A Unified Semantic Framework for Fully-structural Propositional Sequent Systems

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We identify a large family of fully-structural propositional sequent systems, which we call *basic systems*. We present a general uniform method for providing (potentially, non-deterministic) strongly sound and complete Kripke-style semantics, which is applicable for every system of this family. In addition, this method can also be applied when (i) some formulas are not allowed to appear in derivations, (ii) some formulas are not allowed to serve as cut-formulas, and (iii) some instances of the identity axiom are not allowed to be used. This naturally leads to new semantic characterizations of analyticity (global subformula property), cut-admissibility and axiom-expansion in basic systems. We provide a large variety of examples showing that many soundness and completeness theorems for different sequent systems, as well as analyticity, cut-admissibility and axiom-expansion results, easily follow using the general method of this paper.

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1. INTRODUCTION

Many times in mathematical and applied logic one faces a new proof system. At first it usually has no evident semantics. For example, intuitionistic logic was initially formulated only as a calculus, and semantics for it were proposed much later. The same applies to the first modal logics. The lack of semantics makes it very difficult to understand the logic induced by a new proof system. Most importantly, an effective sound semantics for a given proof system is useful to obtain "negative" results, namely that some conclusion cannot be derived in the proof system from a given set of assumptions.

The main goal of this paper is to uniformly obtain useful semantics for a (new or existing) given proof system. Obviously, it would be too ambitious to deal with the huge variety of possible proof systems. Here we focus on a family of Gentzen-type sequent systems, which turns out to be well-behaved for our purposes, and is also sufficiently wide to include a variety of important existing calculi. We call the sequent systems of this family *basic systems*. Generally, basic systems are multiple-conclusion propositional sequent calculi that include all of Gentzen's original structural rules (in fact, for the aspects of proof systems discussed in this paper, every such calculus that we know is equivalent to a basic system, see Remark 3.6 for further clarifications).

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Various sequent calculi that seem to have completely different natures belong to the family of basic systems. For example, this includes standard sequent calculi for modal logics, as well as the usual *multiple*-conclusion systems for intuitionistic logic, its dual, and bi-intuitionistic logic.

On the semantic side, we consider a generalization of Kripke-style semantics for modal and intuitionistic logic, and show that each basic system induces some sets of (generalized) Kripke valuations for which it is strongly sound and complete. In fact, we provide a uniform method to obtain these sets of Kripke valuations. In many important cases, the usual well-known soundness and completeness theorems for known calculi are simple corollaries of this general method (see e.g., Example 4.33).

An important property of the Kripke valuations semantics introduced in this paper is the fact that it is not necessarily truth-functional. Thus, the proposed semantics belongs to the framework of *non-deterministic semantics*, where the truth-value of a compound formula may not be uniquely determined by the truth-values of its subformulas (see [Avron and Zamansky 2011]). Relaxing the truth-functionality principle is a major key for providing semantics for *every* basic system (see e.g., Example 4.41). In fact, it is also necessary for the *modularity* of the proposed method. By allowing non truth-functional valuations, we are able to separately analyse the semantic effect of each component of the syntactic machinery (each derivation rule, and in fact, also each ingredient of a rule). The full semantics of the calculus is then obtained by joining the semantic effects of all of its components.

An illuminating contribution of a semantic study of proof systems is the ability to provide semantic proofs (or refutations) of important proof-theoretic properties. In many cases these proofs are much simpler and easier to verify than their proof-theoretic counterparts. This is the topic of the second part of this paper, where we extend the semantics and use its extension to provide semantic characterizations for three important proof-theoretic properties: generalized analyticity, (strong) cut-admissibility, and axiom-expansion. We demonstrate in a variety of examples how these characterizations can be applied to prove that a given basic system either enjoys or lacks each of these properties. For some well-known sequent systems this provides new simple *semantic* proofs of known proof-theoretic properties.

The structure of this paper is as follows. After providing some preliminaries and notations in Section 2, Section 3 is dedicated to precisely define the family of basic systems, and to provide examples of some known proof systems that can be presented in this framework. The semantic framework is presented in Section 4, where we also present the method to obtain semantics for a given basic system. Examples of applications of this method are provided in Section 4.1. Section 5 presents semantic characterizations for proof-theoretic properties of basic systems.

Related Works

The present paper substantially extends the family of calculi studied in our previous works (see e.g., [Avron and Zamansky 2011],[Avron 2007]). In particular, here we do not require the derivation rules to be "canonical". The price to pay is of course that the proposed semantics is more complicated, and is not always effective. Related works of different type are those that were devoted to characterize cut-free sequent systems. For example, a semantic characterization of cut-admissibility was the subject of [Ciabattoni and Terui 2006]. There, however, the authors consider substructural systems, and use phase semantics, which is significantly more abstract and complex than Kripke-style semantics.

Finally, we note that a preliminary short version of this paper is included in [Avron and Lahav 2011]. Besides the addition of full proofs, we significantly generalized the semantic framework of [Avron and Lahav 2011] in order to obtain stronger soundness and completeness results, and uniformly handle arbitrary proof-specifications (see Section 5.2).

2. PRELIMINARIES

In what follows $\mathcal{L}$ is a an arbitrary propositional language, and $wff_{\mathcal{L}}$ stands for its set of formulas. Without loss of generality, we assume that $at_{\mathcal{L}} = \{p_1, p_2, \dots\}$ is the set of atomic formulas of $\mathcal{L}$. Given a unary connective $\diamond$ and a set Γ of formulas, we denote the set $\{\diamond\psi \mid \psi \in \Gamma\}$ by $\diamond\Gamma$.

In this paper we only consider two-sided fully-structural sequent systems, and so a *signed formula* is defined to be an expression of the form $\texttt{T}{:}\psi$ or $\texttt{F}{:}\psi$ where $\psi \in wff_{\mathcal{L}}$, and a *sequent* is defined to be a finite set of signed formulas.

We shall usually employ the usual sequent notation $\Gamma \Rightarrow \Delta$, where Γ and Δ are (possibly empty) finite sets of formulas. $\Gamma \Rightarrow \Delta$ is interpreted as the sequent $\{\texttt{F}{:}\psi \mid \psi \in \Gamma\} \cup \{\texttt{T}{:}\psi \mid \psi \in \Delta\}$.[1] We also employ the standard abbreviations, e.g., $\Gamma, \varphi \Rightarrow \psi$ instead of $\Gamma \cup \{\varphi\} \Rightarrow \{\psi\}$, and $\Gamma \Rightarrow$ instead of $\Gamma \Rightarrow \emptyset$.

Definition 2.1. A *substitution* is a function $\sigma : wff_{\mathcal{L}} \to wff_{\mathcal{L}}$, such that $\sigma(\diamond(\psi_1, \dots, \psi_n)) = \diamond(\sigma(\psi_1), \dots, \sigma(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$. A substitution is extended to signed formulas, sequents, etc. in the obvious way.

3. BASIC SYSTEMS

In this section we precisely define the family of *basic systems*. For doing so, we define the general structure of derivation rules that are allowed to appear in basic systems. Rules of this structure will be called *basic rules*. Two key ideas are applied in this definition. First, we explicitly differentiate between a rule and its applications. Derivations in a certain proof system consist of *applications* of rules, and the rules themselves are just succinct formulations of their sets of applications. Rules are often given as schemas involving meta-variables for formulas and sets of formulas. However, we shall use a less standard formulation, that will later allow to isolate the semantic effect of the different ingredients of each rule. Second, in the formulation of the rules, we differentiate between two parts of their applications, namely the *context* part and the *non-context* part (see [Troelstra and Schwichtenberg 1996]). The non-context part is obtained by instantiating a rigid structure that is given in the rule. In turn, the structure of the context part is determined using *context-relations*. This structure is less restrictive, as the number of context formulas is completely free. Next we turn to the formal definitions.

Definition 3.1. A *context-relation* is a finite binary relation on the set of signed formulas. Given a context-relation π, we denote by $\bar{\pi}$ the binary relation between signed formulas $\bar{\pi} = \{\langle \sigma(\alpha), \sigma(\beta) \rangle \mid \sigma \textit{ is a substitution, and } \langle \alpha, \beta \rangle \in \pi\}$. A π-*instance* is a pair of sequents $\langle s_1, s_2 \rangle$ for which there exist (not necessarily distinct) signed formulas $\alpha_1, \dots, \alpha_n$, and $\beta_1, \dots, \beta_n$ such that $s_1 = \{\alpha_1, \dots, \alpha_n\}$, $s_2 = \{\beta_1, \dots, \beta_n\}$, and $\alpha_i \bar{\pi} \beta_i$ for every $1 \leq i \leq n$.

Definition 3.2. The context-relation π_0 is the relation $\{\langle \texttt{F}{:}p_1, \texttt{F}{:}p_1 \rangle, \langle \texttt{T}{:}p_1, \texttt{T}{:}p_1 \rangle\}$.

By definition, $\alpha \bar{\pi}_0 \beta$ iff $\alpha = \beta$, and π_0-instances are the pairs of the form $\langle s, s \rangle$.

Definition 3.3.

[1] Our signs should not be confused with those usually used in tableau calculi, where T-signed formulas correspond to the formulas on the left side of the sequent, and F-signed formulas correspond to those on the right side.

(1) A *basic premise* is an ordered pair of the form $\langle s, \pi \rangle$, where s is a sequent and π is a context-relation.
(2) A *basic rule* is an expression of the form S/C, where S is a finite set of basic premises, and C is a sequent. C is called the *conclusion* of the rule. To improve readability, we usually drop the set braces of the set of premises.
(3) An *application* of the basic rule $\langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / C$ is any inference step of the following form:

$$\frac{\sigma(s_1) \cup c_1 \quad \ldots \quad \sigma(s_n) \cup c_n}{\sigma(C) \cup c'_1 \cup \ldots \cup c'_n}$$

where σ is a substitution, and $\langle c_i, c'_i \rangle$ is a π_i-instance for every $1 \leq i \leq n$. The sequents $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n$ are called the *premises* of the application, while $\sigma(C) \cup c'_1 \cup \ldots \cup c'_n$ is called the *conclusion* of the application.

Note that the language of the proof system (the one used in its proofs) is also used in the formulation of the basic rules of the system. In particular, meta-variables are not needed. Roughly speaking, applications of some rule are obtained by applying a substitution on the premises $s_1, \ldots, s_n$ and the conclusion C of the rule, and (optionally) adding context-formulas according to the context-relations.

Table I provides some examples of basic rules and the forms of their applications. The names given to the rules in this table will be used below.

Definition 3.4. A *basic system* is a set of basic rules in which (cut), (id), and the two weakening rules, $(W \Rightarrow)$ and $(\Rightarrow W)$, are all included. We denote by $\Upsilon_{\mathbf{G}}$ all other rules of a basic system $\mathbf{G}$, and by $\Pi_{\mathbf{G}}$ the set of context-relations appearing in the basic rules of $\mathbf{G}$ (in particular, since (cut) is always included, $\pi_0 \in \Pi_{\mathbf{G}}$ for every basic system $\mathbf{G}$).

Definition 3.5. A *proof* in a basic system $\mathbf{G}$ of a sequent s from a set $\mathcal{S}$ of sequents is a list[2] of sequents ending with s, such that every sequent in the list is either an element of $\mathcal{S}$, or a conclusion of some application of some rule of $\mathbf{G}$, provided that all premises of this application appear before. We shall write $\mathcal{S} \vdash_{\mathbf{G}} s$ if such a proof exists.

Remark 3.6. For our purposes, we find it most convenient to define sequents using *sets*, so that the structural rules of contraction and exchange are built-in. One can choose to work with *lists* (as in the original work of Gentzen) or with *multisets*, and explicitly include contraction and exchange in the definition of a basic system. Obviously, this would not affect the derivability relation. In fact, for all aspects of basic systems studied in this paper (semantics, cut-admissibility, analyticity, etc.) this choice is immaterial, since any result in one formulation trivially holds in the other. Of course, this might not be the case when studying other (structural) properties (like e.g. in [Dyckhoff 1992]). Similarly, we formulated basic rules as *multiplicative* rules (context-independent in the terms of [Troelstra and Schwichtenberg 1996]), rather than additive (context-sharing) rules. Clearly, in the presence of all structural rules, the multiplicative version and the additive one are interderivable. Again, this decision does not affect any property we study below.

3.1. Examples of Basic Systems

The above notion of a basic rule is sufficiently general, so that many known sequent systems for various propositional logics can be easily presented in the framework of basic systems. In this section, we list some of these sequent systems, and present their

[2] Similarly, one can use trees or DAGs.

Table I. Basic Rules Examples

Name	Basic Rule	Application
(id)	$\emptyset / p_1 \Rightarrow p_1$	$\psi \Rightarrow \psi$
(cut)	$\langle p_1 \Rightarrow\ , \pi_0\rangle, \langle\ \Rightarrow p_1, \pi_0\rangle / \emptyset$	$\dfrac{\Gamma_1, \psi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$
$(W \Rightarrow)$	$\langle \emptyset, \pi_0\rangle / p_1 \Rightarrow$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \psi \Rightarrow \Delta}$
$(\Rightarrow W)$	$\langle \emptyset, \pi_0\rangle /\ \Rightarrow p_1$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \psi, \Delta}$
(T)	$\langle p_1 \Rightarrow, \pi_0\rangle / \Box p_1 \Rightarrow$	$\dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta}$
$(S4)$	$\langle \Rightarrow p_1, \pi\rangle / \Rightarrow \Box p_1$ where $\pi = \{\langle \texttt{F}{:}\Box p_1, \texttt{F}{:}\Box p_1\rangle\}$	$\dfrac{\Box\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi}$
$(K4)$	$\langle \Rightarrow p_1, \pi\rangle / \Rightarrow \Box p_1$ where $\pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}\Box p_1\rangle, \langle \texttt{F}{:}\Box p_1, \texttt{F}{:}\Box p_1\rangle\}$	$\dfrac{\Gamma_1, \Box\Gamma_2 \Rightarrow \psi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi}$
(D_0)	$\langle p_1 \Rightarrow\ , \emptyset\rangle / \Box p_1 \Rightarrow$	$\dfrac{\psi \Rightarrow}{\Box\psi \Rightarrow}$
(D)	$\langle \emptyset, \pi\rangle /\ \Rightarrow$ where $\pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}\Box p_1\rangle\}$	$\dfrac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}$
$(\supset\Rightarrow)$	$\langle\ \Rightarrow p_1, \pi_0\rangle, \langle p_2 \Rightarrow\ , \pi_0\rangle / p_1 \supset p_2 \Rightarrow$	$\dfrac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \supset \varphi \Rightarrow \Delta_1, \Delta_2}$
$(\supset\Rightarrow_0)$	$\langle\ \Rightarrow p_1, \pi_0\rangle, \langle p_2 \Rightarrow\ , \pi\rangle / p_1 \supset p_2 \Rightarrow$ where $\pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}p_1\rangle\}$	$\dfrac{\Gamma_1 \Rightarrow \psi, \Delta \quad \Gamma_2, \varphi \Rightarrow}{\Gamma_1, \Gamma_2, \psi \supset \varphi \Rightarrow \Delta}$
$(\Rightarrow\supset^*)$	$\langle p_1 \Rightarrow p_2, \pi\rangle /\ \Rightarrow p_1 \supset p_2$ where $\pi = \{\langle \texttt{F}{:}p_1 \supset p_2, \texttt{F}{:}p_1 \supset p_2\rangle\}$	$\dfrac{\psi_1 \supset \varphi_1, \ldots, \psi_n \supset \varphi_n, \psi \Rightarrow \varphi}{\psi_1 \supset \varphi_1, \ldots, \psi_n \supset \varphi_n \Rightarrow \psi \supset \varphi}$

formulation as basic systems. In the sequel, we will return to some of these basic systems, provide a semantics for them, and use it to study their proof-theoretic properties.

Note: *This paper deals only with propositional logics. Henceforth, when we mention a known Gentzen-type system, we refer only to its propositional part.*

Example 3.7 (**LK**). The most important example of a multiple-conclusion sequent system is of-course Gentzen's system LK for classical logic [Gentzen 1964]. This system can be straightforwardly presented as a basic system (which we denote by **LK**), in which the only context-relation is π_0 (see Definition 3.2). The rules in $\Upsilon_{\mathbf{LK}}$ are all "well-behaved", as they introduce exactly one connective on exactly one side of a sequent. Each rule is either a right introduction rule or a left introduction rule associated with some connective. For example, the rules for implication are $(\supset\Rightarrow)$ (see Table I) and $(\Rightarrow\supset)$, which is the basic rule $\langle p_1 \Rightarrow p_2, \pi_0\rangle /\ \Rightarrow p_1 \supset p_2$. Applications of $(\Rightarrow\supset)$ allow to infer $\Gamma \Rightarrow \psi \supset \varphi, \Delta$ from $\Gamma, \psi \Rightarrow \varphi, \Delta$. Similarly, the rules for conjunction are the following rules:

$(\wedge \Rightarrow) \quad \langle p_1, p_2 \Rightarrow\ , \pi_0\rangle / p_1 \wedge p_2 \Rightarrow \qquad (\Rightarrow \wedge) \quad \langle \Rightarrow p_1, \pi_0\rangle, \langle \Rightarrow p_2, \pi_0\rangle / \Rightarrow p_1 \wedge p_2$

Applications of these rules have the form:

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2 \Rightarrow \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \psi \wedge \varphi, \Delta_1, \Delta_2}$$

The relation $\vdash_{\mathbf{LK}}$ is the usual consequence relation of classical logic between sets of sequents and sequents.

Remark 3.8. The family of canonical systems (of which **LK** is just an important example) was defined and studied in [Avron and Lev 2005]. The sequent systems of this family are all "well-behaved". In particular, they allow context formulas on both sides of the sequent. Clearly, every canonical system can be presented as a basic system $\mathbf{G}$ where $\Pi_{\mathbf{G}} = \{\pi_0\}$.

Example 3.9 (**LJ**). The most famous sequent system for intuitionistic logic is of-course Gentzen's **LJ** [Gentzen 1964]. This system manipulates *single-conclusion* sequents, and thus it does not fall in our framework. However, there is an equivalent multiple-conclusion system, called LJ' in [Takeuti 1975]. This system is naturally presented as a basic system, which we call **LJ**. In addition to π_0, **LJ** uses another context-relation, π_{int} which is the relation $\{\langle \mathtt{F}{:}p_1, \mathtt{F}{:}p_1\rangle\}$, so that π_{int}-instances are all pairs of the form $\langle \Gamma \Rightarrow\ , \Gamma \Rightarrow\ \rangle$. The rules of **LJ** are the same rules of **LK**, except for $(\Rightarrow\supset)$, in which π_{int} is used instead of π_0.[3] This rule has now the form $\langle p_1 \Rightarrow p_2, \pi_{\mathsf{int}}\rangle / \Rightarrow p_1 \supset p_2$, and its applications allow to infer sequents of the form $\Gamma \Rightarrow \psi \supset \varphi$ from $\Gamma, \psi \Rightarrow \varphi$ (note that right context-formulas are forbidden).

Example 3.10 (**BLJ**). Bi-intuitionistic logic is the extension of intuitionistic logic with a binary connective dual to implication (denoted here by $\prec$) (see e.g., [Goré and Postniece 2010]). A sequent system for this logic (see [Pinto and Uustalu 2009]) can be presented as the basic system, which we call **BLJ**, obtained from **LJ** by adding the following rules for $\prec$:

$(\prec\Rightarrow) \quad \langle\ p_1 \Rightarrow p_2, \pi_{\mathsf{d}}\rangle / p_1 \prec p_2 \Rightarrow \qquad (\Rightarrow\prec) \quad \langle \Rightarrow p_1, \pi_0\rangle, \langle p_2 \Rightarrow\ , \pi_0\rangle / \Rightarrow p_1 \prec p_2$

where π_{d} is the relation $\{\langle \mathtt{T}{:}p_1, \mathtt{T}{:}p_1\rangle\}$. π_{d}-instances are all pairs of the form $\langle \Rightarrow \Delta, \Rightarrow \Delta\rangle$. Applications of these rules have the forms:

$$(\prec\Rightarrow) \quad \frac{\psi \Rightarrow \varphi, \Delta}{\psi \prec \varphi \Rightarrow \Delta} \qquad (\Rightarrow\prec) \quad \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \psi \prec \varphi, \Delta_1, \Delta_2}$$

Example 3.11 (**PLJ**). Sequent systems for several families of paraconsistent logics are defined and studied in [Avron 2007]. All of these sequent systems belong to the family of basic systems. For example, we present the system $PLJ(\{(\Rightarrow \neg \supset)\})$ from [Avron 2007] as a basic system, which we call **PLJ**. **PLJ** is obtained from **LJ** by adding the following rules (the language of **LJ** is augmented with a unary connective $\neg$):

$(\Rightarrow \neg) \quad \langle p_1 \Rightarrow\ , \pi_0\rangle / \Rightarrow \neg p_1 \qquad (\Rightarrow \neg \supset) \quad \langle \Rightarrow p_1, \pi_0\rangle, \langle \Rightarrow \neg p_2, \pi_0\rangle / \Rightarrow \neg(p_1 \supset p_2)$

[3]Similar modification is needed in the right introduction rule of negation. However, here we take the language of **LJ** to consist of $\wedge, \vee, \supset$ and $\perp$, and $\neg\varphi$ is defined by $\varphi \supset\perp$.

