Non-deterministic Connectives in Propositional Gödel Logic

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Current fuzzy logics follow the principle of truth-functionality, and fuzziness is limited to the level of the atomic formulas.

Non-deterministic semantics [Avron, Lev ’01] relaxes the truth-functionality principle, and allows uncertainty also on the level of the connectives.

However, non-deterministic semantics has not yet been applied for fuzzy logics.

We provide a first step towards a theory of non-deterministic semantics for fuzzy logics, by identifying a family of non-deterministic connectives that can be added to Gödel logic.
Proof-theoretically, the meaning of a connective is determined by its derivation rules in some “ideal” deduction system.
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\[
\begin{align*}
\Gamma, \varphi, \psi & \Rightarrow \Delta \\
\Gamma, \varphi \land \psi & \Rightarrow \Delta
\end{align*}
\]
Proof-theoretically, the meaning of a connective is determined by its derivation rules in some “ideal” deduction system.

\[
\begin{array}{ccc}
\Gamma, \varphi, \psi & \Rightarrow & \Delta \\
\hline
\Gamma, \varphi \land \psi & \Rightarrow & \Delta \\
\hline
\Gamma \Rightarrow & \varphi, \Delta & \Gamma \Rightarrow & \psi, \Delta \\
\hline
\Gamma \Rightarrow & \varphi \land \psi, \Delta \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
T & T & T \\
T & F & F \\
F & T & F \\
F & F & F \\
\end{array}
\]
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\[
\begin{array}{c|c|c|c}
\Gamma & \varphi & \psi & \Delta \\
\hline
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\[
\begin{array}{c}
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\end{array}
\]

\[
\begin{array}{c|c|c}
\hline
\varphi & \psi & \tilde{\varphi} \\
\hline
T & T & T \\
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\]

\[
\begin{array}{c|ccc}
\hline
\hline
 & \varpi & \hline
\Gamma & T & T & T \\
\Gamma, \psi & T & F & F \\
\Gamma & F & T & F \\
\Gamma & F & F & F \\
\hline
\end{array}
\]

\[
\begin{align*}
\Gamma, \varphi & \Rightarrow \Delta \\
\Gamma, \psi & \Rightarrow \Delta \\
\Gamma & \Rightarrow \varphi, \psi, \Delta \\
\Gamma & \Rightarrow \varphi \lor \psi, \Delta
\end{align*}
\]

\[
\begin{array}{c|ccc}
\hline
\hline
 & \vartriangle & \hline
\Gamma & T & T & T \\
\Gamma, \psi & T & F & F \\
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\Gamma & \Rightarrow \varphi \land \psi, \Delta \\
\Gamma, \varphi & \Rightarrow \Delta \\
\Gamma, \psi & \Rightarrow \Delta \\
\Gamma, \varphi \lor \psi & \Rightarrow \Delta \\
\Gamma & \Rightarrow \varphi \lor \psi, \Delta \\
\end{align*}
\]
This naturally leads to non-deterministic semantics.

\[
\begin{align*}
\Gamma, \varphi &\implies \Delta & \Gamma, \psi &\implies \Delta & \Gamma, \varphi \circ \psi &\implies \Delta \\
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\hline
\Gamma, \varphi \circ \psi & \Rightarrow \Delta
\end{align*}
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\Gamma & \Rightarrow \varphi, \Delta \\
\Gamma & \Rightarrow \psi, \Delta \\
\hline
\Gamma & \Rightarrow \varphi \circ \psi, \Delta
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\[ \Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta \]
\[\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta \]
\[\Gamma \Rightarrow \varphi \circ \psi, \Delta \]

\[\begin{array}{c|c|c}
\top & \top & \top \\
\top & \bot & \top, \top \\
\bot & \top & \top, \top \\
\bot & \bot & \bot \\
\end{array}\]

\[\sim\]
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\Gamma, \varphi &\Rightarrow \Delta & \Gamma, \psi &\Rightarrow \Delta \\
\Delta &\Rightarrow \Gamma, \varphi \circ \psi \\
\Gamma &\Rightarrow \varphi \circ \psi, \Delta \\
\Gamma &\Rightarrow \varphi, \Delta & \Gamma &\Rightarrow \psi, \Delta \\
\end{align*}
\]

\[v(\varphi \circ \psi) \in \tilde{o}(v(\varphi), v(\psi))\]
The only known “ideal” (in the above sense) system for Gödel logic is the **single-conclusion hypersequent** systemHG [Avron ’91].

