Automata on S-adic words

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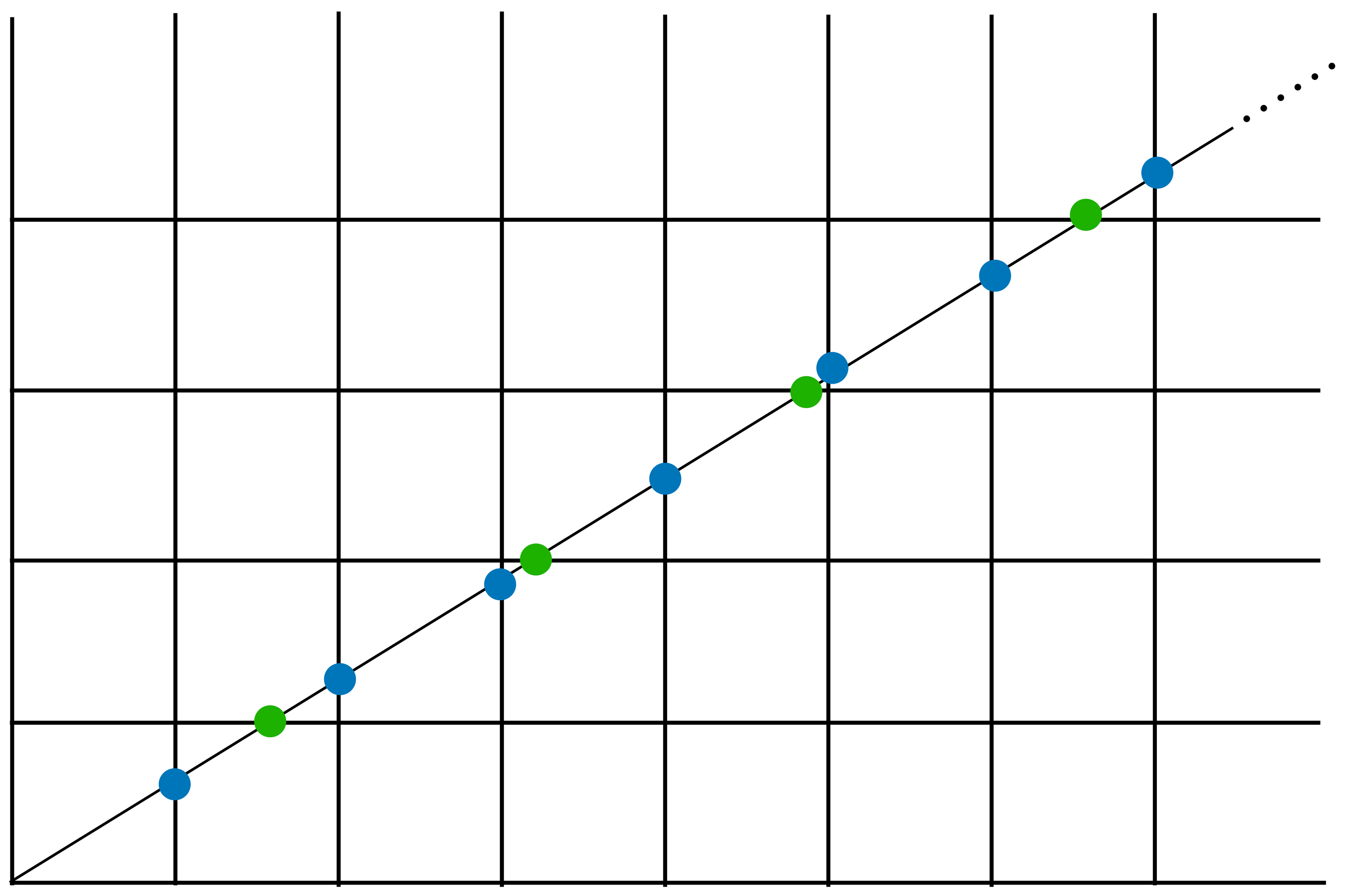
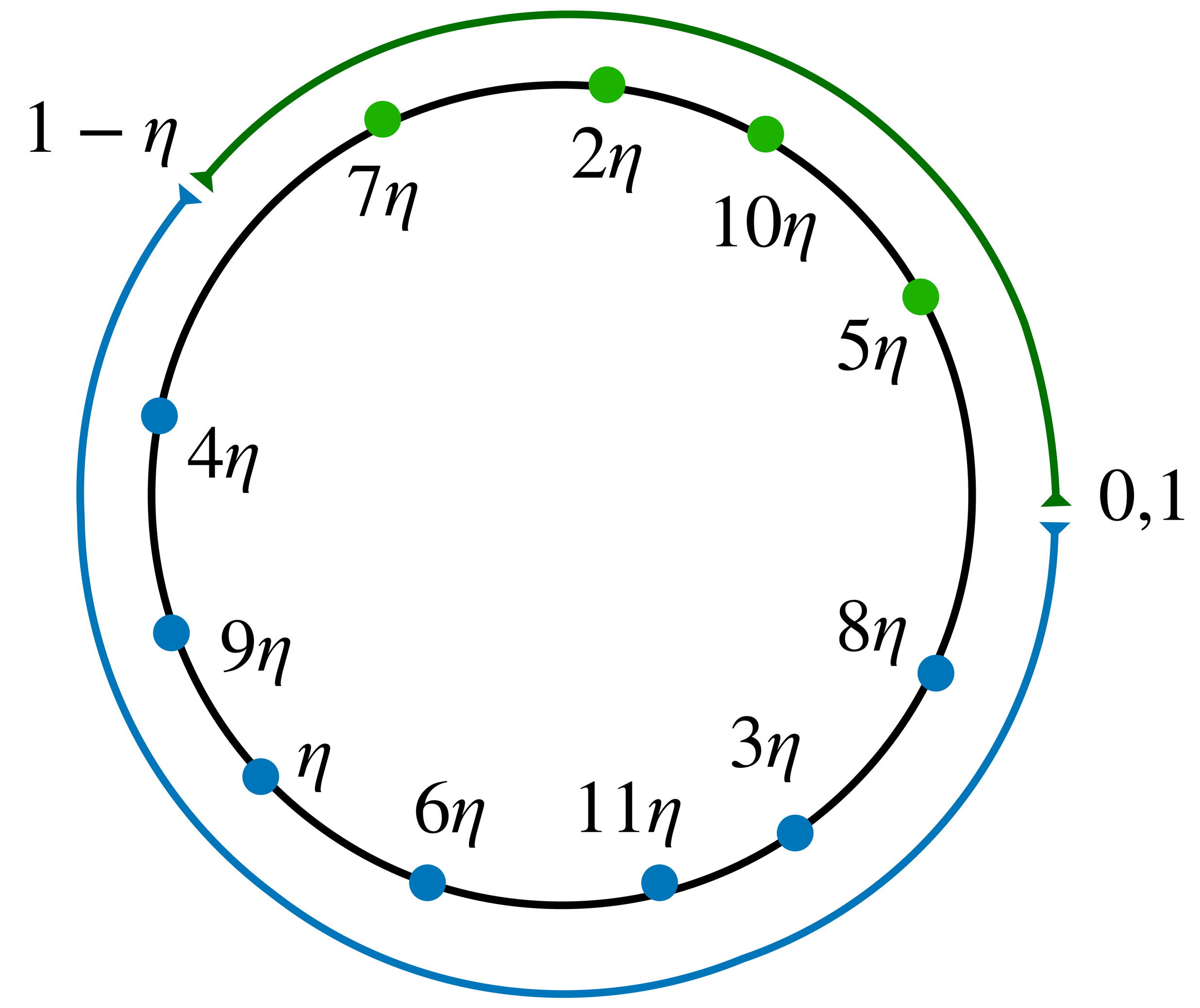
Some families of symbolic dynamical systems, such as **Sturmian words**, have several equivalent characterisations and enjoy neat combinatorial properties that reflect **self-similarity**