Applications of these rules have the forms:

$$(\Rightarrow \neg)\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\psi, \Delta} \qquad (\Rightarrow \neg \supset)\ \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2 \Rightarrow \neg\varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \neg(\psi \supset \varphi), \Delta_1, \Delta_2}$$

Example 3.12 (*Systems for Modal Logics*). Ordinary sequent systems for modal logics are surveyed in [Wansing 2002] and [Poggiolesi 2010]. All of them belong to the family of basic systems. As examples we present as basic systems six of them (used later to demonstrate certain semantic phenomena). For this purpose, we use the rules (K), (B) and $(S5)$ (in addition to some of the rules presented in Table I). These three rules all have the form $\langle \Rightarrow p_1, \pi\rangle / \ \Rightarrow \Box p_1$, where π is as follows:

$(K)\quad \pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}\Box p_1\rangle\}$
$(B)\quad \pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}\Box p_1\rangle, \langle \texttt{T}{:}\Box p_1, \texttt{T}{:}p_1\rangle\}$
$(S5)\quad \pi = \{\langle \texttt{F}{:}\Box p_1, \texttt{F}{:}\Box p_1\rangle, \langle \texttt{T}{:}\Box p_1, \texttt{T}{:}\Box p_1\rangle\}$

Applications of these rules have the form:

$$(K)\ \frac{\Gamma \Rightarrow \psi}{\Box\Gamma \Rightarrow \Box\psi} \qquad (B)\ \frac{\Gamma \Rightarrow \psi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\psi, \Delta} \qquad (S5)\ \frac{\Box\Gamma \Rightarrow \psi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\psi, \Box\Delta}$$

Based on **LK**, six basic systems are defined as follows:

$$\mathbf{K} = \mathbf{LK} + (K) \qquad \mathbf{K4} = \mathbf{LK} + (K4) \qquad \mathbf{KD} = \mathbf{K} + (D)$$

$$\mathbf{KB} = \mathbf{LK} + (B) \qquad \mathbf{S4} = \mathbf{LK} + (S4) + (T) \qquad \mathbf{S5} = \mathbf{LK} + (S5) + (T)$$

Note that these systems are used for a language in which $\Box$ is the only primitive modal operator and $\Diamond\psi$ is defined as $\neg\Box\neg\psi$. For a language with two dual primitive modal operators, one should modify some of the context-relations in the rules for $\Box$, and add dual rules for $\Diamond$. For example, for the logic $S4$ the following four schemas are used:

$$\frac{\Box\Gamma \Rightarrow \psi, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box\psi, \Diamond\Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box\psi \Rightarrow \Delta} \qquad \frac{\Box\Gamma, \psi \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond\psi \Rightarrow \Diamond\Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Diamond\psi, \Delta}$$

Example 3.13 (**GL**). The logic GL (the modal logic of provability, see [Verbrugge 2010]) is obtained by adding the axiom $\Box(\Box\psi \supset \psi) \supset \Box\psi$ to the usual Hilbert system for the modal logic K. In addition, GL has a well-known sequent system (see e.g., [Leivant 1981; Sambin and Valentini 1982; Avron 1984]), that can be presented as a basic system, which we call **GL**. **GL** is obtained from **LK** by adding (GL) – the basic rule $\langle \Box p_1 \Rightarrow p_1, \pi\rangle / \ \Rightarrow \Box p_1$, where $\pi = \{\langle \texttt{F}{:}p_1, \texttt{F}{:}\Box p_1\rangle, \langle \texttt{F}{:}\Box p_1, \texttt{F}{:}\Box p_1\rangle\}$. Applications of (GL) allow to infer sequents of the form $\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\psi$ from $\Gamma_1, \Box\Gamma_2, \Box\psi \Rightarrow \psi$.

Example 3.14 (**IS5**). Sequent systems for intuitionistic modal logics provide an interesting source of examples to be studied in the framework of basic systems, as they naturally employ more than one (non-trivial) context-relation. For example, the system G_3 from [Ono 1977] can be presented as the basic system obtained from **LJ** by adding the rules $(S5)$ and (T) (mentioned above) (the language of **LJ** is augmented with a unary connective $\Box$). In the sequel, we refer to this basic system as **IS5**.

Example 3.15. In [Lavendhomme and Lucas 2000] several sequent calculi for weak modal logics are introduced. All of them belong to the family of basic systems. For example, the first system from [Lavendhomme and Lucas 2000] (called

$Mseq$ there), is the basic system obtained from **LK** by adding (M) – the basic rule $\langle p_1 \Rightarrow p_2, \emptyset\rangle / \Box p_1 \Rightarrow \Box p_2$. Its applications allow to infer sequents of the form $\Box\psi \Rightarrow \Box\varphi$ from $\psi \Rightarrow \varphi$.

Example 3.16. Several sequent systems for logics of strict implication are provided in [Ishigaki and Kashima 2008], and can be presented as basic systems. For example, $\mathbf{GS4}^I$ (from [Ishigaki and Kashima 2008]) is equivalent to the basic system, obtained from **LK** by replacing the rule $(\Rightarrow\supset)$ with the rule $(\Rightarrow\supset^*)$ (see Table I).[4]

Example 3.17 (**GP**). Primal logic was defined and studied in [Beklemishev and Gurevich 2012]. As explained there, this logic is used in the context of the access control language DKAL. We consider here primal logic with disjunction and quotations, and present the sequent system GP from [Beklemishev and Gurevich 2012] as a basic system, that we call **GP**. The language of the system **GP** consists of the classical connectives $\wedge, \vee, \bot, \top$, a binary connective $\rightarrow$, and a set of unary connectives denoted by q `said` and q `implied` for every q in some set Q. The rules of this system are the rules of **LK** for $\wedge, \vee, \bot, \top$, and the following rules for the other connectives (for every $q \in Q$):

$$(\rightarrow\Rightarrow) \quad \langle \Rightarrow p_1, \pi_0\rangle, \langle p_2 \Rightarrow\ , \pi_0\rangle / p_1 \rightarrow p_2 \Rightarrow \qquad (\Rightarrow\rightarrow) \quad \langle \Rightarrow p_2, \pi_0\rangle / \ \Rightarrow p_1 \rightarrow p_2$$

$$(\mathtt{Said}_q) \quad \langle \Rightarrow p_1, \pi_s^q\rangle / \ \Rightarrow q\ \mathtt{said}\ p_1 \qquad (\mathtt{Implied}_q) \quad \langle \Rightarrow p_1, \pi_i^q\rangle / \ \Rightarrow q\ \mathtt{implied}\ p_1$$

where $\pi_s^q = \{\langle \mathtt{F}{:}p_1, \mathtt{F}{:}q\ \mathtt{said}\ p_1\rangle\}$, and $\pi_i^q = \pi_s^q \cup \{\langle \mathtt{F}{:}p_1, \mathtt{F}{:}q\ \mathtt{implied}\ p_1\rangle\}$ for every $q \in Q$. Applications of these rules have the form:

$$(\rightarrow\Rightarrow) \ \frac{\Gamma_1 \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \psi \rightarrow \varphi \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\rightarrow) \ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \rightarrow \varphi, \Delta}$$

$$(\mathtt{Said}_q) \ \frac{\Gamma \Rightarrow \psi}{q\ \mathtt{said}\ \Gamma \Rightarrow q\ \mathtt{said}\ \psi} \qquad (\mathtt{Implied}_q) \ \frac{\Gamma, \Delta \Rightarrow \psi}{q\ \mathtt{said}\ \Gamma, q\ \mathtt{implied}\ \Delta \Rightarrow q\ \mathtt{implied}\ \psi}$$

4. KRIPKE-STYLE SEMANTICS FOR BASIC SYSTEMS

In this section we introduce a method for providing Kripke-style semantics for any given basic system. In fact, we show how to uniformly recognize classes of "Kripke valuations" for which a given basic system is sound and complete. Note that we consider here proofs from a set of assumptions (or "non-logical axioms"), and so we actually obtain *strong* soundness and completeness. We begin with a general notion of a (Kripke) valuation, and the consequence relation associated with a given set of such valuations.

Definition 4.1. A *(Kripke) valuation* is a function v from the Cartesian product of some set W_v (of "worlds") and $wff_{\mathcal{L}}$ to $\{\mathtt{T}, \mathtt{F}\}$ (i.e., $v : W_v \times wff_{\mathcal{L}} \rightarrow \{\mathtt{T}, \mathtt{F}\}$).

Signed formulas and sequents are interpreted as follows:

Definition 4.2. Let v be a valuation.

(1) A signed formula $\mathtt{X}{:}\psi$ (for $\mathtt{X} \in \{\mathtt{T}, \mathtt{F}\}$) is *true* in some $w \in W_v$ with respect to v (denoted by: $w \models^v \mathtt{X}{:}\psi$) if $v(w, \psi) = \mathtt{X}$.
(2) A sequent s is *true* in $w \in W_v$ with respect to v (denoted by: $w \models^v s$) if $w \models^v \alpha$ for some $\alpha \in s$.

[4]Note that $\mathbf{GS4}^I$ includes also a less standard rule (denoted by $(\rightarrow \mathsf{K}^I)$ in [Ishigaki and Kashima 2008]) that cannot be presented as one basic rule. However, one can show that it is redundant in $\mathbf{GS4}^I$.

(3) A sequent s is *true* in $W \subseteq W_v$ with respect to v (denoted by: $W \models^v s$) if $w \models^v s$ for every $w \in W$.
(4) v is a *model* of:
   (a) a *sequent* s (denoted by: $\models^v s$) if $W_v \models^v s$.
   (b) a *set* $\mathcal{S}$ *of sequents* (denoted by: $\models^v \mathcal{S}$) if $\models^v s$ for every $s \in \mathcal{S}$.

The consequence relation induced by a set $\mathbf{M}$ of valuations is defined as follows:

Definition 4.3. A sequent s *follows* from a set $\mathcal{S}$ of sequents with respect to a set $\mathbf{M}$ of valuations (denoted by:[5] $\mathcal{S} \vdash_{\mathbf{M}} s$) if for every $v \in \mathbf{M}$, $\models^v s$ whenever $\models^v \mathcal{S}$.

Next we turn to identify sets of valuations for which a given basic system $\mathbf{G}$ is sound and complete. Here the idea is that each syntactic ingredient of $\mathbf{G}$ imposes a certain constraint on valuations. Taking all of these constraints together, we get a set of valuations for which $\mathbf{G}$ is sound and complete. The exact constraints are formulated below.

First, we associate with each context-relation π of $\mathbf{G}$ a binary ("accessibility") relation on W_v, and enforce certain conditions on the associated accessibility relations.

Definition 4.4. Given a set W, a $\langle \mathbf{G}, W\rangle$-coupling is a function assigning a binary relation on W to every $\pi \in \Pi_{\mathbf{G}}$.

Definition 4.5. Let v be a valuation.

(1) Given a context-relation π, R^v_π denotes the binary relation on W_v defined by: $w_1 R^v_\pi w_2$ iff for every signed formulas α and β, if $\alpha\bar{\pi}\beta$ and $w_2 \models^v \alpha$ then $w_1 \models^v \beta$.
(2) Let $\mathbf{G}$ be a basic system, and let $\pi \in \Pi_{\mathbf{G}}$. v *respects* π for a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$, if $\Re(\pi) \subseteq R^v_\pi$.

Example 4.6. Let $\mathbf{G}$ be a basic system and v be a valuation. Consider the context-relation π_0. By definition, $\alpha\bar{\pi}_0\beta$ iff $\alpha = \beta$. Thus $w_1 R^v_{\pi_0} w_2$ iff for every signed formula α such that $w_2 \models^v \alpha$, we have $w_1 \models^v \alpha$. Equivalently, $w_1 R^v_{\pi_0} w_2$ iff $v(w_1, \psi) = v(w_2, \psi)$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$. Thus v respects π_0 for a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$ iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi_0) w_2$, we have that $v(w_1, \psi) = v(w_2, \psi)$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$.

Notation 4.7. Given a valuation v, we denote by $\mathbf{\mathit{Id}}_v$ the identity relation on W_v. Following Example 4.6, observe that $\mathbf{\mathit{Id}}_v \subseteq R^v_{\pi_0}$ for every valuation v, and that v respects π_0 for $\Re$ if $\Re(\pi_0) = \mathbf{\mathit{Id}}_v$.

Example 4.8. Let $\mathbf{G}$ be a basic system and v be a valuation. Suppose that $\pi = \{\langle \mathtt{F}{:}p_1, \mathtt{F}{:}p_1\rangle\}$ appears in $\Pi_{\mathbf{G}}$ (as happens in $\mathbf{LJ}$ for example). Here, $\alpha\bar{\pi}\beta$ iff $\alpha = \beta = \mathtt{F}{:}\psi$ for some $\psi \in \mathit{wff}_{\mathcal{L}}$. Thus $w_1 R^v_\pi w_2$ iff for every signed formula α of the form $\mathtt{F}{:}\psi$, if $w_2 \models^v \alpha$, then $w_1 \models^v \alpha$. Equivalently, $w_1 R^v_\pi w_2$ iff $v(w_2, \psi) = \mathtt{F}$ implies that $v(w_1, \psi) = \mathtt{F}$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$. It follows that v respects π for a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$ iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi) w_2$, we have that $v(w_1, \psi) = \mathtt{T}$ implies $v(w_2, \psi) = \mathtt{T}$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$. This is the persistence (or monotonicity) condition used in intuitionistic Kripke semantics.

Example 4.9. Let $\mathbf{G}$ be a basic system and v be a valuation. Suppose that $\pi = \{\langle \mathtt{F}{:}p_1, \mathtt{F}{:}\Box p_1\rangle\}$ appears in $\Pi_{\mathbf{G}}$ (as happens in $\mathbf{K}$ for example). Here, $\alpha\bar{\pi}\beta$ iff $\alpha = \mathtt{F}{:}\psi$ and $\beta = \mathtt{F}{:}\Box\psi$ for some $\psi \in \mathit{wff}_{\mathcal{L}}$. Thus $w_1 R^v_\pi w_2$ iff $w_2 \models^v \mathtt{F}{:}\psi$ implies $w_1 \models^v \mathtt{F}{:}\Box\psi$ for every formula ψ. Equivalently, $w_1 R^v_\pi w_2$ iff for every $\psi \in \mathit{wff}_{\mathcal{L}}$, if $v(w_1, \Box\psi) = \mathtt{T}$ then $v(w_2, \psi) = \mathtt{T}$. It follows that v respects π for a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$ iff for every $w_1, w_2 \in W_v$ such that $w_1 \Re(\pi) w_2$, we have that $v(w_1, \Box\psi) = \mathtt{T}$ implies $v(w_2, \psi) = \mathtt{T}$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$. Roughly speaking, this provides "one half" of the usual semantics of $\Box$.

[5] We reserve the symbol $\models$ for the satisfaction relation, and use $\vdash$ for consequence relations.

Example 4.10. Let $\mathbf{G}$ be a basic system and v be a valuation. Suppose that $\pi = \emptyset$ appears in $\Pi_{\mathbf{G}}$ (this context-relation is used in the rule (M), see Example 3.15). Since there do not exist signed formulas α, β such that $\alpha\bar{\emptyset}\beta$, $wR^v_\pi u$ trivially holds for every $w, u \in W_v$. Thus $R^v_\pi = W_v \times W_v$, and v trivially respects π for every $\langle \mathbf{G}, W_v \rangle$-coupling.

Next we formulate the effect of the basic rules appearing in a basic system.

Notation 4.11. Given a set W, a binary relation $R \subseteq W \times W$, and an element $w \in W$, we denote the set $\{u \in W \mid wRu\}$ by $R[w]$.

Definition 4.12. Let $r = \langle s_1, \pi_1 \rangle, \ldots, \langle s_n, \pi_n \rangle / C$ be a basic rule of a basic system $\mathbf{G}$. A valuation v *respects* r for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$, if the following condition holds for every $w \in W_v$ and substitution σ: if $\Re(\pi_i)[w] \models^v \sigma(s_i)$ for every $1 \leq i \leq n$, then $w \models^v \sigma(C)$.

Example 4.13. Suppose that a basic system $\mathbf{G}$ contains a rule r of the form $\langle \Rightarrow p_1, \pi \rangle / \Rightarrow \Box p_1$ (such a rule appears in various basic systems for modal logics presented in Section 3.1). A valuation v respects r for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$ iff for every $w \in W_v$ and substitution σ: if $\Re(\pi)[w] \models^v \sigma(\Rightarrow p_1)$, then $w \models^v \sigma(\Rightarrow \Box p_1)$. Hence v respects r for $\Re$ iff for every $w \in W_v$ and formula ψ: if $v(u, \psi) = \mathtt{T}$ for every $u \in \Re(\pi)[w]$, then $v(w, \Box\psi) = \mathtt{T}$. Roughly speaking, this provides the "other half" of the usual semantics of $\Box$ (see Example 4.9).

Example 4.14. Suppose that a basic system $\mathbf{G}$ contains a rule r of the form $\langle \Rightarrow p_1, \pi \rangle, \langle p_2 \Rightarrow\ , \pi \rangle / p_1 \supset p_2 \Rightarrow$ (a rule of this form appears in $\mathbf{LK}$ and $\mathbf{LJ}$ with $\pi = \pi_0$). A valuation v respects r for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$ iff for every $w \in W_v$ and substitution σ: if $\Re(\pi)[w] \models^v \sigma(\Rightarrow p_1)$ and $\Re(\pi)[w] \models^v \sigma(p_2 \Rightarrow\ )$, then $w \models^v \sigma(p_1 \supset p_2 \Rightarrow\ )$. Hence v respects r for $\Re$ iff for every $w \in W_v$ and two formulas ψ, φ: if $v(u, \psi) = \mathtt{T}$ and $v(u, \varphi) = \mathtt{F}$ for every $u \in \Re(\pi)[w]$, then $v(w, \psi \supset \varphi) = \mathtt{F}$.

Example 4.15. Suppose that a basic system $\mathbf{G}$ contains a rule r of the form $\langle p_1 \Rightarrow p_2, \pi \rangle / \Rightarrow p_1 \supset p_2$ (for example, a rule of this form appears in $\mathbf{LK}$ with $\pi = \pi_0$, and in $\mathbf{LJ}$ with $\pi = \pi_{\mathsf{int}}$). A valuation v respects r for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$ iff for every $w \in W_v$ and substitution σ: if $\Re(\pi)[w] \models^v \sigma(p_1 \Rightarrow p_2)$, then $w \models^v \sigma(\Rightarrow p_1 \supset p_2)$. Equivalently, v respects r for $\Re$ iff for every $w \in W_v$ and two formulas ψ, φ: if $v(u, \psi) = \mathtt{F}$ or $v(u, \varphi) = \mathtt{T}$ for every $u \in \Re(\pi)[w]$, then $v(w, \psi \supset \varphi) = \mathtt{T}$.

Example 4.16. Suppose that a basic system $\mathbf{G}$ contains a rule r of the form $\langle \emptyset, \pi \rangle / \emptyset$ (for example, this is the form of the rule (D), see Table I). Applications of this rule allow to infer a sequent s' from a sequent s whenever $\langle s, s' \rangle$ is a π-instance. A valuation v respects r for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$ iff for every $w \in W_v$ and substitution σ: if $\Re(\pi)[w] \models^v \emptyset$, then $w \models^v \emptyset$ (note that $\sigma(\emptyset) = \emptyset$ for every substitution σ). Since the empty sequent is not true in any world, this condition would hold iff for every world w there exists some $u \in \Re(\pi)[w]$. In other words, v respects this rule for $\Re$ iff $\Re(\pi)$ is a serial relation.

Now we identify a set of valuations for which a given basic system is sound and complete, and state our first soundness and completeness theorem. We omit its proof, since in Section 5.2 we prove a generalization of this result (see Remark 5.18).

Definition 4.17. Let $\mathbf{G}$ be a basic system, and v be a valuation.