A single-conclusion hypersequent is a set of single-conclusion sequents denoted by:

\[
\Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n
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The \textit{communication} rule:

\[
\frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2} \quad \text{(com)}
\]
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\hline
H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2
\end{align*}
\] (com)

All logical rules are the single-version hypersequent version of classical rules. E.g.

\[
\begin{align*}
H \mid \Gamma, \varphi, \psi \Rightarrow E & \quad H \mid \Gamma, \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi \\
\hline
H \mid \Gamma, \varphi \land \psi \Rightarrow E & \quad H \mid \Gamma \Rightarrow \varphi \land \psi
\end{align*}
\]
New Connectives in Gödel Logic

Example:

\[
\begin{align*}
H \mid \Gamma, \varphi \Rightarrow E & \quad \frac{H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \circ \psi \Rightarrow E} \\
H \mid \Gamma \Rightarrow \varphi & \quad \frac{H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \circ \psi}
\end{align*}
\]
Example:

\[
\frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, \psi \Rightarrow E} \quad \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \varphi \circ \psi}
\]

In general, new connectives can be added to Gödel logic by adding to \(HG\) rules of the following forms:

\[
\begin{align*}
\{ H \mid \Gamma, \Pi_i \Rightarrow E_i \}_{1 \leq i \leq m} & \quad \{ H \mid \Gamma, \Sigma_i \Rightarrow E \}_{1 \leq i \leq k} \\
H \mid \Gamma, \varphi \Rightarrow E & \quad H \mid \Gamma \Rightarrow \psi
\end{align*}
\]

\[
\begin{align*}
H \mid \Gamma, \Pi_i \Rightarrow E_i & \quad \{ H \mid \Gamma, \Pi_i \Rightarrow E_i \}_{1 \leq i \leq m} \\
H \mid \Gamma \Rightarrow \psi & \quad H \mid \Gamma \Rightarrow \varphi \circ \psi
\end{align*}
\]

where \(\Pi_i, E_i, \Sigma_i \subseteq \{ \psi_1, \ldots, \psi_n \}\)
Many-valued Semantics

Gödel valuation

- Non-empty linearly ordered set \( \langle V, \leq \rangle \) with a maximal element 1 and a minimal element 0
- A valuation function \( v : Frm_L \rightarrow V \)
Many-valued Semantics

Gödel valuation

- Non-empty linearly ordered set \( \langle V, \leq \rangle \) with a maximal element 1 and a minimal element 0
- A valuation function \( v : Frm_L \rightarrow V \)

The set of rules for each connective imposes interval-restrictions on \( v \).
Many-valued Semantics

Gödel valuation

- Non-empty linearly ordered set $⟨V, \leq⟩$ with a maximal element 1 and a minimal element 0
- A valuation function $v : Frm_L \rightarrow V$

The set of rules for each connective imposes interval-restrictions on $v$. For example:

$$H \mid \Gamma, \varphi \Rightarrow E \quad H \mid \Gamma, \psi \Rightarrow E \quad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \circ \psi}$$

$$v(\varphi \circ \psi) \in [\min(v(\varphi), v(\psi)), \max(v(\varphi), v(\psi))]$$
In general:

\[ v(\Diamond(\psi_1, \ldots, \psi_n)) \in [F_{\Diamond}(v(\psi_1), \ldots, v(\psi_n)), G_{\Diamond}(v(\psi_1), \ldots, v(\psi_n))] \]

where \( F_{\Diamond} \) and \( G_{\Diamond} \) involve min, max, and \( \rightarrow \)
Many-valued Semantics

In general:

$$\nu(\Diamond(\psi_1, \ldots, \psi_n)) \in [F_\Diamond(\nu(\psi_1), \ldots, \nu(\psi_n)), G_\Diamond(\nu(\psi_1), \ldots, \nu(\psi_n))]$$

where $F_\Diamond$ and $G_\Diamond$ involve min, max, and $\rightarrow$

$$\frac{\{H \mid \Gamma, \Pi_i \Rightarrow E_i\}_{1 \leq i \leq m}}{H \mid \Gamma \Rightarrow \Diamond(\psi_1, \ldots, \psi_n)} \quad \frac{\{H \mid \Gamma, \Theta_i \Rightarrow F_i\}_{1 \leq i \leq l}}{H \mid \Gamma, \Diamond(\psi_1, \ldots, \psi_n) \Rightarrow E} \quad \frac{\{H \mid \Gamma, \Sigma_i \Rightarrow E\}_{1 \leq i \leq k}}{H \mid \Gamma \Rightarrow \Diamond(\psi_1, \ldots, \psi_n) \Rightarrow E}$$

$$F_\Diamond(x_1, \ldots, x_n) = \min_{1 \leq i \leq m} (\min x(\Pi_i) \rightarrow \max x(E_i))$$

$$G_\Diamond(x_1, \ldots, x_n) = \min_{1 \leq i \leq l} (\min x(\Theta_i) \rightarrow \max x(F_i)) \rightarrow \max_{1 \leq i \leq k} (\min x(\Sigma_i))$$

where for every set $\Delta \subseteq \{\psi_1, \ldots, \psi_n\}$, $x(\Delta) = \{x_i \mid \psi_i \in \Delta\}$
Usual implication:

\[
\begin{align*}
H | \Gamma, \varphi & \Rightarrow \psi \\
H | \Gamma & \Rightarrow \varphi \supset \psi
\end{align*}
\]

\[
\begin{align*}
H | \Gamma & \Rightarrow \varphi \\
H | \Gamma, \psi & \Rightarrow E
\end{align*}
\]

\[
\begin{align*}
H | \Gamma, \varphi \supset \psi & \Rightarrow E
\end{align*}
\]

\[
\nu(\varphi \supset \psi) \in [\nu(\varphi) \rightarrow \nu(\psi), (1 \rightarrow \nu(\varphi)) \rightarrow \nu(\psi)]
\]
**Example**

