On the Monadic Second-Order Theory of Arithmetic Predicates

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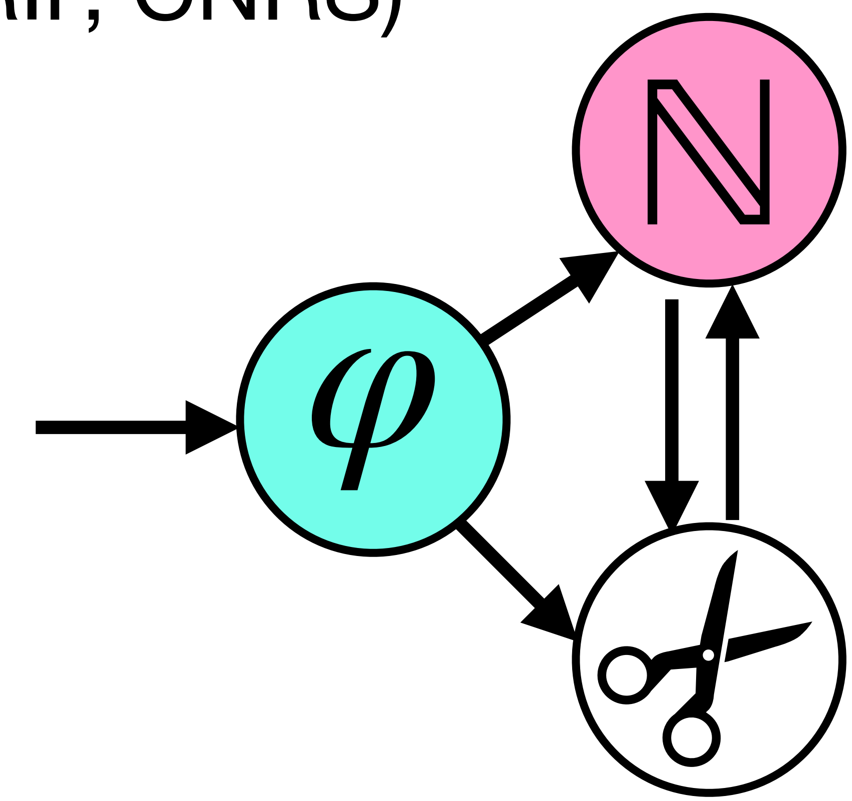

Automata Seminar, IRIF

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Research questions can often be exposed through playful riddles

Are there infinitely many n, m such that:

1. n is a power of 3; m is a power of 2
2. The units place digits of n, m are 9, 8 respectively
3. m is the smallest power of 2 larger than n, and their difference is at least 100

$$n = 4782969, m = 8388608$$

… ?

The riddle is an example of
how we can push the expressive limits of
Monadic Second-Order (MSO) Theory of
the natural numbers with order

$$\langle \mathbb{N}; < \rangle$$

But what is MSO Logic?

(over the structure of the natural numbers with order)

Statements in MSO logic have two kinds of variables: those that refer to numbers, and those that refer to sets of numbers

So why is MSO Logic important?

To practitioners:
for its ability to serve as a framework to reason about systems' execution traces

To theoreticians:
for its profound connections to formal language theory, and its place at the frontiers of decidability

Our research question

What expressive power can be added to the MSO Theory of the natural numbers with order while retaining its decidability?

MSO Theory of $\langle \mathbb{N}; < \rangle$

$$x = y$$

$$\neg(x < y) \wedge \neg(y < x)$$

MSO Theory of $\langle \mathbb{N}; < \rangle$

$$x = 0$$

$$\forall y \, . \, x \leq y$$

$$y = x + 1$$

$$x < y \wedge \neg \exists z \, . \, (x < z < y)$$

MSO Theory of $\langle \mathbb{N}; < \rangle$

Variables can refer to numbers $x, y, \ldots$
or to sets $X, Y, \ldots$ of numbers

The logic allows us to express that
x is an element of X

X is the empty set

$$\forall y \,.\, y \notin X$$

MSO Theory of $\langle \mathbb{N}; < \rangle$

$$X \subseteq Y$$

$$\forall x\,.\,(x \in X \Rightarrow x \in Y)$$

X has infinitely many elements

$$\forall x\,.\,\exists y\,.\,(x < y \land y \in X)$$

MSO Theory of $\langle \mathbb{N}; < \rangle$

Second-Order variables $X, Y, \ldots$ allow us to define some interesting unary *predicates*

x is even

$$\exists X .\ (x \in X \wedge 0 \in X \wedge \forall y .\ (y \in X \Leftrightarrow y+1 \notin X))$$

MSO Theory of $\langle \mathbb{N}; < \rangle$: Sentences

Variables occurring in a formula are either free or bound to a quantifier

A formula with only bound variables is called a **sentence**

$$\forall X . (\exists x . x \in X) \Rightarrow \big(\exists x . x \in X \wedge \forall y . (y \in X \Rightarrow x \leq y)\big)$$

Every non-empty set has a minimum element

Deciding an MSO Theory

$$\forall X . (\exists x . x \in X) \Rightarrow \big(\exists x . x \in X \wedge \forall y . (y \in X \Rightarrow x \leq y)\big)$$

Every non-empty set has a minimum element

Büchi (1962) showed how to decide:

Context	MSO Theory of $\langle \mathbb{N}; < \rangle$
Input	A sentence
Output	Whether the input sentence is true

Büchi's work


$\langle \mathbb{N}; < \rangle$
Automata
Expressive Power
Decision Techniques

Expanding the MSO Theory of $\langle \mathbb{N}; < \rangle$

However, the expressive power is not enough to assert, for instance:

x is a perfect square

x is a power of 2

Adding such predicates results in an expanded theory $\langle \mathbb{N}; <, P_1, \ldots, P_d \rangle$

Deciding expanded MSO Theories
[Elgot and Rabin, 1966]

It is known how to decide:

Context	MSO Theory of $\langle \mathbb{N}; <, \mathrm{Pow}_2 \rangle$
Input	A sentence
Output	Whether the input sentence is true

and also:

Context	MSO Theory of $\langle \mathbb{N}; <, \mathrm{Pow}_3 \rangle$
Input	A sentence
Output	Whether the input sentence is true

State of the art
[Carton and Thomas, 2002]

P

$\langle \mathbb{N}; < \rangle$

Algebra

Automata

Expressive Power

Decision Techniques

Sentence in MSO Theory of $\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3 \rangle$

