varphi], R[\varphi]]$. To see that $\Omega^*[\varphi]$ is a non-empty interval for every $\mathcal{L}$-formula $\varphi$, note that in the presence of $(id)$, either $\varphi \notin \mathtt{L}$ or $\varphi \notin \mathtt{R}$ for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ and $\mathcal{L}$-formula $\varphi$ (otherwise, $\{\varphi \Rightarrow \varphi\} \sqsubseteq \Omega$, contradicting the unprovability of $\Omega$), and consequently, $L[\varphi] \subseteq R[\varphi]$. Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$ be the Herbrand quasi-domain for $\mathcal{L}$, and define $P$ and $v$ as follows:

- For every $n$-ary predicate symbol $p$ of $\mathcal{L}$, $P[p] = \lambda t_1, \dots, t_n \in \mathcal{D}_i.\ \Omega^*[\{p(t_1, \dots, t_n)\}]$.
- For every $\mathcal{L}$-abstract $\tau \in \mathcal{D}_s$, $P[\tau] = \lambda t \in \mathcal{D}_i.\ \Omega^*[\tau[t]]$.
- For every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$, $v[\varphi, \sigma] = \Omega^*[\sigma[\varphi]]$.

It is easy to verify that conditions (a) and (b) from Definition 5.3 hold. Indeed, Lemma 5.22 ensures that if $y \notin fv[\varphi]$, then for every first-order $\mathcal{L}$-term we have $\sigma_{x:=t}[\varphi] = \sigma_{y:=t}[\varphi\{y/x\}]$. This implies that $v[\varphi, \sigma_{x:=t}] = v[\varphi\{y/x\}, \sigma_{y:=t}]$ for every $t \in \mathcal{D}_i$. Condition (b) holds for a similar reason using the second part of Lemma 5.22.

We show that $\mathcal{Q}$ is not a model of $H_0$. Consider the $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma_{id}$ defined by $\sigma_{id}[x] = x$ for every individual variable $x$ of $\mathcal{L}$, and $\sigma_{id}[X] = X_{abs}$ for every set variable $X$ of $\mathcal{L}$ (see Notation 3.9). Let $\Gamma \Rightarrow E \in H_0$. Since $H_0 \sqsubseteq \Omega^*$, there exists some $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega^*$, such that $\Gamma \Rightarrow E \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$. We claim that $\mathtt{L} \Rightarrow \mathtt{R} \in v^l[\varphi, \sigma_{id}]$ for every $\varphi \in \Gamma$, and $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\varphi, \sigma_{id}]$ for every $\varphi \in E$. To see this, it suffices to note that $\sigma_{id}[\varphi] = \varphi$ for every $\mathcal{L}$-formula $\varphi$. This fact follows from the definition of $\sigma_{id}[\varphi]$. As a consequence, we obtain that $\bigcap_{\varphi \in \Gamma} v^l[\varphi, \sigma_{id}] \not\subseteq \bigcup_{\varphi \in E} v^r[\varphi, \sigma_{id}]$, and so $\sigma_{id} \not\models^{\mathcal{Q}} \Gamma \Rightarrow E$.

It remains to prove that $\mathcal{Q}$ is legal and comprehensive. We first show that it is legal, namely that $\mathcal{Q}[\varphi, \sigma] \subseteq v[\varphi, \sigma]$ for every $\mathcal{L}$-formula $\varphi$ and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$. Let $\varphi$ be an $\mathcal{L}$-formula, and $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. Then, exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some $n$-ary predicate symbol $p$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. By definition, $\mathcal{Q}[\varphi, \sigma] = P[p][\sigma[t_1], \dots, \sigma[t_n]] = \Omega^*[\{p(\sigma[t_1], \dots, \sigma[t_n])\}] = \Omega^*[\sigma[\varphi]] = v[\varphi, \sigma]$.
- $\varphi = \{(t \varepsilon T)\}$ for some first-order $\mathcal{L}$-term $t$, and second-order $\mathcal{L}$-term $T$. By definition, we have $\mathcal{Q}[\varphi, \sigma] = P[\sigma[T]][\sigma[t]] = \Omega^*[\sigma[T][\sigma[t]]] = \Omega^*[\sigma[\varphi]] = v[\varphi, \sigma]$.
- $\varphi = \{\bot\}$. By definition, $\mathcal{Q}[\varphi, \sigma] = \{\emptyset\}$. To see that $\mathcal{Q}[\varphi, \sigma] \subseteq v[\varphi, \sigma]$, it suffices to note that $v^l[\varphi, \sigma] = \emptyset$. This follows from the fact that $\sigma[\varphi] = \{\bot\} \notin \mathtt{L}$ for every $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega^*$. (Otherwise, $\{\{\bot\} \Rightarrow \} \sqsubseteq \Omega^*$, but $\vdash \{\bot\} \Rightarrow$ by applying the rule $(\bot \Rightarrow)$.)
- $\varphi = (\varphi_1 \wedge \varphi_2)$ for some $\varphi_1$ and $\varphi_2$. By definition, $\mathcal{Q}[\varphi, \sigma] = [v^l[\varphi_1, \sigma] \cap v^l[\varphi_2, \sigma], v^r[\varphi_1, \sigma] \cap v^r[\varphi_2, \sigma]]$. We first prove that $v^l[\varphi, \sigma] \subseteq v^l[\varphi_1, \sigma] \cap v^l[\varphi_2, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_1, \sigma]$, and so $\sigma[\varphi_1] \notin \mathtt{L}$. We prove that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi, \sigma]$. (The case that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_2, \sigma]$ is symmetric.) By Proposition 5.16, since $\sigma[\varphi_1] \notin \mathtt{L}$, there exist an $\mathcal{L}$-hypersequent $H_1 \sqsubseteq \Omega^*$, and $\mathcal{L}$-sequents $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$, such that $\vdash H_1 \mid \Gamma, \sigma[\varphi_1] \Rightarrow E_1 \mid \dots \mid \Gamma, \sigma[\varphi_1] \Rightarrow E_n$. The availability of $(\wedge \Rightarrow_1)$ entails that $\vdash H$ for $H = H_1 \mid \Gamma, (\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \Rightarrow E_1 \mid \dots \mid \Gamma, (\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \Rightarrow E_n$. Since $\Omega^*$ is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $(\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \notin \mathtt{L}$. By definition, $(\sigma[\varphi_1] \wedge \sigma[\varphi_2]) = \sigma[\varphi]$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi, \sigma]$.
  Next, we prove that $v^r[\varphi_1, \sigma] \cap v^r[\varphi_2, \sigma] \subseteq v^r[\varphi, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi_1, \sigma] \cap v^r[\varphi_2, \sigma]$. Then we have $\sigma[\varphi_1] \notin \mathtt{R}$ and $\sigma[\varphi_2] \notin \mathtt{R}$. By Proposition 5.16, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$,

and finite sets $\Gamma_1, \Gamma_2 \subseteq \mathtt{L}$, such that $\vdash H_1 \mid \Gamma_1 \Rightarrow \sigma[\varphi_1]$ and $\vdash H_2 \mid \Gamma_2 \Rightarrow \sigma[\varphi_2]$. The availability of $(\Rightarrow\wedge)$ entails that $\vdash H$ for $H = H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow (\sigma[\varphi_1] \wedge \sigma[\varphi_2])$. Since $\Omega^*$ is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\sigma[\varphi] = (\sigma[\varphi_1] \wedge \sigma[\varphi_2]) \notin \mathtt{R}$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi, \sigma]$.