- **Usual implication:**

\[ \frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \supset \psi} \quad \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \supset \psi \Rightarrow E} \]

\[ \nu(\varphi \supset \psi) \in [\nu(\varphi) \rightarrow \nu(\psi), (1 \rightarrow \nu(\varphi)) \rightarrow \nu(\psi)] \]

\[ \nu(\varphi \supset \psi) = \nu(\varphi) \rightarrow \nu(\psi) \]
Example

- **Usual implication:**

\[
\begin{align*}
\frac{H | \Gamma, \varphi \Rightarrow \psi}{H | \Gamma \Rightarrow \varphi \supset \psi} \quad \frac{H | \Gamma \Rightarrow \varphi \ H | \Gamma, \psi \Rightarrow E}{H | \Gamma, \varphi \supset \psi \Rightarrow E}
\end{align*}
\]

\[\nu(\varphi \supset \psi) \in [\nu(\varphi) \rightarrow \nu(\psi), (1 \rightarrow \nu(\varphi)) \rightarrow \nu(\psi)]\]

\[\nu(\varphi \supset \psi) = \nu(\varphi) \rightarrow \nu(\psi)\]

- **Semi-implication [Gurevich, Neeman ’09]:**

\[
\begin{align*}
\frac{H | \Gamma \Rightarrow \psi}{H | \Gamma \Rightarrow \varphi \bowtie \psi} \quad \frac{H | \Gamma \Rightarrow \varphi \ H | \Gamma, \psi \Rightarrow E}{H | \Gamma, \varphi \bowtie \psi \Rightarrow E}
\end{align*}
\]
Example

- **Usual implication:**

  \[
  \frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \supset \psi}
  \]

  \[
  \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \supset \psi \Rightarrow E}
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  \[\nu(\varphi \supset \psi) \in [\nu(\varphi) \rightarrow \nu(\psi), (1 \rightarrow \nu(\varphi)) \rightarrow \nu(\psi)]\]

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- **Semi-implication [Gurevich, Neeman ’09]:**

  \[
  \frac{H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \bowtie \psi}
  \]

  \[
  \frac{H \mid \Gamma \Rightarrow \varphi \quad H \mid \Gamma, \psi \Rightarrow E}{H \mid \Gamma, \varphi \bowtie \psi \Rightarrow E}
  \]

  \[\nu(\varphi \bowtie \psi) \in [1 \rightarrow \nu(\psi), (1 \rightarrow \nu(\varphi)) \rightarrow \nu(\psi)]\]
Example

- Usual implication:

\[
\begin{align*}
H \mid \Gamma, \varphi & \Rightarrow \psi & H \mid \Gamma \Rightarrow \varphi, \varphi \supset \psi & \Rightarrow \psi \\
H \mid \Gamma & \Rightarrow \varphi \supset \psi & H \mid \Gamma, \psi & \Rightarrow E
\end{align*}
\]

\[
v(\varphi \supset \psi) \in [v(\varphi) \rightarrow v(\psi), (1 \rightarrow v(\varphi)) \rightarrow v(\psi)]
\]

\[
v(\varphi \supset \psi) = v(\varphi) \rightarrow v(\psi)
\]

- Semi-implication [Gurevich, Neeman ’09]:

\[
\begin{align*}
H \mid \Gamma & \Rightarrow \psi & H \mid \Gamma \Rightarrow \varphi \leadsto \psi & \Rightarrow \psi \\
H \mid \Gamma & \Rightarrow \varphi \leadsto \psi & H \mid \Gamma, \psi & \Rightarrow E
\end{align*}
\]

\[
v(\varphi \leadsto \psi) \in [1 \rightarrow v(\psi), (1 \rightarrow v(\varphi)) \rightarrow v(\psi)]
\]

\[
v(\varphi \leadsto \psi) \in [v(\psi), v(\varphi) \rightarrow v(\psi)]
\]

\[
v(\varphi \leadsto \psi) \in \begin{cases} 
\{v(\psi)\} & v(\varphi) > v(\psi) \\
[v(\psi), 1] & \text{otherwise}
\end{cases}
\]
We characterize proof-theoretically and semantically a family of (non-deterministic) connectives that can be added to propositional Gödel logic.

The paper also provides:

- General strong cut-admissibility results.
- Decidability results.
- Non-deterministic Kripke-style semantics.

Further Research:

- Provide an independent semantic characterization of this family of connectives.
- Apply these methods for other fuzzy logics.
Thank you!