(1) v *respects* $\mathbf{G}$ for a $\langle \mathbf{G}, W_v \rangle$-coupling $\Re$, if it respects for $\Re$ every $\pi \in \Pi_{\mathbf{G}}$ and $r \in \Upsilon_{\mathbf{G}}$.
(2) v is called $\mathbf{G}$-*legal* if it respects $\mathbf{G}$ for some $\langle \mathbf{G}, W_v \rangle$-coupling.

THEOREM 4.18. *Let* $\mathbf{G}$ *be a basic system, and let* $\mathbf{M}$ *be the set of all* $\mathbf{G}$*-legal valuations. Then* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$. *In other words: there exists a proof in* $\mathbf{G}$ *of a sequent* s *from a set* $\mathcal{S}$ *of sequents, iff every* $\mathbf{G}$*-legal valuation which is a model of* $\mathcal{S}$*, is also a model of* s.

Remark 4.19. To show that applications of some basic rule r are sound (i.e., that if a valuation v is a model of the premises of an application of r then it is also a model of its conclusion), we use the fact that **G**-legal valuations respect r and every context-relation π occurring in the premises of r for $\Re$. Note that we do not require that v respects for $\Re$ the predetermined structural rules (id), (cut), $(W \Rightarrow)$, and $(\Rightarrow W)$: their soundness is a priori guaranteed for every valuation.

In order to obtain a very general soundness result, we chose above the set M to be as large as possible. On the other hand, a stronger completeness result can be obtained by considering a smaller set of valuations:

Definition 4.20. Given a basic system **G** and a valuation v, $\Re^v_{\mathbf{G}}$ denotes the $\langle \mathbf{G}, W_v \rangle$-coupling defined by $\Re^v_{\mathbf{G}}(\pi_0) = \boldsymbol{Id}_v$, and $\Re^v_{\mathbf{G}}(\pi) = R^v_\pi$ for every other $\pi \in \Pi_{\mathbf{G}}$.

Definition 4.21. Let **G** be a basic system. A valuation v is called *strongly* **G***-legal* if it respects for $\Re^v_{\mathbf{G}}$ every $r \in \Upsilon_{\mathbf{G}}$.

Remark 4.22. Following Example 4.6, every valuation v respects π_0 for $\Re^v_{\mathbf{G}}$. In addition, every valuation v obviously respects every other $\pi \in \Pi_{\mathbf{G}}$ for $\Re^v_{\mathbf{G}}$. Thus, a strongly **G**-legal valuation is **G**-legal.

Definition 4.23. A valuation v is called *differentiated* if $R^v_{\pi_0} = \boldsymbol{Id}_v$.

Remark 4.24. Following Example 4.6, a valuation v is differentiated iff $w_1 = w_2$ whenever $v(w_1, \psi) = v(w_2, \psi)$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$. The name of this property is taken from [Chagrov and Zakharyaschev 1997].

THEOREM 4.25. *Let* **G** *be a basic system, and let* M *be the set of all* strongly **G**-legal differentiated *valuations. Then* $\vdash_{\mathbf{G}} = \vdash_{\mathrm{M}}$. *In other words: there exists a proof in* **G** *of a sequent* s *from a set* $\mathcal{S}$ *of sequents, iff every strongly* **G***-legal differentiated valuation which is a model of* $\mathcal{S}$*, is also a model of* s*.*

Again, the proof is omitted since we prove a more general result in Section 5.2 (see Remark 5.18). The two last theorems are combined in the following corollary, that provides an “interval” of possible semantics for a given basic system.

COROLLARY 4.26. *Let* **G** *be a basic system. Then,* $\vdash_{\mathbf{G}} = \vdash_{\mathrm{M}}$ *for every set* M *of* **G***-legal valuations containing all strongly* **G***-legal differentiated valuations.*

Corollary 4.26 provides a general soundness and completeness result applicable to every basic system **G**. Its exact content depends on the choice of set of valuations M. M should meet two conditions: first, it should contain only **G**-legal valuations; and second, it should contain *all* strongly **G**-legal differentiated valuations. This easily entails various different soundness and completeness theorems for different families of basic systems. Indeed, using the structure of the context-relations in $\Pi_{\mathbf{G}}$, it is possible to recognize some properties common to all strongly **G**-legal differentiated valuations, and derive specific soundness and completeness results with respect to the set of all **G**-legal valuations satisfying these properties. The following proposition is particularly useful for this purpose.

Notation 4.27. Given a signed formula of the form F:ψ, we denote by $\overline{\text{F:}\psi}$ the signed formula T:ψ. Similarly, $\overline{\text{T:}\psi}$ denotes the signed formula F:ψ.

PROPOSITION 4.28. *Let* v *be a valuation, and* π_1, π_2, π_3 *be context-relations.*

(*1*) *Suppose that* $\bar{\pi}_3 = \bar{\pi}_1 \cup \bar{\pi}_2$. *Then* $R^v_{\pi_3} = R^v_{\pi_2} \cap R^v_{\pi_1}$. *In particular, if* $\bar{\pi}_1 \subseteq \bar{\pi}_2$ *then* $R^v_{\pi_2} \subseteq R^v_{\pi_1}$.

(2) *Suppose that* $\bar{\pi}_3 \subseteq \bar{\pi}_1 \circ \bar{\pi}_2$.[6] *Then* $R^v_{\pi_2} \circ R^v_{\pi_1} \subseteq R^v_{\pi_3}$. *In particular, if* $\bar{\pi}_1 \subseteq \bar{\pi}_1 \circ \bar{\pi}_1$ *then* $R^v_{\pi_1}$ *is a transitive relation.*

(3) *Suppose that* $\overline{\beta}\bar{\pi}_1\overline{\alpha}$ *whenever* $\alpha\bar{\pi}_2\beta$. *Then* $R^v_{\pi_1} \subseteq (R^v_{\pi_2})^{-1}$. *In particular,* (i) *if* $\alpha\bar{\pi}_1\beta$ *implies* $\overline{\beta}\bar{\pi}_2\overline{\alpha}$ *and vice-versa, then* $R^v_{\pi_2} = (R^v_{\pi_1})^{-1}$, *and* (ii) *if* $\alpha\bar{\pi}_1\beta$ *implies* $\overline{\beta}\bar{\pi}_1\overline{\alpha}$ *and vice-versa, then* $R^v_{\pi_1}$ *is a symmetric relation.*

PROOF.

(1) Suppose first that $uR^v_{\pi_3}w$. By definition, this means that for every signed formulas α, β, if $\alpha\bar{\pi}_3\beta$ and $w \models^v \alpha$ then $u \models^v \beta$. Since $\bar{\pi}_1 \subseteq \bar{\pi}_3$, this implies that for every signed formulas α, β, if $\alpha\bar{\pi}_1\beta$ and $w \models^v \alpha$ then $u \models^v \beta$. Hence, $uR^v_{\pi_1}w$. Similarly, $uR^v_{\pi_2}w$. For the converse, suppose that $uR^v_{\pi_1}w$ and $uR^v_{\pi_2}w$. By definition, this means that for every signed formulas α, β, if $\alpha\bar{\pi}_1\beta$ or $\alpha\bar{\pi}_2\beta$, and $w \models^v \alpha$ then $u \models^v \beta$. Since $\bar{\pi}_3 \subseteq \bar{\pi}_1 \cup \bar{\pi}_2$, this implies that for every signed formulas α, β, if $\alpha\bar{\pi}_3\beta$ and $w \models^v \alpha$ then $u \models^v \beta$. Hence, $uR^v_{\pi_3}w$.
(2) Let $w, u \in W_v$ such that $wR^v_{\pi_2} \circ R^v_{\pi_1}u$. Then there exists $z \in W_v$, such that $wR^v_{\pi_2}z$ and $zR^v_{\pi_1}u$. We show that $wR^v_{\pi_3}u$. Let α, β be signed formulas, such that $\alpha\bar{\pi}_3\beta$, and $u \models^v \alpha$. Therefore, there exists a signed formula γ such that $\alpha\bar{\pi}_1\gamma$, and $\gamma\bar{\pi}_2\beta$. Since $zR^v_{\pi_1}u$, we have $z \models^v \gamma$. Since $wR^v_{\pi_2}z$, we have $w \models^v \beta$.
(3) Let $wR^v_{\pi_1}u$. We show that $uR^v_{\pi_2}w$. Let α, β be signed formulas, such that $\alpha\bar{\pi}_2\beta$ and $w \models^v \alpha$. This implies that $\overline{\beta}\bar{\pi}_1\overline{\alpha}$. Now, since $w \models^v \alpha$, we have that $w \not\models^v \overline{\alpha}$. Since $wR^v_{\pi_1}u$, we have $u \not\models^v \overline{\beta}$. This entails that $u \models^v \beta$. □

The following soundness and completeness results are easily obtained using Proposition 4.28:

COROLLARY 4.29. *Let* $\mathbf{G}$ *be a basic system, and suppose* $\bar{\pi} = \bar{\pi} \circ \bar{\pi}$ *for some* $\pi \in \Pi_{\mathbf{G}}$. *Let* M *be the set of all valuations* v *for which there exists a* $\langle \mathbf{G}, W_v\rangle$*-coupling* $\Re$ *such that* $\Re(\pi)$ *is a transitive relation, and* v *respects* $\mathbf{G}$ *for* $\Re$. *Then,* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

PROOF. Clearly, $\mathbf{M}$ is a set of $\mathbf{G}$-legal valuations. By Corollary 4.26 it suffices to show that $\mathbf{M}$ contains all strongly $\mathbf{G}$-legal valuations. Let v be such a valuation. Then v respects $\mathbf{G}$ for $\Re^v_{\mathbf{G}}$. By Proposition 4.28 (Item 2), $\Re^v_{\mathbf{G}}(\pi) = R^v_{\pi}$ is transitive. It follows that $v \in \mathbf{M}$. □

COROLLARY 4.30. *Let* $\mathbf{G}$ *be a basic system, and suppose that for some* $\pi \in \Pi_{\mathbf{G}}$, $\bar{\pi}$ *includes only pairs of the form* $\langle \alpha, \alpha\rangle$. *Let* $\mathbf{M}$ *be the set of all valuations* v *for which there exists a* $\langle \mathbf{G}, W_v\rangle$*-coupling* $\Re$ *such that* $\Re(\pi)$ *is a reflexive relation, and* v *respects* $\mathbf{G}$ *for* $\Re$. *Then,* $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M}}$.

PROOF. Clearly, $\mathbf{M}$ is a set of $\mathbf{G}$-legal valuations. By Corollary 4.26 it suffices to show that $\mathbf{M}$ contains all strongly $\mathbf{G}$-legal valuations. Let v be such a valuation. Then v respects $\mathbf{G}$ for $\Re^v_{\mathbf{G}}$. Proposition 4.28 (Item 1) entails that $R^v_{\pi_0} \subseteq R^v_{\pi}$. Since $\mathit{Id}_v \subseteq R^v_{\pi_0}$, $\Re^v_{\mathbf{G}}(\pi) = R^v_{\pi}$ is reflexive. It follows that $v \in \mathbf{M}$. □

4.1. Examples of Semantics for Basic Systems

Corollary 4.26 is a central result of this paper. We devote this section to provide various examples of its consequences, by applying it to some of the basic systems presented in Section 3.1. In particular, many fundamental strong soundness and completeness theorems for known logics and systems are easily obtained as special cases.[7]

[6] Given two relations $R, S \subseteq A^2$, $aR \circ Sb$ if there exists some $c \in A$ such that aRc and cSb.
[7] Everywhere in this paper we use the terms "strong soundness" and "strong completeness" in their general sense. Thus a sequent system $\mathbf{G}$ is strongly complete for a semantics $\mathbf{Sem}$ (i.e. a set of "models"), if $\mathcal{S} \vdash_{\mathbf{G}} s$

Example 4.31 (**LK**). Let $\mathbf{M_{LK}}$ be the set of valuations that respect the usual truth-tables of the classical connectives in each world (e.g., $v(w, \psi \supset \varphi) = \mathtt{T}$ iff $v(w, \psi) = \mathtt{F}$ or $v(w, \varphi) = \mathtt{T}$). Corollary 4.26 entails that **LK** is strongly sound and complete with respect to the set $\mathbf{M_{LK}}$. To see this, we prove that $v \in \mathbf{M_{LK}}$ iff v is strongly **LK**-legal. It follows that $\mathbf{M_{LK}}$ is a set of **LK**-legal valuations containing all strongly **LK**-legal differentiated valuations (and so $\vdash_{\mathbf{LK}} = \vdash_{\mathbf{M_{LK}}}$):

(⇒) Let $v \in \mathbf{M_{LK}}$. We show that v respects **LK** for $\Re^v_{\mathbf{LK}}$. Following Example 4.6, since $\Re^v_{\mathbf{LK}}(\pi_0) = \mathit{Id}_v$, v respects π_0 for $\Re^v_{\mathbf{LK}}$. We claim that v respects for $\Re^v_{\mathbf{LK}}$ every $r \in \Upsilon_{\mathbf{LK}}$. We show it for the rules of $\supset$. The other rules are treated similarly. Following Example 4.14 and since $\Re^v_{\mathbf{LK}}(\pi_0) = \mathit{Id}_v$, v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LK}}$ if $v(w, \psi \supset \varphi) = \mathtt{F}$ whenever $v(w, \psi) = \mathtt{T}$ and $v(w, \varphi) = \mathtt{F}$. Following Example 4.15, v respects $(\Rightarrow\supset)$ for $\Re^v_{\mathbf{LK}}$ if $v(w, \psi \supset \varphi) = \mathtt{T}$ whenever either $v(w, \psi) = \mathtt{F}$ or $v(w, \varphi) = \mathtt{T}$. Obviously, these two requirements hold since v respects the truth-table of $\supset$ in each world.

(⇐) Let v be a strongly **LK**-legal valuation. We claim that v respects the usual truth-tables of the classical connectives in each world. Again, we show it only for $\supset$. Since v is strongly **LK**-legal, v respects for $\Re^v_{\mathbf{LK}}$ every $r \in \Upsilon_{\mathbf{LK}}$. Following Example 4.14, since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LK}}$ and $\Re^v_{\mathbf{LK}}(\pi_0) = \mathit{Id}_v$, $v(w, \psi \supset \varphi) = \mathtt{F}$ whenever $v(w, \psi) = \mathtt{T}$ and $v(w, \varphi) = \mathtt{F}$. Similarly, following Example 4.15, since v respects $(\Rightarrow\supset)$ for $\Re^v_{\mathbf{LK}}$, $v(w, \psi \supset \varphi) = \mathtt{T}$ whenever either $v(w, \psi) = \mathtt{F}$ or $v(w, \varphi) = \mathtt{T}$. Together, we have that v respects the truth-table of $\supset$ in each world.

Remark 4.32. It is easy to see that in any basic system **G** like **LK**, in which $\Pi_{\mathbf{G}} = \{\pi_0\}$, it suffices to consider only valuations consisting of a single world. This leads to the usual two-valued semantics of **LK**.

Example 4.33 (**LJ**). Using Corollary 4.26, we obtain a strongly sound and complete semantics for **LJ**, which is practically identical to the usual Kripke semantics for intuitionistic logic. For this purpose, let $\mathbf{M_{LJ}}$ be the set of valuations v that respect the usual truth-tables of $\wedge, \vee, \perp$ in each world, and in addition there exists a partial order $\leq$ on W_v satisfying the following conditions:

(persistence) If $v(w, \psi) = \mathtt{T}$ then $v(u, \psi) = \mathtt{T}$ for every $u \in \leq[w]$.
(implication) $v(w, \psi \supset \varphi) = \mathtt{T}$ iff $v(u, \psi) = \mathtt{F}$ or $v(u, \varphi) = \mathtt{T}$ for every $u \in \leq[w]$.

We show that (1) $\mathbf{M_{LJ}}$ is a set of **LJ**-legal valuations, and (2) $\mathbf{M_{LJ}}$ contains all strongly **LJ**-legal differentiated valuations. Corollary 4.26 implies then that $\vdash_{\mathbf{LJ}} = \vdash_{\mathbf{M_{LJ}}}$.

(1) Let $v \in \mathbf{M_{LJ}}$, and let $\leq$ be a partial order on W_v satisfying *(persistence)* and *(implication)*. Recall that $\Pi_{\mathbf{LJ}} = \{\pi_0, \pi_{\mathsf{int}}\}$. Choose $\Re$ to be the $\langle \mathbf{LJ}, W_v \rangle$-coupling assigning Id_v to π_0, and $\leq$ to π_{int}. Clearly, v respects π_0 for $\Re$. By Example 4.8, condition *(persistence)* ensures that v respects π_{int} for $\Re$. It is straightforward to show that v respects for $\Re$ every $r \in \Upsilon_{\mathbf{LJ}}$. For example, *(implication)* above immediately implies that v respects $(\Rightarrow\supset)$ (see Example 4.15) for $\Re$.
(2) Let v be a strongly **LJ**-legal differentiated valuation. Similarly to Example 4.31, it is easy to show that v respects the usual truth-tables of $\wedge, \vee, \perp$ in each world. We show that $R^v_{\pi_{\mathsf{int}}}$ is a partial order satisfying *(persistence)* and *(implication)*. Since $\bar{\pi_{\mathsf{int}}} \subseteq \bar{\pi_{\mathsf{int}}} \circ \bar{\pi_{\mathsf{int}}}$, Proposition 4.28 (Item 2) entails that $R^v_{\pi_{\mathsf{int}}}$ is transitive. Next, note that $\pi_{\mathsf{int}} \subseteq \pi_0$, hence, by Proposition 4.28 (Item 1), $R^v_{\pi_0} \subseteq R^v_{\pi_{\mathsf{int}}}$; since $\mathit{Id}_v \subseteq R^v_{\pi_0}$,

whenever every model of $\mathcal{S}$ in **Sem** is also a model of s. We note that in the context of modal logics there are related notions of "Kripke completeness". These are particular instances of the general notion we use, in which **Sem** is fully *determined* by classes of Kripke *frames* (but such a frame is not an element of **Sem**).

$R^v_{\pi_{\mathsf{int}}}$ is reflexive. To see that $R^v_{\pi_{\mathsf{int}}}$ is anti-symmetric, suppose that $w_1 R^v_{\pi_{\mathsf{int}}} w_2$ and $w_2 R^v_{\pi_{\mathsf{int}}} w_1$. This implies that $v(w_1, \psi) = v(w_2, \psi)$ for every $\psi \in wff_{\mathcal{L}}$. Since v is differentiated, $w_1 = w_2$. It remains to show that *(persistence)* and *(implication)* hold for $R^v_{\pi_{\mathsf{int}}}$. Following Example 4.8, since v respects π_{int} for $\Re^v_{\mathbf{LJ}}$, condition *(persistence)* holds. By Example 4.14, since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LJ}}$, we have that for every $w \in W_v$, if $v(w, \psi) = \mathtt{T}$ and $v(w, \varphi) = \mathtt{F}$ then $v(w, \psi \supset \varphi) = \mathtt{F}$. By Example 4.15, since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LJ}}$, we have that for every $w \in W_v$, if $v(u, \psi) = \mathtt{F}$ or $v(u, \varphi) = \mathtt{T}$ for every $u \in R^v_{\pi_{\mathsf{int}}}[w]$, then $v(w, \psi \supset \varphi) = \mathtt{T}$. These two facts together with *(persistence)* establish *(implication)*.

Now, what happens if we simply apply Theorem 4.18 for **LJ** (perhaps without knowing about the usual Kripke semantics for intuitionistic logic)? In this case, we obtain that **LJ** is strongly sound and complete for the set of **LJ**-legal valuations. This set can be defined exactly like $\mathrm{M}_{\mathbf{LJ}}$ without restricting $\leq$ to be a partial order. Thus, we obtain a semantics which is less restrictive than the previous one. On the other hand, we can apply Theorem 4.25 to obtain that **LJ** is strongly sound and complete for the set of strongly **LJ**-legal differentiated valuations. This set is a subset of $\mathrm{M}_{\mathbf{LJ}}$ obtained by imposing also the converse of *(persistence)* (if $v(w, \psi) = \mathtt{T}$ implies $v(u, \psi) = \mathtt{T}$ for every ψ, then $w \leq u$). Here we obtain a more restrictive semantics than the usual one.

Example 4.34 (**BLJ**). Using Corollary 4.26, we obtain a strongly sound and complete semantics for **BLJ**, which is practically the usual Kripke semantics for bi-intuitionistic logic. For this purpose, let $\mathrm{M}_{\mathbf{BLJ}}$ be the set of all valuations v satisfying the conditions from Example 4.33, and the following additional condition:

(exclusion) $v(w, \psi \prec \varphi) = \mathtt{F}$ iff $v(u, \psi) = \mathtt{F}$ or $v(u, \varphi) = \mathtt{T}$ for every $u \in W_v$ such that $w \geq u$.