There are infinitely many n, m such that:

$$\forall x \exists n \exists m . x < n < m \wedge \ldots$$

n is a power of 3; m is a power of 2

$$n \in \mathsf{Pow}_3 \wedge m \in \mathsf{Pow}_2 \wedge \ldots$$

The units place digits of n, m are of $9, 8$ respectively

$$n \in \mathsf{Units}_9 \wedge m \in \mathsf{Units}_8 \wedge \ldots$$

m is the smallest power of 2 larger than n,
and their difference is at least 100

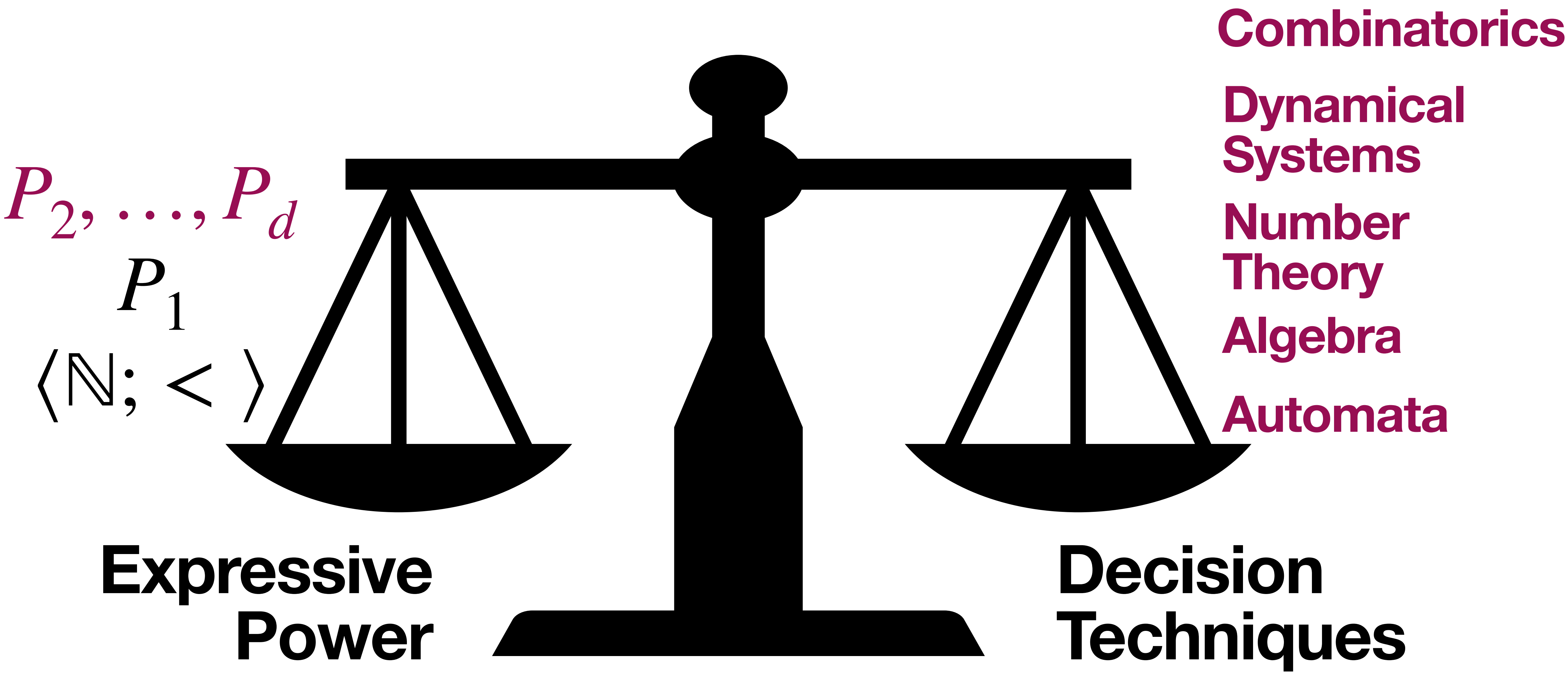
$$(n + 100 \leq m) \wedge \neg \exists k . (k \in \mathsf{Pow}_2 \wedge n < k < m)$$

We show that…

The MSO Theory of $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$ is decidable.

Context	MSO Theory of $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$
Input	A sentence
Output	Whether the input sentence is true

Our contribution



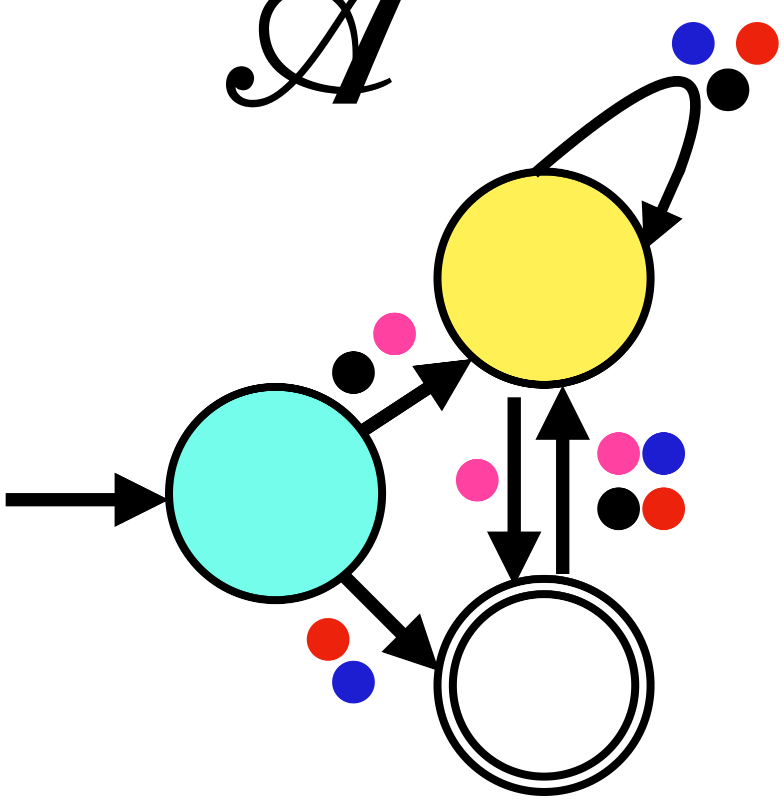
Sentence holds in Theory

$$\varphi \qquad \langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$$

Turing-equivalent

Due to Büchi, McNaughton, ...