- $\varphi = (\varphi_1 \vee \varphi_2)$ for some $\varphi_1$ and $\varphi_2$. By definition, $\mathcal{Q}[\varphi, \sigma] = [v^l[\varphi_1, \sigma] \cup v^l[\varphi_2, \sigma], v^r[\varphi_1, \sigma] \cup v^r[\varphi_2, \sigma]]$. We first prove that $v^l[\varphi, \sigma] \subseteq v^l[\varphi_1, \sigma] \cup v^l[\varphi_2, \sigma]$. Assume that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_1, \sigma] \cup v^l[\varphi_2, \sigma]$. We prove that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi, \sigma]$. Our assumption entails that $\sigma[\varphi_1] \notin \mathtt{L}$ and $\sigma[\varphi_2] \notin \mathtt{L}$. By Proposition 5.16, there are $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, and $\mathcal{L}$-sequents $\Gamma_1 \Rightarrow E_1, \dots, \Gamma_1 \Rightarrow E_n, \Gamma_2 \Rightarrow F_1, \dots, \Gamma_2 \Rightarrow F_m \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$, such that $\vdash H_1 \mid \Gamma_1, \sigma[\varphi_1] \Rightarrow E_1 \mid \dots \mid \Gamma_1, \sigma[\varphi_1] \Rightarrow E_n$ and $\vdash H_2 \mid \Gamma_2, \sigma[\varphi_2] \Rightarrow F_1 \mid \dots \mid \Gamma_2, \sigma[\varphi_2] \Rightarrow F_m$. Using Proposition 3.16 (note that $(\sigma[\varphi_1] \vee \sigma[\varphi_2]) = \sigma[\varphi]$), we obtain that $\vdash H_1 \mid H_2 \mid H_3 \mid H_4$, where $H_3 = \Gamma_1, \sigma[\varphi] \Rightarrow E_1 \mid \dots \mid \Gamma_1, \sigma[\varphi] \Rightarrow E_n$ and $H_4 = \Gamma_2, \sigma[\varphi] \Rightarrow F_1 \mid \dots \mid \Gamma_2, \sigma[\varphi] \Rightarrow F_m$. Since $\Omega^*$ is unprovable, $H_1 \mid H_2 \mid H_3 \mid H_4 \not\sqsubseteq \Omega^*$, and thus $\sigma[\varphi] \notin \mathtt{L}$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi, \sigma]$.
  Next, we prove that $v^r[\varphi_1, \sigma] \cup v^r[\varphi_2, \sigma] \subseteq v^r[\varphi, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi_1, \sigma]$, and so $\sigma[\varphi_1] \notin \mathtt{R}$. (The case that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi_2, \sigma]$ is symmetric.) By Proposition 5.16, there exist an $\mathcal{L}$-hypersequent $H_1 \sqsubseteq \Omega^*$, and a finite set $\Gamma \subseteq \mathtt{L}$, such that $\vdash H_1 \mid \Gamma \Rightarrow \sigma[\varphi_1]$. The availability of $(\Rightarrow\vee_1)$ entails that $\vdash H$ for $H = H_1 \mid \Gamma \Rightarrow (\sigma[\varphi_1] \vee \sigma[\varphi_2])$. Since $\Omega^*$ is unprovable, $H \not\sqsubseteq \Omega^*$, and thus $\sigma[\varphi] = (\sigma[\varphi_1] \vee \sigma[\varphi_2]) \notin \mathtt{R}$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi, \sigma]$.
- $\varphi = (\varphi_1 \supset \varphi_2)$ for some $\varphi_1$ and $\varphi_2$. Then, $\mathcal{Q}[\varphi, \sigma] = [v^r[\varphi_1, \sigma] \to v^l[\varphi_2, \sigma], v^l[\varphi_1, \sigma] \to v^r[\varphi_2, \sigma]]$. We first prove that $v^l[\varphi, \sigma] \subseteq v^r[\varphi_1, \sigma] \to v^l[\varphi_2, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\varphi_1, \sigma] \to v^l[\varphi_2, \sigma]$. Then, $v^r[\varphi_1, \sigma] \not\subseteq v^l[\varphi_2, \sigma]$ and $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_2, \sigma]$. Let $\mathtt{L}' \Rightarrow \mathtt{R}' \in \Omega^*$ such that $\mathtt{L}' \Rightarrow \mathtt{R}' \in v^r[\varphi_1, \sigma]$, and $\mathtt{L}' \Rightarrow \mathtt{R}' \notin v^l[\varphi_2, \sigma]$. Hence, $\sigma[\varphi_1] \notin \mathtt{R}'$ and $\sigma[\varphi_2] \notin \mathtt{L}'$. By Proposition 5.16, there exist $\mathcal{L}$-hypersequents $H_1, H_2 \sqsubseteq \Omega^*$, a finite set $\Gamma_1 \subseteq \mathtt{L}'$, and $\mathcal{L}$-sequents $\Gamma_2 \Rightarrow E_1, \dots, \Gamma_2 \Rightarrow E_n \sqsubseteq \mathtt{L}' \Rightarrow \mathtt{R}'$, such that $\vdash H_1 \mid \Gamma_1 \Rightarrow \sigma[\varphi_1]$, and $\vdash H_2 \mid \Gamma_2, \sigma[\varphi_2] \Rightarrow E_1 \mid \dots \mid \Gamma_2, \sigma[\varphi_2] \Rightarrow E_n$. By $n$ consecutive applications of $(\supset\Rightarrow)$ (note that $(\sigma[\varphi_1] \supset \sigma[\varphi_2]) = \sigma[\varphi]$), we obtain that
  $$\vdash H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \sigma[\varphi] \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2, \sigma[\varphi] \Rightarrow E_n. \tag{1}$$
  Since $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_2, \sigma]$, we have $\sigma[\varphi_2] \notin \mathtt{L}$. By Proposition 5.16, there exist $\mathcal{L}$-hypersequent $H_3 \sqsubseteq \Omega^*$, and $\mathcal{L}$-sequents $\Gamma_3 \Rightarrow F_1, \dots, \Gamma_3 \Rightarrow F_m \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$, such that $\vdash H_3 \mid \Gamma_3, \sigma[\varphi_2] \Rightarrow F_1 \mid \dots \mid \Gamma_3, \sigma[\varphi_2] \Rightarrow F_m$. By another $m$ applications of $(\supset\Rightarrow)$, we obtain that
  $$\vdash H_1 \mid H_3 \mid \Gamma_1, \Gamma_3, \sigma[\varphi] \Rightarrow F_1 \mid \dots \mid \Gamma_1, \Gamma_3, \sigma[\varphi] \Rightarrow F_m. \tag{2}$$
  By Proposition 3.15, from (1) and (2) above, we have:
  $$\vdash H_1 \mid H_2 \mid H_3 \mid \Gamma_1, \Gamma_2 \Rightarrow E_1 \mid \dots \mid \Gamma_1, \Gamma_2 \Rightarrow E_n \mid \Gamma_3, \sigma[\varphi] \Rightarrow F_1 \mid \dots \mid \Gamma_3, \sigma[\varphi] \Rightarrow F_m.$$
  Now, if $\sigma[\varphi] \in \mathtt{L}$, then $\Omega^*$ extends this hypersequent, and this contradicts $\Omega^*$'s unprovability. Therefore, $\sigma[\varphi] \notin \mathtt{L}$, and consequently $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\sigma[\varphi]]$.
  Next, we prove that $v^l[\varphi_1, \sigma] \to v^r[\varphi_2, \sigma] \subseteq v^r[\varphi, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\varphi, \sigma]$, and so $\sigma[\varphi] \in \mathtt{R}$. To show that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi_1, \sigma] \to v^r[\varphi_2, \sigma]$, we first show that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\varphi_2, \sigma]$ and then we show that $v^l[\varphi_1, \sigma] \not\subseteq v^r[\varphi_2, \sigma]$:
  1. Assume for contradiction that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi_2, \sigma]$, and thus $\sigma[\varphi_2] \notin \mathtt{R}$. Then by Proposition 5.16, there exist an $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega^*$, and a finite set $\Gamma \subseteq \mathtt{L}$, such that $\vdash H \mid \Gamma \Rightarrow \sigma[\varphi_2]$. By applying internal weakening we obtain $\vdash H \mid \Gamma, \sigma[\varphi_1] \Rightarrow \sigma[\varphi_2]$. Using $(\Rightarrow\supset)$ we obtain $\vdash H \mid \Gamma \Rightarrow \sigma[\varphi]$. This contradicts $\Omega^*$'s unprovability (because $H \mid \Gamma \Rightarrow \sigma[\varphi] \sqsubseteq \Omega^*$).
  2. Note that $\Omega^*$'s unprovability and the availability of $(\Rightarrow\supset)$ also entail that $\not\vdash H \mid \sigma[\varphi_1] \Rightarrow \sigma[\varphi_2]$. Therefore, by Proposition 5.17, $\Omega^*$'s external maximality entails that $\sigma[\varphi_1] \Rightarrow \sigma[\varphi_2] \sqsubseteq \Omega^*$. Thus there exists an extended $\mathcal{L}$-sequent $\mathtt{L}' \Rightarrow \mathtt{R}' \in \Omega^*$, such that $\sigma[\varphi_1] \in \mathtt{L}'$ and $\sigma[\varphi_2] \in \mathtt{R}'$. Consequently, $\mathtt{L}' \Rightarrow \mathtt{R}' \in v^l[\varphi_1, \sigma]$ and $\mathtt{L}' \Rightarrow \mathtt{R}' \notin v^r[\varphi_2, \sigma]$. Hence $v^l[\varphi_1, \sigma] \not\subseteq v^r[\varphi_2, \sigma]$.
- $\varphi = (\exists^i x \psi)$ for some individual variable $x \notin fv[\sigma[\varphi]]$ and $\mathcal{L}$-formula $\psi$. By definition, we have $\mathcal{Q}[\varphi, \sigma] = [\sup_{t \in \mathcal{D}_i} v^l[\psi, \sigma_{x:=t}], \sup_{t \in \mathcal{D}_i} v^r[\psi, \sigma_{x:=t}]]$.
  We first prove that $v^l[\varphi, \sigma] \subseteq \sup_{t \in \mathcal{D}_i} v^l[\psi, \sigma_{x:=t}]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \in v^l[\varphi, \sigma]$. Thus $\sigma[\varphi] \in \mathtt{L}$. By definition, $\sigma[\varphi] = (\exists^i x \sigma_{x:=x}[\psi])$. By $\Omega^*$'s witness property, there exists an individual variable $y$

of $\mathcal{L}$, such that $\sigma_{x:=x}[\psi]\{y/x\} \in \mathtt{L}$. By Lemma 5.23, $\sigma_{x:=x}[\psi]\{y/x\} = \sigma_{x:=y}[\psi]$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in v^l[\psi, \sigma_{x:=y}]$, and therefore $\mathtt{L} \Rightarrow \mathtt{R} \in \bigcup_{t\in\mathcal{D}_i} v^l[\psi, \sigma_{x:=t}]$.
Next, we prove that $\sup_{t\in\mathcal{D}_i} v^r[\psi, \sigma_{x:=t}] \subseteq v^r[\varphi, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \in \bigcup_{t\in\mathcal{D}_i} v^r[\psi, \sigma_{x:=t}]$. Thus $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\psi, \sigma_{x:=t}]$ for some $t \in \mathcal{D}_i$. By definition, we have $\sigma_{x:=t}[\psi] \notin \mathtt{R}$. By Lemma 5.23, $\sigma_{x:=t}[\psi] = \sigma_{x:=x}[\psi]\{t/x\}$. By Proposition 5.16, there exist an $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega^*$, and a finite set $\Gamma \subseteq \mathtt{L}$, such that $\vdash H \mid \Gamma \Rightarrow \sigma_{x:=x}[\psi]\{t/x\}$. By applying $(\Rightarrow\exists^i)$, we obtain $\vdash H \mid \Gamma \Rightarrow (\exists^i x \sigma_{x:=x}[\psi])$. Since $\Omega^*$ is unprovable, $(\exists^i x \sigma_{x:=x}[\psi]) \notin \mathtt{R}$. By definition, $(\exists^i x \sigma_{x:=x}[\psi]) = \sigma[\varphi]$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \in v^r[\varphi, \sigma]$.

- $\varphi = (\forall^s X \psi)$ for some set variable $X \notin fv[\sigma[\varphi]]$ and $\mathcal{L}$-formula $\psi$. By definition, in this case we have $\mathcal{Q}[\varphi, \sigma] = [\inf_{\tau\in\mathcal{D}_s} v^l[\psi, \sigma_{X:=\tau}], \inf_{\tau\in\mathcal{D}_s} v^r[\psi, \sigma_{X:=\tau}]]$.
We first prove that $v^l[\varphi, \sigma] \subseteq \inf_{\tau\in\mathcal{D}_s} v^l[\psi, \sigma_{X:=\tau}]$. Thus we show that $v^l[\varphi, \sigma] \subseteq v^l[\psi, \sigma_{X:=\tau}]$ for every $\tau \in \mathcal{D}_s$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\psi, \sigma_{X:=\tau}]$ for some $\tau \in \mathcal{D}_s$. By definition, $\sigma_{X:=\tau}[\psi] \notin \mathtt{L}$. By Lemma 5.23, $\sigma_{X:=\tau}[\psi] = \sigma_{X:=X_{abs}}[\psi]\{\tau/X\}$. By Proposition 5.16, there exist an $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega^*$, and $\mathcal{L}$-sequents $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$, such that
$$\vdash H \mid \Gamma, \sigma_{X:=X_{abs}}[\psi]\{\tau/X\} \Rightarrow E_1 \mid \dots \mid \Gamma, \sigma_{X:=X_{abs}}[\psi]\{\tau/X\} \Rightarrow E_n.$$
By $n$ consecutive applications of $(\forall^s\Rightarrow)$, we obtain
$$\vdash H \mid \Gamma, (\forall^s X \sigma_{X:=X_{abs}}[\psi]) \Rightarrow E_1 \mid \dots \mid \Gamma, (\forall^s X \sigma_{X:=X_{abs}}[\psi]) \Rightarrow E_n.$$
Since $\Omega^*$ is unprovable, $(\forall^s X \sigma_{X:=X_{abs}}[\psi]) \notin \mathtt{L}$. By definition, $(\forall^s X \sigma_{X:=X_{abs}}[\psi]) = \sigma[\varphi]$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^l[\varphi, \sigma]$.
Next, we prove that $\inf_{\tau\in\mathcal{D}_s} v^r[\psi, \sigma_{X:=\tau}] \subseteq v^r[\varphi, \sigma]$. Suppose that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\varphi, \sigma]$. By definition, $\sigma[\varphi] = (\forall^s X \sigma_{X:=X_{abs}}[\psi])$. By $\Omega^*$'s witness property, there exists a set variable $Y$ of $\mathcal{L}$, such that $\sigma_{X:=X_{abs}}[\psi]\{Y/X\} \in \mathtt{R}$. By Lemma 3.10, $\sigma_{X:=X_{abs}}[\psi]\{Y/X\} = \sigma_{X:=X_{abs}}[\psi]\{Y_{abs}/X\}$. By Lemma 5.23, $\sigma_{X:=X_{abs}}[\psi]\{Y_{abs}/X\} = \sigma_{X:=Y_{abs}}[\psi]$. Thus, $\sigma_{X:=Y_{abs}}[\psi] \in \mathtt{R}$. It follows that $\mathtt{L} \Rightarrow \mathtt{R} \notin v^r[\psi, \sigma_{X:=\tau}]$, for $\tau = Y_{abs} \in \mathcal{D}_s$. and therefore $\mathtt{L} \Rightarrow \mathtt{R} \notin \bigcap_{\tau\in\mathcal{D}_s} v^r[\psi, \sigma_{X:=\tau}]$.
- The cases $\varphi = (\forall^i x \psi)$ and $\varphi = (\exists^s X \psi)$ are handled similarly.

