Kripke Semantics for Basic Sequent Systems

Arnon Avron   Ori Lahav

Tel Aviv University

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Main Contributions

- A correspondence between a wide class of proof-systems (called basic systems) and Kripke semantics.
- More precisely, a general soundness and completeness result which uniformly provides Kripke semantics for each basic system.
- Extension of the previous result to obtain semantic characterizations of crucial syntactic properties of basic systems:
  - Analyticity
  - Cut-admissibility

Basic Systems: General Framework

1. Propositional sequent systems
2. Manipulate two-sided multiple-conclusion sequents
3. Fully structural :
   - Sequents are finite sets of signed formulas, e.g.

$$\psi, \varphi \Rightarrow \varphi, \psi \wedge \varphi \quad \equiv \quad \{f{:}\psi, f{:}\varphi, t{:}\varphi, t{:}(\psi \wedge \varphi)\}$$

   - Identity axioms, cut, weakening rules always present
4. The logical rules are all basic rules

Basic Rules - Examples

$$\frac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$$

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- Distinction between active and context formulas

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- Distinction between active and context formulas
- The structure of the active part:

$$\frac{\Rightarrow \psi}{\Rightarrow \Box\psi} \quad \rightsquigarrow \quad \Rightarrow p_1 / \Rightarrow \Box p_1 \qquad\qquad \frac{\psi \Rightarrow}{\Box\psi \Rightarrow} \quad \rightsquigarrow \quad p_1 \Rightarrow / \Box p_1 \Rightarrow$$

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- Introducing context-relations to handle the context part:

$$\frac{\Box\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} \rightsquigarrow \pi_1 = \{\langle f{:}\Box p_1, f{:}\Box p_1 \rangle\}$$

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- The final formulation:

$$\langle \Rightarrow p_1, \pi_1\rangle / \Rightarrow \Box p_1 \qquad \langle p_1 \Rightarrow, \pi_0\rangle / \Box p_1 \Rightarrow$$

Basic Rules

- A basic rule:

$$\langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle / C$$

  - Premises: sequents $s_1, \ldots, s_n$
  - Corresponding context-relations: $\pi_1, \ldots, \pi_n$
  - Conclusion: sequent C

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- Its application:

$$\frac{\sigma(s_1) \cup c_1 \quad \ldots \quad \sigma(s_n) \cup c_n}{\sigma(C) \cup c'_1 \cup \ldots \cup c'_n}$$

where :

  - σ is a substitution
  - for every $1 \leq i \leq n$, $\langle c_i, c'_i \rangle$ is a π_i-instance

Basic Rules - More Examples

Basic Rule	Application
$\langle p_1 \Rightarrow, \pi_0 \rangle, \langle \Rightarrow p_1, \pi_0 \rangle / \Rightarrow$	$\dfrac{\Gamma_1, \psi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$
$\langle p_1 \Rightarrow p_2, \pi_0 \rangle / \Rightarrow p_1 \supset p_2$	$\dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}$
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Basic Systems

Many sequent systems are basic.

This includes systems for (the propositional fragments of):

- Classical logic
- Intuitionistic logic, its dual, and bi-intuitionistic logic
- Variety of modal logics
- Intuitionistic modal logics
- Many-valued logics
- Variety of paraconsistent logics

Kripke Semantics in General

Definition

A Kripke frame consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times Frm_{\mathcal{L}} \to \{\mathsf{T}, \mathsf{F}\}$

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- A signed formula $x{:}\psi$ is true in a world w if $v(w, \psi) = \mathsf{X}$
- A sequent s is true in a world w if it contains at least one signed formula which is true in w
- Accordingly, a sequent $\Gamma \Rightarrow \Delta$ is true in w iff $v(w, \psi) = \mathsf{F}$ for some $\psi \in \Gamma$ or $v(w, \psi) = \mathsf{T}$ for some $\psi \in \Delta$
- A frame is a model of a sequent s if it is true in every world

Kripke Semantics for Basic Systems

- To obtain Kripke semantics for a proof system **G**, we identify a set of **G**-legal frames for which **G** is sound and complete, i.e. $\mathcal{C} \vdash_{\mathbf{G}} s$ iff every **G**-legal frame which is a model of $\mathcal{C}$ is also a model of s.
- For a basic system **G**:
  - Each context-relation in **G** and each basic rule of **G** imposes a constraint on the set of frames.
  - Joining all of these constraints, we obtain the set of **G**-legal frames.
- It might produce non-deterministic semantics.

G-legal Frames

- For every context-relation π in **G** there is a corresponding accessibility relation R_π, where R_{π_0} is the identity relation.
- The constraint imposed by the context-relation π:
  if $wR_\pi u$ then for every π-instance $\langle x{:}\psi, y{:}\varphi\rangle$, either $v(u,\psi) \neq$ X or $v(w,\varphi) =$ Y.
- The constraint imposed by the basic rule $\langle s_1, \pi_1\rangle, \dots, \langle s_n, \pi_n\rangle / C$:
  For every world w, substitution σ, if for every $1 \leq i \leq n$, $\sigma(s_i)$ is true in every u such that $wR_{\pi_i}u$, then $\sigma(C)$ is true in w.

G-legal Frames

Reminder: $\pi_0 = \{\langle f{:}p_1, f{:}p_1\rangle, \langle t{:}p_1, t{:}p_1\rangle\}$

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- The constraint imposed by the basic rule $\langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle / C$:
  For every world w, substitution σ, if for every $1 \leq i \leq n$, $\sigma(s_i)$ is true in every u such that $wR_{\pi_i}u$, then $\sigma(C)$ is true in w.