$\mathscr{A}$



$$\overset{0}{\binom{0}{0}} \overset{1}{\binom{1}{1}} \overset{2}{\binom{1}{0}} \overset{3}{\binom{0}{1}} \overset{4}{\binom{1}{0}} \overset{5}{\binom{0}{0}} \overset{6}{\binom{0}{0}} \overset{7}{\binom{0}{0}} \overset{8}{\binom{1}{0}} \overset{9}{\binom{0}{1}} \overset{10}{\binom{0}{0}} \cdots$$

Automaton accepts characteristic word

Characteristic word $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$

$$\beta = \binom{0}{0}\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\binom{0}{0}\binom{1}{0}\binom{0}{1}\binom{0}{0}\cdots$$

Acceptance Problem

$\mathrm{Acc}_\beta :=$ Is the run of a given automaton $\mathscr{A}$ on β (Muller) accepting?

↑ trivially reduces to

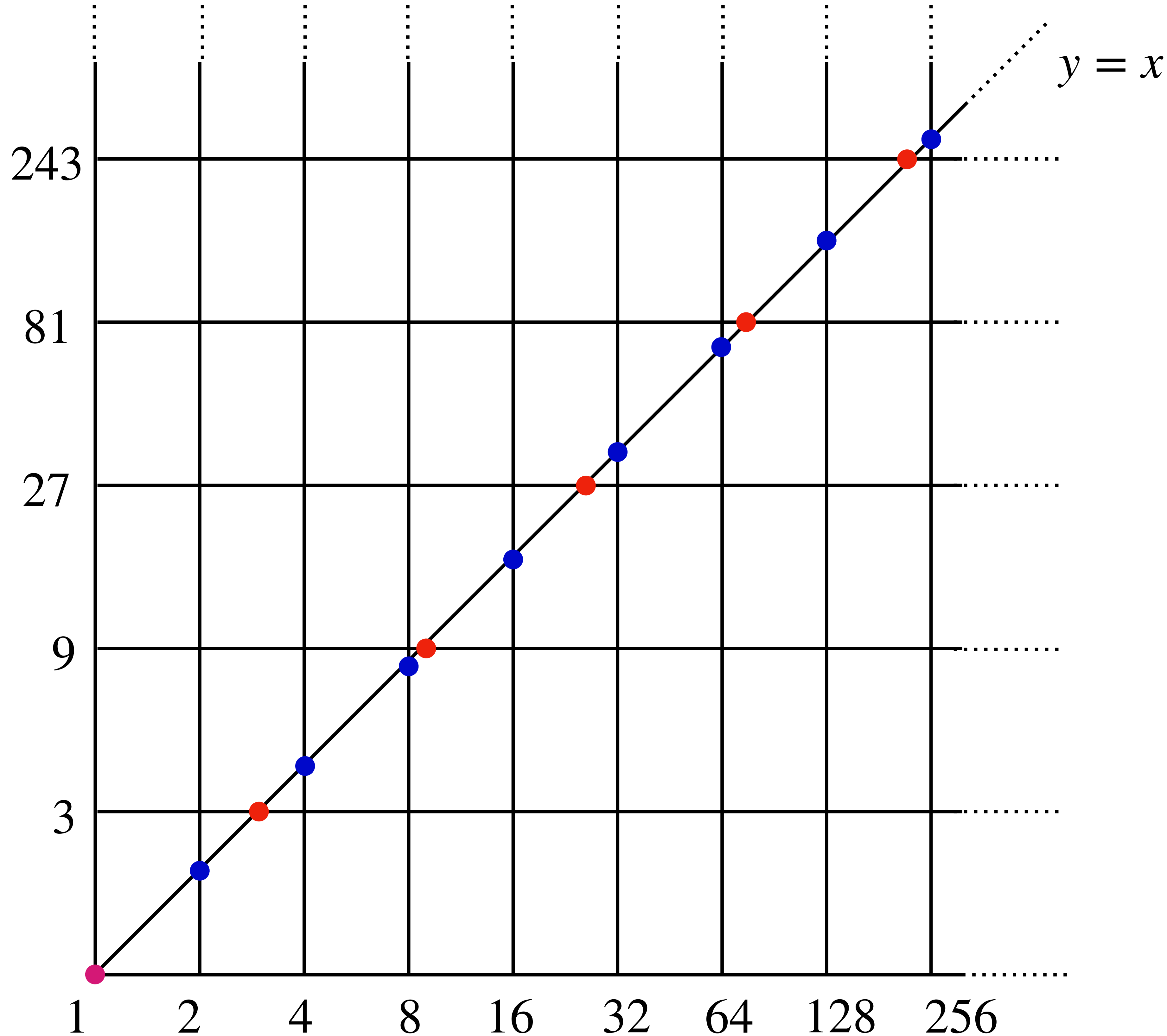
Acc_α

Order word

$$\alpha = \binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\cdots$$

Our order word is a cutting sequence

$$\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\cdots$$

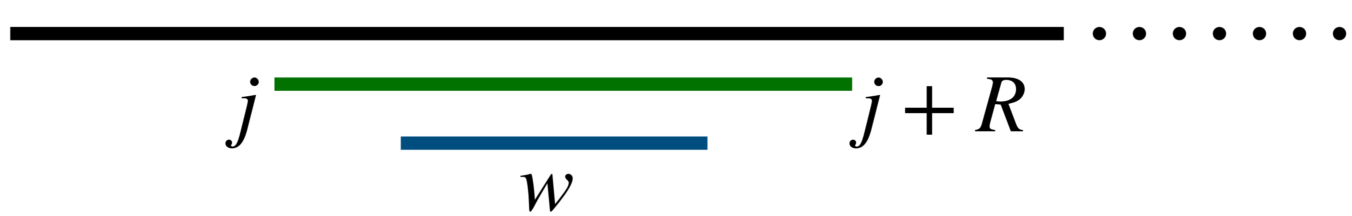


Cutting sequences are almost-periodic: a crucial combinatorial property

For every finite word w, there exists $R \in \mathbb{N}$ such that either:

1) w does not occur in the suffix $\alpha(R)\cdots$

2) For all $j \in \mathbb{N}$, w occurs in the segment $\alpha(j)\cdots\alpha(j+R-1)$



Our α is effectively almost-periodic, because we can compute $R(w)$

Theorem (Semenov)

If α is effectively almost-periodic, then Acc_α is decidable.

Characteristic word $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$

$$\beta = \binom{0}{0}\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\binom{0}{0}\binom{1}{0}\binom{0}{1}\binom{0}{0}\cdots$$

Acceptance Problem

$\mathrm{Acc}_\beta :=$ Is the run of a given automaton $\mathscr{A}$ on β (Muller) accepting?

trivially reduces to ↑

↓ **when is this reduction possible?**

Acc_α

Order word

$$\alpha = \binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\cdots$$

$\alpha \in \Sigma^\omega, \beta \in \Gamma^\omega$, and Samson v. Delilah

I will give a finite monoid M and a morphism $h : \Gamma* \to M$

I respond with a transducer $\mathscr{B}$ and a factorisation $\beta = u_0 u_1 \cdots$

I consider the words $\gamma_1 = \mathscr{B}(\alpha)$, $\gamma_2 = h(u_0)h(u_1)\cdots$

I win if $\gamma_1 = \gamma_2$

If Delilah always wins, then Acc_β reduces to Acc_α.