Finally, we show that $\mathcal{Q}$ is comprehensive. Let $\varphi$ be an $\mathcal{L}$-formula, $x$ be an individual variable of $\mathcal{L}$, and $\sigma$ be an $\langle\mathcal{L}, \mathcal{D}\rangle$-assignment. Let $y \notin fv[\varphi] \cup fv[\sigma[\varphi]]$ be an individual variable of $\mathcal{L}$, and let $\tau = \{\!\{y \mid \sigma_{y:=y}[\varphi\{y/x\}]\}\!\}$. Then $\tau \in \mathcal{D}_s$. We show that $P[\tau] = \lambda t \in \mathcal{D}_i.\ v[\varphi, \sigma_{x:=t}]$. Let $t \in \mathcal{D}_i$. We have $P[\tau][t] = \Omega^*[\tau[t]] = \Omega^*[\sigma_{y:=y}[\varphi\{y/x\}]\{t/y\}]$. By Lemma 5.22, $\sigma_{y:=y}[\varphi\{y/x\}] = \sigma_{x:=y}[\varphi]$. By Lemma 5.23, $\sigma_{x:=y}[\varphi]\{t/y\} = \sigma_{x:=t}[\varphi]$. Thus, $P[\tau][t] = \Omega^*[\sigma_{x:=t}[\varphi]] = v[\varphi, \sigma_{x:=t}]$. □

## 6. Completeness for the Ordinary Semantics

In this section, we use the complete semantics of quasi-structures to prove the completeness of $\mathbf{HIF}^2$ for the (ordinary) structures of Henkin-style second-order Gödel logic. To do so, we show that from every legal quasi-structure which is a counter-model of some hypersequent $H$, it is possible to extract an (ordinary) structure, which is also not a model of $H$, without losing comprehension.

**Definition 6.1.** Let $\mathcal{D} = \langle\mathcal{D}_i, \mathcal{D}_s, I\rangle$ be a quasi-domain for $\mathcal{L}$, $\mathcal{D}' = \langle\mathcal{D}_i, \mathcal{D}'_s, I'\rangle$ be a domain for $\mathcal{L}$ and $\mathcal{V}$, and $\delta$ be a function from $\mathcal{D}_s$ to $\wp^+[\mathcal{D}'_s]$.[6] A pair $\langle\sigma, \sigma'\rangle$ of an $\langle\mathcal{L}, \mathcal{D}\rangle$-assignment and an $\langle\mathcal{L}, \mathcal{D}'\rangle$-assignment (respectively) is called a *δ-pair* if (*i*) $\sigma[x] = \sigma'[x]$ for every individual variable; and (*ii*) $\sigma'[X] \in \delta[\sigma[X]]$ for every set variable.

**Theorem 6.2.** Let $\mathcal{Q} = \langle\mathcal{V}, \mathcal{D}, P, v\rangle$ be a legal and comprehensive quasi-$\mathcal{L}$-structure, where $\mathcal{D} = \langle\mathcal{D}_i, \mathcal{D}_s, I\rangle$. Then there exists a comprehensive $\mathcal{L}$-structure $\mathcal{U} = \langle\mathcal{V}, \mathcal{D}', P'\rangle$, where $\mathcal{D}' = \langle\mathcal{D}_i, \mathcal{D}'_s, I'\rangle$, and a function $\delta : \mathcal{D}_s \to \wp^+[\mathcal{D}'_s]$, such that $\mathcal{U}[\varphi, \sigma'] \in v[\varphi, \sigma]$ for every $\mathcal{L}$-formula $\varphi$ and $\delta$-pair $\langle\sigma, \sigma'\rangle$ (of an $\langle\mathcal{L}, \mathcal{D}\rangle$-assignment and an $\langle\mathcal{L}, \mathcal{D}'\rangle$-assignment).

[6] $\wp^+[\mathcal{D}'_s]$ denotes the set of all non-empty subsets of $\mathcal{D}'_s$.

PROOF. First, we define $\mathcal{D}'_s$ (the second component in the domain of $\mathcal{U}$). For every $D \in \mathcal{D}_s$, denote by $F_D$ the set of fuzzy subsets $D'$ of $\mathcal{D}_i$ over $\mathcal{V}$ (i.e. $D' : \mathcal{D}_i \to \mathcal{V}$), such that $D'[d] \in P[D][d]$ for every $d \in \mathcal{D}_i$. Note that for every $D \in \mathcal{D}_s$, $F_D$ is non-empty, since $P[D][d]$ is non-empty for every $d \in \mathcal{D}_i$. Define $\mathcal{D}'_s$ to be $\bigcup_{D \in \mathcal{D}_s} F_D$. Next, $I'$ and $P'$ are defined as follows:

- For every individual constant symbol $c$ of $\mathcal{L}$, $I'[c] = I[c]$.
- For every set constant symbol $C$ of $\mathcal{L}$, $I'[C]$ is defined to be an arbitrary element in $F_{I[C]}$.
- For every function symbol $f$ of $\mathcal{L}$, $I'[f] = I[f]$.
- For every predicate symbol $p$ of $\mathcal{L}$, $P'[p] = P[p]^l$.

Let $\delta : \mathcal{D}_s \to \wp^+[\mathcal{D}'_{\mathbf{s}}]$ defined by $\delta = \lambda D \in \mathcal{D}_s.\ F_D$. We prove that $\mathcal{U}$ and $\delta$ satisfy the requirement in the theorem: $\mathcal{U}[\varphi, \sigma'] \in v[\varphi, \sigma]$ for every $\mathcal{L}$-formula $\varphi$ and $\delta$-pair $\langle \sigma, \sigma' \rangle$. Let $\mathcal{V} = \langle V, \leq \rangle$. We use induction on the complexity of $\varphi$. Note that since $\mathcal{Q}$ is legal, it suffices to show that $\mathcal{U}[\varphi, \sigma'] \in \mathcal{Q}[\varphi, \sigma]$ for every $\delta$-pair $\langle \sigma, \sigma' \rangle$.

First, suppose that $cp[\varphi] = 1$, and let $\langle \sigma, \sigma' \rangle$ be a $\delta$-pair of an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment and an $\langle \mathcal{L}, \mathcal{D}' \rangle$-assignment. Exactly one of the following holds:

- $\varphi = \{p(t_1, \dots, t_n)\}$ for some $n$-ary predicate symbol $p$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. By definition, $\mathcal{U}[\varphi, \sigma'] = P'[p][\sigma'[t_1], \dots, \sigma'[t_n]] = P[p]^l[\sigma'[t_1]], \dots, \sigma'[t_n]]]$. Now, since $\sigma$ and $\sigma'$ agree on all individual variables, we have $\sigma'[t] = \sigma[t]$ for every first-order $\mathcal{L}$-term $t$. Hence, we have $\mathcal{U}[\varphi, \sigma'] = P[p]^l[\sigma[t_1]], \dots, \sigma[t_n]]] \in \mathcal{Q}[\varphi, \sigma]$.
- $\varphi = \{(t \varepsilon T)\}$ for some first-order $\mathcal{L}$-term $t$, and second-order $\mathcal{L}$-term $T$. By definition, we have $\mathcal{U}[\varphi, \sigma'] = \sigma'[T][\sigma'[t]]$. As in the previous case, we have $\sigma'[t] = \sigma[t]$. We also have that $\sigma'[T] \in F_{\sigma[T]}$ (in case $T$ is a variable, this holds since $\langle \sigma, \sigma' \rangle$ is a $\delta$-pair, and if $T$ is a constant then it holds by definition). Therefore, $\sigma'[T][\sigma'[t]] = \sigma'[T][\sigma[t]] \in P[\sigma[T]][\sigma[t]] = \mathcal{Q}[\varphi, \sigma]$.
- $\varphi = \{\bot\}$. Then by definition, $\mathcal{U}[\varphi, \sigma'] = 0 \in \{0\} = \mathcal{Q}[\varphi, \sigma]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\langle \sigma, \sigma' \rangle$ be a $\delta$-pair. Exactly one of the following holds:

- $\varphi = (\varphi_1 \wedge \varphi_2)$ for $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. By the induction hypothesis, we have $\mathcal{U}[\varphi_1, \sigma'] \in [v^l[\varphi_1, \sigma], v^r[\varphi_1, \sigma]]$, and $\mathcal{U}[\varphi_2, \sigma'] \in [v^l[\varphi_2, \sigma], v^r[\varphi_2, \sigma]]$. Therefore:

$$\mathcal{U}[\varphi, \sigma'] = \min\{\mathcal{U}[\varphi_1, \sigma'], \mathcal{U}[\varphi_2, \sigma']\} \in [\min\{v^l[\varphi_1, \sigma], v^l[\varphi_2, \sigma]\}, \min\{v^r[\varphi_1, \sigma], v^r[\varphi_2, \sigma]\}] = \mathcal{Q}[\varphi, \sigma].$$

- $\varphi = (\varphi_1 \vee \varphi_2)$ for $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. This case is similar to the previous case (replace min by max).
- $\varphi = (\varphi_1 \supset \varphi_2)$ for $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. By the induction hypothesis, we have $\mathcal{U}[\varphi_1, \sigma'] \in [v^l[\varphi_1, \sigma], v^r[\varphi_1, \sigma]]$, and $\mathcal{U}[\varphi_2, \sigma'] \in [v^l[\varphi_2, \sigma], v^r[\varphi_2, \sigma]]$. Therefore:

$$\mathcal{U}[\varphi, \sigma'] = \mathcal{U}[\varphi_1, \sigma'] \to \mathcal{U}[\varphi_2, \sigma'] \in [v^r[\varphi_1, \sigma] \to v^l[\varphi_2, \sigma], v^l[\varphi_1, \sigma] \to v^r[\varphi_2, \sigma]] = \mathcal{Q}[\varphi, \sigma]$$

(here we use the fact that if $u_1 \leq u' \leq u_2$ and $u_3 \leq u'' \leq u_4$, then $u_2 \to u_3 \leq u' \to u'' \leq u_1 \to u_4$).
- $\varphi = (Q^i x \psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, individual variable $x$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. We continue with $Q^i = \forall^i$ (the proof is similar for $\exists^i$). Clearly, for every $d \in \mathcal{D}_i$, $\langle \sigma_{x:=d}, \sigma'_{x:=d} \rangle$ is a $\delta$-pair. Thus by the induction hypothesis, for every $d \in \mathcal{D}_i$, $\mathcal{U}[\psi, \sigma'_{x:=d}] \in v[\psi, \sigma_{x:=d}]$. Hence,

$$\mathcal{U}[\varphi, \sigma'] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi, \sigma'_{x:=d}] \in [\inf_{d \in \mathcal{D}_i} v^l[\psi, \sigma_{x:=d}], \inf_{d \in \mathcal{D}_i} v^r[\psi, \sigma_{x:=d}]] = \mathcal{Q}[\varphi, \sigma].$$

- $\varphi = (Q^s X \psi)$ for some $Q^s \in \{\forall^i, \exists^i\}$, set variable $X$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. We continue with $Q^s = \forall^s$ (the proof is similar for $\exists^s$). In this case, we should prove that:

$$\inf_{D' \in \mathcal{D}'_s} \mathcal{U}[\psi, \sigma'_{X:=D'}] \in [\inf_{D \in \mathcal{D}_s} v^l[\psi, \sigma_{X:=D}], \inf_{D \in \mathcal{D}_s} v^r[\psi, \sigma_{X:=D}]].$$