The semantics induced by $\mathrm{M}_{\mathbf{BLJ}}$ is practically identical to the usual Kripke semantics of bi-intuitionistic logic (see e.g., [Goré and Postniece 2010]). Now, $\mathrm{M}_{\mathbf{BLJ}}$ is a set of **BLJ**-legal valuations, that contains all strongly **BLJ**-legal differentiated valuations, and so Corollary 4.26 implies that $\vdash_{\mathbf{BLJ}} = \vdash_{\mathrm{M}_{\mathbf{BLJ}}}$. This is shown similarly as for **LJ**. In particular, the rules of $\prec$ correspond to *(exclusion)*, and Proposition 4.28 (Item 3) entails that in strongly **BLJ**-legal valuations $R^v_{\pi_{\mathsf{d}}} = (R^v_{\pi_{\mathsf{int}}})^{-1}$ (since $\alpha \bar{\pi}_{\mathsf{d}} \beta$ iff $\overline{\beta} \bar{\pi}_{\mathsf{int}} \overline{\alpha}$).

Example 4.35 (**PLJ**). Using Corollary 4.26, **PLJ** is sound and complete with respect to the set M of valuations v satisfying the conditions from Example 4.33, and the following two conditions concerning $\neg$:

— If $v(w, \psi) = \mathtt{F}$ then $v(w, \neg\psi) = \mathtt{T}$.
— If $v(w, \psi) = \mathtt{T}$ and $v(w, \neg\varphi) = \mathtt{T}$ then $v(w, \neg(\psi \supset \varphi)) = \mathtt{T}$.

To see this it suffices to show that M is a set of **PLJ**-legal valuations containing all strongly **PLJ**-legal differentiated valuations. This is done straightforwardly. Clearly, this semantics is non-deterministic, as the truth-values of ψ in every world may not determine the truth-values of $\neg\psi$. For example, in a valuation with a single world w, if $v(w, p_1) = \mathtt{T}$, then $v(w, \neg p_1)$ can be either T or F. Note that this semantics is different from the (three-valued) semantics given in [Avron 2007] for this system.

Remark 4.36. A similar study for simpler sequent systems for paraconsistent logics (based on **LK**, rather than on **LJ**) leads to a non-deterministic two-valued semantics, known also as a bivaluation semantics (see e.g., [Carnielli et al. 2007]).

Example 4.37 (**K**). The usual Kripke semantics of the modal logic K can be described using the set $\mathrm{M}_{\mathbf{K}}$ of valuations defined as follows: $v \in \mathrm{M}_{\mathbf{K}}$ iff v respects the usual truth-tables of the classical connectives in each world, and there exists a binary relation R on W_v such that the following condition holds:

(necessity) $v(w, \Box\psi) = \texttt{T}$ iff $v(u, \psi) = \texttt{T}$ for every $u \in R[w]$.

Now Corollary 4.26 implies that $\vdash_{\mathbf{K}} = \vdash_{\mathbf{M_K}}$. To see this, we prove that $\mathbf{M_K}$ is a set of $\mathbf{K}$-legal valuations that contains all strongly $\mathbf{K}$-legal valuations:

(1) Let $v \in \mathbf{M_K}$, and let R be a relation on W_v satisfying *(necessity)*. Choose $\Re$ to be the $\langle \mathbf{K}, W_v \rangle$-coupling assigning Id_v to π_0, and R to π. Following Examples 4.6 and 4.9, v respects π_0 and π for $\Re$. It remains to show that v respects for $\Re$ the rules in $\Upsilon_{\mathbf{K}}$. We show it here for (K). The other rules are treated as in Example 4.31. Following Example 4.13, it suffices to see that for every $w \in W_v$ and formula ψ: if $v(u, \psi) = \texttt{T}$ for every $u \in R[w]$, then $v(w, \Box\psi) = \texttt{T}$. This follows from the definition of $\mathbf{M_K}$.
(2) Let v be a strongly $\mathbf{K}$-legal valuation. Similarly to Example 4.31, it is easy to show that v respects the usual truth-tables of the classical connectives in each world. We claim that R^v_π is a relation satisfying *(necessity)*. To see this, note that since v respects (K) for $\Re^v_{\mathbf{K}}$, we have that, if $v(u, \psi) = \texttt{T}$ for every $u \in R^v_\pi[w]$, then $v(w, \Box\psi) = \texttt{T}$. The converse is obtained from the fact that (by definition) $w_1 R^v_\pi w_2$ iff $v(w_2, \psi) = \texttt{F}$ implies $v(w_1, \Box\psi) = \texttt{F}$ for every $\psi \in \mathit{wff}_{\mathcal{L}}$.

Example 4.38 (*Systems for modal logics*). The usual Kripke semantics of the modal logics $K4$, KD, KB, $S4$ and $S5$ can be described using the following variations on the set $\mathbf{M_K}$ (from Example 4.37):

— $\mathbf{M_{K4}}$ is defined as $\mathbf{M_K}$ with the addition that R is transitive.
— $\mathbf{M_{KD}}$ is defined as $\mathbf{M_K}$ with the addition that R is serial.
— $\mathbf{M_{KB}}$ is defined as $\mathbf{M_K}$ with the addition that R is symmetric.
— $\mathbf{M_{S4}}$ is defined as $\mathbf{M_K}$ with the addition that R is reflexive and transitive.
— $\mathbf{M_{S5}}$ is defined as $\mathbf{M_K}$ with the addition that R is an equivalence relation.

For every $\mathbf{G} \in \{\mathbf{K4}, \mathbf{KD}, \mathbf{KB}, \mathbf{S4}, \mathbf{S5}\}$, Corollary 4.26 implies that $\vdash_{\mathbf{G}} = \vdash_{\mathbf{M_G}}$. Indeed, we prove that in each of these cases $\mathbf{M_G}$ is a set of $\mathbf{G}$-legal valuations that contains all strongly $\mathbf{G}$-legal valuations. Let $\mathbf{G} \in \{\mathbf{K4}, \mathbf{KD}, \mathbf{KB}\}$ (the proofs for $\mathbf{S4}$ and $\mathbf{S5}$ are similar and left for the reader).

(1) Let $v \in \mathbf{M_G}$, and R be a relation on W_v satisfying *(necessity)* and the required additional condition of $\mathbf{M_G}$ (transitivity, seriality, or symmetry). Choose $\Re$ to be the $\langle \mathbf{G}, W_v \rangle$-coupling assigning Id_v to π_0, and R to π. We show that v is $\mathbf{G}$-legal:
   K4 As in Example 4.37, v respects π_0, and every $r \in \Upsilon_{\mathbf{K4}}$ for $\Re$. It remains to show that $R \subseteq R^v_\pi$ (and so v respects π for $\Re$). Suppose that $w_1 R w_2$. Let α and β be signed formulas such that $\alpha \bar{\pi} \beta$ and $w_2 \models^v \alpha$. The structure of π ensures that either $\alpha = \texttt{F}{:}\psi$ and $\beta = \texttt{F}{:}\Box\psi$ for some formula ψ, or $\alpha = \texttt{F}{:}\Box\psi$ and $\beta = \texttt{F}{:}\Box\psi$ for some formula ψ. In the first case, *(necessity)* directly implies that $w_1 \models^v \beta$. Suppose now that $\alpha = \texttt{F}{:}\Box\psi$ and $\beta = \texttt{F}{:}\Box\psi$ for some formula ψ. Since $w_2 \models^v \alpha$ (i.e., $v(w_2, \Box\psi) = \texttt{F}$), *(necessity)* entails that $w \models^v \texttt{F}{:}\psi$ (i.e., $v(w, \psi) = \texttt{F}$) for some $w \in R[w_2]$. The transitivity of R then ensures that $w_1 R w$. Again *(necessity)* implies that $w_1 \models^v \texttt{F}{:}\Box\psi$. It follows that $w_1 R^v_\pi w_2$.
   KD As in Example 4.37, v respects π_0, π, and every $r \in \Upsilon_{\mathbf{KD}} \setminus \{(D)\}$ for $\Re$. In addition, following Example 4.16, the seriality of R ensures that v respects (D) for $\Re$. Therefore v is $\mathbf{KD}$-legal.
   KB As in Example 4.37, v respects π_0, and every $r \in \Upsilon_{\mathbf{KB}}$ for $\Re$. It remains to show that $R \subseteq R^v_\pi$ (and so v respects π for $\Re$). Suppose that $w_1 R w_2$. Let α and β be signed formulas such that $\alpha \bar{\pi} \beta$ and $w_2 \models^v \alpha$. The structure of π ensures that either $\alpha = \texttt{F}{:}\psi$ and $\beta = \texttt{F}{:}\Box\psi$ for some formula ψ, or $\alpha = \texttt{T}{:}\Box\psi$

and $\beta = \mathtt{T}{:}\psi$ for some formula ψ. In the first case, *(necessity)* directly implies that $w_1 \models^v \beta$. Suppose now that $\alpha = \mathtt{T}{:}\Box\psi$ and $\beta = \mathtt{T}{:}\psi$ for some formula ψ. Since $w_2 \models^v \alpha$ (i.e., $v(w_2, \Box\psi) = \mathtt{T}$), *(necessity)* entails $w \models^v \beta$ for every $w \in R[w_2]$. The symmetry of R ensures that $w_2 R w_1$, and so $w_1 \models^v \beta$. It follows that $w_1 R^v_\pi w_2$.

(2) Let v be a strongly **G**-legal valuation. Similarly to Example 4.37, one shows that $v \in \mathbf{M_K}$. In addition:
- **K4** Since $\bar{\pi} \subseteq \bar{\pi} \circ \bar{\pi}$, Proposition 4.28 (Item 2) entails that R^v_π is transitive.
- **KD** Since v respects (D) for $\Re^v_{\mathbf{KD}}$, R^v_π is serial (see Example 4.16).
- **KB** Since $\alpha\bar{\pi}\beta$ iff $\overline{\beta}\bar{\pi}\overline{\alpha}$, Proposition 4.28 (Item 3) entails that R^v_π is symmetric.

Example 4.39 (**GL**). Semantically, the modal logic GL is characterized by the set of modal Kripke frames whose accessibility relation is transitive and conversely well-founded. However, GL is not *strongly* complete with respect to models built on this set of frames (indeed, compactness fails for the logic induced by this semantics, see [Verbrugge 2010]). Using our method, starting from the basic system **GL**, we obtain a (different) *strongly* sound and complete semantics for GL. Indeed, by Corollary 4.26, **GL** is (strongly) sound and complete with respect to the set $\mathbf{M_{GL}}$ of valuations, defined similarly to $\mathbf{M_K}$ (see Example 4.37), with two additional requirements: (1) R is transitive; (2) If $v(u, \psi) = \mathtt{F}$ for some $u \in R[w]$, then there is some $u' \in R[w]$, such that $v(u', \psi) = \mathtt{F}$ and $v(u', \Box\psi) = \mathtt{T}$. To see this we prove that $\mathbf{M_{GL}}$ is a set of **GL**-legal valuations that contains all strongly **GL**-legal valuations:

(1) Let $v \in \mathbf{M_{GL}}$, and let R be a transitive relation on W_v, satisfying *(necessity)* and condition (2) above. Choose $\Re$ to be the $\langle \mathbf{GL}, W_v \rangle$-coupling assigning Id_v to π_0, and R to π. Similarly to Example 4.38, one can prove that $R \subseteq R^v_\pi$ (using the transitivity of R), and so v respects π for $\Re$. It remains to show that v respects for $\Re$ the rules in $\Upsilon_{\mathbf{GL}}$. We show it here for (GL). The other rules are treated as in Example 4.31. Let $w \in W_v$, and let σ be a substitution. Suppose that $\Re(\pi)[w] \models^v \sigma(\Box p_1 \Rightarrow p_1)$. We show that $w \models^v \sigma(\Rightarrow \Box p_1)$. Assume otherwise. Then $v(w, \Box\sigma(p_1)) = \mathtt{F}$. Thus *(necessity)* implies that there exists some $u \in R[w]$, such that $v(u, \sigma(p_1)) = \mathtt{F}$. By condition (2), there is some $u' \in R[w]$, such that $v(u', \sigma(p_1)) = \mathtt{F}$ and $v(u', \Box\sigma(p_1)) = \mathtt{T}$. Clearly, $u' \not\models^v \sigma(\Box p_1 \Rightarrow p_1)$. But, since $\Re(\pi) = R$, this contradicts the fact that $\Re(\pi)[w] \models^v \sigma(\Box p_1 \Rightarrow p_1)$.

(2) Let v be a strongly **GL**-legal valuation. Similarly to Example 4.31, it is possible to show that v respects the usual truth-tables of the classical connectives in each world. It remains to show that there exists a transitive relation R on W_v satisfying *(necessity)* and condition (2) above. We show that R^v_π has this property (its transitivity is proved exactly as in Example 4.38):
   (a) Since v respects (GL) for $\Re^v_{\mathbf{GL}}$, we have that, if $v(u, \psi) = \mathtt{T}$ for every $u \in R^v_\pi[w]$, then $v(w, \Box\psi) = \mathtt{T}$. The converse holds since v respects π for $\Re^v_{\mathbf{GL}}$.
   (b) We prove that R^v_π satisfies condition (2) above. Suppose for contradiction that there exist some $\psi \in \mathit{wff}_\mathcal{L}$ and $w \in W_v$, such that $v(u, \psi) = \mathtt{F}$ for some $u \in R^v_\pi[w]$, and there does not exist $u' \in R^v_\pi[w]$, such that $v(u', \psi) = \mathtt{F}$ and $v(u', \Box\psi) = \mathtt{T}$. It follows that $\Re(\pi)[w] \models^v \Box\psi \Rightarrow \psi$. Since v respects (GL) for $\Re$, $v(w, \Box\psi) = \mathtt{T}$. But, this contradicts *(necessity)*.

Remark 4.40. Note that while the semantics provided above is strongly sound and complete for GL, it is not clear whether it is useful (in particular, whether it leads to a decision procedure). We leave this question to a future work.

Example 4.41 (**GP**). Using Corollary 4.26, we obtain a strongly sound and complete semantics for **GP**, which is practically identical to the semantics presented in

[Beklemishev and Gurevich 2012]. For this purpose, let M be the set of valuations v that respect the usual truth-tables of $\wedge, \vee, \perp, \top$ in each world, and satisfy the following conditions:

(1) If $v(w, \psi) = \mathtt{T}$ and $v(w, \varphi) = \mathtt{F}$ then $v(w, \psi \to \varphi) = \mathtt{F}$.
(2) If $v(w, \varphi) = \mathtt{T}$ then $v(w, \psi \to \varphi) = \mathtt{T}$.
(3) For every $q \in Q$, there exist binary relations on W_v, S^q and I^q, satisfying the following conditions:
  (a) $I^q \subseteq S^q$.
  (b) $v(w, q\ \mathtt{said}\ \psi) = \mathtt{T}$ iff $v(u, \psi) = \mathtt{T}$ for every $u \in S^q[w]$.
  (c) $v(w, q\ \mathtt{implied}\ \psi) = \mathtt{T}$ iff $v(u, \psi) = \mathtt{T}$ for every $u \in I^q[w]$

Clearly, this semantics is *non-deterministic*, as the truth-values of ψ and φ in every world may not determine the truth-value of $\psi \to \varphi$. As in previous examples, it is straightforward to show that M is a set of **GP**-legal valuations, that contains all strongly **GP**-legal valuations (in particular, the fact that in all strongly **GP**-legal valuations $\Re(\pi_i^q) \subseteq \Re(\pi_s^q)$ for every $q \in Q$ follows from Proposition 4.28).

Example 4.42 (**IS5**). Using Corollary 4.26, we obtain a strongly sound and complete Kripke semantics for **IS5**. For this purpose, let M be the set of valuations v satisfying the conditions from Example 4.33, and in addition, there exists an equivalence relation $\sim$, such that $v(w, \Box\psi) = \mathtt{T}$ iff $v(u, \psi) = \mathtt{T}$ for every $u \in\sim[w]$. (Note that if $v \in$ M, then for every $w, u \in W_v$, we have that, if $w \leq u$ and $v(w', \psi) = \mathtt{T}$ for every $w' \in\sim[w]$, then $v(u', \psi) = \mathtt{T}$ for every $u' \in\sim[u]$.) As in previous examples, it is straightforward to show that M is a set of **IS5**-legal valuations, that contains all strongly **IS5**-legal valuations. Interestingly, the Kripke semantics presented in [Ono 1977] is not identical to this one. In particular, in our semantics $\sim$ should be an equivalence relation, and no direct conditions bind $\leq$ and $\sim$.

5. SEMANTIC CHARACTERIZATIONS OF PROOF-THEORETIC PROPERTIES

In this section we extend the results of the previous part in order to provide semantic characterizations of three important proof-theoretic properties of basic systems. We begin with precise definitions of these properties.

5.1. Analyticity, Cut-admissibility, and Axiom-Expansion

5.1.1. Analyticity. Analyticity is perhaps the most important property of fully-structural propositional proof systems, as it usually implies its consistency and decidability. Roughly speaking, a sequent system is *analytic* if whenever a sequent s is provable in it from a set of assumptions, then s can be proven using only the syntactic material available within the assumptions and the sequent s. Now, there is more than one way to precisely define the "material available within some sequent". Usually, it is taken to consist of all subformulas occurring in the sequent, and then analyticity is just another name for *the global subformula property* (i.e., if there exists a proof of s from $\mathcal{S}$, then there exists such a proof using only subformulas of the formulas in $\mathcal{S}$ and s). However, it is also possible (and sometimes necessary, see Example 5.52) to consider analyticity based on different relations defining the "material available within sequents". While these substitutes might be weaker than the global subformula property, they may still suffice to imply consistency and decidability of a proof system. Next we define a generalized analyticity property, based on an arbitrary partial order.

Notation 5.1. Given a signed formula α, we denote by $frm[\alpha]$ the (ordinary) formula appearing in α. frm is extended to sets of signed formulas and to sets of sets of signed formulas in the obvious way. In addition, given a set $\mathcal{E}$ of formulas, a formula ψ (respectively, sequent s) is called an *$\mathcal{E}$-formula* (*$\mathcal{E}$-sequent*) if $\psi \in \mathcal{E}$ ($frm[s] \subseteq \mathcal{E}$).

Notation 5.2. Let $\leq$ be a partial order on $wff_{\mathcal{L}}$. For every formula ψ, we denote by $\downarrow^{\leq}[\psi]$ the set $\{\varphi \in wff_{\mathcal{L}} \mid \varphi \leq \psi\}$. This notation is extended to sets of formulas, sequents, and sets of sequents in the obvious way (e.g., for a sequent s, $\downarrow^{\leq}[s] = \downarrow^{\leq}[frm[s]]$).

Definition 5.3. Let $\leq$ be a partial order on $wff_{\mathcal{L}}$. A basic system $\mathbf{G}$ is $\leq$-*analytic* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that there is a proof in $\mathbf{G}$ of s from $\mathcal{S}$ consisting of $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-formulas only.

Notation 5.4. We denote by sub the subformula relation between formulas. In the case of sub, we simply write $sub[\,\cdot\,]$ instead of $\downarrow^{sub}[\,\cdot\,]$. Note that sub-analyticity is equivalent to the global subformula property.

The following are three major consequences of analyticity.