Monoids describe Automata

Word $u(0)u(1)\cdots u(\ell - 1) \in \Gamma^*$

Run $q_i = r_0 \xrightarrow{u(0)} r_1 \xrightarrow{u(1)} \cdots \xrightarrow{u(\ell-1)} r_\ell = q_j$

Journey $\langle q_i, q_j, \{r_1, \ldots, r_\ell\} \rangle$

from to via set of states

A word defines a function from states to journeys

$$\frac{u(q_i) = \langle q_i, q_j, V_1 \rangle \qquad v(q_j) = \langle q_j, q_k, V_2 \rangle}{v \circ u(q_i) = uv(q_i) = \langle q_i, q_k, V_1 \cup V_2 \rangle} \text{ Composition}$$

The functions form a finite monoid M, the map h from words to functions naturally defines a morphism

From natural monoids to automaton acceptance

$\beta = u_0 u_1 \cdots \in \Gamma^\omega, \alpha \in \Sigma^\omega$ $\qquad$ $\mathscr{A} = (Q, q_{\mathsf{init}}, \Gamma, \delta, \mathsf{Muller}(\mathscr{F}))$

$\mathscr{B} = (R, r_{\mathsf{init}}, \Sigma, M, \delta'')$ $\qquad$ $\gamma = \mathscr{B}(\alpha) = h(u_0)h(u_1)\cdots \in M^\omega$

$$\mathscr{A}' = (Q', q'_{\mathsf{init}}, \Sigma, \delta', \mathsf{Muller}(\mathscr{F}'))$$

The run of $\mathscr{A}'$ on α gives a sequence of journeys made by $\mathscr{A}$ on β

$$Q' = Q \times 2^Q \times R$$

$$q'_{\mathsf{init}} = \langle q_{\mathsf{init}}, \{\}, r_{\mathsf{init}} \rangle$$

$\delta'(\langle q_1, V_1, r_1 \rangle, \sigma) = \langle q_2, V_2, r_2 \rangle$ where

$\mathscr{B}$ prints $m_0 \cdots m_{\ell-1} \in M^*$ upon reading σ in state r_1 and moves to r_2

$m_0 \cdot m_1 \cdot \cdots \cdot m_{\ell-1} = m \in M$ $\qquad$ $m(q_1) = \langle q_1, q_2, V_2 \rangle$

The run of $\mathscr{A}'$ on α gives a sequence of journeys made by $\mathscr{A}$ on β

$$\langle q_{\text{init}}, \{\,\}, r_{\text{init}}\rangle \xrightarrow[h(u_0)\cdots h(u_{i_0-1})]{\alpha(0)} \langle q_1, V_1, r_1\rangle \xrightarrow[h(u_{i_0})\cdots h(u_{i_1-1})]{\alpha(1)} \langle q_2, V_2, r_2\rangle$$

apply to q_{init} apply to q_1

$$\langle q_{\text{init}}, q_1, V_1\rangle \qquad \langle q_1, q_2, V_2\rangle$$

A state is visited infinitely often if and only if infinitely many journeys traverse through it

$$F' \in \mathscr{F}' \Leftrightarrow \left(\bigcup_{\langle q,V,r\rangle \in F'} V \right) \in \mathscr{F}$$

$\mathscr{A}$ accepts β iff $\mathscr{A}'$ accepts α

$\alpha \in \Sigma^\omega, \beta \in \Gamma^\omega$, and Samson v. Delilah

I will give a finite monoid M and a morphism $h : \Gamma* \to M$

I respond with a transducer $\mathscr{B}$ and a factorisation $\beta = u_0 u_1 \cdots$

I consider the words $\gamma_1 = \mathscr{B}(\alpha)$, $\gamma_2 = h(u_0)h(u_1)\cdots$

I win if $\gamma_1 = \gamma_2$

If Delilah always wins, then Acc_β reduces to Acc_α.

Winning Ways: Transduction

$\beta = \mathscr{A}(\alpha)$ for some transducer $\mathscr{A}$

I will give a finite monoid M and a morphism $h : \Gamma^* \to M$

I factorise $\beta = u_0 u_1 \cdots$ such that $\mathscr{A}$ prints u_n upon reading $\alpha(n)$

and construct $\mathscr{B}$ as $h \circ \mathscr{A}$

Indeed, $\mathscr{B}(\alpha) = h(u_0)h(u_1)\cdots$

Acc_β reduces to Acc_α.

Winning Ways: Effectively Profinitely Ultimately Periodic Words

β is effectively profinitely ultimately periodic

I will give a finite monoid M and a morphism $h : \Gamma^* \to M$

By definition, I can factorise $\beta = u_0 u_1 \cdots$ such that $h(u_0)h(u_1)\cdots$ is effectively ultimately periodic

It is then straightforward for you to construct $\mathscr{B}$ such that $\mathscr{B}(0^\omega) = h(u_0)h(u_1)\cdots$

Acc_β is decidable.

Increasing function $f : \mathbb{N} \to \mathbb{N}$ is Effectively PUP if the sequence $0^{f(1)-f(0)}, 0^{f(2)-f(1)}, \ldots$ is Effectively PUP

Delilah's Dream Run: Composition

Consider effectively PUP functions $f_1, \ldots, f_d$

$$\beta \in \{0,1,\ldots,d\}^\omega$$

$\beta(n) = j$ if $n \in \text{Im}(f_1 \circ \cdots \circ f_j)$ but $n \notin \text{Im}(f_1 \circ \cdots \circ f_{j+1})$

E.g. $f_1(n) = n^2$, $f_2(n) = 2^n$, $\beta(n) \geq 1$ for squares,

$\beta(n) = 2$ for powers of 4, $\beta = 12002000010\cdots$

How do your victories for n^2, 2^n compose?