– First, we show that $\inf_{D\in\mathcal{D}_s} v^l[\psi,\sigma_{X:=D}] \leq \inf_{D'\in\mathcal{D}'_s} \mathcal{U}[\psi,\sigma'_{X:=D'}]$, by showing that we have $\inf_{D\in\mathcal{D}_s} v^l[\psi,\sigma_{X:=D}] \leq \mathcal{U}[\psi,\sigma'_{X:=D'}]$ for every $D' \in \mathcal{D}'_s$. Let $D' \in \mathcal{D}'_s$, and let $D$ be an arbitrary element in $\mathcal{D}_s$ such that $D' \in F_D$. Then $\langle\sigma_{X:=D},\sigma'_{X:=D'}\rangle$ is a $\delta$-pair. By the induction hypothesis, $v^l[\psi,\sigma_{X:=D}] \leq \mathcal{U}[\psi,\sigma'_{X:=D'}]$. Thus, $\inf_{D\in\mathcal{D}_s} v^l[\psi,\sigma_{X:=D}] \leq \mathcal{U}[\psi,\sigma'_{X:=D'}]$.
– Next, we show that $\inf_{D'\in\mathcal{D}'_s} \mathcal{U}[\psi,\sigma'_{X:=D'}] \leq \inf_{D\in\mathcal{D}_s} v^r[\psi,\sigma_{X:=D}]$, by showing that we have $\inf_{D'\in\mathcal{D}'_s} \mathcal{U}[\psi,\sigma'_{X:=D'}] \leq v^r[\psi,\sigma_{X:=D}]$ for every $D \in \mathcal{D}_s$. Let $D \in \mathcal{D}_s$, and let $D'$ be an arbitrary element in $F_D$. Then $D' \in \mathcal{D}'_s$, and $\langle\sigma_{X:=D},\sigma'_{X:=D'}\rangle$ is a $\delta$-pair. By the induction hypothesis, $\mathcal{U}[\psi,\sigma'_{X:=D'}] \leq v^r[\psi,\sigma_{X:=D}]$. Thus, $\inf_{D'\in\mathcal{D}'_s} \mathcal{U}[\psi,\sigma'_{X:=D'}] \leq v^r[\psi,\sigma_{X:=D}]$.

Finally, we show that $\mathcal{U}$ is comprehensive. Let $\varphi$ be an $\mathcal{L}$-formula, $x$ be an individual variable, and $\sigma'$ be an $\langle\mathcal{L},\mathcal{D}'\rangle$-assignment. We show that $\mathcal{U}[\varphi,\sigma',x] \in \mathcal{D}'_s$. Define an $\langle\mathcal{L},\mathcal{D}\rangle$-assignment $\sigma$ as follows: $(i)$ $\sigma[x] = \sigma'[x]$ for every individual variable $x$; and $(ii)$ for every set variable $X$, $\sigma[X]$ is an (arbitrary) element of $\mathcal{D}_s$ such that $\sigma'[X] \in F_{\sigma[X]}$. Since $\mathcal{Q}$ is comprehensive, there exists some $D \in \mathcal{D}_s$ such that $P[D] = \lambda d \in \mathcal{D}_i.\ v[\varphi,\sigma_{x:=d}]$. We claim that $\mathcal{U}[\varphi,\sigma',x] \in F_D$ (and so, $\mathcal{U}[\varphi,\sigma',x] \in \mathcal{D}'_s$). By definition, we should show that $\mathcal{U}[\varphi,\sigma',x][d] \in P[D][d]$ for every $d \in \mathcal{D}_i$. Let $d \in \mathcal{D}_i$. Obviously, $\langle\sigma_{x:=d},\sigma'_{x:=d}\rangle$ is a $\delta$-pair, and thus by the claim proved above, we have $\mathcal{U}[\varphi,\sigma',x][d] = \mathcal{U}[\varphi,\sigma'_{x:=d}] \in v[\varphi,\sigma_{x:=d}] = P[D][d]$. □

**Corollary 6.3.** If $\nvdash H$, then there exists a comprehensive $\mathcal{L}$-structure which is not a model of $H$.

Proof. Suppose that $\nvdash H$. Then, by Theorem 5.10, there exists a legal and comprehensive quasi-$\mathcal{L}$-structure $\mathcal{Q} = \langle\mathcal{V},\mathcal{D},P,v\rangle$, which is not a model of $H$. This implies that there exists an $\langle\mathcal{L},\mathcal{D}\rangle$-assignment $\sigma$, such that $\sigma \not\models^{\mathcal{Q}} \Gamma\Rightarrow E$ for every $\Gamma\Rightarrow E \in H$. Let $\mathcal{U} = \langle\mathcal{V},\mathcal{D}',P'\rangle$ be a comprehensive $\mathcal{L}$-structure and $\delta$ be a function, satisfying the requirement in Theorem 6.2. Let $\sigma'$ be an $\langle\mathcal{L},\mathcal{D}'\rangle$-assignment such that $\langle\sigma,\sigma'\rangle$ is a $\delta$-pair (there exists such an assignment since the range of $\delta$ does not include the empty set). We show that $\sigma' \not\models^{\mathcal{U}} H$. Let $\Gamma\Rightarrow E \in H$. Since $\sigma \not\models^{\mathcal{Q}} \Gamma\Rightarrow E$, we have $\min_{\varphi\in\Gamma} v^l[\varphi,\sigma] > \max_{\varphi\in E} v^r[\varphi,\sigma]$. The fact that $\mathcal{U}[\varphi,\sigma'] \in v[\varphi,\sigma]$ for every $\varphi$ entails that $\min_{\varphi\in\Gamma}\mathcal{U}[\varphi,\sigma'] > \max_{\varphi\in E}\mathcal{U}[\varphi,\sigma']$, and so $\sigma' \not\models^{\mathcal{U}} \Gamma\Rightarrow E$. □

**Corollary 6.4.** For every $\mathcal{L}$-formula $\varphi$, if $\vdash^{\mathbf{G}^2_{\mathcal{L}}} \varphi$ then $\vdash \Rightarrow\varphi$.

Finally, recall that $\mathbf{HIF}^2$ is cut-free, so we automatically obtain the admissibility of the cut rule:

**Corollary 6.5.** Let $\mathbf{HIF}^2_c$ be the extension of **HIF** with the rule:

$$(cut)\quad \frac{H \mid \Gamma\Rightarrow\varphi \quad H \mid \Gamma,\varphi\Rightarrow E}{H \mid \Gamma\Rightarrow E}$$

Then, an $\mathcal{L}$-hypersequent $H$ is provable in $\mathbf{HIF}^2_c$ iff it is provable in $\mathbf{HIF}^2$.

Proof. The left to right direction is obvious. For the converse, we first prove that applications of $(cut)$ are sound. Indeed, suppose that $H \mid \Gamma\Rightarrow E$ is derived from $H \mid \Gamma\Rightarrow\varphi$ and $H \mid \Gamma,\varphi\Rightarrow E$ using $(cut)$. Let $\mathcal{U} = \langle\mathcal{V},\mathcal{D},P\rangle$ be an $\mathcal{L}$-structure, and $\sigma$ be an $\langle\mathcal{L},\mathcal{D}\rangle$-assignment. If $\sigma \models^{\mathcal{U}} s$ for some component $s \in H$, then we are done. Otherwise, $\sigma \models^{\mathcal{U}} \Gamma\Rightarrow\varphi$ and $\sigma \models^{\mathcal{U}} \Gamma,\varphi\Rightarrow E$. Hence, $\min_{\psi\in\Gamma}\mathcal{U}[\psi,\sigma] \leq \mathcal{U}[\varphi,\sigma]$ and $\min_{\psi\in\Gamma\cup\{\varphi\}}\mathcal{U}[\psi,\sigma] \leq \max_{\psi\in E}\mathcal{U}[\psi,\sigma]$. Thus $\min_{\psi\in\Gamma}\mathcal{U}[\psi,\sigma] \leq \max_{\psi\in E}\mathcal{U}[\psi,\sigma]$, and so $\sigma \models^{\mathcal{U}} \Gamma\Rightarrow E$. Consequently, $\sigma \models^{\mathcal{U}} H \mid \Gamma\Rightarrow E$. Now, it follows that if an $\mathcal{L}$-hypersequent $H$ is provable in $\mathbf{HIF}^2_c$, then every comprehensive $\mathcal{L}$-structure is a model of $H$. By Corollary 6.3, this implies that $\vdash H$. □

*Remark* 6.6. While we allowed any Gödel set to serve as the set of truth values in $\mathcal{L}$-structures, we could equivalently take the real interval $[0,1]$. Obviously, soundness for $[0,1]$ is a particular instance. Completeness for $[0,1]$ can be obtained by embedding $\langle V_0,\subseteq\rangle$ in the proof of Theorem 5.10 into the rational numbers in $[0,1]$ (note that $\langle V_0,\subseteq\rangle$ is countable). The result $\langle V,\subseteq\rangle$ of the completion would then be the real interval $[0,1]$ with its standard order. Finally, the extracted ordinary counter-model employs the same Gödel set.

## 7. Further Work

The properties proved in [16] for **HIF** are slightly stronger than those shown in this paper for $\mathbf{HIF}^2$. Obtaining these stronger results for $\mathbf{HIF}^2$ seems to be straightforward. This includes:

1. In [16] we considered derivations from (possibly) non-empty sets of hypersequents, serving as assumptions. In this case the cut rule must be added to the calculus, and obviously one cannot have full cut-admissibility. However, it was (semantically) proved that in **HIF**-proofs cuts can be confined to formulas appearing in the set of assumptions (this property is called *strong cut-admissibility*).
2. For applications, it is sometimes useful to enrich Gödel logic with a globalization connective (also known as Baaz Delta connective, see [19]). [16] studies the extension of **HIF** with rules for this connective, and the same can be done for $\mathbf{HIF}^2$.

In addition, the following extensions of the current result are left for a future work:

1. It is interesting to consider equality, both between first-order terms and second-order ones. In this case, rules for extensionality should be added.
2. Extending the calculus for richer second order signatures, that include arbitrary predicate symbols that take sets as arguments, as well as quantification over $n$-ary predicates seem to be possible. Additionally, we believe that our approach can be straightforwardly generalized to handle full type theory. In the case of classical logic, cut-free completeness for the extended system was proved shortly after Tait's proof for the second-order one by Takahashi and Prawitz, [20, 21]. This extension is necessary in order to obtain a proof system for (the Gödel fragment) of fuzzy set theory (see [12]).

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## Appendix A. Some Proofs

In this appendix we provide the proofs for some technical lemmas that appear above. The following lemmas will be useful:

**Lemma Appendix A.1.** Let $\mathcal{U} = \langle \mathcal{V}, \mathcal{D}, P \rangle$ be an $\mathcal{L}$-structure, where $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$.

1. Let $x$ be an individual variable of $\mathcal{L}$ and let $d \in \mathcal{D}_i$. For every $\mathcal{L}$-formula $\varphi$ such that $x \notin fv[\varphi]$, and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$: $\mathcal{U}[\varphi, \sigma_{x:=d}] = \mathcal{U}[\varphi, \sigma]$.
2. Let $X$ be a set variable of $\mathcal{L}$ and let $D \in \mathcal{D}_s$. For every $\mathcal{L}$-formula $\varphi$ such that $X \notin fv[\varphi]$, and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$: $\mathcal{U}[\varphi, \sigma_{X:=D}] = \mathcal{U}[\varphi, \sigma]$.

Proof. After proving the claim for first-order $\mathcal{L}$-terms (that $\sigma[t] = \sigma_{x:=d}[t]$ for every first-order $\mathcal{L}$-term such that $x \notin fv[t]$, and $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment $\sigma$), the claim is obtained by usual induction on the complexity of $\varphi$. The second item is similar. □

**Lemma Appendix A.2.** Let $\tau$ be an $\mathcal{L}$-abstract, $t$ and $t'$ be first-order $\mathcal{L}$-terms, and $x$ be an individual variable such that $x \notin fv[\tau]$. Then, $\tau[t']\{t/x\} = \tau[t'\{t/x\}]$.