PROPOSITION 5.5 (CONSISTENCY). *Let $\mathbf{G}$ be basic system, which is $\leq$-analytic for some partial order $\leq$. Assume that the basic rule $\emptyset/\emptyset$ is not in $\mathbf{G}$. Then, $\not\vdash_{\mathbf{G}} \emptyset$.*

PROOF. Assume that $\vdash_{\mathbf{G}} \emptyset$. Since $\mathbf{G}$ is $\leq$-analytic, there exists a proof of the empty sequent using no formulas at all. The only way to have this is using the rule $\emptyset/\emptyset$. □

PROPOSITION 5.6 (CONSERVATIVITY). *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be basic systems in languages $\mathcal{L}_1$ and $\mathcal{L}_2$ (respectively). Assume that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$ by some set of connectives, and that $\mathbf{G_2}$ is obtained from $\mathbf{G_1}$ by adding to the latter rules involving connectives in $\mathcal{L}_2 \setminus \mathcal{L}_1$. (i.e., at least one connective in $\mathcal{L}_2 \setminus \mathcal{L}_1$ appears in any application of a rule from $\mathbf{G_2} \setminus \mathbf{G_1}$). Let $\leq$ be a partial order on $wff_{\mathcal{L}_2}$, such that $wff_{\mathcal{L}_1}$ is closed under $\leq$. If $\mathbf{G_2}$ is $\leq$-analytic, then $\mathbf{G_2}$ is a conservative extension of $\mathbf{G_1}$ (i.e., if $frm[\mathcal{S} \cup \{s\}] \subseteq wff_{\mathcal{L}_1}$ then $\mathcal{S} \vdash_{\mathbf{G_1}} s$ iff $\mathcal{S} \vdash_{\mathbf{G_2}} s$).*

PROOF. Obviously, $\mathcal{S} \vdash_{\mathbf{G_1}} s$ implies $\mathcal{S} \vdash_{\mathbf{G_2}} s$. For the converse, assume that $\mathcal{S} \vdash_{\mathbf{G_2}} s$. Since $\mathbf{G_2}$ is $\leq$-analytic, there exists a proof in $\mathbf{G_2}$ of s from $\mathcal{S}$ consisting of $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-formulas only. Since $frm[\mathcal{S} \cup \{s\}] \subseteq wff_{\mathcal{L}_1}$, and $wff_{\mathcal{L}_1}$ is closed under $\leq$, this is also a proof in $\mathbf{G_1}$, and so $\mathcal{S} \vdash_{\mathbf{G_1}} s$. □

PROPOSITION 5.7 (DECIDABILITY). *Let $\mathbf{G}$ be a* finite *basic system (i.e., $\mathbf{G}$ consists of a finite number of basic rules).*[8] *Suppose that $\mathbf{G}$ is $\leq$-analytic for some partial order $\leq$. Furthermore, assume that $\leq$ is* safe*, i.e., $\downarrow^{\leq}[\varphi]$ is finite for every $\varphi \in wff_{\mathcal{L}}$, and $\lambda\varphi \in wff_{\mathcal{L}}.\downarrow^{\leq}[\varphi]$ is computable. Then, given a finite set $\mathcal{S}$ of sequents and a sequent s, it is decidable whether $\mathcal{S} \vdash_{\mathbf{G}} s$ or not.*

PROOF. Exhaustive proof-search is possible. Since $\mathbf{G}$ is $\leq$-analytic, $\mathcal{S} \vdash_{\mathbf{G}} s$ iff there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ consisting of $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-sequents only. Since $\leq$ is safe, one can construct the (finite) set $\mathcal{S}'$ of all $\downarrow^{\leq}[\mathcal{S} \cup \{s\}]$-sequents. Clearly, $\mathcal{S} \vdash_{\mathbf{G}} s$ iff there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$ of length lower or equal to $|\mathcal{S}'|$, consisting only of sequents from $\mathcal{S}'$. Thus one can construct all possible candidates. The fact that $\mathbf{G}$ is finite entails that it is possible to check whether a certain candidate is indeed a proof in $\mathbf{G}$ of s from $\mathcal{S}$. □

5.1.2. Strong Cut-Admissibility. While analyticity of a proof system suffices for many desirable properties, cut-admissibility is traditionally preferred. Cut-admissibility is sometimes required to obtain space-complexity bounds on proof-search in basic systems (see e.g., [Beklemishev and Gurevich 2012]). Since we deal with proofs from arbitrary sets of assumptions (not necessarily the empty one), we consider a stronger property, called *strong* cut-admissibility in [Avron 1993]:

[8] In fact, it suffices to assume that $\mathbf{G}$ is *verifiable*, i.e., that given sequents $s_1, \dots, s_n$ and a sequent c, it is decidable whether c can be inferred from $s_1, \dots, s_n$ by applying one of the rules of $\mathbf{G}$.

Definition 5.8. A basic system $\mathbf{G}$ enjoys *strong cut-admissibility* if $\mathcal{S} \vdash_{\mathbf{G}} s$ implies that there exists a proof in $\mathbf{G}$ of s from $\mathcal{S}$, in which only $frm[\mathcal{S}]$-formulas serve as cut formulas (a formula ψ serve as a *cut-formula* in some proof, if the proof contains an application of (cut) in which $\sigma(p_1) = \psi$).

Usual *cut-admissibility* (whenever there exists a proof of a sequent s in $\mathbf{G}$, then there exists a cut-free proof in $\mathbf{G}$ of s) is weaker than strong cut-admissibility (obtained by choosing $\mathcal{S} = \emptyset$). Note that if all rules in a basic system (except for (cut)) admit *the local subformula property* (the premises of every application consist of subformulas of formulas occurring in the conclusion of the application), then strong cut-admissibility implies sub-analyticity.

5.1.3. Axiom-Expansion. Roughly speaking, axiom-expansion means that non-atomic applications of (id) (deriving sequents of the form $\psi \Rightarrow \psi$ where ψ is compound) are redundant. This property is sometimes considered crucial when designing "well-behaved" sequent systems. Following [Ciabattoni and Terui 2006], we define this property for a given connective as follows:

Definition 5.9. An n-ary connective $\diamond$ admits *axiom-expansion* in a basic system $\mathbf{G}$ if there exists a cut-free proof of $\diamond(p_1, \dots, p_n) \Rightarrow \diamond(p_1, \dots, p_n)$ in $\mathbf{G}$, in which only atomic applications of (id) are used (applications in which $\sigma(p_1) \in at_{\mathcal{L}}$).

Clearly, if $\diamond$ admits axiom-expansion in $\mathbf{G}$, then there exists a cut-free proof in $\mathbf{G}$ of every sequent of the form $\diamond(\psi_1, \dots, \psi_n) \Rightarrow \diamond(\psi_1, \dots, \psi_n)$ from $\psi_1 \Rightarrow \psi_1, \dots, \psi_n \Rightarrow \psi_n$ without any applications of (id). If every connective admits axiom-expansion in $\mathbf{G}$, then all non-atomic applications of (id) are redundant.

5.2. Proof-Specifications and Their Semantics

By definition, proofs in basic systems allow all applications of (cut) and (id). Moreover, any formula of the language can appear in proofs, regardless of the assumptions and the proven sequent. However, each of the proof-theoretic properties defined in Section 5.1 deals with restricted proofs, in which either some formulas are not allowed to appear (as in $\leq$-analyticity), or some applications of (cut) and (id) are forbidden (as in strong cut-admissibility and axiom-expansion). To uniformly handle these kinds of restricted proofs, we introduce the following notion of a *proof-specification*.

Definition 5.10. A *proof-specification* is a triple of sets of formulas $\langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$. Given a proof-specification $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$, a proof P in a basic system $\mathbf{G}$ is called a ρ-proof if (i) it contains only $\mathcal{E}$-sequents; (ii) $\sigma(p_1) \in \mathcal{C}$ for every application of (cut) in P; and (iii) $\sigma(p_1) \in \mathcal{A}$ for every application of (id) in P. We write $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ if there exists a ρ-proof in $\mathbf{G}$ of s from $\mathcal{S}$.

Remark 5.11. Note that by definition:

(1) $\vdash_{\mathbf{G}}$ is a special case of $\vdash^{\rho}_{\mathbf{G}}$, obtained by choosing $\rho = \langle wff_{\mathcal{L}}, wff_{\mathcal{L}}, wff_{\mathcal{L}} \rangle$.
(2) $\vdash^{\rho}_{\mathbf{G}} \subseteq \vdash^{\rho'}_{\mathbf{G}}$ whenever $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$, $\rho' = \langle \mathcal{E}', \mathcal{C}', \mathcal{A}' \rangle$ and $\mathcal{E} \subseteq \mathcal{E}'$, $\mathcal{C} \subseteq \mathcal{C}'$, $\mathcal{A} \subseteq \mathcal{A}'$.
(3) $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ for $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$ iff $\{s' \in \mathcal{S} \mid frm[s'] \subseteq \mathcal{E}\} \vdash^{\rho'}_{\mathbf{G}} s$ where $\rho' = \langle \mathcal{E}, \mathcal{C} \cap \mathcal{E}, \mathcal{A} \cap \mathcal{E} \rangle$.
(4) $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ for $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$ iff $\mathcal{S} \cup \{\psi \Rightarrow \psi \mid \psi \in \mathcal{A}\} \vdash^{\rho'}_{\mathbf{G}} s$ where $\rho' = \langle \mathcal{E}, \mathcal{C}, \emptyset \rangle$.

PROPOSITION 5.12 (DECIDABILITY). *Let* $\mathbf{G}$ *be a* finite *basic system (see Proposition 5.7). Let* $\mathcal{C}$ *and* $\mathcal{A}$ *be decidable sets of formulas. Given a finite set* $\mathcal{S}$ *of sequents, a sequent* s*, and a finite set* $\mathcal{E}$ *of formulas, it is decidable whether* $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ *or not, for* $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$.

PROOF. By exhaustive proof-search (see the proof of Proposition 5.7). □

The notion of a proof-specification provides alternative formulations of the three properties introduced in Section 5.1:

PROPOSITION 5.13. *Let* $\mathbf{G}$ *be a basic system.*

(*1*) *Let* $\leq$ *be a partial order on* $wff_{\mathcal{L}}$. $\mathbf{G}$ *is* $\leq$*-analytic if* $\mathcal{S} \vdash_{\mathbf{G}} s$ *implies* $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ *for* $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], wff_{\mathcal{L}}, wff_{\mathcal{L}} \rangle$.
(*2*) $\mathbf{G}$ *enjoys strong cut-admissibility if* $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ *for* $\rho = \langle wff_{\mathcal{L}}, frm[\mathcal{S}], wff_{\mathcal{L}} \rangle$ *whenever* $\mathcal{S} \vdash_{\mathbf{G}} s$.
(*3*) *An* n*-ary connective* $\diamond$ *admits axiom-expansion in* $\mathbf{G}$ *if* $\vdash^{\rho}_{\mathbf{G}} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ *for* $\rho = \langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}} \rangle$.

Next we generalize the semantics of Section 4, to obtain an adequate semantics in the presence of arbitrary proof-specifications. The generalized semantics naturally leads to semantic characterizations of the three proof-theoretic properties. It is based on the following notion of a *quasi-valuation*.

Definition 5.14. A *quasi-valuation* is a function v from the Cartesian product of some set W_v and some set $D_v \subseteq wff_{\mathcal{L}}$ to $2^{\{\mathbf{T},\mathbf{F}\}}$ (i.e., $v : W_v \times D_v \to 2^{\{\mathbf{T},\mathbf{F}\}}$).

Quasi-valuations are different from valuations in two respects. First, while valuations are total (defined for every formula), a quasi-valuation v assigns truth-values only to formulas in a specific domain D_v. To handle proof-specifications that allow only formulas from some set $\mathcal{E}$ to appear in proofs, we will use $D_v = \mathcal{E}$. Second, the assigned truth-values in quasi-valuations are *subsets* of $\{\mathtt{T}, \mathtt{F}\}$, rather than *elements* of $\{\mathtt{T}, \mathtt{F}\}$ that are used in valuations. This is the key for providing sound and complete semantics for proofs matching some proof-specification, in which (cut) and (id) are restricted.

Signed formulas and sequents are interpreted as follows:

Definition 5.15. Let v be a quasi-valuation.

(1) A signed formula $\mathtt{X}{:}\psi$ (for $\mathtt{X} \in \{\mathtt{T}, \mathtt{F}\}$) is *true* in some $w \in W_v$ with respect to v (denoted by: $w \models^v \mathtt{X}{:}\psi$) if $\psi \in D_v$ and $\mathtt{X} \in v(w, \psi)$.
(2) A sequent s is *true* in $w \in W_v$ with respect to v (denoted by: $w \models^v s$) if $w \models^v \alpha$ for some $\alpha \in s$.
(3) A sequent s is *true* in $W \subseteq W_v$ with respect to v (denoted by: $W \models^v s$) if $w \models^v s$ for every $w \in W$.
(4) v is a *model* of:
    (a) a *sequent* s (denoted by: $\models^v s$) if s is a D_v-sequent and $W_v \models^v s$.
    (b) a *set* $\mathcal{S}$ *of sequents* (denoted by: $\models^v \mathcal{S}$) if $\models^v s$ for every D_v-sequent $s \in \mathcal{S}$.

The differences between the previous definition and the corresponding definition for valuations (Definition 4.2) are: (i) here we have $\mathtt{X} \in v(w, \psi)$ instead of $v(w, \psi) = \mathtt{X}$; (ii) we added the requirement that s is a D_v-sequent in Item 4a; and (iii) only D_v-sequents are considered in Item 4b. Note that a quasi-valuation can only be a model of sequents consisting solely of formulas in its domain. However, it can be a model of a *set* of sequents containing formulas which are not in its domain.

Using the previous definition, the consequence relation (between sets of sequents and sequents) induced by a set of quasi-valuations is defined exactly like the consequence relation induced by a set of valuations (see Definition 4.3).

We can now explain how proof-specifications, in which (cut) and (id) are restricted, are handled using subsets of $\{\mathtt{T}, \mathtt{F}\}$ as truth-values. Consider first (cut). Ignoring the context formulas, (cut) allows to infer the empty sequent from two sequents of the form $\psi \Rightarrow$ and $\Rightarrow \psi$. Semantically, this means that $\mathtt{F}{:}\psi$ and $\mathtt{T}{:}\psi$ cannot be both “true” at the same time. However, if cut is not allowed on some formula ψ, then nothing in the proof system disallows this possibility. In this case, for completeness, we need some mechanism that will make both $\texttt{F}{:}\psi$ and $\texttt{T}{:}\psi$ "true". Here we do this by choosing $v(w,\psi)=\{\texttt{T},\texttt{F}\}$. The same applies to (id), which allows to infer $\psi\Rightarrow\psi$ for any formula ψ. Semantically, this means that either $\texttt{F}{:}\psi$ or $\texttt{T}{:}\psi$ should be always "true". Again, if the use of $\psi\Rightarrow\psi$ is forbidden, it should be possible that both $\texttt{F}{:}\psi$ and $\texttt{T}{:}\psi$ are not "true". Here we do this by choosing $v(w,\psi)=\emptyset$. Clearly, the use of $\{\texttt{T},\texttt{F}\}$ and $\emptyset$ should be limited according to the given proof-specification. When cut is allowed on some formula ψ ($\psi\in\mathcal{C}$), the value $\{\texttt{T},\texttt{F}\}$ should be forbidden for ψ. Similarly, when $\psi\Rightarrow\psi$ is allowed to be used ($\psi\in\mathcal{A}$), the value $\emptyset$ should be forbidden for ψ. Next we formulate these requirements.

Definition 5.16. Let $\rho=\langle\mathcal{E},\mathcal{C},\mathcal{A}\rangle$ be a proof-specification. A quasi-valuation v is called a ρ-quasi-valuation if $D_v=\mathcal{E}$, $v(w,\psi)\neq\{\texttt{T},\texttt{F}\}$ for every $w\in W_v$ and $\psi\in\mathcal{C}$, and $v(w,\psi)\neq\emptyset$ for every $w\in W_v$ and $\psi\in\mathcal{A}$.

Remark 5.17. Obviously, an $\langle\mathcal{E},\mathcal{C},\mathcal{A}\rangle$-quasi-valuation is also an $\langle\mathcal{E},\mathcal{C}',\mathcal{A}'\rangle$-quasi-valuation for every $\mathcal{C}'\subseteq\mathcal{C}$ and $\mathcal{A}'\subseteq\mathcal{A}$.

Remark 5.18. Note that in a $\langle \mathit{wff}_\mathcal{L},\mathit{wff}_\mathcal{L},\mathit{wff}_\mathcal{L}\rangle$-quasi-valuation, the only allowed truth-values are the singletons $\{\texttt{T}\}$ and $\{\texttt{F}\}$. In addition, the domain of these quasi-valuations is $\mathit{wff}_\mathcal{L}$. Clearly, such quasi-valuations are practically valuations (see Definition 4.1). For this reason, we say that quasi-valuations generalize valuations. Moreover, the results about valuations of Section 4 are all obtained by applying the corresponding results of this section with $\langle \mathit{wff}_\mathcal{L},\mathit{wff}_\mathcal{L},\mathit{wff}_\mathcal{L}\rangle$-quasi-valuations.

All definitions concerning valuations are straightforwardly generalized for quasi-valuations. To assist the reader, we provide these generalizations, and some examples.

Definition 5.19. Let v be a quasi-valuation, and π be a context-relation. R^v_π denotes the binary relation on W_v defined as follows: $w_1R^v_\pi w_2$ iff for every signed D_v-formulas α and β, if $\alpha\bar{\pi}\beta$ and $w_2\models^v\alpha$ then $w_1\models^v\beta$.

Example 5.20. Let v be a quasi-valuation. Like in Example 4.6, it is easy to see that $w_1R^v_{\pi_0}w_2$ iff $v(w_2,\psi)\subseteq v(w_1,\psi)$ for every $\psi\in D_v$. In addition, for $\pi=\{\langle\texttt{F}{:}p_1,\texttt{F}{:}\Box p_1\rangle\}$, we have $w_1R^v_\pi w_2$ iff $\texttt{F}\in v(w_2,\psi)$ implies $\texttt{F}\in v(w_1,\Box\psi)$ whenever $\psi\in D_v$ and $\Box\psi\in D_v$.

The following proposition concerning the relations induced by context-relations in quasi-valuations generalizes Proposition 4.28. Its proof is similar and left to the reader.

PROPOSITION 5.21. *Let v be a quasi-valuation, and π_1,π_2,π_3 be context-relations.*

(1) *Suppose that $\bar{\pi}_3=\bar{\pi}_1\cup\bar{\pi}_2$. Then $R^v_{\pi_3}=R^v_{\pi_2}\cap R^v_{\pi_1}$. In particular, if $\bar{\pi}_1\subseteq\bar{\pi}_2$ then $R^v_{\pi_2}\subseteq R^v_{\pi_1}$.*
(2) *Suppose that for every signed D_v-formulas α,β, if $\alpha\bar{\pi}_3\beta$ then there exists $\gamma\in D_v$ such that $\alpha\bar{\pi}_1\gamma$ and $\gamma\bar{\pi}_2\beta$. Then $R^v_{\pi_2}\circ R^v_{\pi_1}\subseteq R^v_{\pi_3}$. In particular, if for every signed D_v-formulas α,β, $\alpha\bar{\pi}_1\beta$ implies that there exists $\gamma\in D_v$ such that $\alpha\bar{\pi}_1\gamma$ and $\gamma\bar{\pi}_1\beta$, then $R^v_{\pi_1}$ is a transitive relation.*
(3) *Assume that v is a $\langle\mathcal{E},\mathit{wff}_\mathcal{L},\mathit{wff}_\mathcal{L}\rangle$-quasi-valuation for some set $\mathcal{E}$ of formulas. Suppose that $\overline{\beta}\bar{\pi}_1\overline{\alpha}$ whenever $\alpha\bar{\pi}_2\beta$. Then $R^v_{\pi_1}\subseteq(R^v_{\pi_2})^{-1}$. In particular, (i) if $\alpha\bar{\pi}_1\beta$ implies $\overline{\beta}\bar{\pi}_2\overline{\alpha}$ and vice-versa, then $R^v_{\pi_2}=(R^v_{\pi_1})^{-1}$, and (ii) if $\alpha\bar{\pi}_1\beta$ implies $\overline{\beta}\bar{\pi}_1\overline{\alpha}$ and vice-versa, then $R^v_{\pi_1}$ is a symmetric relation.*

Definition 5.22. Let $\mathbf{G}$ be a basic system, v be a quasi-valuation, and $\Re$ be a $\langle\mathbf{G},W_v\rangle$-coupling.

— Let $\pi\in\Pi_\mathbf{G}$. v *respects* π for a $\Re$ if $\Re(\pi)\subseteq R^v_\pi$.

— Let $r = \langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle / C$ be a basic rule of $\mathbf{G}$. v *respects* r for $\Re$ if the following condition holds for every $w \in W_v$ and substitution σ: if $frm[\sigma(s_1) \cup \ldots \cup \sigma(s_n) \cup \sigma(C)] \subseteq D_v$, and $\Re(\pi_i)[w] \models^v \sigma(s_i)$ for every $1 \leq i \leq n$, then $w \models^v \sigma(C)$.

Example 5.23. Suppose that a basic system $\mathbf{G}$ contains a rule r of the form $\langle \Rightarrow p_1, \pi\rangle / \Rightarrow \Box p_1$. A quasi-valuation v respects r for a $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$ iff for every $w \in W_v$ and substitution σ: if $frm[\sigma(\Rightarrow p_1) \cup \sigma(\Rightarrow \Box p_1)] \subseteq D_v$, and $\Re(\pi)[w] \models^v \sigma(\Rightarrow p_1)$, then $w \models^v \sigma(\Rightarrow \Box p_1)$. Hence, we obtain that v respects r for $\Re$ iff for every $w \in W_v$ and formula ψ: if $\{\psi, \Box\psi\} \subseteq D_v$, and $\mathrm{T} \in v(u, \psi)$ for every $u \in \Re(\pi)[w]$, then $\mathrm{T} \in v(w, \Box\psi)$.