Let $\beta^{(0)} = \beta$, $\beta^{(j)} \in \{j, \ldots, d\}^\omega$, and express

$$\beta^{(j)} = j^{f_{j+1}(0)} \cdot \beta^{(j+1)}(0) \cdot j^{f_{j+1}(1) - f_{j+1}(0) - 1} \cdot \beta^{(j+1)}(1) \cdot \cdots$$

e.g. $\beta^{(1)} = 122121112\cdots$, $\beta^{(2)} = 2222\cdots$

$\text{Acc}_{\beta^{(j)}}$ reduces to $\text{Acc}_{\beta^{(j+1)}}$, and the chain proves Acc_{β} decidable

Characteristic word $\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3 \rangle$

$$\beta = \binom{0}{0}\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\binom{0}{0}\binom{1}{0}\binom{0}{1}\binom{0}{0}\cdots$$

Acceptance Problem

$\mathsf{Acc}_\beta :=$ Is the run of a given automaton $\mathscr{A}$ on β (Muller) accepting?

trivially reduces to

I can do this reduction

Acc_α

Order word

$$\alpha = \binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\cdots$$

$\alpha \in \Sigma^\omega, \beta \in \Gamma^\omega$, and Samson v. Delilah

I will give a finite monoid M and a morphism $h : \Gamma * \to M$

The factorisation is $\beta = 0^{k_0}\alpha(0) \cdot 0^{k_1}\alpha(1) \cdot \cdots$

I respond with a transducer $\mathscr{B}$

To win, you need $\mathscr{B}(\alpha) = h\left(0^{k_0}\alpha(0)\right) \cdot h\left(0^{k_1}\alpha(1)\right)\cdots$

What is your plan to track $h\left(0^{k_j}\right)$?

Modular arithmetic! For any fixed p, I know what the current letter of α corresponds to, mod p

I can thus deduce how many intervening 0's there are, modulo p

But not all monoids are cyclic

I can easily find N, p such that

$h(0^n) = h(0^{n+p})$ for all $n \geq N$

The plan to use periodicity can come undone...

But don't you need a sparsity condition? What if $k_j < N$?

This happens only finitely many times,

which I can moreover enumerate

Corollary of Baker's Theorem

For all $N \in \mathbb{N}$, the inequality in n, m

$$|2^n - 3^m| \leq N$$

has finitely many solutions which can moreover be effectively enumerated.

Characteristic word

$$\beta = \binom{0}{0}\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{0}\binom{0}{0}\binom{0}{0}\binom{1}{0}\binom{0}{1}\binom{0}{0}\cdots$$

Acceptance Problem

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trivially reduces to ↑

↓ when is this reduction possible?

predicates periodic and sparse

Acc_α

Order word

$$\alpha = \binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{0}{1}\binom{1}{0}\binom{1}{0}\binom{0}{1}\cdots$$

Deciding whether…

A given sentence holds in a theory

↕ Turing-equivalent

A given automaton accepts the characteristic word

↕ Turing-equivalent for sparse, periodic predicates

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3, \mathrm{Pow}_6 \rangle$

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Fibonacci} \rangle$

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3, \mathrm{Pow}_5 \rangle$

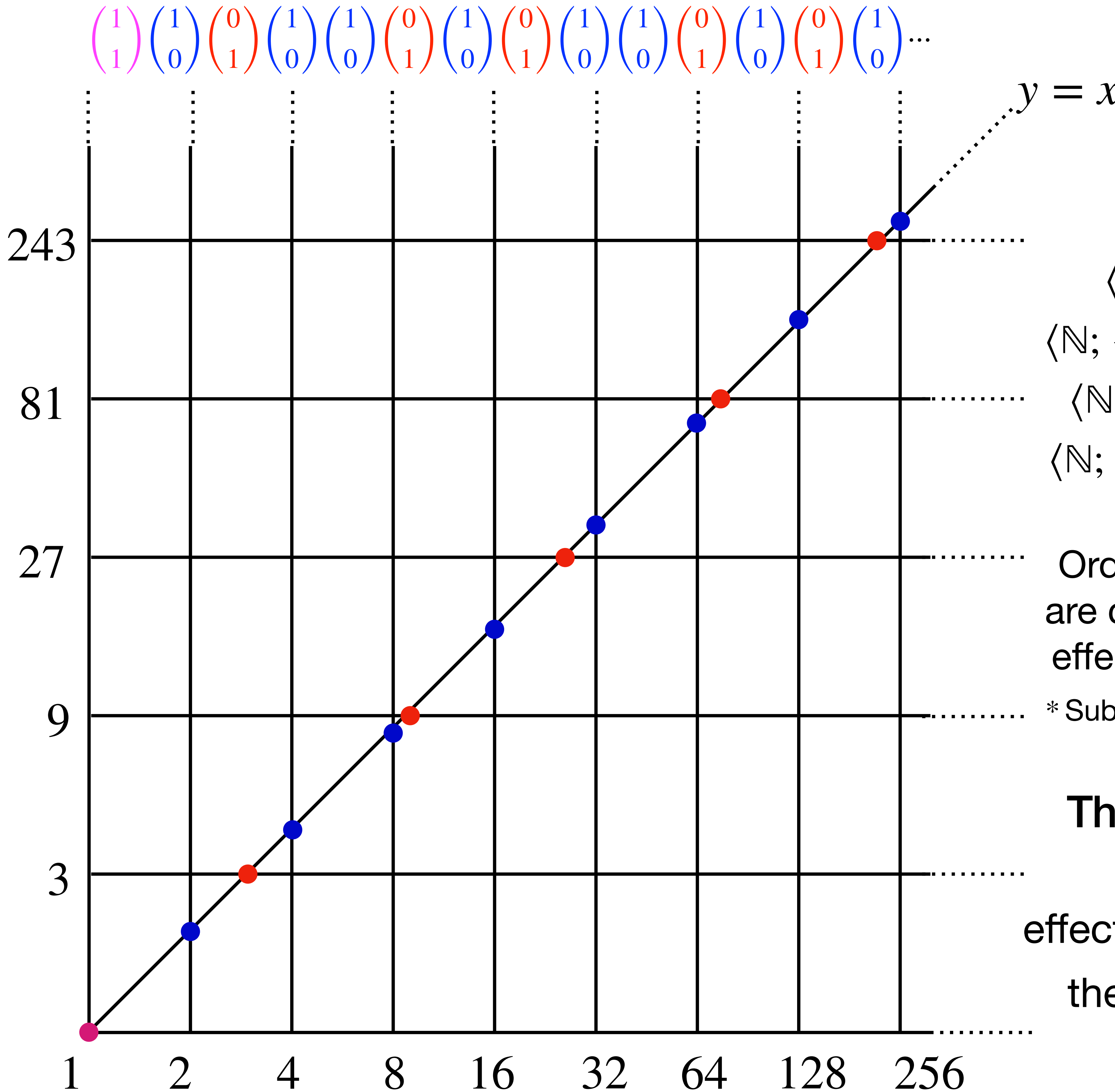
$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Squares} \rangle$

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Factorials} \rangle$

A given automaton accepts the order word

Order words are often traces of dynamical systems, and have nice combinatorial properties