Proof. It is straightforward to prove that $\varphi\{t'/y\}\{t/x\} = \varphi\{t'\{t/x\}/y\}$ for every $\mathcal{L}$-formula $\varphi$, first-order $\mathcal{L}$-terms $t$ and $t'$, and individual variables $x$ and $y$, such that $x \notin fv[\varphi]$. The claim then easily follows from our definitions. □

In addition, to prove Lemma 4.4, we use the following lemma:

**Lemma Appendix A.3.** Let $\mathcal{D}$ be a domain for $\mathcal{L}$ and $\mathcal{V}$, $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment, $t$ be a first-order $\mathcal{L}$-term, and $x$ be an individual variable of $\mathcal{L}$. For every first-order $\mathcal{L}$-term $t'$: $\sigma[t'\{t/x\}] = \sigma_{x:=\sigma[t]}[t']$.

Proof. By usual induction on the structure of $t'$. □

Proof (Lemma 4.4). Suppose that $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I \rangle$. We prove the claim by induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$, and let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = \{\perp\}$. In this case the claim obviously holds.
- $\varphi = \{p(t_1, \dots, t_n)\}$. In this case, $\varphi\{{}^t/_x\} = \{p(t_1\{{}^t/_x\}, \dots, t_n\{{}^t/_x\})\}$. Thus $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma]$ is equal to $P[p][\sigma[t_1\{{}^t/_x\}], \dots, \sigma[t_n\{{}^t/_x\}]]$. By Lemma Appendix A.3,

$$P[p][\sigma[t_1\{{}^t/_x\}], \dots, \sigma[t_n\{{}^t/_x\}] = P[p][\sigma_{x:=\sigma[t]}[t_1], \dots, \sigma_{x:=\sigma[t]}[t_n]].$$

  By definition, this is equal to $\mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}]$.
- $\varphi = \{(t'\varepsilon T)\}$. In this case, $\varphi\{{}^t/_x\} = \{(t'\{{}^t/_x\}\varepsilon T)\}$. Thus $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma] = \sigma[T][\sigma[t'\{{}^t/_x\}]]$. By Lemma Appendix A.3, $\sigma[t'\{{}^t/_x\}] = \sigma_{x:=\sigma[t]}[t']$. Clearly, $\sigma[T] = \sigma_{x:=\sigma[t]}[T]$. Hence, $\sigma[T][\sigma[t'\{{}^t/_x\}]] = \sigma_{x:=\sigma[t]}[T][\sigma_{x:=\sigma[t]}[t']]$. By definition, this is equal to $\mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. By definition, $\varphi\{{}^t/_x\} = (\varphi_1\{{}^t/_x\} \diamond \varphi_2\{{}^t/_x\})$. We continue with $\diamond = \supset$ (the proof is similar for $\wedge$ and $\vee$). Thus, $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma] = \mathcal{U}[\varphi_1\{{}^t/_x\}, \sigma] \to \mathcal{U}[\varphi_2\{{}^t/_x\}, \sigma]$. By the induction hypothesis, $\mathcal{U}[\varphi_1\{{}^t/_x\}, \sigma] \to \mathcal{U}[\varphi_2\{{}^t/_x\}, \sigma] = \mathcal{U}[\varphi_1, \sigma_{x:=\sigma[t]}] \to \mathcal{U}[\varphi_2, \sigma_{x:=\sigma[t]}]$. By definition, this is equal to $\mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}]$.
- $\varphi = (Q^i y \psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, individual variable $y \notin \{x\} \cup fv[t]$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. By definition, $\varphi\{{}^t/_x\} = (Q^i y \psi\{{}^t/_x\})$. We continue with $Q^i = \forall^i$ (the proof is similar for $\exists^i$). Thus, $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi\{{}^t/_x\}, \sigma_{y:=d}]$. By the induction hypothesis, $\mathcal{U}[\psi\{{}^t/_x\}, \sigma_{y:=d}] = \mathcal{U}[\psi, \sigma_{y:=d, x:=\sigma[t]}]$ for every $d \in \mathcal{D}_i$ (note that $y \neq x$), and so $\mathcal{U}[\varphi\{{}^t/_x\}] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi, \sigma_{y:=d, x:=\sigma[t]}]$. By definition, $\inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi, \sigma_{y:=d, x:=\sigma[t]}] = \mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}]$.
- $\varphi = (Q^s X \psi)$ for some $Q^s \in \{\forall^s, \exists^s\}$, set variable $X$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. By definition, $\varphi\{{}^t/_x\} = (Q^s X \psi\{{}^t/_x\})$. We continue with $Q^s = \forall^s$ (the proof is similar for $\exists^s$). Thus, $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{U}[\psi\{{}^t/_x\}, \sigma_{X:=D}]$. By the induction hypothesis, $\mathcal{U}[\psi\{{}^t/_x\}, \sigma_{X:=D}] = \mathcal{U}[\psi, \sigma_{X:=D, x:=\sigma[t]}]$ for every $D \in \mathcal{D}_s$, and so $\mathcal{U}[\varphi\{{}^t/_x\}, \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{U}[\psi, \sigma_{X:=D, x:=\sigma[t]}]$. The claim follows, since by definition $\inf_{D \in \mathcal{D}_s} \mathcal{U}[\psi, \sigma_{X:=D, x:=\sigma[t]}] = \mathcal{U}[\varphi, \sigma_{x:=\sigma[t]}]$. □

Proof (Lemma 4.5). If $X \notin fv[\varphi]$, then $\varphi\{{}^\tau/_X\} = \varphi$ and the claim follows by Lemma Appendix A.1. Suppose otherwise. We prove the claim by induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment, and let $D_0 = \mathcal{U}[\tau[x], \sigma, x]$. Suppose that $D_0 \in \mathcal{D}_s$. Since $X \in fv[\varphi]$, we must have $\varphi = \{(t \varepsilon X)\}$ for some first-order $\mathcal{L}$-term $t$. In this case, $\varphi\{{}^\tau/_X\} = \tau[t]$. Then, by Lemma Appendix A.2, $\tau[t] = \tau[x\{{}^t/_x\}] = \tau[x]\{{}^t/_x\}$. Thus $\mathcal{U}[\varphi\{{}^\tau/_X\}, \sigma] = \mathcal{U}[\tau[x]\{{}^t/_x\}, \sigma]$. By Lemma 4.4, $\mathcal{U}[\tau[x]\{{}^t/_x\}, \sigma] = \mathcal{U}[\tau[x], \sigma_{x:=\sigma[t]}]$. Now, by definition, $\mathcal{U}[\tau[x], \sigma_{x:=\sigma[t]}] = D_0[\sigma[t]] = \mathcal{U}[\varphi, \sigma_{X:=D_0}]$.

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment, and again let $D_0 = \mathcal{U}[\tau[x], \sigma, x]$. Suppose that $D_0 \in \mathcal{D}_s$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for some $\diamond \in \{\wedge, \vee, \supset\}$, and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. By definition, $\varphi\{{}^\tau/_X\} = (\varphi_1\{{}^\tau/_X\} \diamond \varphi_2\{{}^\tau/_X\})$. We continue with $\diamond = \supset$ (the proof is similar for $\wedge$ and $\vee$). Thus, $\mathcal{U}[\varphi\{{}^\tau/_X\}, \sigma] = \mathcal{U}[\varphi_1\{{}^\tau/_X\}, \sigma] \to [\varphi_2\{{}^\tau/_X\}, \sigma]$. By the induction hypothesis (and the case in which $X \notin fv[\varphi]$), $\mathcal{U}[\varphi_1\{{}^\tau/_X\}, \sigma] = \mathcal{U}[\varphi_1, \sigma_{X:=D_0}]$ and $\mathcal{U}[\varphi_2\{{}^\tau/_X\}, \sigma] = \mathcal{U}[\varphi_2, \sigma_{X:=D_0}]$. By definition, $\mathcal{U}[\varphi_1, \sigma_{X:=D_0}] \to \mathcal{U}[\varphi_2, \sigma_{X:=D_0}] = \mathcal{U}[\varphi, \sigma_{X:=D_0}]$.
- $\varphi = (Q^i y \psi)$ for some $Q^i \in \{\forall^i, \exists^i\}$, individual variable $y \notin \{x\} \cup fv[\tau]$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. By definition, $\varphi\{{}^\tau/_X\} = (Q^i y \psi\{{}^\tau/_X\})$. We continue with $Q^i = \forall^i$ (the proof is similar for $\exists^i$). Thus, $\mathcal{U}[\varphi\{{}^\tau/_X\}, \sigma] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi\{{}^\tau/_X\}, \sigma_{y:=d}]$. Now, using Lemma Appendix A.1, we have $D_0 = \mathcal{U}[\tau[x], \sigma_{y:=d}, x]$ for every $d \in \mathcal{D}_i$ (since $y \notin fv[\tau[x]]$). Therefore, by the induction hypothesis, $\inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi\{{}^\tau/_X\}, \sigma_{y:=d}] = \inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi, \sigma_{y:=d, X:=D_0}]$. By definition, $\inf_{d \in \mathcal{D}_i} \mathcal{U}[\psi, \sigma_{y:=d, X:=D_0}] = \mathcal{U}[\varphi, \sigma_{X:=D_0}]$.
- $\varphi = (Q^s Y \psi)$ for some $Q^s \in \{\forall^s, \exists^s\}$, set variable $Y \notin \{X\} \cup fv[\tau]$ of $\mathcal{L}$, and $\mathcal{L}$-formula $\psi$ of lower complexity. By definition, $\varphi\{{}^\tau/_X\} = (Q^s Y \psi\{{}^\tau/_X\})$. We continue with $Q^s = \forall^s$ (the proof is similar for $\exists^s$). Thus, $\mathcal{U}[\varphi\{{}^\tau/_X\}, \sigma] = \inf_{D \in \mathcal{D}_s} \mathcal{U}[\psi\{{}^\tau/_X\}, \sigma_{Y:=D}]$. Now, using Lemma Appendix A.1, we

have $D_0 = \mathcal{U}[\tau[x], \sigma_{Y:=D}, x]$ for every $D \in \mathcal{D}_s$ (since $Y \notin fv[\tau[x]]$). Therefore, by the induction hypothesis, $\inf_{D\in\mathcal{D}_s} \mathcal{U}[\psi\{\tau/X\}, \sigma_{Y:=D}] = \inf_{D\in\mathcal{D}_s} \mathcal{U}[\psi, \sigma_{Y:=D,X:=D_0}]$ (note that $Y \neq X$). By definition, $\inf_{D\in\mathcal{D}_s} \mathcal{U}[\psi, \sigma_{Y:=D,X:=D_0}] = \mathcal{U}[\varphi, \sigma_{X:=D_0}]$. □

Proof (Lemma 5.22). Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I\rangle$. First, we show that for every first-order $\mathcal{L}$-terms $t'$ and $t$, $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment $\sigma$, and individual variables $x, y$ such that $y \notin fv[t']$, $\sigma_{x:=t}[t'] = \sigma_{y:=t}[t'\{y/x\}]$. This claim is proved by induction on the structure of $t'$:

- Suppose that $t' = c$ for some individual constant symbol $c$ of $\mathcal{L}$, or $t' = z$ for some individual variable $z \notin \{x, y\}$. Then $t'\{y/x\} = t'$, and $\sigma_{x:=t}[t'] = \sigma_{y:=t}[t']$.
- Suppose that $t' = x$. Then $\sigma_{x:=t}[t'] = t$, and $\sigma_{y:=t}[t'\{y/x\}] = \sigma_{y:=t}[y] = t$.
- Suppose that $t' = f(t_1, \dots, t_n)$ for some $n$-ary function symbol $f$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. Then, $\sigma_{x:=t}[t'] = f(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])$. By the induction hypotheses this term equals $f(\sigma_{y:=t}[t_1\{y/x\}], \dots, \sigma_{y:=t}[t_n\{y/x\}])$, which in turn equals $\sigma_{y:=t}[f(t_1, \dots, t_n)\{y/x\}]$.

