Automata on Self-Similar words

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Highlights 2025, Saarbrücken

Automata on S-adic words

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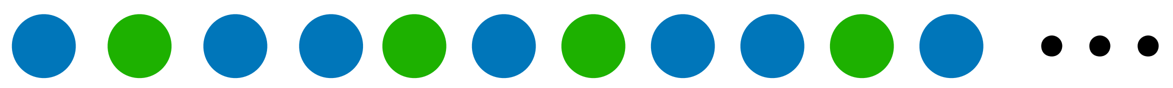
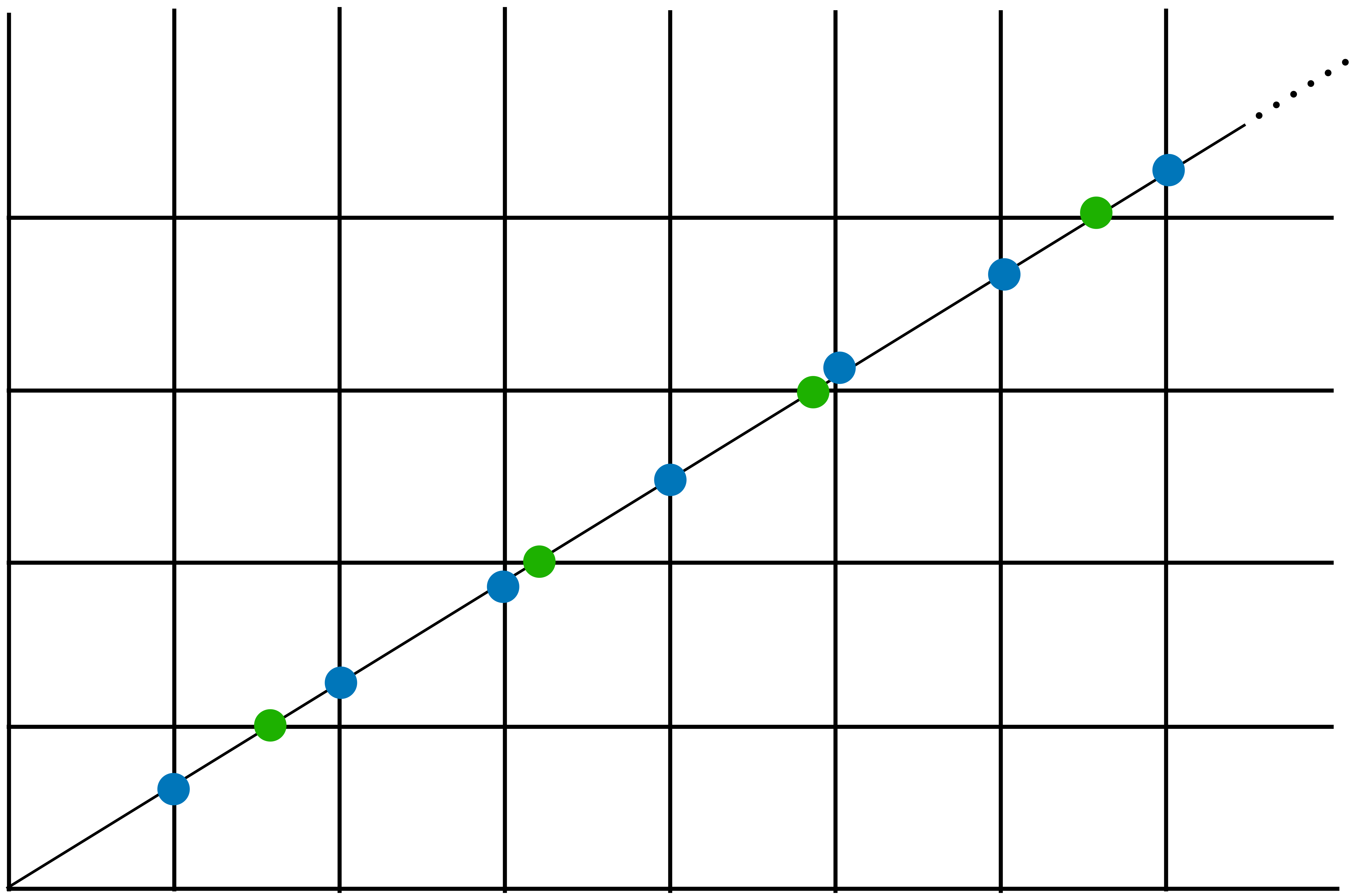
Some families of symbolic dynamical systems, such as **Sturmian words**, have several equivalent characterisations and enjoy neat combinatorial properties that reflect **self-similarity**

In this talk, for simplicity, the term Sturmian words will refer to what are technically characteristic Sturmian words