Some order words are cutting sequences



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$\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3, \mathsf{Pow}_5 \rangle$ *

Order words of the above are cutting sequences, and effectively almost-periodic

* Subject to Schanuel's conjecture

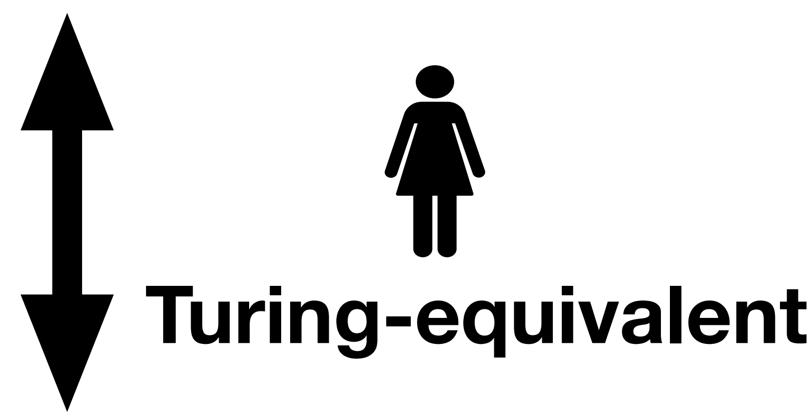
Theorem (Semenov)

If α is effectively almost-periodic, then Acc_α is decidable.

Some order words are driven by numeration systems

α is the order word of $\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Squares} \rangle$

Acc_α



Acc_γ

γ is the binary expansion of $\sqrt{2}$

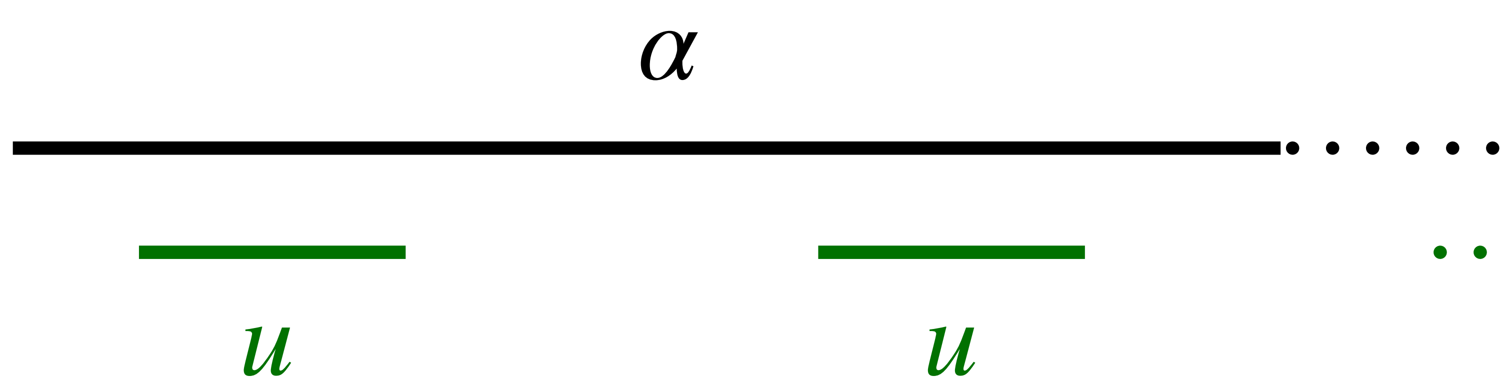
$$\sqrt{2} = 1.0110101000001001111001...$$

This chaotic-looking string is conjectured to be **weakly normal**

Weak Normality: Predictability in Chaos

$$\sqrt{2} = 1.0110101000001001111001\ldots$$

A word $\alpha \in \Sigma^\omega$ is weakly normal if every $u \in \Sigma^+$ occurs as a factor of α infinitely often.



If α is weakly normal then Acc_α is decidable.

Proof: Murphy's Law

Anything that can happen, will happen.

Consider the graph induced by the automaton

Fact 1

For any bottom strongly connected component (SCC) S, there exists $u_{\text{tour}} \in \Sigma^+$ such that starting in any $q \in S$ and reading u_{tour} is guaranteed to visit all states in S

Fact 2

For any non-bottom SCC S, there exists $u_{\text{esc}} \in \Sigma^+$ such that starting in any $q \in S$ and reading u_{esc} is guaranteed to end in a state not in S

Inference

The set of states visited infinitely often by the run on a weakly normal word is precisely a bottom SCC.

Deciding whether…

A given sentence holds in a theory

↕ **Turing-equivalent**

A given automaton accepts the characteristic word

↕ **Turing-equivalent for sparse, periodic predicates**

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$

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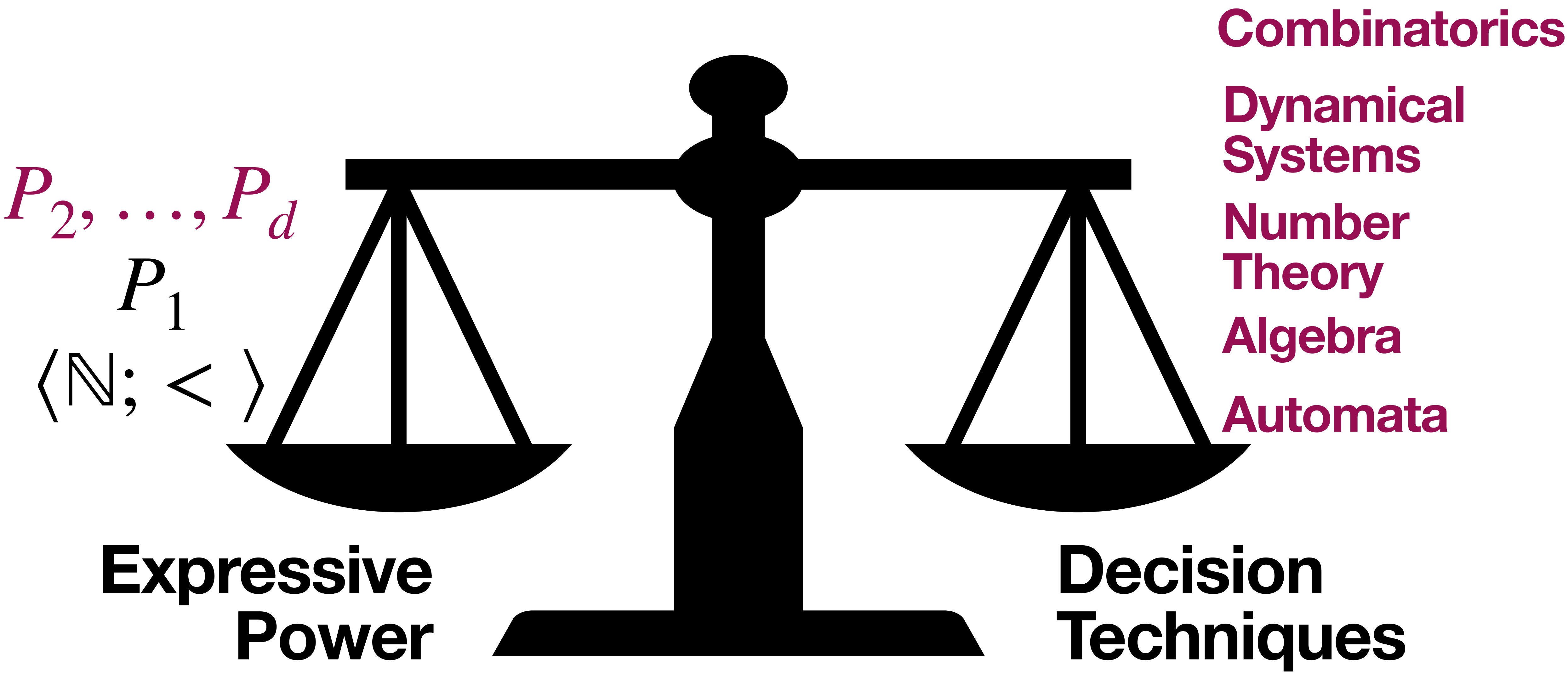
$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Squares} \rangle$ **