Next, we prove the claims in Lemma 5.22:

1. We use induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $x, y$ be individual variables such that $y \notin fv[\varphi]$. Exactly one of the following holds:
   - $\varphi = \{p(t_1, \dots, t_n)\}$ for some $n$-ary predicate symbol $p$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. Then, by definition $\sigma_{x:=t}[\varphi] = \{p(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])\}$. Since $y \notin fv[t_i]$ for every $1 \leq i \leq n$, this formula equals $\{p(\sigma_{y:=t}[t_1\{y/x\}], \dots, \sigma_{y:=t}[t_n\{y/x\}])\}$, which is, by definition, $\sigma_{y:=t}[\varphi\{y/x\}]$.
   - $\varphi = \{(t'\varepsilon T)\}$ for some first-order $\mathcal{L}$-term $t'$, and second-order $\mathcal{L}$-term $T$. Then, $\sigma_{x:=t}[\varphi] = \sigma_{x:=t}[T][\sigma_{x:=t}[t']]$. Since $y \notin fv[t']$, this formula equals $\sigma_{x:=t}[T][\sigma_{y:=t}[t'\{y/x\}]]$. Since $x$ and $y$ does not occur in $T$, this is equal to $\sigma_{y:=t}[T][\sigma_{y:=t}[t'\{y/x\}]]$. By definition, this formula is equal to $\sigma_{y:=t}[\{(t'\{y/x\}\varepsilon T)\}]$, which is $\sigma_{y:=t}[\{(t'\varepsilon T)\}\{y/x\}]$.
   - $\varphi = \{\bot\}$. Then, $\sigma_{x:=t}[\varphi] = \{\bot\} = \sigma_{y:=t}[\varphi\{y/x\}]$.

   Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $x, y$ be individual variables such that $y \notin fv[\varphi]$. Exactly one of the following holds:
   - $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. Then, $\sigma_{x:=t}[\varphi] = (\sigma_{x:=t}[\varphi_1] \diamond \sigma_{x:=t}[\varphi_2])$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(\sigma_{y:=t}[\varphi_1\{y/x\}] \diamond \sigma_{y:=t}[\varphi_2\{y/x\}])$. And, by definition, this is equal to $\sigma_{y:=t}[\varphi\{y/x\}]$.
   - $\varphi = (Q^i z\psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and individual variable $z$ of $\mathcal{L}$ such that $z \notin \{x, y\} \cup \sigma[\varphi] \cup fv[t]$. Then, $\sigma_{x:=t}[\varphi] = (Q^i z\sigma_{x:=t,z:=z}[\psi])$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^i z\sigma_{y:=t,z:=z}[\psi\{y/x\}])$. And this is (by definition) equal to $\sigma_{y:=t}[(Q^i z\psi\{y/x\})]$, which is equal to $\sigma_{y:=t}[\varphi\{y/x\}]$.
   - $\varphi = (Q^s X\psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and set variable $X$ of $\mathcal{L}$ such that $X \notin fv[\sigma[\varphi]]$. Then, $\sigma_{x:=t}[\varphi] = (Q^s X\sigma_{x:=t,X:=X_{abs}}[\psi])$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^s X\sigma_{y:=t,X:=X_{abs}}[\psi\{y/x\}])$. And this is (by definition) equal to $\sigma_{y:=t}[(Q^s X\psi\{y/x\})]$, which in turn equals $\sigma_{y:=t}[(\varphi\{y/x\}]$.
2. If $X \notin fv[\varphi]$, then $\sigma_{X:=\tau}[\varphi] = \sigma[\varphi] = \sigma_{Y:=\tau}[\varphi\{Y/X\}]$. Suppose now that $X \in fv[\varphi]$. We use induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $X, Y$ be set variables such that $Y \notin fv[\varphi]$. Assume that $X \in fv[\varphi]$. Thus we have $\varphi = \{(t\varepsilon X)\}$ for some first-order $\mathcal{L}$-term $t$. Then, $\sigma_{X:=\tau}[\varphi] = \tau[\sigma_{X:=\tau}[t]]$. Since $X$ and $Y$ do not occur in $t$, this is equal to $\tau[\sigma_{Y:=\tau}[t]]$, which in turn equals $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$. Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $X, Y$ be set variables such that $Y \notin fv[\varphi]$. Exactly one of the following holds:
   - $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. Then, $\sigma_{X:=\tau}[\varphi] = (\sigma_{X:=\tau}[\varphi_1] \diamond \sigma_{X:=\tau}[\varphi_2])$. By the induction hypothesis (and the case in which $X \notin fv[\varphi]$), this $\mathcal{L}$-formula is equal to $(\sigma_{Y:=\tau}[\varphi_1\{Y/X\}] \diamond \sigma_{Y:=\tau}[\varphi_2\{Y/X\}])$. And, by definition, this is equal to $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$.

- $\varphi = (Q^i x\psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and individual variable $x$ of $\mathcal{L}$ such that $x \notin fv[\tau] \cup fv[\sigma[\varphi]]$. Then, $\sigma_{X:=\tau}[\varphi] = (Q^i x\sigma_{X:=\tau,x:=x}[\psi])$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^i x\sigma_{Y:=\tau,x:=x}[\psi\{Y/X\}])$. And this is (by definition) equal to $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$.
- $\varphi = (Q^s Z\psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and set variable $Z$ of $\mathcal{L}$ such that $Z \notin fv[\tau] \cup \{X, Y\} \cup fv[\sigma[\varphi]]$. Then, $\sigma_{X:=\tau}[\varphi] = (Q^s Z\sigma_{X:=\tau,Z:=Z_{abs}}[\psi])$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^s Z\sigma_{Y:=\tau,Z:=Z_{abs}}[\psi\{Y/X\}])$. And this is (by definition) equal to $\sigma_{Y:=\tau}[\varphi\{Y/X\}]$. □

Proof (Lemma 5.23). Let $\mathcal{D} = \langle \mathcal{D}_i, \mathcal{D}_s, I\rangle$. First, we show that for every first-order $\mathcal{L}$-terms $t'$ and $t$, $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment $\sigma$, and individual variables $x, z$ such that $z \notin fv[\sigma[t']]$, $\sigma_{x:=z}[t']\{t/z\} = \sigma_{x:=t}[t']$. This claim is proved by induction on the structure of $t'$:

- Suppose that $t' = c$ for some individual constant symbol $c$ of $\mathcal{L}$. Then:
$$\sigma_{x:=z}[t']\{t/z\} = I[c]\{t/z\} = c\{t/z\} = c = I[c] = \sigma_{x:=t}[t'].$$
- Suppose that $t' = y$ for some individual variable $y \neq x$. Then $\sigma_{x:=z}[t']\{t/z\} = \sigma[y]\{t/z\}$. Since $z \notin fv[\sigma[y]]$, we have $\sigma[y]\{t/z\} = \sigma[y]$. The claim then follows since $\sigma[y] = \sigma_{x:=t}[y]$.
- Suppose that $t' = x$. Then, $\sigma_{x:=z}[t']\{t/z\} = z\{t/z\} = t = \sigma_{x:=t}[x]$.
- Suppose that $t' = f(t_1, \dots, t_n)$ for some $n$-ary function symbol $f$ of $\mathcal{L}$, and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. Then, $\sigma_{x:=z}[t']\{t/z\} = f(\sigma_{x:=z}[t_1], \dots, \sigma_{x:=z}[t_n])\{t/z\} = f(\sigma_{x:=z}[t_1]\{t/z\}, \dots, \sigma_{x:=z}[t_n]\{t/z\})$. By the induction hypotheses this term equals $f(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])$, which in turn equals $\sigma_{x:=t}[t']$.

Next, we prove the claims in Lemma 5.23:

1. We use induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $x, z$ be individual variables such that $z \notin fv[\sigma[\varphi]]$. Exactly one of the following holds:
   - $\varphi = \{p(t_1, \dots, t_n)\}$ for some $n$-ary predicate symbol $p$ of $\mathcal{L}$ and first-order $\mathcal{L}$-terms $t_1, \dots, t_n$. Then, $\sigma_{x:=z}[\varphi]\{t/z\} = \{p(\sigma_{x:=z}[t_1], \dots, \sigma_{x:=z}[t_n])\}\{t/z\} = \{p(\sigma_{x:=z}[t_1]\{t/z\}, \dots, \sigma_{x:=z}[t_n]\{t/z\})\}$. Since $z \notin fv[\sigma[t_i]]$ for every $1 \leq i \leq n$, the claim above for terms entails that this formula equals $\{p(\sigma_{x:=t}[t_1], \dots, \sigma_{x:=t}[t_n])\}$, which is, by definition, $\sigma_{x:=t}[\varphi]$.
   - $\varphi = \{(t'\varepsilon T)\}$ for some first-order $\mathcal{L}$-term $t'$ and second-order $\mathcal{L}$-term $T$. In this case, we have $\sigma_{x:=z}[\varphi]\{t/z\} = \sigma_{x:=z}[T][\sigma_{x:=z}[t']]\{t/z\} = \sigma[T][\sigma_{x:=z}[t']]\{t/z\}$. By Lemma Appendix A.2, since $z \notin fv[\sigma[T]]$, this formula equals $\sigma[T][\sigma_{x:=z}[t']\{t/z\}]$. Since $z \notin fv[\sigma[t']]$, the proof above for terms entails that this formula equals $\sigma[T][\sigma_{x:=t}[t']]$. Since $x$ does not occur in $T$, this is equal to $\sigma_{x:=t}[T][\sigma_{x:=t}[t']]$, which is, by definition, $\sigma_{x:=t}[\varphi]$.
   - $\varphi = \{\bot\}$. Then, $\sigma_{x:=z}[\varphi]\{t/z\} = \{\bot\} = \sigma_{x:=t}[\varphi]$.

   Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D}\rangle$-assignment, and let $x, z$ be individual variables such that $z \notin fv[\sigma[\varphi]]$. Exactly one of the following holds:

   - $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. Then, we have $\sigma_{x:=x}[\varphi]\{t/x\} = (\sigma_{x:=z}[\varphi_1]\{t/z\} \diamond \sigma_{x:=z}[\varphi_2]\{t/z\})$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(\sigma_{x:=t}[\varphi_1] \diamond \sigma_{x:=t}[\varphi_2])$. And, by definition, this is equal to $\sigma_{x:=t}[\varphi]$.
   - $\varphi = (Q^i y\psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and individual variable $y$ of $\mathcal{L}$ such that $y \notin fv[t] \cup \{x, z\} \cup fv[\sigma[\varphi]]$. Then, $\sigma_{x:=z}[\varphi]\{t/z\} = (Q^i y\sigma_{x:=z,y:=y}[\psi])\{t/z\} = (Q^i y\sigma_{x:=z,y:=y}[\psi]\{t/z\})$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^i y\sigma_{x:=t,y:=y}[\psi])$. And this is (by definition) equal to $\sigma_{x:=t}[\varphi]$.
   - $\varphi = (Q^s X\psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and set variable $X$ of $\mathcal{L}$ such that $X \notin fv[\sigma[\varphi]]$. Then, $\sigma_{x:=z}[\varphi]\{t/z\} = (Q^s X\sigma_{x:=z,X:=X_{abs}}[\psi])\{t/z\} = (Q^i X\sigma_{x:=z,X:=X_{abs}}[\psi]\{t/z\})$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $(Q^i y\sigma_{x:=t,X:=X_{abs}}[\psi])$. And this is (by definition) equal to $\sigma_{x:=t}[\varphi]$.