$\mathbf{G}$-legal quasi-valuations are defined exactly like $\mathbf{G}$-legal valuations (see Definition 4.17). Next, we prove the following strong soundness theorem:

THEOREM 5.24 (STRONG SOUNDNESS). *Let $\mathbf{G}$ be a basic system, and ρ be a proof-specification. Let $\mathbf{M}$ be the set of all $\mathbf{G}$-legal ρ-quasi-valuations. Then $\vdash^{\rho}_{\mathbf{G}} \subseteq \vdash_{\mathbf{M}}$.*

For the proof we use the following simple lemmas.

LEMMA 5.25. *Let v be a quasi-valuation, let $w \in W_v$, and let s_1 and s_2 be two sequents. Then, $w \models^v s_1 \cup s_2$ iff either $w \models^v s_1$ or $w \models^v s_2$.*

LEMMA 5.26. *Let $\langle s, s'\rangle$ be a π-instance of some context-relation π. Let v be a quasi-valuation, and let $w \in W_v$. Suppose that $u \models^v s$ for some $u \in R^v_\pi[w]$. Then either $frm[s'] \not\subseteq D_v$ or $w \models^v s'$.*

PROOF. Suppose that $frm[s'] \subseteq D_v$, we show that $w \models^v s'$. Since $u \models^v s$, we have $u \models^v \alpha$ for some $\alpha \in s$. Since $\langle s, s'\rangle$ is a π-instance, there exists $\beta \in s'$ such that $\alpha\bar{\pi}\beta$. Note that $frm[\alpha] \in D_v$ (because $u \models^v \alpha$) and $frm[\beta] \in D_v$ (because $frm[s'] \subseteq D_v$). Then since $wR^v_\pi u$, $w \models^v \beta$. It follows that $w \models^v s'$. $\Box$

PROOF OF THEOREM 5.24. Assume that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ where $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A}\rangle$. Thus there exists a ρ-proof P in $\mathbf{G}$ of s from $\mathcal{S}$. We prove that $\mathcal{S} \vdash_{\mathbf{M}} s$. Let $v \in \mathbf{M}$. Then, v respects $\mathbf{G}$ for some $\langle \mathbf{G}, W_v\rangle$-coupling $\Re$. Suppose that $\models^v \mathcal{S}$. Using induction on the length of P, we show that $\models^v s'$ for every sequent s' appearing in P. It then follows that $\models^v s$. Note first that since v is a ρ-quasi-valuation, $D_v = \mathcal{E}$, and so every sequent in P is a D_v-sequent. Thus it suffices to prove that for every sequent s' appearing in P, we have $W_v \models^v s'$. This trivially holds for the sequents of $\mathcal{S}$ that appear in P. We show that the property of being true in W_v is preserved by applications of the rules of $\mathbf{G}$. Consider such an application in P, and assume that for every premise s' of this application we have $W_v \models^v s'$. We show that its conclusion is also true in W_v. Let $w \in W_v$.

(1) Suppose that $\psi \Rightarrow \psi$ is derived using (id). In this case, $\psi \in \mathcal{A}$. Since v is a ρ-quasi-valuation, $v(w, \psi) \neq \emptyset$. This easily implies that $w \models^v \psi \Rightarrow \psi$.
(2) Suppose that $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is derived from $\Gamma_1, \psi \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \psi, \Delta_2$ using (cut). In this case, $\psi \in \mathcal{C}$. Since v is a ρ-quasi-valuation, $v(w, \psi) \neq \{\mathrm{T}, \mathrm{F}\}$. This easily implies that either $w \not\models^v \psi \Rightarrow$ or $w \not\models^v \Rightarrow \psi$. Since $w \models^v \Gamma_1, \psi \Rightarrow \Delta_1$, Lemma 5.25 entails that either $w \models^v \Gamma_1 \Rightarrow \Delta_1$ or $w \models^v \psi \Rightarrow$. Similarly, either $w \models^v \Gamma_2 \Rightarrow \Delta_2$ or $w \models^v \Rightarrow \psi$. This entails that either $w \models^v \Gamma_1 \Rightarrow \Delta_1$ or $w \models^v \Gamma_2 \Rightarrow \Delta_2$. Therefore Lemma 5.25 entails that $w \models^v \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.
(3) Suppose that $\Gamma \Rightarrow \psi, \Delta$ is derived using from $\Gamma \Rightarrow \Delta$, using $(\Rightarrow W)$ (dealing with $(W \Rightarrow)$ is similar). Since $w \models^v \Gamma \Rightarrow \Delta$, Lemma 5.25 entails that $w \models^v \Gamma \Rightarrow \psi, \Delta$.
(4) Suppose that $\sigma(C) \cup c'_1 \cup \ldots \cup c'_n$ is derived from $\sigma(s_1) \cup c_1, \ldots, \sigma(s_n) \cup c_n$ using a rule $r \in \Upsilon_{\mathbf{G}}$, where $r = \langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle / C$. Thus $\langle c_i, c'_i\rangle$ is a π_i-instance for every $1 \leq i \leq n$. Now, if $w \models^v c'_i$ for some $1 \leq i \leq n$, then by

Lemma 5.25, $w \models^v \sigma(C) \cup c'_1 \cup \ldots \cup c'_n$, and we are done. Assume otherwise. We show that $\Re(\pi_i)[w] \models^v \sigma(s_i)$ for every $1 \leq i \leq n$. Let $1 \leq i \leq n$, and let $u \in \Re(\pi_i)[w]$. Since v respects π_i for $\Re$, we have that $wR^v_{\pi_i}u$. Now, since $\langle c_i, c'_i \rangle$ is a π_i-instance, Lemma 5.26 entails that $u \not\models^v c_i$. By Lemma 5.25 (since we assumed that $u \models^v \sigma(s_i) \cup c_i$), $u \models^v \sigma(s_i)$. Finally, since v respects r for $\Re$, and since $frm[\sigma(s_1) \cup \ldots \cup \sigma(s_n) \cup \sigma(C)] \subseteq D_v$, we have $w \models^v \sigma(C)$. Again by Lemma 5.25, $w \models^v \sigma(C) \cup c'_1 \cup \ldots \cup c'_n$. □

An interesting possible application of the last theorem is for providing relatively simple semantic arguments for the failure of certain proof-theoretic properties. Thus we can easily obtain semantic proofs of theorems which say that there does not exist a proof of certain sequents without using cuts on certain formulas. Indeed, if one finds a **G**-legal $\langle wff_\mathcal{L}, wff_\mathcal{L} \setminus \mathcal{E}, wff_\mathcal{L} \rangle$-quasi-valuation which is a model of $\mathcal{S}$ but not a model of s, then the last theorem implies that some $\mathcal{E}$-formula is serving as a cut-formula in every proof in **G** of s from $\mathcal{S}$. The same applies to the set of formulas allowed to appear in proofs, and to the allowed applications of the identity axiom (choosing $\langle wff_\mathcal{L} \setminus \mathcal{E}, wff_\mathcal{L}, wff_\mathcal{L} \rangle$ and $\langle wff_\mathcal{L}, wff_\mathcal{L}, wff_\mathcal{L} \setminus \mathcal{E} \rangle$ as proof-specifications). Note that proving facts of this kind using proof-theoretic methods is sometimes very challenging! Next we provide some examples of such applications.

Example 5.27. Let s be the sequent $p_1 \Rightarrow p_2, p_1 \supset (p_1 \prec p_2)$. We show that $\not\vdash^\rho_{\mathbf{BLJ}} s$ for $\rho = \langle wff_\mathcal{L}, \emptyset, wff_\mathcal{L} \rangle$ (i.e., there does not exist a proof of s in **BLJ** without cuts). By Theorem 5.24, it suffices to find a **BLJ**-legal ρ-quasi-valuation which is not a model of s. Let v be a ρ-quasi-valuation, where $W_v = \{w_1, w_2\}$, and $v(w, \psi) = \{\mathtt{T}, \mathtt{F}\}$ for every $w \in W_v$ and ψ except for: $v(w_1, p_1) = v(w_2, p_1) = v(w_2, p_2) = \{\mathtt{T}\}$, and $v(w_1, p_2) = v(w_1, p_1 \supset (p_1 \prec p_2)) = v(w_2, p_1 \prec p_2) = \{\mathtt{F}\}$. Let $\Re$ be the $\langle \mathbf{BLJ}, \{w_1, w_2\} \rangle$-coupling defined by $\Re(\pi_0) = \{\langle w_2, w_2 \rangle\}$, $\Re(\pi_{\mathsf{int}}) = \{\langle w_1, w_2 \rangle\}$, and $\Re(\pi_{\mathsf{d}}) = \emptyset$. One can straightforwardly verify that v respects **BLJ** for $\Re$, and clearly, $\not\models^v s$. However, it is easy to verify that every valuation in $\mathbf{M_{BLJ}}$ (see Example 4.34) is a model of s, and so $\vdash_{\mathbf{BLJ}} s$. This provides a semantic demonstration of the fact that **BLJ** does not enjoy cut-admissibility (the sequent s is a simplified version of the one used in [Pinto and Uustalu 2009] to syntactically prove this fact).

Example 5.28. It is well-known that **S5** does not enjoy cut-admissibility. We provide a semantic demonstration of this fact. Let s be the sequent $\Rightarrow p_1, \Box\neg\Box p_1$. It is easy to see that s is provable in **S5** (using a cut on $\Box p_1$), Let $\rho = \langle wff_\mathcal{L}, \{p_1, \Box\neg\Box p_1\}, wff_\mathcal{L} \rangle$. We show that $\not\vdash^\rho_{\mathbf{S5}} s$ (and so, in particular, there does exist a cut-free proof of s in **S5**). Define $W_v = \{w_1, w_2\}$, and $v(w, \psi) = \{\mathtt{T}, \mathtt{F}\}$ for every $w \in W_v$ and ψ except for: $v(w_2, p_1) = v(w_2, \Box p_1) = \{\mathtt{T}\}$, and $v(w_1, p_1) = v(w_1, \Box\neg\Box p_1) = v(w_2, \neg\Box p_1) = v(w_2, \Box\neg\Box p_1) = \{\mathtt{F}\}$. Clearly, v is a ρ-quasi-valuation and $\not\models^v s$. We show that v is **S5**-legal. Let $\Re$ be the $\langle \mathbf{S5}, \{w_1, w_2\} \rangle$-coupling defined by $\Re(\pi_0) = \{\langle w_2, w_2 \rangle\}$ and $\Re(\pi) = \{\langle w_1, w_2 \rangle, \langle w_2, w_2 \rangle\}$. One can straightforwardly verify that v respects **S5** for $\Re$. For example:

— v respects π for $\Re$ since the following two conditions are met: (1) if $w\Re(\pi)u$ and $\mathtt{T} \in v(u, \Box\psi)$ then $\mathtt{T} \in v(w, \Box\psi)$; (2) if $w\Re(\pi)u$ and $\mathtt{F} \in v(u, \Box\psi)$ then $\mathtt{F} \in v(w, \Box\psi)$.
— v respects $(S5)$ for $\Re$ since the following condition is met: if $\mathtt{T} \in v(u, \psi)$ for every $u \in \Re(\pi)[w]$, then $\mathtt{T} \in v(w, \Box\psi)$.
— v respects (T) for $\Re$ since the following condition is met: if $\mathtt{F} \in v(u, \psi)$ for every $u \in \Re(\pi_0)[w]$, then $\mathtt{F} \in v(w, \Box\psi)$.

Example 5.29. **PLJ** does not enjoy the global subformula property. This is shown in [Avron 2007], by proving that the sequent $s = \Rightarrow p_1, p_2 \supset \neg(p_2 \supset p_1)$ is provable, but every proof of it must include a formula that does not occur in $sub[s]$. Using Theorem 5.24, we can provide a semantic demonstration of this fact. Let $\rho = \langle sub[s], wff_{\mathcal{L}}, wff_{\mathcal{L}}\rangle$. Consider the ρ-quasi-valuation v, where $W_v = \{w_1, w_2\}$, $v(w_1, p_1) = v(w_1, p_2) = v(w_1, \neg(p_2 \supset p_1)) = v(w_1, p_2 \supset \neg(p_2 \supset p_1)) = \{\mathtt{F}\}$, $v(w_1, p_2 \supset p_1) = \{\mathtt{T}\}$, $v(w_2, \neg(p_2 \supset p_1)) = v(w_2, p_2 \supset \neg(p_2 \supset p_1)) = \{\mathtt{F}\}$, and $v(w_2, p_1) = v(w_2, p_2) = v(w_2, p_2 \supset p_1) = \{\mathtt{T}\}$. Let $\Re$ be the $\langle \mathbf{PLJ}, \{w_1, w_2\}\rangle$-coupling defined by $\Re(\pi_0) = \boldsymbol{Id}_v$ and $\Re(\pi_{\mathsf{int}}) = \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle, \langle w_1, w_2\rangle\}$. It is straightforward to show that v respects **PLJ** for $\Re$, and so it is a **PLJ**-legal ρ-quasi-valuation. Clearly, $\not\models^v s$. By Theorem 5.24, $\not\vdash^{\rho}_{\mathbf{PLJ}} s$. In other words, there does not exist a proof of s in **PLJ** consisting solely of $sub[s]$-sequents.

Example 5.30. Let s be the sequent $p_1 \to p_2 \Rightarrow p_1 \to p_2$. We show that s is not cut-free provable in **GP** using atomic applications of (id) only. In other words, we show that $\not\vdash^{\rho}_{\mathbf{GP}} s$ for $\rho = \langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$. This implies that $\to$ does not admit axiom-expansion in **GP**. Using Theorem 5.24, it suffices to provide a **GP**-legal ρ-quasi-valuation v, such that $\not\models^v s$. Let $W_v = \{w_0\}$, $v(w_0, p) = \{\mathtt{F}\}$ for every atomic formula p, $v(w_0, p_1 \to p_2) = \emptyset$, and $v(w_0, \psi) = \{\mathtt{T}, \mathtt{F}\}$ for every other formula ψ. Clearly, v is a ρ-quasi-valuation, and $\not\models^v s$. It is easy to verify that v is **GP**-legal (take $\Re(\pi_0) = \boldsymbol{Id}_v$ and $\Re(\pi) = \emptyset$ for every other context-relation).

Next we turn to completeness. As we did for valuations, we obtain a stronger completeness result by considering *strongly* **G**-*legal differentiated* quasi-valuations. The coupling $\Re^v_{\mathbf{G}}$, strongly **G**-legal quasi-valuations and differentiated quasi-valuations are defined exactly as for usual valuations (see Definitions 4.20,4.21, and 4.23).

THEOREM 5.31 (STRONG COMPLETENESS). *Let* **G** *be a basic system, and* ρ *be a proof-specification. Let* M *be the set of all strongly* **G**-*legal differentiated* ρ-*quasi-valuations. Then* $\vdash_{\mathrm{M}} \subseteq \vdash^{\rho}_{\mathbf{G}}$.

The proof of this theorem is given below. Taken together, the last two theorems lead to the following corollary.

COROLLARY 5.32 (STRONG SOUNDNESS AND COMPLETENESS). *Let* **G** *be a basic system, and* ρ *be a proof-specification. Then,* $\vdash^{\rho}_{\mathbf{G}} = \vdash_{\mathrm{M}}$ *for every set* M *of* **G**-*legal* ρ-*quasi-valuations containing all strongly* **G**-*legal differentiated* ρ-*quasi-valuations.*

Semantic characterizations of $\leq$-analyticity, strong cut-elimination, and axiom-expansion easily follow from the last corollary. This is the topic of Section 5.3. To end this section, we prove a useful property of differentiated ρ-quasi-valuations:

PROPOSITION 5.33. *Let* v *be a differentiated* ρ-*quasi-valuation, where* $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A}\rangle$.

(*1*) *If* $v(w, \psi) = v(u, \psi)$ *for every* $\psi \in \mathcal{E}$ *then* $w = u$.
(*2*) $|W_v| \leq 2^{|\mathcal{E}\cap\mathcal{C}\cap\mathcal{A}|} \cdot 3^{|\mathcal{E}\cap\overline{\mathcal{C}}\cap\mathcal{A}|+|\mathcal{E}\cap\mathcal{C}\cap\overline{\mathcal{A}}|} \cdot 4^{|\mathcal{E}\cap\overline{\mathcal{C}}\cap\overline{\mathcal{A}}|}$.

PROOF. Item 2 directly follows from Item 1 by counting the number of possible functions from $\mathcal{E}$ to $2^{\{\mathtt{T},\mathtt{F}\}}$ that can be used in a ρ-quasi-valuation (see Definition 5.16). For 1, suppose that $v(w, \psi) = v(u, \psi)$ for every $\psi \in \mathcal{E}$. It follows that $wR^v_{\pi_0}u$. Since v is differentiated, $R^v_{\pi_0} = \boldsymbol{Id}_v$, and so $w = u$. □

Together with Corollary 5.32, the last proposition makes it possible to have a *semantic decision procedure* for the problem described in Proposition 5.12. Given a finite set $\mathcal{S}$ of sequents, a sequent s, and a finite set $\mathcal{E}$ of formulas, it is possible (under the assumptions of Proposition 5.12) to check all functions of the form $v : W \times \mathcal{E} \to 2^{\{\mathtt{T},\mathtt{F}\}}$, where $|W|$ is bounded according to the last proposition. Corollary 5.32 and the last proposition entail that $\mathcal{S} \not\vdash^{\rho}_{\mathbf{G}} s$ iff one of these functions is a strongly **G**-legal ρ-quasi-valuation, which is a model of $\mathcal{S}$ but not of s. In this case the semantics is effective, leading to a counter-model search procedure. Consequently, we obtain a semantic decision procedure for $\leq$-analytic finite basic systems, provided $\leq$ is safe (see Proposition 5.7, and compare with the syntactic procedure described in its proof). Indeed, in this case $\mathcal{S} \vdash_{\mathbf{G}} s$ iff $\mathcal{S} \vdash_{\mathbf{G}}^{\rho} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$.

Remark 5.34. Semantics for sequent systems without cut or identity-axiom was studied in some previous works. Following [Schütte 1960], [Girard 1987] studied the cut-free fragment of **LK**, and provided semantics for this fragment using (non-deterministic) three-valued valuations. Together with better understanding of the semantic role of the cut rule, this three-valued semantics was applied for proving several generalizations of the cut-elimination theorem (such as Takeuti's conjecture). Later, the axiom-free fragment of **LK** was studied in [Hösli and Jäger 1994]. As noted there, axiom-free systems play an important role in the proof-theoretic analysis of logic programming and in connection with the so called negation as failure. Hösli and Jäger provided a dual (non-deterministic) three-valued valuation semantics for axiom-free derivability in **LK**. The current work generalizes [Girard 1987] and [Hösli and Jäger 1994] in two main aspects:[9]

(1) Our results apply to the much broader family of basic systems, of which **LK** is just a particular example. For the cut-free and the axiom-free fragments of **LK**, the semantics that we obtain is practically identical to the one suggested in [Girard 1987] and [Hösli and Jäger 1994].
(2) The notion of proof-specification makes it possible to allow cuts and identity axioms on some formulas, and disallow it on others. In contrast, in terms of proof-specifications [Girard 1987] and [Hösli and Jäger 1994] only handle $\rho = \langle \mathit{wff}_{\mathcal{L}}, \emptyset, \mathit{wff}_{\mathcal{L}} \rangle$ and $\rho = \langle \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}}, \emptyset \rangle$ (respectively). In fact, from the presentation in [Hösli and Jäger 1994], it is unclear whether the two dual kinds of three-valued valuation semantics can be combined. From our results, it follows that such a combination is obtained using four-valued non-deterministic semantics.

5.2.1. A Proof of the Strong Completeness Theorem. We prove Theorem 5.31. For the rest of this section, let $\mathbf{G}$ be an arbitrary basic system, $\rho = \langle \mathcal{E}, \mathcal{C}, \mathcal{A} \rangle$ be some proof-specification, and $\mathcal{S}$ be a set of sequents.

Definition 5.35. A $\langle \mathbf{G}, \mathcal{S}, \rho \rangle$-*maximal set* is a (possibly infinite) set of signed $\mathcal{E}$-formulas μ such that $\mathcal{S} \nvdash_{\mathbf{G}}^{\rho} s$ for every sequent $s \subseteq \mu$, but for every signed $\mathcal{E}$-formula $\alpha \notin \mu$, there exists a sequent $s \subseteq \mu$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\rho} s \cup \{\alpha\}$.