$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Factorials} \rangle$?

* Subject to Schanuel's conjecture

* Assuming the binary expansion of $\sqrt{2}$ is weakly normal

Our contribution



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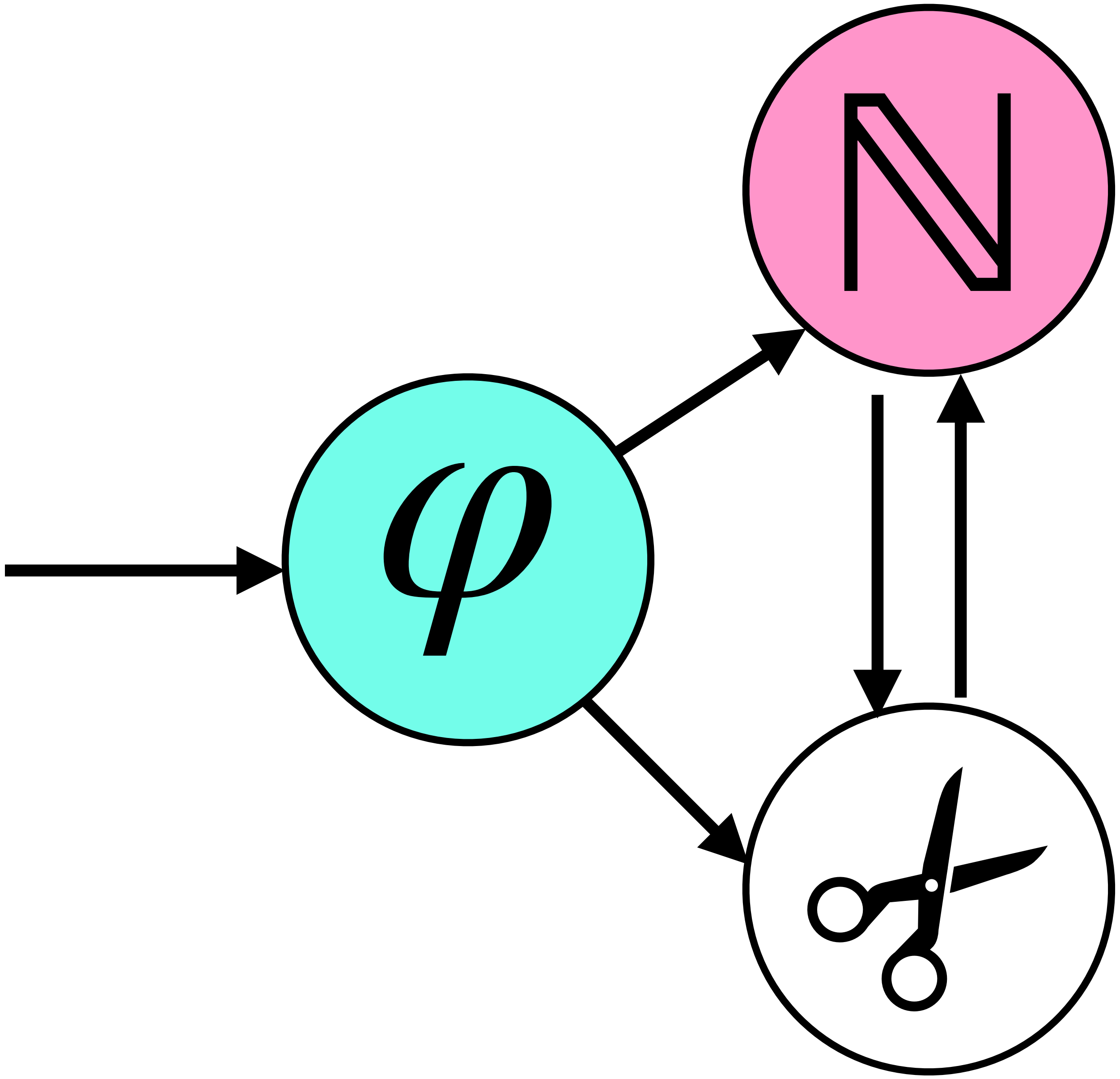

Thank You!

A given sentence holds in a theory

Turing-equivalent ⇕

A given automaton accepts the characteristic word

Turing-equivalent for sparse, periodic predicates ⇕

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$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3, \mathrm{Pow}_5 \rangle$

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$\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Factorials} \rangle$?