2. If $X \notin fv[\varphi]$, then $\sigma_{X:=X_{abs}}[\varphi] = \sigma[\varphi] = \sigma_{X:=\tau}[\varphi]$. Since $X \notin fv[\sigma[\varphi]]$, we also have $\sigma[\varphi]\{\tau/X\} = \sigma[\varphi]$ as well. Suppose now that $X \in fv[\varphi]$. We use induction on the complexity of $\varphi$. First, suppose that $cp[\varphi] = 1$. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment, and let $X \notin fv[\sigma[\varphi]]$. Since $X \in fv[\varphi]$, we must have $\varphi = \{(t \varepsilon X)\}$ for some first-order $\mathcal{L}$-term $t$. Then,

$$\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = X_{abs}[\sigma[t]]\{\tau/X\} = \{(\sigma[t]\varepsilon X)\}\{\tau/X\} = \tau[\sigma[t]] = \sigma_{X:=\tau}[\varphi].$$

Next, suppose that $cp[\varphi] > 1$, and that the claim holds for $\mathcal{L}$-formulas of lower complexity. Let $\sigma$ be an $\langle \mathcal{L}, \mathcal{D} \rangle$-assignment, and let $X \notin fv[\sigma[\varphi]]$. Exactly one of the following holds:

- $\varphi = (\varphi_1 \diamond \varphi_2)$ for $\diamond \in \{\wedge, \vee, \supset\}$ and $\mathcal{L}$-formulas $\varphi_1$ and $\varphi_2$ of lower complexity. Then, we have $\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (\sigma_{X:=X_{abs}}[\varphi_1]\{\tau/X\} \diamond \sigma_{X:=X_{abs}}[\varphi_2]\{\tau/X\})$. By the induction hypothesis (and the case in which $X \notin fv[\varphi]$), this $\mathcal{L}$-formula is equal to $(\sigma_{X:=\tau}[\varphi_1] \diamond \sigma_{X:=\tau}[\varphi_2])$. And, by definition, this is equal to $\sigma_{X:=\tau}[\varphi]$.
- $\varphi = (Q^i x \psi)$ for $Q^i \in \{\forall^i, \exists^i\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and individual variable $x$ of $\mathcal{L}$ such that $x \notin fv[\tau] \cup fv[\sigma[\varphi]]$. Then, $\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (Q^i x \sigma_{X:=X_{abs}, x:=x}[\psi])\{\tau/X\} = (Q^i x \sigma_{X:=X_{abs}, x:=x}[\psi]\{\tau/X\})$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $\sigma_{X:=\tau}[\varphi]$.
- $\varphi = (Q^s Y \psi)$ for $Q^s \in \{\forall^s, \exists^s\}$, $\mathcal{L}$-formula $\psi$ of lower complexity, and set variable $Y$ of $\mathcal{L}$ such that $Y \notin fv[\tau] \cup \{X\} \cup fv[\sigma[\varphi]]$. Then, $\sigma_{X:=X_{abs}}[\varphi]\{\tau/X\} = (Q^s Y \sigma_{X:=X_{abs}, Y:=Y_{abs}}[\psi])\{\tau/X\} = (Q^i Y \sigma_{X:=X_{abs}, Y:=Y_{abs}}[\psi]\{\tau/X\})$. By the induction hypothesis, this $\mathcal{L}$-formula is equal to $\sigma_{X:=\tau}[\varphi]$. □

## Appendix B. Proofs for Section 5.1

Proof (Proposition 5.16).

1. Let $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ such that $\varphi \notin \mathtt{L}$. By internal maximality, $\Omega \mid \mathtt{L}, \varphi \Rightarrow \mathtt{R}$ is provable, and so there exists an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega \mid \mathtt{L}, \varphi \Rightarrow \mathtt{R}$, such that $\vdash H'$. Let $H = \{s \in H' : \{s\} \sqsubseteq \Omega\}$. Note that for every $\mathcal{L}$-sequent $\Gamma \Rightarrow E \in H' \setminus H$, we have $\varphi \in \Gamma$, $\Gamma \setminus \{\varphi\} \subseteq \mathtt{L}$, and $E \subseteq \mathtt{R}$. Let $\Gamma_1 \Rightarrow E_1, \dots, \Gamma_n \Rightarrow E_n$ be an enumeration of these sequents, and let $\Gamma = \bigcup \Gamma_i \setminus \{\varphi\}$. By applying internal weakenings on $H'$, we obtain $\vdash H \mid \Gamma, \varphi \Rightarrow E_1 \mid \dots \mid \Gamma, \varphi \Rightarrow E_n$. Clearly, $H \sqsubseteq \Omega$ and $\Gamma \Rightarrow E_1, \dots, \Gamma \Rightarrow E_n \sqsubseteq \mathtt{L} \Rightarrow \mathtt{R}$.
2. Let $\mathtt{L} \Rightarrow \mathtt{R} \in \Omega$ such that $\varphi \notin \mathtt{R}$. By internal maximality, $\Omega \mid \mathtt{L} \Rightarrow \varphi, \mathtt{R}$ is provable, and so there exists an $\mathcal{L}$-hypersequent $H' \sqsubseteq \Omega \mid \mathtt{L} \Rightarrow \varphi, \mathtt{R}$, such that $\vdash H'$. Let $H = \{s \in H' : \{s\} \sqsubseteq \Omega\}$. Note that for every $\mathcal{L}$-sequent $\Gamma \Rightarrow E \in H' \setminus H$, we have $E = \{\varphi\}$ and $\Gamma \subseteq \mathtt{L}$. Let $\Gamma_1 \Rightarrow \varphi, \dots, \Gamma_n \Rightarrow \varphi$ be an enumeration of theses sequents. Let $\Gamma = \bigcup \Gamma_i$. By applying internal weakenings on $H'$, we obtain $\vdash H \mid \Gamma \Rightarrow \varphi$. Clearly, $H \sqsubseteq \Omega$ and $\Gamma \subseteq \mathtt{L}$. □

To prove Lemma 5.18, we need some additional lemmas:

**Lemma Appendix B.1.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be an unprovable finite extended $\mathcal{L}$-hypersequent. Then there exists an unprovable finite extended $\mathcal{L}$-hypersequent $H'$ of the form $\Gamma'_1 \Rightarrow \Delta'_1 \mid \dots \mid \Gamma'_n \Rightarrow \Delta'_n$, such that $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$, and $H'$ admits the witness property.

Proof. This extension is done in steps.[7] In every step, we take some extended $\mathcal{L}$-sequent $\Gamma \Rightarrow \Delta \in H$, and proceed as follows:

- If $\Gamma$ contains a formula of the form $(\exists^i x \varphi)$, we take an individual variable $y$ of $\mathcal{L}$, which is not a free variable in the current hypersequent, and add the formula $\varphi\{y/x\}$ to $\Gamma$.
- If $\Delta$ contains a formula of the form $(\forall^i x \varphi)$, we take an individual variable $y$ of $\mathcal{L}$, which is not a free variable in the current hypersequent, and add the formula $\varphi\{y/x\}$ to $\Delta$.
- If $\Gamma$ contains a formula of the form $(\exists^s X \varphi)$, we take a set variable $Y$ of $\mathcal{L}$, which is not a free variable in the current hypersequent, and add the formula $\varphi\{Y/X\}$ to $\Gamma$.

[7] Formally, this extension should be defined inductively, but the intention should be clear.

- If $\Delta$ contains a formula of the form $(\forall^s X\varphi)$, we take a set variable $Y$ of $\mathcal{L}$, which is not a free variable in the current hypersequent, and add the formula $\varphi\{Y/X\}$ to $\Delta$.

We continue this procedure until the obtained extended $\mathcal{L}$-hypersequent admits the witness property. Note that since the number of formulas in $H$ is finite, and the complexity of the formulas which are added is decreasing, this procedure would terminate after a finite number of steps. $H'$ is the finite extended $\mathcal{L}$-hypersequent obtained from $H$ by this procedure. We show that every such extension keeps the extended $\mathcal{L}$-hypersequent unprovable (and thus $H'$ is unprovable):

- Suppose that an unprovable extended $\mathcal{L}$-hypersequent $H_1$ contains an extended $\mathcal{L}$-sequent $\Gamma\Rightarrow\Delta$, where $\Gamma$ contains a formula of the form $(\exists^i x\varphi)$. Let $H_2$ be the extended $\mathcal{L}$-hypersequent obtained from $H_1$ by adding $\varphi\{y/x\}$ to $\Gamma$, where $y$ is an individual variable which does not occur in $fv[H_1]$. Assume for contradiction that $H_2$ is provable. Hence there exist an $\mathcal{L}$-hypersequent $H \sqsubseteq H_2$, and $\mathcal{L}$-sequents $\Gamma'\Rightarrow E_1,\dots,\Gamma'\Rightarrow E_n \sqsubseteq \Gamma\Rightarrow\Delta$, such that $\vdash H \mid \Gamma',\varphi\{y/x\}\Rightarrow E_1 \mid\dots\mid \Gamma',\varphi\{y/x\}\Rightarrow E_n$. Proposition 3.17 entails that $\vdash H \mid \Gamma',(\exists^i x\varphi)\Rightarrow E_1 \mid\dots\mid \Gamma',(\exists^i x\varphi)\Rightarrow E_n$. This contradicts the fact the $H_1$ is unprovable.
- Suppose that an unprovable extended $\mathcal{L}$-hypersequent $H_1$ contains an extended $\mathcal{L}$-sequent $\Gamma\Rightarrow\Delta$, where $\Delta$ contains a formula of the form $(\forall^i x\varphi)$. Let $H_2$ be the extended $\mathcal{L}$-hypersequent obtained from $H_1$ by adding $\varphi\{y/x\}$ to $\Delta$, where $y$ is an individual variable which does not occur in $fv[H_1]$. Assume for contradiction that $H_2$ is provable. Hence there exist an $\mathcal{L}$-hypersequent $H \sqsubseteq H_2$, and a finite set $\Gamma' \subseteq \Gamma$, such that $\vdash H \mid \Gamma'\Rightarrow\varphi\{y/x\}$. By applying $(\Rightarrow\forall^i)$, we obtain $\vdash H \mid \Gamma'\Rightarrow(\forall^i x\varphi)$. This contradicts the fact the $H_1$ is unprovable.
- The set quantifiers are handled similarly, using Proposition 3.18 in the case of $\exists^s$. □

**Lemma Appendix B.2.** Let $H = \Gamma_1\Rightarrow\Delta_1 \mid\dots\mid \Gamma_n\Rightarrow\Delta_n$ be an unprovable finite extended $\mathcal{L}$-hypersequent. Let $\varphi$ be an $\mathcal{L}$-formula, and $s$ be an $\mathcal{L}$-sequent. Then there exists an unprovable finite extended $\mathcal{L}$-hypersequent $H'$, such that:

- $H' = \Gamma'_1\Rightarrow\Delta'_1 \mid\dots\mid \Gamma'_{n'}\Rightarrow\Delta'_{n'}$, where $n' \in \{n, n+1\}$, $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \le i \le n$.
- $H'$ is internally maximal with respect to $\varphi$.
- $H'$ is externally maximal with respect to $s$.
- $H'$ admits the witness property.

Proof. Suppose $s = \Gamma^* \Rightarrow E$. First, if $H \mid s$ is unprovable, let $n' = n+1$ and define $\Gamma_{n+1} = \Gamma^*$ and $\Delta_{n+1} = E$. Otherwise, let $n' = n$. We recursively define a finite sequence of finite extended $\mathcal{L}$-hypersequents, $H_0 = \Gamma^0_1\Rightarrow\Delta^0_1 \mid\dots\mid \Gamma^0_{n'}\Rightarrow\Delta^0_{n'},\dots,H_{n'} = \Gamma^{n'}_1\Rightarrow\Delta^{n'}_1 \mid\dots\mid \Gamma^{n'}_{n'}\Rightarrow\Delta^{n'}_{n'}$, in which $\Gamma^i_j \subseteq \Gamma^{i+1}_j$ and $\Delta^i_j \subseteq \Delta^{i+1}_j$ for every $1 \le j \le n'$ and $0 \le i \le n'-1$.