LEMMA 5.36. *Let μ be a set of signed $\mathcal{E}$-formulas. If $\mathcal{S} \nvdash_{\mathbf{G}}^{\rho} s$ for every sequent $s \subseteq \mu$, then there exists a $\langle \mathbf{G}, \mathcal{S}, \rho \rangle$-maximal set μ' such that $\mu \subseteq \mu'$.*

PROOF. Let $\alpha_1, \alpha_2 \dots$ be an enumeration of all signed $\mathcal{E}$-formulas which are not in μ. We recursively define a sequence of sets of signed formulas, $\{\mu_k\}_{k=0}^{k=\infty}$. Let $\mu_0 = \mu$. For $k \geq 1$, let $\mu_k = \mu_{k-1}$ iff there exists a sequent $s \subseteq \mu_{k-1}$ such that $\mathcal{S} \vdash_{\mathbf{G}}^{\rho} s \cup \{\alpha_k\}$. Otherwise, let $\mu_k = \mu_{k-1} \cup \{\alpha_k\}$. Finally, let $\mu' = \bigcup_{k \geq 0} \mu_k$. It is easy to verify that μ' has all required properties. □

Definition 5.37. The $\langle \mathbf{G}, \mathcal{S}, \rho \rangle$-*canonical quasi-valuation* is defined by:

— W_v is the set of all $\langle \mathbf{G}, \mathcal{S}, \rho \rangle$-maximal sets.
— $D_v = \mathcal{E}$ and $v(\mu, \psi) = \{\mathrm{X} \in \{\mathrm{T}, \mathrm{F}\} \mid \mathrm{X}{:}\psi \notin \mu\}$ for every $\mu \in W_v$ and $\psi \in \mathcal{E}$.

[9]It should be mentioned, however, that [Girard 1987] and [Hösli and Jäger 1994] handled also the usual quantifiers of **LK**, while we only investigate *propositional* logics, leaving the more complicated first-order case (and beyond) to a future work.

LEMMA 5.38. *The $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation is a ρ-quasi-valuation.*

PROOF. Let v be the $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation. By definition $D_v = \mathcal{E}$. It remains to prove that the following conditions hold for every $\mu \in W_v$:

(1) If $\psi \in \mathcal{C}$ then $v(\mu, \psi) \neq \{\mathtt{T}, \mathtt{F}\}$. To see this, it suffices to prove that if $\psi \in \mathcal{E} \cap \mathcal{C}$ then $\mathtt{F}{:}\psi \in \mu$ or $\mathtt{T}{:}\psi \in \mu$. Assume by way of contradiction that $\mathtt{F}{:}\psi \notin \mu$ and $\mathtt{T}{:}\psi \notin \mu$ for some $\psi \in \mathcal{E} \cap \mathcal{C}$. It follows that there exist sequents $s_1, s_2 \subseteq \mu$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s_1 \cup \{\mathtt{F}{:}\psi\}$ and $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s_2 \cup \{\mathtt{T}{:}\psi\}$. Since $\psi \in \mathcal{C}$, a (legal) application of (cut) yields $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s_1 \cup s_2$. But this contradicts the properties of μ.
(2) If $\psi \in \mathcal{A}$ then $v(\mu, \psi) \neq \emptyset$. To see this, it suffices to prove that $\psi \in \mathcal{E} \cap \mathcal{A}$ implies that $\mathtt{F}{:}\psi \notin \mu$ or $\mathtt{T}{:}\psi \notin \mu$. Note that if $\psi \in \mathcal{E} \cap \mathcal{A}$ then $\psi \Rightarrow \psi$ is a (legal) application of (id), and so $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} \psi \Rightarrow \psi$. Since μ is a $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-maximal set, it follows that either $\mathtt{F}{:}\psi \notin \mu$ or $\mathtt{T}{:}\psi \notin \mu$. □

Definition 5.39. Let π be a context-relation. $R^{\mathcal{E}}_{\pi}$ denotes the binary relation between sets of signed $\mathcal{E}$-formulas defined as follows: $\mu R^{\mathcal{E}}_{\pi} \mu'$ iff for every signed $\mathcal{E}$-formulas α, β, if $\alpha \bar{\pi} \beta$ and $\beta \in \mu$ then $\alpha \in \mu'$.

LEMMA 5.40. *Let v be the $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation. The following hold:*

(*1*) *For every signed $\mathcal{E}$-formula α and $\mu \in W_v$: $\mu \models^v \alpha$ iff $\alpha \notin \mu$.*
(*2*) *$R^{\mathcal{E}}_{\pi} = R^{v}_{\pi}$ for every context-relation π, and $R^{\mathcal{E}}_{\pi_0} = \mathbf{Id}_v$.*
(*3*) *For every $\mathcal{E}$-sequent s and $\mu \in W_v$:*
  (*a*) *$s \not\subseteq \mu$ iff $\mu \models^v s$.*
  (*b*) *If there exists a sequent $s' \subseteq \mu$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s \cup s'$, then $\mu \models^v s$.*
  (*c*) *For every context-relation π, if $R^{\mathcal{E}}_{\pi}[\mu] \models^v s$, then there exists a π-instance $\langle c, c'\rangle$ such that $c' \subseteq \mu$ and $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s \cup c$.*
(*4*) *For every sequent s: $\models^v s$ iff $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$.*

PROOF.

(1) Suppose that $\alpha = \mathtt{X}{:}\psi$ where $\mathtt{X} \in \{\mathtt{T}, \mathtt{F}\}$. Then, $\mathtt{X} \in v(\mu, \psi)$ iff $\alpha \notin \mu$. Equivalently, $\mu \models^v \alpha$ iff $\alpha \notin \mu$.
(2) For every context-relation π, $R^{\mathcal{E}}_{\pi} = R^{v}_{\pi}$ follows from Item 1 (see Definitions 5.19 and 5.39). To see that $R^{\mathcal{E}}_{\pi_0} = \mathbf{Id}_v$, note that $\alpha \bar{\pi}_0 \beta$ iff $\alpha = \beta$. By Definition 5.39, $\mu R^{\mathcal{E}}_{\pi_0} \mu'$ iff for every signed $\mathcal{E}$-formula α, $\alpha \in \mu$ implies that $\alpha \in \mu'$. Equivalently, $\mu R^{\mathcal{E}}_{\pi_0} \mu'$ iff $\mu \subseteq \mu'$. Therefore, obviously, $\mu R^{\mathcal{E}}_{\pi_0} \mu$ for every $\mu \in W_v$. For the converse, we show that if $\mu, \mu' \in W_v$ and $\mu \subseteq \mu'$, then $\mu = \mu'$. Assume (by way of contradiction) that $\mu \subseteq \mu'$ and there exists $\alpha \in \mu' \setminus \mu$. Since μ is a $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-maximal set, there exists a sequent $s \subseteq \mu$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s \cup \{\alpha\}$. But, $s \cup \{\alpha\} \subseteq \mu'$, and this contradicts the fact that $\mathcal{S} \not\vdash^{\rho}_{\mathbf{G}} s$ for every $s \subseteq \mu'$.
(3) (a) Easily follows from Item 1.
  (b) Assume that there exists a sequent $s' \subseteq \mu$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s \cup s'$. Since μ is a $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-maximal set, $s \not\subseteq \mu$. Therefore, Item 3a entails that $\mu \models^v s$.
  (c) Assume that there does not exist a π-instance $\langle c, c'\rangle$ such that $c' \subseteq \mu$ and $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s \cup c$. We show that $R^{\mathcal{E}}_{\pi}[\mu] \not\models^v s$. Let $\mu^* = \{\alpha \mid frm[\alpha] \in \mathcal{E} \textit{ and } \exists \beta \in \mu . \alpha \bar{\pi} \beta\}$. Because of the presence of $(\Rightarrow W)$ and $(W \Rightarrow)$, our assumption implies that there does not exist $s' \subseteq s \cup \mu^*$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s'$. Since $frm[s \cup \mu^*] \subseteq \mathcal{E}$, Lemma 5.36 entails that there exists a $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-maximal set μ', such that $s \cup \mu^* \subseteq \mu'$. Item 3a entails that $\mu' \not\models^v s$. By definition, $\mu R^{\mathcal{E}}_{\pi} \mu'$. Hence, $R^{\mathcal{E}}_{\pi}[\mu] \not\models^v s$.
(4) Note first that if $frm[s] \not\subseteq \mathcal{E}$, then by definition, $\not\models^v s$ and $\mathcal{S} \not\vdash^{\rho}_{\mathbf{G}} s$. Assume now that $frm[s] \subseteq \mathcal{E}$. One direction easily follows from Item 3b. For the converse, assume that $\mathcal{S} \not\vdash^{\rho}_{\mathbf{G}} s$. We show that $\not\models^v s$. Because of the presence of $(\Rightarrow W)$ and $(W \Rightarrow)$, there

does not exist $s' \subseteq s$ such that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s'$. By Lemma 5.36, there exists a $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-maximal set $\mu \in W_v$, such that $s \subseteq \mu$. Item 3a entails that $\mu \not\models^{v} s$. Hence, $\not\models^{v} s$. □

LEMMA 5.41. *The $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation is strongly $\mathbf{G}$-legal.*

PROOF. Let v be the $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation. We show that v respects every $r \in \Upsilon_{\mathbf{G}}$ for $\Re^{v}_{\mathbf{G}}$. Let $r = \langle s_1, \pi_1\rangle, \ldots, \langle s_n, \pi_n\rangle / C$ be a rule in $\Upsilon_{\mathbf{G}}$. Let $\mu \in W_v$, and let σ be a substitution. Suppose that $frm[\sigma(s_1) \cup \ldots \cup \sigma(s_n) \cup \sigma(C)] \subseteq \mathcal{E}$, and that $\Re^{v}_{\mathbf{G}}(\pi_i)[\mu] \models^{v} \sigma(s_i)$ for every $1 \leq i \leq n$. We prove that $\mu \models^{v} \sigma(C)$. By Lemma 5.40 (Item 2), $\Re^{v}_{\mathbf{G}}(\pi_i) = R^{\mathcal{E}}_{\pi_i}$ for every $1 \leq i \leq n$. Thus Lemma 5.40 (Item 3c) entails that for every $1 \leq i \leq n$, there exists a π-instance $\langle c_i, c'_i\rangle$ such that $c'_i \subseteq \mu$ and $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} \sigma(s_i) \cup c_i$. Now we can use these proofs, and the rule r to obtain $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} \sigma(C) \cup c'_1 \cup \ldots \cup c'_n$, where $c'_1 \cup \ldots \cup c'_n \subseteq \mu$. Lemma 5.40 (Item 3b) entails that $\mu \models^{v} \sigma(C)$. □

PROOF OF THEOREM 5.31. Assume that $\mathcal{S} \vdash_{\mathbf{M}} s$, where $\mathbf{M}$ is the set of all strongly $\mathbf{G}$-legal differentiated ρ-quasi-valuations. Let v be the $\langle \mathbf{G}, \mathcal{S}, \rho\rangle$-canonical quasi-valuation. By Lemmas 5.38, 5.40 (Item 2), and 5.41, $v \in \mathbf{M}$. Since obviously $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s'$ for every $\mathcal{E}$-sequent $s' \in \mathcal{S}$, Lemma 5.40 (Item 4) implies that $\models^{v} s'$ for every such s'. By definition, we have $\models^{v} \mathcal{S}$, and so $\models^{v} s$. Finally, Lemma 5.40 (Item 4) implies that $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$. □

5.3. Semantic Characterizations of Analyticity, Cut-admissibility, and Axiom-Expansion

In this section we use Corollary 5.32 to derive semantic characterizations of the three proof-theoretic properties of basic systems discussed in Section 5.1. We will use the following new notion:

Definition 5.42. An *instance* of a quasi-valuation v is a valuation v' such that $W_{v'} = W_v$ and $v'(w, \psi) \in v(w, \psi)$ for every $w \in W_v$ and $\psi \in D_v$.

The following proposition immediately follows from the definitions.

PROPOSITION 5.43. *Let v be a $\langle \mathcal{E}, \mathcal{C}, \mathcal{A}\rangle$-quasi-valuation, and let v' be an instance of v. Then, for every D_v-sequent s: if $\models^{v'} s$ then $\models^{v} s$. In addition, if $frm[s] \subseteq \mathcal{C}$, the converse holds as well.*

The following characterization of analyticity follows from the previous results.

COROLLARY 5.44 (CHARACTERIZATION OF ANALYTICITY). *Let $\leq$ be a partial order on $wff_{\mathcal{L}}$. A basic system $\mathbf{G}$ is $\leq$-analytic iff for every finite $\mathcal{S}$ and s, $\mathcal{S} \vdash_{\mathbf{M}_1} s$ implies $\mathcal{S} \vdash_{\mathbf{M}_2} s$ where $\mathbf{M}_1$ is the set of all $\mathbf{G}$-legal valuations, and $\mathbf{M}_2$ is the set of all strongly $\mathbf{G}$-legal differentiated $\langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], wff_{\mathcal{L}}, wff_{\mathcal{L}}\rangle$-quasi-valuations.*

PROOF. Suppose that $\mathbf{G}$ is $\leq$-analytic. Assume that $\mathcal{S} \vdash_{\mathbf{M}_1} s$ for some finite set $\mathcal{S}$ of sequents and sequent s. By Theorem 4.18, $\mathcal{S} \vdash_{\mathbf{G}} s$, and so $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], wff_{\mathcal{L}}, wff_{\mathcal{L}}\rangle$. By Theorem 5.24, $\mathcal{S} \vdash_{\mathbf{M}_2} s$.

For the converse, suppose that $\mathcal{S} \vdash_{\mathbf{G}} s$. Obviously, there exists a finite subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}' \vdash_{\mathbf{G}} s$. By Theorem 4.18, $\mathcal{S}' \vdash_{\mathbf{M}_1} s$, and so our assumption entails that $\mathcal{S}' \vdash_{\mathbf{M}_2} s$. By Theorem 5.31, $\mathcal{S}' \vdash^{\rho}_{\mathbf{G}} s$ for $\rho = \langle \downarrow^{\leq}[\mathcal{S} \cup \{s\}], wff_{\mathcal{L}}, wff_{\mathcal{L}}\rangle$. Thus $\mathcal{S} \vdash^{\rho}_{\mathbf{G}} s$ for the same ρ. □

The above characterization might be quite complicated to be used in practice. Therefore, we now present a simpler semantic criterion, that turns out to be useful for many basic systems.

COROLLARY 5.45. *Let $\mathbf{G}$ be a basic system, and $\leq$ be a partial order on $wff_{\mathcal{L}}$, such that $\downarrow^{\leq}[\varphi]$ is finite for every $\varphi \in wff_{\mathcal{L}}$. Suppose that for every finite set $\mathcal{E}$ of formulas* *closed under $\leq$ (i.e., $\downarrow^{\leq}[\mathcal{E}] \subseteq \mathcal{E}$), every strongly* $\mathbf{G}$*-legal differentiated* $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$*-quasi-valuation has a* $\mathbf{G}$*-legal instance. Then* $\mathbf{G}$ *is* $\leq$*-analytic.*

PROOF. We use Corollary 5.44. Let $\mathcal{S}$ be a finite set of sequents, s be a single sequent, and $\mathcal{E} = \downarrow^{\leq}[\mathcal{S} \cup \{s\}]$ ($\mathcal{E}$ is finite and closed under $\leq$). Let $\mathbf{M}_1$ be the set of all $\mathbf{G}$-legal valuations, and $\mathbf{M}_2$ be the set of all strongly $\mathbf{G}$-legal differentiated $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuations. Assume that $\mathcal{S} \vdash_{\mathbf{M}_1} s$. We prove that $\mathcal{S} \vdash_{\mathbf{M}_2} s$. Let $v \in \mathbf{M}_2$, and suppose that $\models^{v} \mathcal{S}$. Our assumption entails that there exists a $\mathbf{G}$-legal instance v' of v. Thus $v' \in \mathbf{M}_1$. By Proposition 5.43, since $\mathit{frm}[\mathcal{S}] \subseteq \mathcal{E}$, we have $\models^{v'} \mathcal{S}$. Since $\mathcal{S} \vdash_{\mathbf{M}_1} s$, we have $\models^{v'} s$. Proposition 5.43 entails that $\models^{v} s$. $\square$

Remark 5.46. In previous papers (see e.g., [Avron and Zamansky 2011]) analyticity was defined as an extension property, according to which every partial valuation, whose domain is closed under subformulas, can be extended to a total valuation. Note that an instance of an $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation v is actually an extension of v to a full valuation. Thus, the last corollary establishes a connection between this semantic notion of analyticity and the proof-theoretic analyticity defined in this paper. It shows that in our context this extension property is sufficient for sub-analyticity.

Before turning to some applications of the criterion above, we present a characterization of strong cut-admissibility. Its proof is similar to the proof of Corollary 5.44.

COROLLARY 5.47 (CHARACTERIZATION OF STRONG CUT-ADMISSIBILITY). *A basic system* $\mathbf{G}$ *enjoys strong cut-admissibility iff for every finite* $\mathcal{S}$ *and* s*,* $\mathcal{S} \vdash_{\mathbf{M}_1} s$ *implies* $\mathcal{S} \vdash_{\mathbf{M}_2} s$ *where* $\mathbf{M}_1$ *is the set of all* $\mathbf{G}$*-legal valuations, and* $\mathbf{M}_2$ *is the set of all strongly* $\mathbf{G}$*-legal differentiated* $\langle \mathit{wff}_{\mathcal{L}}, \mathit{frm}[\mathcal{S}], \mathit{wff}_{\mathcal{L}} \rangle$*-quasi-valuations.*

Like in the case of analyticity, we provide a simpler *sufficient* criterion:

COROLLARY 5.48. *Let* $\mathbf{G}$ *be a basic system. Suppose that for every finite set* $\mathcal{E}$ *of formulas, every strongly* $\mathbf{G}$*-legal differentiated* $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$*-quasi-valuation has a* $\mathbf{G}$*-legal instance. Then* $\mathbf{G}$ *enjoys strong cut-admissibility.*

PROOF. We use Corollary 5.47. Let $\mathcal{S}$ be a finite set of sequents, s be a single sequent, and $\mathcal{E} = \mathit{frm}[\mathcal{S}]$ ($\mathcal{E}$ is finite). Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be like in Corollary 5.47. Assume that $\mathcal{S} \vdash_{\mathbf{M}_1} s$. We prove that $\mathcal{S} \vdash_{\mathbf{M}_2} s$. Let $v \in \mathbf{M}_2$. Suppose that $\models^{v} \mathcal{S}$. Our assumption entails that there exists a $\mathbf{G}$-legal instance v' of v. Thus $v' \in \mathbf{M}_1$. By Proposition 5.43, since $\mathit{frm}[\mathcal{S}] \subseteq \mathcal{E}$, we have $\models^{v'} \mathcal{S}$. Since $\mathcal{S} \vdash_{\mathbf{M}_1} s$, we have $\models^{v'} s$. Proposition 5.43 entails that $\models^{v} s$. $\square$

Next we apply the previous semantic criteria to prove analyticity and/or strong cut-admissibility for some of the basic systems defined in Section 3.1.

Example 5.49 (**LK**). We prove that **LK** enjoys strong cut-admissibility (consequently, it has the global subformula property). Using Corollary 5.48, it suffices to show that for every finite set $\mathcal{E} \subseteq \mathit{wff}_{\mathcal{L}}$, every strongly **LK**-legal $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation has an **LK**-legal instance. Let $\mathcal{E}$ be a finite set of formulas, and v be a strongly **LK**-legal $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation. We recursively construct an instance of v, $v' : W_v \times \mathit{wff}_{\mathcal{L}} \to \{\mathtt{T}, \mathtt{F}\}$. For every $w \in W_v$ and for every atomic formula p, $v'(w, p) = \mathtt{X}$ if $v(w, p) = \{\mathtt{X}\}$, and otherwise $v'(w, p) = \mathtt{T}$ (say). Now suppose that $v'(w, \psi)$ and $v'(w, \varphi)$ were defined, define $v'(w, \psi \supset \varphi)$ as follows (similar definitions for the other connectives): if $v(w, \psi \supset \varphi) = \{\mathtt{X}\}$ then $v'(w, \psi \supset \varphi) = \mathtt{X}$, and otherwise $v'(w, \psi \supset \varphi) = \mathtt{T}$ iff either $v'(w, \psi) = \mathtt{F}$ or $v'(w, \varphi) = \mathtt{T}$. Clearly, v' is an instance of v. Based on the fact that v is a strongly **LK**-legal $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation, it is immediate to prove that $v' \in \mathbf{M}_{\mathbf{LK}}$ (see Example 4.31). It follows that v' is **LK**-legal.