Example

$\langle \Rightarrow p_1, \pi_K \rangle / \Rightarrow \Box p_1$

$\pi_K = \{\langle f{:}p_1, f{:}\Box p_1 \rangle\}$

$$\frac{\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$$

- A relation $R_{\pi_K} \in \mathcal{R}$.
- If $wR_{\pi_K}u$ then for every ψ, either $v(w, \Box\psi) = \mathsf{F}$ or $v(u, \psi) \neq \mathsf{F}$, i.e. if $v(w, \Box\psi) = \mathsf{T}$, then $v(u, \psi) = \mathsf{T}$ for every u such that $wR_{\pi_K}u$.
- If $v(u, \psi) = \mathsf{T}$ for every u such that $wR_{\pi_K}u$, then $v(w, \Box\psi) = \mathsf{T}$.

Kripke Semantics for Basic Systems

Theorem

Every basic system **G** *is sound and complete with respect to the semantics of* **G***-legal frames.*

Kripke Semantics for Basic Systems

Theorem

Every basic system **G** *is sound and complete with respect to the semantics of* **G***-legal frames.*

- General and uniform:
  - Various known soundness and completeness results are specific cases of this general theorem
  - There are some known systems for which it provides Kripke semantics for the first time, e.g. systems for weak modal logics
- Modular

Analyticity

- A basic system is (strongly) analytic iff it has the subformula property, i.e. $\mathcal{C} \vdash_{\mathbf{G}} s$ implies that there exists a proof of s from $\mathcal{C}$ in **G** that contains only subformulas of the formulas in $\mathcal{C} \cup \{s\}$.
- Analyticity implies decidability and consistency.
- Q: semantic meaning of analyticity?

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- Analyticity implies decidability and consistency.
- Q: semantic meaning of analyticity?

Next, we strengthen the soundness and completeness theorem to characterize proofs containing only formulas from a given set $\mathcal{E}$.

For this we introduce $\mathcal{E}$-semiframes.

Frames

Definition

A frame consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times Frm_{\mathcal{L}} \to \{\mathsf{T}, \mathsf{F}\}$

Theorem

There exists a proof in **G** *of s from* $\mathcal{C}$

if and only if

every **G***-legal frame which is a model of* $\mathcal{C}$ *is also a model of s.*

Semiframes

Definition

A $\mathcal{E}$-semiframe consists of:

- A set of worlds W
- A set of accessibility relations $\mathcal{R}$
- A valuation $v : W \times \mathcal{E} \to \{\mathsf{T}, \mathsf{F}\}$

Theorem

There exists a proof in **G** *of s from $\mathcal{C}$ containing only formulas from $\mathcal{E}$*

if and only if

every **G***-legal $\mathcal{E}$-semiframe which is a model of $\mathcal{C}$ is also a model of s.*

Semantic Characterization of Analyticity

- The last theorem leads to a semantic decision procedure for analytic basic systems (just check all possible semiframes).
- Semantic sufficient condition for analyticity: If every **G**-legal $\mathcal{E}$-semiframe can be extended to a **G**-legal frame for every set $\mathcal{E}$ of formulas closed under subformulas, then **G** is analytic.
- Both the procedure and the criterion are applicable for many interesting basic systems.

Strong Cut-Admissibility

- A basic system enjoys strong cut-admissibility if whenever $\mathcal{C} \vdash_{\mathbf{G}} s$, then there exists a proof of s from $\mathcal{C}$ in which all cuts are on formulas from $\mathcal{C}$.
- In particular, if $\mathcal{C}$ is empty, then no cuts are allowed (usual cut-admissibility).

We strengthen the soundness and completeness theorem to handle proofs in which cut is only allowed on formulas from a given set $\mathcal{E}$.

Quasiframes

Intuition

An application of cut: $$\frac{\psi \Rightarrow \quad \Rightarrow \psi}{\Rightarrow}$$

If cut on ψ is forbidden, we need a frame which is a model of both $\psi \Rightarrow$ and $\Rightarrow \psi$.

Quasiframes

Intuition

An application of cut: $\dfrac{\psi \Rightarrow \quad \Rightarrow \psi}{\Rightarrow}$

If cut on ψ is forbidden, we need a frame which is a model of both $\psi \Rightarrow$ and $\Rightarrow \psi$.

Definition

- A $\mathcal{E}$-quasiframe consists of:
  - A set of worlds W
  - A set of accessibility relations $\mathcal{R}$
  - A valuation $v : W \times Frm_{\mathcal{L}} \to \{\mathsf{T}, \mathsf{F}, \mathsf{I}\}$ such that $v(w, \psi) \neq \mathsf{I}$ for every $w \in W$ and $\psi \in \mathcal{E}$
- A sequent $\Gamma \Rightarrow \Delta$ is *true* in some $w \in W$ if $v(w, \psi) \in \{\mathsf{F}, \mathsf{I}\}$ for some $\psi \in \Gamma$ or $v(w, \psi) \in \{\mathsf{T}, \mathsf{I}\}$ for some $\psi \in \Delta$.

Semantic Characterization of Cut-Admissibility

- Semantic sufficient condition for strong cut-admissibility:
  If every **G**-legal $\mathcal{E}$-quasiframe can be refined into a **G**-legal frame for every set $\mathcal{E}$ of formulas, then **G** enjoys strong cut-admissibility (by refinement, we mean changing all I's to T's or F's).
- Provides a uniform basis for semantic proofs of strong cut-admissibility in basic systems.

Thank you!