First, define $\Gamma^0_j = \Gamma_j$, $\Delta^0_j = \Delta_j$ for every $1 \le j \le n'$. Let $0 \le i \le n'-1$. Assume that the hypersequent $H_i = \Gamma^i_1\Rightarrow\Delta^i_1 \mid\dots\mid \Gamma^i_{n'}\Rightarrow\Delta^i_{n'}$ is defined. We show how to construct $H_{i+1} = \Gamma^{i+1}_1\Rightarrow\Delta^{i+1}_1 \mid\dots\mid \Gamma^{i+1}_{n'}\Rightarrow\Delta^{i+1}_{n'}$:

1. If $\Gamma^i_1\Rightarrow\Delta^i_1 \mid\dots\mid \Gamma^i_{i+1},\varphi\Rightarrow\Delta^i_{i+1} \mid\dots\mid \Gamma^i_{n'}\Rightarrow\Delta^i_{n'}$ is unprovable, then $\Gamma^{i+1}_{i+1} = \Gamma^i_{i+1} \cup \{\varphi\}$, $\Delta^{i+1}_{i+1} = \Delta^i_{i+1}$, and $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \Delta^i_j$ for every $j \ne i+1$.
2. Otherwise, if $\Gamma^i_1 \Rightarrow \Delta^i_1 \mid \dots \mid \Gamma^i_{i+1} \Rightarrow \Delta^i_{i+1},\varphi \mid \dots \mid \Gamma^i_{n'} \Rightarrow \Delta^i_{n'}$ is unprovable, then $\Gamma^{i+1}_{i+1} = \Gamma^i_{i+1}$, $\Delta^{i+1}_{i+1} = \Delta^i_{i+1} \cup \{\varphi\}$, and $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \Delta^i_j$ for every $j \ne i+1$.
3. If both do not hold, then $\Gamma^{i+1}_j = \Gamma^i_j$ and $\Delta^{i+1}_j = \Delta^i_j$ for every $1 \le j \le n'$.

It is easy to verify that $H_{n'} = \Gamma^{n'}_1\Rightarrow\Delta^{n'}_1 \mid\dots\mid \Gamma^{n'}_{n'}\Rightarrow\Delta^{n'}_{n'}$ is an unprovable finite extended $\mathcal{L}$-hypersequent. By Lemma Appendix B.1, there exists an unprovable finite extended $\mathcal{L}$-hypersequent, $H'$ of the form $\Gamma'_1\Rightarrow\Delta'_1 \mid\dots\mid \Gamma'_{n'}\Rightarrow\Delta'_{n'}$, such that $\Gamma^{n'}_j \subseteq \Gamma'_j$ and $\Delta^{n'}_j \subseteq \Delta'_j$ for every $1 \le j \le n'$, and $H'$ admits the witness property. It is again easy to see that $H'$ has all the required properties. For example, we show that $H'$ is internally maximal with respect to $\varphi$. Let $\Gamma\Rightarrow\Delta \in H'$. Suppose that $\Gamma = \Gamma'_j$ and $\Delta = \Delta'_j$.

- Assume that $\varphi \notin \Gamma$. Since $\Gamma^j_j \subseteq \Gamma$, this implies that $\Gamma^{j-1}_1\Rightarrow\Delta^{j-1}_1 \mid\dots\mid \Gamma^{j-1}_j,\varphi\Rightarrow\Delta^{j-1}_j \mid\dots\mid \Gamma^{j-1}_{n'}\Rightarrow\Delta^{j-1}_{n'}$ is provable. It easily follows that $H' \mid \Gamma,\varphi\Rightarrow\Delta$ (which extends this finite extended $\mathcal{L}$-hypersequent) is provable.

- Assume that $\varphi \notin \Delta$. If $\varphi \in \Gamma$, then since $\varphi \Rightarrow \varphi$ is an axiom of $\mathbf{HIF}^2$, $H' \mid \Gamma \Rightarrow \varphi, \Delta$ is provable. Otherwise, $\varphi \notin \Gamma$, and since $\Gamma^j_j \subseteq \Gamma$ and $\Delta^j_j \subseteq \Delta$, our construction ensures that the hypersequent $\Gamma^{j-1}_1 \Rightarrow \Delta^{j-1}_1 \mid ... \mid \Gamma^{j-1}_j \Rightarrow \Delta^{j-1}_j, \varphi \mid ... \mid \Gamma^{j-1}_{n'} \Rightarrow \Delta^{j-1}_{n'}$ is provable. It easily follows that $H' \mid \Gamma \Rightarrow \varphi, \Delta$ (which extends this finite extended $\mathcal{L}$-hypersequent) is provable. □

Proof (Lemma 5.18). Suppose that $H = \Gamma_1 \Rightarrow E_1 \mid ... \mid \Gamma_n \Rightarrow E_n$. Let $\varphi_0, \varphi_1 ...$ be an enumeration of all $\mathcal{L}$-formulas, in which every formula occurs infinitely often. Let $s_0, s_1 ...$ be an enumeration of all $\mathcal{L}$-sequents. We recursively define an infinite sequence of unprovable finite extended $\mathcal{L}$-hypersequents, $H_0 = \Gamma^0_1 \Rightarrow \Delta^0_1 \mid ... \mid \Gamma^0_{n_0} \Rightarrow \Delta^0_{n_0}, H_1 = \Gamma^1_1 \Rightarrow \Delta^1_1 \mid ... \mid \Gamma^1_{n_1} \Rightarrow \Delta^1_{n_1}, ...$ such that: $n_0 \leq n_1 \leq ...$, and $\Gamma^i_j \subseteq \Gamma^{i+1}_j$ and $\Delta^i_j \subseteq \Delta^{i+1}_j$ for every $i \geq 0$ and $1 \leq j \leq n_i$.

First, let $n_0 = n$ and $\Gamma^0_j = \Gamma_j$, $\Delta^0_j = E_j$ for every $1 \leq j \leq n_0$. Let $i \geq 0$. Assume that the hypersequent $H_i = \Gamma^i_1 \Rightarrow \Delta^i_1 \mid ... \mid \Gamma^i_{n_i} \Rightarrow \Delta^i_{n_i}$ is defined. By Lemma Appendix B.2, there exists an unprovable $\mathcal{L}$-hypersequent $H'$ such that:

- $H' = \Gamma'_1 \Rightarrow \Delta'_1 \mid ... \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$ where $n' \in \{n_i, n_i + 1\}$, and $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n_i$.
- $H'$ is internally maximal with respect to $\varphi_i$.
- $H'$ is externally maximal with respect to $s_i$.
- $H'$ admits the witness property.

Let $n_{i+1} = n'$, and $\Gamma^{i+1}_j = \Gamma'_j$, $\Delta^{i+1}_j = \Delta'_j$ for every $1 \leq j \leq n'$.

Note that after every step we have an unprovable finite extended $\mathcal{L}$-hypersequent, so Lemma Appendix B.2 can be applied. Finally, let $N$ be $\max\{n_0, n_1, ...\} + 1$, if such a maximum exists, and infinity otherwise. Let $n(j) = \min\{i : j \leq n_i\}$ for every $1 \leq j < N$. Define $\mathtt{L}_j = \cup_{i \geq n(j)} \Gamma^i_j$ and $\mathtt{R}_j = \cup_{i \geq n(j)} \Delta^i_j$ for every $1 \leq j < N$. Let $\Omega$ be the extended $\mathcal{L}$-hypersequent $\mathtt{L}_1 \Rightarrow \mathtt{R}_1 \mid \mathtt{L}_2 \Rightarrow \mathtt{R}_2 \mid ....$ Obviously, $\Omega$ extends $H$. We prove that $\Omega$ is maximal:

**Unprovability** Suppose by way of contradiction that $\vdash H$ for some $\mathcal{L}$-hypersequent $H \sqsubseteq \Omega$. Assume that $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$. The construction of $\Omega$ ensures that for every $1 \leq i \leq n$, there exists $k_i \geq 1$ such that $\Gamma_i \subseteq \mathtt{L}_{k_i}$ and $\Delta_i \subseteq \mathtt{R}_{k_i}$. This entails that for every $1 \leq i \leq n$, there exists $m_i \geq 0$ such that $\Gamma_i \subseteq \Gamma^{m_i}_{k_i}$ and $\Delta_i \subseteq \Delta^{m_i}_{k_i}$. By the construction of the $\Gamma^i_j$'s and $\Delta^i_j$'s, we have that for every $1 \leq i \leq n$ and $l \geq m_i$, $\Gamma_i \subseteq \Gamma^l_{k_i}$ and $\Delta_i \subseteq \Delta^l_{k_i}$. Let $m = \max\{m_1, \dots, m_n\}$. Then, by definition $H \sqsubseteq H_m$. Since $\vdash H$, it follows that $H_m$ is provable. But, this contradicts the fact that $H_0$ is unprovable, and that each application of Lemma Appendix B.2 yields an unprovable extended $\mathcal{L}$-hypersequent.

**Internal Maximality** Let $\varphi$ be an $\mathcal{L}$-formula, and let $\mathtt{L}_j \Rightarrow \mathtt{R}_j \in \Omega$. Since we included $\varphi$ infinite number of times in the enumeration of the formulas, there exists some $i \geq n(j)$ such that $\varphi_i = \varphi$. Our construction ensures that $H_{i+1}$ is internally maximal with respect to $\varphi$, and so if $\varphi \notin \Gamma^{i+1}_j$ then $H_{i+1} \mid \Gamma^{i+1}_j, \varphi \Rightarrow \Delta^{i+1}_j$ is provable, and if $\varphi \notin \Delta^{i+1}_j$ then $H_{i+1} \mid \Gamma^{i+1}_j \Rightarrow \varphi, \Delta^{i+1}_j$ is provable. Since $H_{i+1} \sqsubseteq \Omega$, it follows that if $\varphi \notin \mathtt{L}_j$ then $\Omega \mid \mathtt{L}_j, \varphi \Rightarrow \mathtt{R}_j$ is provable, and if $\varphi \notin \mathtt{R}_j$ then $\Omega \mid \mathtt{L}_j \Rightarrow \varphi, \mathtt{R}_j$ is provable.

**External Maximality** Let $s$ be an $\mathcal{L}$-sequent. Assume that $s = s_i$ $(i \geq 0)$, our construction ensures that $H_{i+1}$ is externally maximal with respect to $s$. Hence, either $\{s\} \sqsubseteq H_{i+1}$, or $H_{i+1} \mid s$ is provable. Since $H_{i+1} \sqsubseteq \Omega$, either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is provable.

**The Witness Property** Let $1 \leq j < N$. We show that $\mathtt{L}_j \Rightarrow \mathtt{R}_j$ admits the witness property. Assume $(\forall^i x\varphi) \in \mathtt{R}_j$. Then $(\forall^i x\varphi) \in \Delta^i_j$ for some $i \geq n(j)$. We can assume that $i > 0$ (if it holds for $i = 0$ then it holds for $i = 1$ as well). Our construction ensures that $H_i$ admits the witness property, and so there exists an individual variable $y$ such that $\varphi\{y/x\} \in \Delta^i_j$. Since $\Delta^i_j \subseteq \mathtt{R}_j$, we have that $\varphi\{y/x\} \in \mathtt{R}_j$. The cases involving the other quantifiers are analogous. □