Example 5.50 (**LJ**). We use Corollary 5.48 to show that **LJ** enjoys strong cut-admissibility. Let $\mathcal{E}$ be a finite set of formulas, and v be a strongly **LJ**-legal differentiated $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation. We recursively construct an instance of v, $v' : W_v \times \mathit{wff}_{\mathcal{L}} \to \{\mathtt{T}, \mathtt{F}\}$. For every $w \in W_v$ and for every atomic formula p, $v'(w, p) = \mathtt{X}$ if $v(w, p) = \{\mathtt{X}\}$, and otherwise $v'(w, p) = \mathtt{T}$ (say). Now suppose that $v'(w, \psi)$ and $v'(w, \varphi)$ were defined for every $w \in W_v$:

— $v'(w, \psi \supset \varphi)$ is defined as follows: if $v(w, \psi \supset \varphi) = \{\mathtt{X}\}$ then $v'(w, \psi \supset \varphi) = \mathtt{X}$. Otherwise $v'(w, \psi \supset \varphi) = \mathtt{T}$ iff for every $u \in R^v_{\pi_{\mathsf{int}}}[w]$, either $v'(u, \psi) = \mathtt{F}$ or $v'(u, \varphi) = \mathtt{T}$.
— $v'(w, \psi \wedge \varphi)$ is defined as follows: if $v(w, \psi \wedge \varphi) = \{\mathtt{X}\}$ then $v'(w, \psi \wedge \varphi) = \mathtt{X}$. Otherwise $v'(w, \psi \wedge \varphi) = \mathtt{T}$ iff $v'(w, \psi) = \mathtt{T}$ and $v'(w, \varphi) = \mathtt{T}$. Similar definitions are used for the other connectives of **LJ**.

Clearly, v' is an instance of v. Based on the fact that v is a strongly **LJ**-legal $\langle \mathit{wff}_{\mathcal{L}}, \mathcal{E}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation, it is easy to prove that v' respects **LJ** for $\Re^v_{\mathbf{LJ}}$ (and so it is strongly **LJ**-legal).

Example 5.51 (**BLJ**). While **BLJ** does not enjoy cut-admissibility (see Example 5.27), we use Corollary 5.45 to show that it still has the global subformula property. This answers a question raised in [Pinto and Uustalu 2009].[10] Let $\mathcal{E}$ be a finite set of formulas closed under subformulas, and v be a strongly **BLJ**-legal differentiated $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation. A construction of an instance of v, $v' : W_v \times \mathit{wff}_{\mathcal{L}} \to \{\mathtt{T}, \mathtt{F}\}$, is done as in Example 5.50 with the following addition:

> If $v(w, \psi \prec \varphi) = \{\mathtt{X}\}$ then $v'(w, \psi \prec \varphi) = \mathtt{X}$. Otherwise $v'(w, \psi \prec \varphi) = \mathtt{F}$ iff $v'(u, \psi) = \mathtt{F}$ or $v'(u, \varphi) = \mathtt{T}$ for every $u \in R^v_{\pi_{\mathsf{d}}}[w]$.

Clearly, v' is an instance of v. Based on the facts that v is a strongly **BLJ**-legal $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation, and that $\mathcal{E}$ is closed under subformulas, it is straightforward to prove that v' respects **BLJ** for $\Re^v_{\mathbf{BLJ}}$.

Example 5.52 (**PLJ**). **PLJ** does not enjoy the global subformula property (see Example 5.29). As a substitute, a weaker property is proved for this system in [Avron 2007] (called the n-subformula property). Roughly speaking, this property means that whenever a sequent s is provable, there also exists a proof of s that includes only formulas from $sub[s]$ and some of their negations. To be more precise, it is equivalent to $nsub$-analyticity, where $nsub$ is the transitive closure of the union of the relation sub and $\{\langle \neg\psi, \neg(\psi \diamond \varphi)\rangle, \langle \neg\varphi, \neg(\psi \diamond \varphi)\rangle \mid \psi, \varphi \in \mathit{wff}_{\mathcal{L}}, \diamond \in \{\wedge, \vee, \supset\}\}$. Note that $nsub$ is safe, and so $nsub$-analyticity suffices to establish decidability (see Proposition 5.7). Next, we prove $nsub$-analyticity for **PLJ** using the semantic criterion in Corollary 5.45. Let $\mathcal{E}$ be a finite set of formulas closed under $nsub$, and v be a strongly **PLJ**-legal differentiated $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation. A construction of an instance of v, $v' : W_v \times \mathit{wff}_{\mathcal{L}} \to \{\mathtt{T}, \mathtt{F}\}$, is done as in Example 5.50 with the following addition: $v'(w, \neg\psi) = \mathtt{X}$ if $v(w, \neg\psi) = \{\mathtt{X}\}$, and $v'(w, \neg\psi) = \mathtt{T}$ otherwise. Clearly, v' is an instance of v. Based on the fact that v is a strongly **PLJ**-legal $\langle \mathcal{E}, \mathit{wff}_{\mathcal{L}}, \mathit{wff}_{\mathcal{L}} \rangle$-quasi-valuation, we show that v' is **PLJ**-legal, since it respects **PLJ** for $\Re^v_{\mathbf{PLJ}}$. To see that v' respects π_{int} for $\Re^v_{\mathbf{PLJ}}$, it suffices to note that for every $\psi \in \mathit{wff}_{\mathcal{L}}$, if $v'(w, \psi) = \mathtt{T}$ then $v'(u, \psi) = \mathtt{T}$ for every $u \in \Re(\pi_{\mathsf{int}})[w]$. We claim that v' respects every $r \in \Upsilon_{\mathbf{PLJ}}$ for $\Re^v_{\mathbf{PLJ}}$. We demonstrate it here only for the rule $(\Rightarrow \neg \supset)$. Thus we show that if $v'(w, \psi) = \mathtt{T}$ and $v'(w, \neg\varphi) = \mathtt{T}$ then $v'(w, \neg(\psi \supset \varphi)) = \mathtt{T}$. Assume that $v'(w, \neg(\psi \supset \varphi)) = \mathtt{F}$. Our construction then ensures that $\neg(\psi \supset \varphi) \in \mathcal{E}$, and $v(w, \neg(\psi \supset \varphi)) = \mathtt{F}$ as well. Since $\{\psi, \neg\varphi\} \subseteq \downarrow^{nsub}[\neg(\psi \supset \varphi)]$

[10]Note that other systems for this logic, that enjoy cut-admissibility, were devised in [Goré and Postniece 2010] and [Pinto and Uustalu 2009]. However, these systems do not employ the standard notion of a sequent used in Gentzen-type systems, but more complicated data-structures.

and $\mathcal{E}$ is closed under $nsub$, we have that $\{\psi, \neg\varphi\} \subseteq \mathcal{E}$. Since v respects $(\Rightarrow \neg \supset)$ for $\Re^v_{\mathbf{PLJ}}$, either $v(w,\psi) = \{\mathtt{F}\}$ or $v(w,\neg\varphi) = \{\mathtt{F}\}$. By our construction, $v'(w,\psi) = \mathtt{F}$ or $v'(w,\neg\varphi) = \mathtt{F}$.[11]

Example 5.53 (**K**, **K4**, **KD**, **S4**). Each of the four basic systems **K**, **K4**, **KD**, and **S4** admits the semantic criterion given in Corollary 5.48 (and so they all enjoy strong cut-admissibility). To see this, let **G** be any one of these systems (here $\Pi_{\mathbf{G}} = \{\pi_0, \pi\}$ where π depends on **G**). Let $\mathcal{E}$ be a finite set of formulas, and v be a strongly **G**-legal differentiated $\langle wff_{\mathcal{L}}, \mathcal{E}, wff_{\mathcal{L}}\rangle$-quasi-valuation. An instance of v, $v' : W_v \times wff_{\mathcal{L}} \to \{\mathtt{T},\mathtt{F}\}$, is constructed as in Example 5.49 with the following addition: $v'(w,\Box\psi) = \mathtt{X}$ if $v(w,\Box\psi) = \{\mathtt{X}\}$, and otherwise $v'(w,\Box\psi) = \mathtt{T}$ iff $v'(u,\psi) = \mathtt{T}$ for every $u \in R^v_\pi[w]$. Clearly, v' is an instance of v. Using the fact that v is a strongly **G**-legal $\langle wff_{\mathcal{L}}, \mathcal{E}, wff_{\mathcal{L}}\rangle$-quasi-valuation, it is easy to show that v' respects **G** for $\Re^v_{\mathbf{G}}$.

Example 5.54 (**KB**, **S5**). While **KB** and **S5** do not enjoy cut-admissibility (for **S5**, see Example 5.28), Corollary 5.45 can be used to show that they still have the global subformula property. We demonstrate it here for **KB**. Let $\mathcal{E}$ be a finite set of formulas closed under subformulas, and v be a strongly **G**-legal differentiated $\langle \mathcal{E}, wff_{\mathcal{L}}, wff_{\mathcal{L}}\rangle$-quasi-valuation. A construction of an instance of v, $v' : W_v \times wff_{\mathcal{L}} \to \{\mathtt{T},\mathtt{F}\}$, is done exactly as in Example 5.53. We show that v' is indeed a **KB**-legal valuation, as it respects **KB** for $\Re^v_{\mathbf{KB}}$. To see that v' respects π for $\Re^v_{\mathbf{KB}}$, we show that $R^v_\pi \subseteq R^{v'}_\pi$. Suppose that $w_1 R^v_\pi w_2$. Note that by Proposition 5.21 (Item 3), we have that $w_2 R^v_\pi w_1$ (because of the structure of π in **KB**). We prove that $w_1 R^{v'}_\pi w_2$. Let α and β be signed formulas such that $\alpha\bar{\pi}\beta$ and $w_2 \models^v \alpha$. The structure of π ensures that there exists some $\psi \in wff_{\mathcal{L}}$ such that either $\alpha = \mathtt{F}{:}\psi$ and $\beta = \mathtt{F}{:}\Box\psi$, or $\alpha = \mathtt{T}{:}\Box\psi$ and $\beta = \mathtt{T}{:}\psi$. If $\Box\psi \in \mathcal{E}$ then α and β are $\mathcal{E}$-formulas (since $\mathcal{E}$ is closed under subformulas). In this case, since $w_1 R^v_\pi w_2$, we have that $w_1 \models^v \beta$, and we are done. Otherwise, for every $w \in W_v$, $v'(w,\Box\psi) = \mathtt{T}$ iff $v'(u,\psi) = \mathtt{T}$ for every $u \in R^v_\pi[w]$. Now, if $\alpha = \mathtt{F}{:}\psi$ and $\beta = \mathtt{F}{:}\Box\psi$, then $w_2 \models^v \alpha$ directly entails that $w_1 \models^v \beta$. Otherwise, $\alpha = \mathtt{T}{:}\Box\psi$ and $\beta = \mathtt{T}{:}\psi$. It follows that $v'(u,\psi) = \mathtt{T}$ for every $u \in R^v_\pi[w_2]$. Since $w_2 R^v_\pi w_1$, $w_1 \models^v \beta$ in this case as well. Finally, we claim that v' respects every $r \in \Upsilon_{\mathbf{KB}}$ for $\Re^v_{\mathbf{KB}}$. We show it here only for the rule $(\Rightarrow \Box)$. Following Example 4.13, we should prove that for every $w \in W_v$ and formula ψ: if $v'(u,\psi) = \mathtt{T}$ for every $u \in R^v_\pi[w]$, then $v'(w,\Box\psi) = \mathtt{T}$. Let $w \in W_v$, and $\psi \in wff_{\mathcal{L}}$. Suppose that $v'(u,\psi) = \mathtt{T}$ for every $u \in R^v_\pi[w]$. If $\Box\psi \in \mathcal{E}$, then the construction of v' directly entails that $v'(w,\Box\psi) = \mathtt{T}$. Otherwise, $\psi \in \mathcal{E}$ as well, and the construction of v' entails that $v(u,\psi) = \{\mathtt{T}\}$ for every $u \in R^v_\pi[w]$. Since v is strongly **KB**-legal, it respects $(\Rightarrow \Box)$ for $\Re^v_{\mathbf{KB}}$. Thus we have that $\mathtt{T} \in v(w,\Box\psi)$ (see Example 5.23). It then follows that $v'(w,\Box\psi) = \mathtt{T}$.

Example 5.55 (**GP**). Using the semantic criterion of Corollary 5.48, it is easy to see that **GP** enjoys strong cut-admissibility (and so it has the global subformula property). The construction of a **GP**-legal instance for every strongly **GP**-legal differentiated $\langle wff_{\mathcal{L}}, \mathcal{E}, wff_{\mathcal{L}}\rangle$-quasi-valuation is done as for **LK** (see Example 5.49), with straightforward modifications for q `said` and q `implied`. In addition we replace the $\{\mathtt{T},\mathtt{F}\}$ values assigned to formulas of the form $\psi \to \varphi$ by the value assigned to φ in each world.

Example 5.56 (**IS5**). **IS5** does not enjoy cut-admissibility, since the sequent $\Box(\Box p_1 \vee p_2) \Rightarrow \Box p_1, (\Box p_2 \supset \bot) \supset \bot$ is provable, but not cut-free provable (see [Ono 1977]). Using Theorem 5.24, one can semantically verify that there is no cut-free proof for this sequent, by constructing an **IS5**-legal $\langle wff_{\mathcal{L}}, \emptyset, wff_{\mathcal{L}}\rangle$-quasi-valuation which is not a

[11] The same construction proves a stronger claim, namely that **PLJ** is $\leq$-analytic, where $\leq$ is the transitive closure of the union of the relation sub and $\{\langle \neg\varphi, \neg(\psi \supset \varphi)\rangle \mid \psi, \varphi \in wff_{\mathcal{L}}\}$.

model of this sequent. In addition, the condition for sub-analyticity given in Corollary 5.45 does not hold for **IS5**. Since this condition is only proven to be sufficient, it does not mean that **IS5** is not sub-analytic, and this question remains open.

Finally, Corollary 5.32 also naturally leads to the following semantic characterization of axiom-expansion.

COROLLARY 5.57 (CHARACTERIZATION OF AXIOM-EXPANSION). *An n-ary connective $\diamond$ admits* axiom-expansion *in a basic system* **G** *iff every strongly* **G***-legal* $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$*-quasi-valuation is a* $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}} \cup \{\diamond(p_1, \ldots, p_n)\}\rangle$*-quasi-valuation.*

PROOF. We prove one direction. The converse is similar. Assume that $\diamond$ admits axiom-expansion in **G**. By definition, $\vdash^{\rho}_{\mathbf{G}} \diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$ for $\rho = \langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$. Corollary 5.32 entails that every strongly **G**-legal ρ-quasi-valuation is a model of $\diamond(p_1, \ldots, p_n) \Rightarrow \diamond(p_1, \ldots, p_n)$. It follows that in every strongly **G**-legal ρ-quasi-valuation v, $v(w, \diamond(p_1, \ldots, p_n)) \neq \emptyset$ for every $w \in W_v$. Thus, every strongly **G**-legal ρ-quasi-valuation is a $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}} \cup \{\diamond(p_1, \ldots, p_n)\}\rangle$-quasi-valuation. □

Example 5.58 (**LK** *and* **LJ**). Using the semantic criterion given in Corollary 5.57, it is straightforward to prove that every connective of **LK** (respectively, **LJ**) admits axiom-expansion in **LK** (**LJ**). We do it here for $\supset$. Let v be a strongly **LK**-legal (**LJ**-legal) $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$-quasi-valuation. We show that $v(w, p_1 \supset p_2) \neq \emptyset$ for every $w \in W_v$, and so v is a $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}} \cup \{p_1 \supset p_2\}\rangle$-quasi-valuation:

LK Since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LK}}$ and $\Re^v_{\mathbf{LK}}(\pi_0) = \mathit{Id}_v$, we have that $\mathtt{F} \in v(w, p_1 \supset p_2)$ whenever $\mathtt{T} \in v(w, p_1)$ and $\mathtt{F} \in v(w, p_2)$. Since v respects $(\Rightarrow\supset)$ for $\Re^v_{\mathbf{LK}}$ and $\Re^v_{\mathbf{LK}}(\pi_0) = \mathit{Id}_v$, we have that $\mathtt{T} \in v(w, p_1 \supset p_2)$ whenever either $\mathtt{F} \in v(w, p_1)$ or $\mathtt{T} \in v(w, p_2)$. Note that since v is a $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$-quasi-valuation, $\mathtt{F} \in v(w, p_1)$ or $\mathtt{T} \in v(w, p_1)$, and similarly $\mathtt{F} \in v(w, p_2)$ or $\mathtt{T} \in v(w, p_2)$. Together, it follows that $v(w, p_1 \supset p_2) \neq \emptyset$ for every $w \in W_v$.

LJ Suppose that $\mathtt{T} \notin v(w, p_1 \supset p_2)$ for some $w \in W_v$. Since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LJ}}$, $\mathtt{F} \notin v(u, p_1)$ and $\mathtt{T} \notin v(u, p_2)$ for some $u \in \Re^v_{\mathbf{LJ}}(\pi_{\mathsf{int}})[w]$. Since v is a $\langle wff_{\mathcal{L}}, \emptyset, at_{\mathcal{L}}\rangle$-quasi-valuation, $v(u, p_1) \neq \emptyset$ and $v(u, p_2) \neq \emptyset$. This entails that $v(u, p_1) = \{\mathtt{T}\}$ and $v(u, p_2) = \{\mathtt{F}\}$. Since v respects $(\supset\Rightarrow)$ for $\Re^v_{\mathbf{LJ}}$ and $\Re^v_{\mathbf{LJ}}(\pi_0) = \mathit{Id}_v$, we have that $\mathtt{F} \in v(u, p_1 \supset p_2)$. Since v respects π_{int} for $\Re^v_{\mathbf{LJ}}$, $\mathtt{F} \in v(w, p_1 \supset p_2)$ as well.

6. CONCLUSIONS AND FURTHER RESEARCH TOPICS

This paper is a part of an on-going project aiming to get a unified semantic theory and understanding of Gentzen-type systems and their proof-theoretic properties. Considering the broad family of basic systems, we substantially extended the scope of previous papers (like, e.g., [Avron and Lev 2005]). Many well-known sequent systems that seem unrelated can now be studied in a general framework. This framework provides (potentially, non-deterministic) Kripke-style semantics for these sequent systems. In turn, the semantics can be extended and used to derive important proof-theoretic properties of the sequent systems. We believe that the results of this paper provide useful tools, that may be applied whenever a (new) fully-structural propositional sequent system is encountered.

On the other hand, it should be admitted that the high generality we aim to is also a source of a certain weakness. Unlike the previous works that considered narrower families of sequent systems (e.g., [Avron and Lev 2005], [Avron and Lahav 2010]), the current paper does not provide *decidable* criteria for proof-theoretic properties. In fact, the given semantic characterizations might not be easy to apply. For example, the system **GL** (see Example 3.13) has the global subformula property and enjoys strong cut-admissibility (this can be shown, e.g., by a straightforward generalization of the proof in [Avron 1984]), but it is still not clear how to obtain these facts using our general semantic characterizations. In addition, the Kripke valuations semantics obtained for arbitrary basic systems might not be effective (i.e., it might not naturally lead to a semantic decision procedure). Indeed, the semantic tools are intended to complement the usual proof-theoretic ones, rather than replace them.

Several extensions of the current work seem interesting:

Single-Conclusion. The current work deals only with multiple-conclusion systems. It can be useful to derive similar results for single-conclusion systems. For *canonical* single-conclusion systems, this was done in [Avron and Lahav 2010]. It should be interesting to check whether every single-conclusion system in this framework has an equivalent multiple-conclusion system.

Hypersequents. Hypersequent systems make it possible to deal with more logics. For example, hypersequent structural rules can be used to bound the width of some accessibility relations (see [Ciabattoni and Ferrari 2001]). General formulations of hypersequent rules and their semantic effect deserves further work.

Many-Sided Sequents. Many-sided sequents, rather than the two-sided sequents that we considered, are particularly useful for finite-valued logics. We believe that it should be straightforward to extend our results for many-sided sequent systems.

First-Order Logics. Extending the framework for general first-order systems is an important goal for a future work.

Substructural Systems. Many important logics have only substructural sequent systems (in particular, contraction-free or weakening-free calculi), that can not be treated in our framework. We believe that dealing with substructural systems would be much more difficult, as the current semantic framework would not be expressive enough.

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