Definition by example: the Fibonacci word

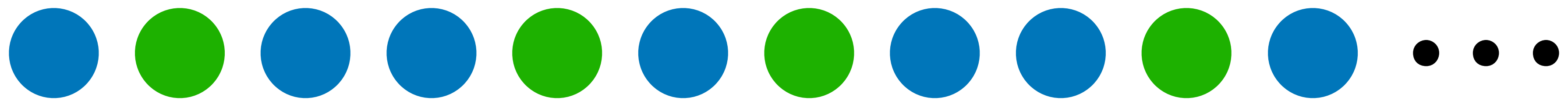




Cutting sequence, obtained by drawing a line of gradient $1/\phi$, and recording the order of crossing grid lines

Here ϕ is the golden ratio

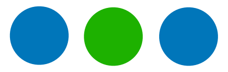
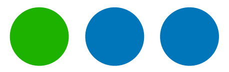
General Sturmian words also allow an intercept

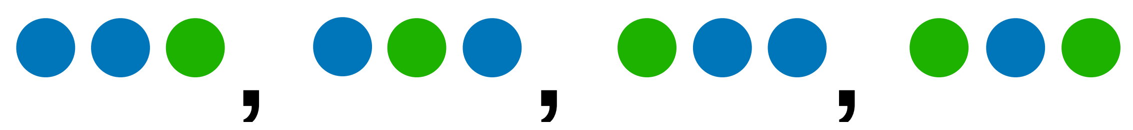


$$\mathbf{x}(k) = \lfloor (k+2)\cdot\eta \rfloor - \lfloor (k+1)\cdot\eta \rfloor$$

Formula for k-th letter of Sturmian word $\mathbf{x} \in \{0, 1\}^\omega$
with slope $\eta \in (0,1)$ irrational

For the Fibonacci word, $\eta = 1/\phi^2 \approx 0.38$

Any Sturmian word has exactly $n+1$ length-n factors, e.g., , , , 

Any Sturmian word has exactly $n+1$ length-n factors, e.g., 

The Problem

Given an ω-regular language L,
effectively characterise
the set of Sturmian words in L

Key Fact reflecting self-similarity

Any Sturmian word $\mathbf{x}$ is the image of a Sturmian word $\mathbf{x}'$ under a Sturmian substitution

Sturmian substitutions $S = \{\lambda_0, \lambda_1\}$, where $\lambda_0(0) = 0$, $\lambda_0(1) = 01$, $\lambda_1(0) = 10$, $\lambda_1(1) = 1$

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Consequently

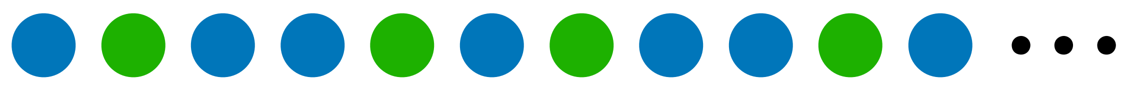
A Sturmian word $\mathbf{x}$ is directed by a sequence $\mathbf{s} = s_0 s_1 \cdots \in S^\omega$, i.e.,

there are Sturmian words $\mathbf{x}^{(0)} = \mathbf{x}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ satisfying $\mathbf{x}^{(n)} = s_n(\mathbf{x}^{(n+1)})$

This allows to further deduce $\mathbf{x} = \lim s_0 s_1 \cdots s_n(0)$

A tale of two spaces

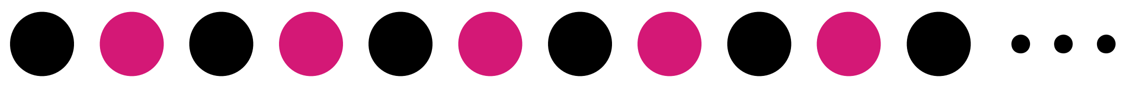
Sturmian word x

 generated from **s**, has sophisticated combinatorial properties

Any Sturmian word has exactly $n+1$ length-n factors

$$\mathbf{x} = \lim s_0 s_1 \cdots s_n(0)$$

Directive sequence s

 obtained from continued fraction of slope

$$\eta = \cfrac{1}{1 + a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = \cfrac{1}{1 + 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

$$\mathbf{s} = \lambda_0^{a_1} \cdot \lambda_1^{a_2} \cdot \lambda_0^{a_3} \cdots = (\lambda_0 \lambda_1)^{\omega}$$

Theorem. Given ω-regular $L \subseteq \{0,1\}^\omega$, we can compute ω-regular $L' \subseteq S^\omega$ such that a Sturmian word $\mathbf{x} \in L$ if and only if its directive sequence $\mathbf{s} \in L'$

Thank You!