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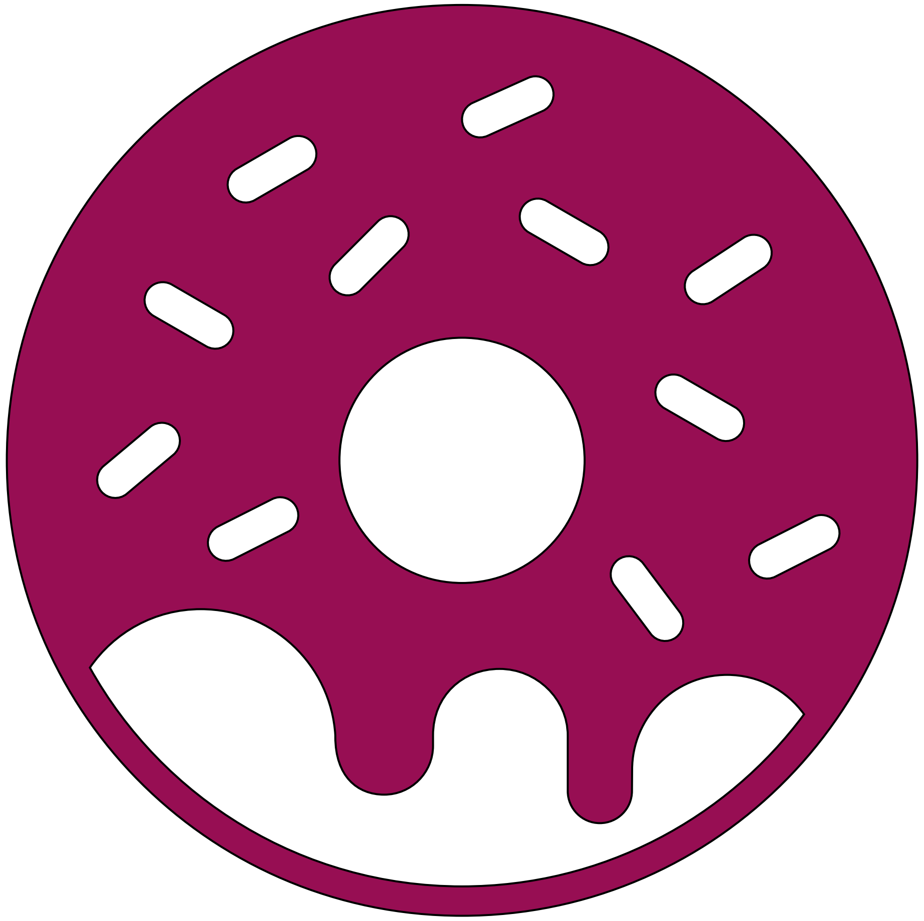
* Assuming the binary expansion of $\sqrt{2}$ is weakly normal

Schanuel's Conjecture

Given any n complex numbers $z_1, ..., z_n$ that are linearly independent over the rational numbers $\mathbb{Q}$, the field extension $\mathbb{Q}(z_1, ..., z_n, e^{z_1}, ..., e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$.

We need a donut.

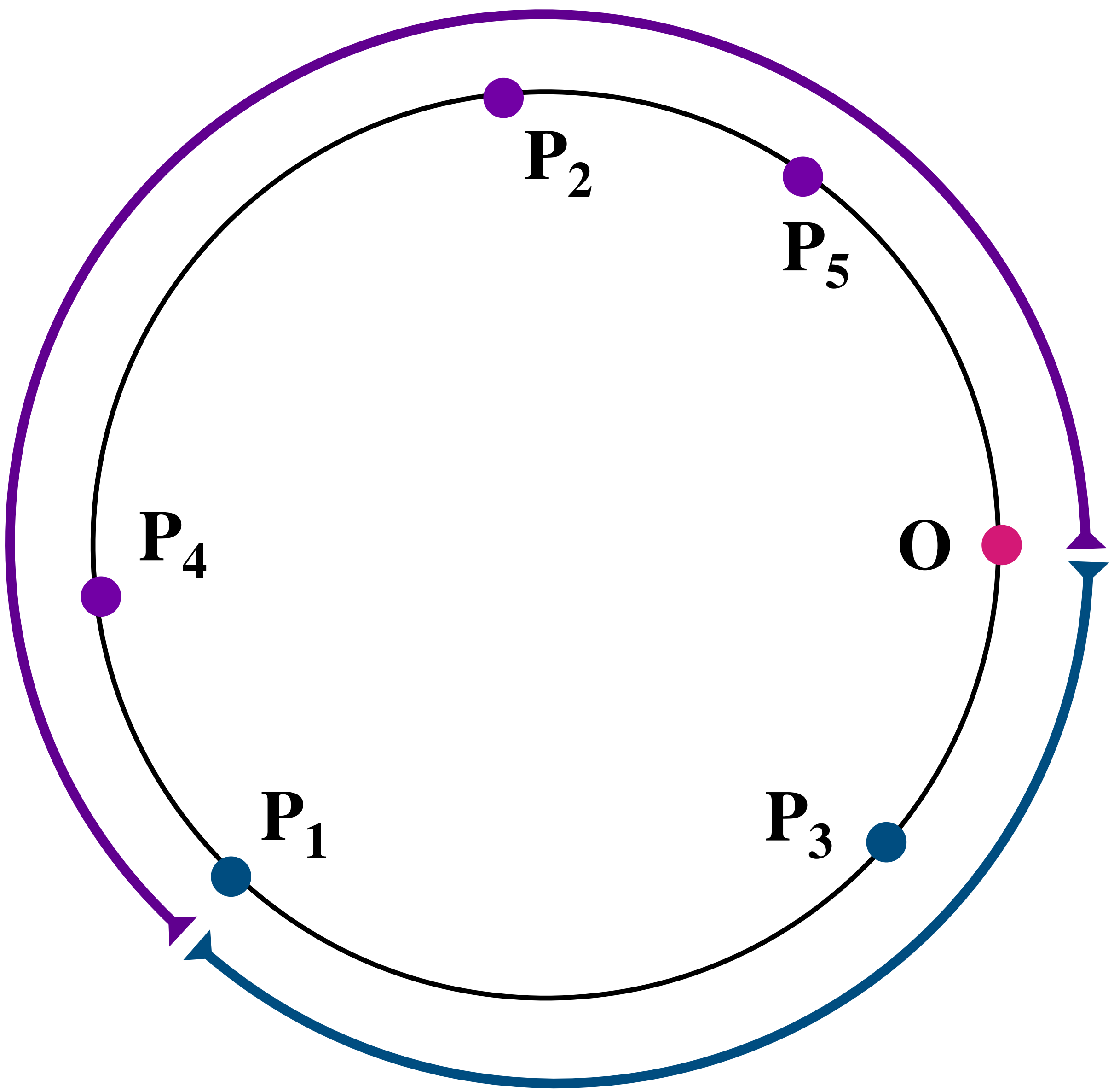


More technically, a torus

Order word through a compact dynamical system

A point starts at $\mathbf{O}$ and travels around torus in steps of $\log_3 2$

arc length $\log_3 2 \approx 0.63$



circumference 1

Number line perspective

one revolution ≡ triple the number

arc $\theta \equiv 3^{\theta} \times$

one step ≡ double the number

trajectory ≡ powers of 2

cross $\mathbf{O}$ ≡ cross a power of 3

purple arc ≡ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

blue arc ≡ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdots$$