Definition by example: the Fibonacci word



Cutting sequence, obtained by drawing a line of slope η, and recording the order of crossing grid lines
Here $\eta = 1/\phi$, where ϕ is the golden ratio



⋯ pattern obtained as the itinerary of a rotation by η around a circle of circumference 1

$$\mathbf{x}(k) = \lfloor (k+2)\cdot\eta \rfloor - \lfloor (k+1)\cdot\eta \rfloor$$

Formula for k-th letter of Sturmian word $\mathbf{x} \in \{0,1\}^\omega$

Any Sturmian word has exactly $n+1$ length-n factors, e.g., , , ,

The Problem

Given an ω-regular language L, effectively characterise the set of Sturmian words in L

Key Fact

Any Sturmian word $\mathbf{x}$ is the image of a Sturmian word $\mathbf{x}'$ under a Sturmian substitution

Sturmian substitutions $S = \{\lambda_0, \lambda_1\}$, where $\lambda_0(0) = 0,\ \lambda_0(1) = 01,\ \lambda_1(0) = 10,\ \lambda_1(1) = 1$

Consequence: A Sturmian word $\mathbf{x}$ is directed by a sequence $\mathbf{s} = s_0 s_1 \cdots \in S^\omega$, i.e., there are Sturmian words $\mathbf{x}^{(0)} = \mathbf{x}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ satisfying $\mathbf{x}^{(n)} = s_n(\mathbf{x}^{(n+1)})$

This allows to further deduce $\mathbf{x} = \lim s_0 s_1 \cdots s_n(0)$

Sturmian word x

⋯ generated from **s**, has sophisticated combinatorial properties

Directive sequence s

⋯ obtained from slope, is subject to simpler constraints

Theorem. Given ω-regular $L \subseteq \{0,1\}^\omega$, we can compute ω-regular $L' \subseteq S^\omega$ such that a Sturmian word $\mathbf{x} \in L$ if and only if its directive sequence $\mathbf{s} \in L'$

We consider only characteristic Sturmian words in this presentation for simplicity. General Sturmian words additionally have an intercept.