Monadic Second-Order Logic with Arithmetic Predicates

Valérie Berthé, Toghrul Karimov, Joris Nieuwveld, Joël Ouaknine, **Mihir Vahanwala**, James Worrell

MAX PLANCK INSTITUTE FOR SOFTWARE SYSTEMS

LICS 2024

A riddle…

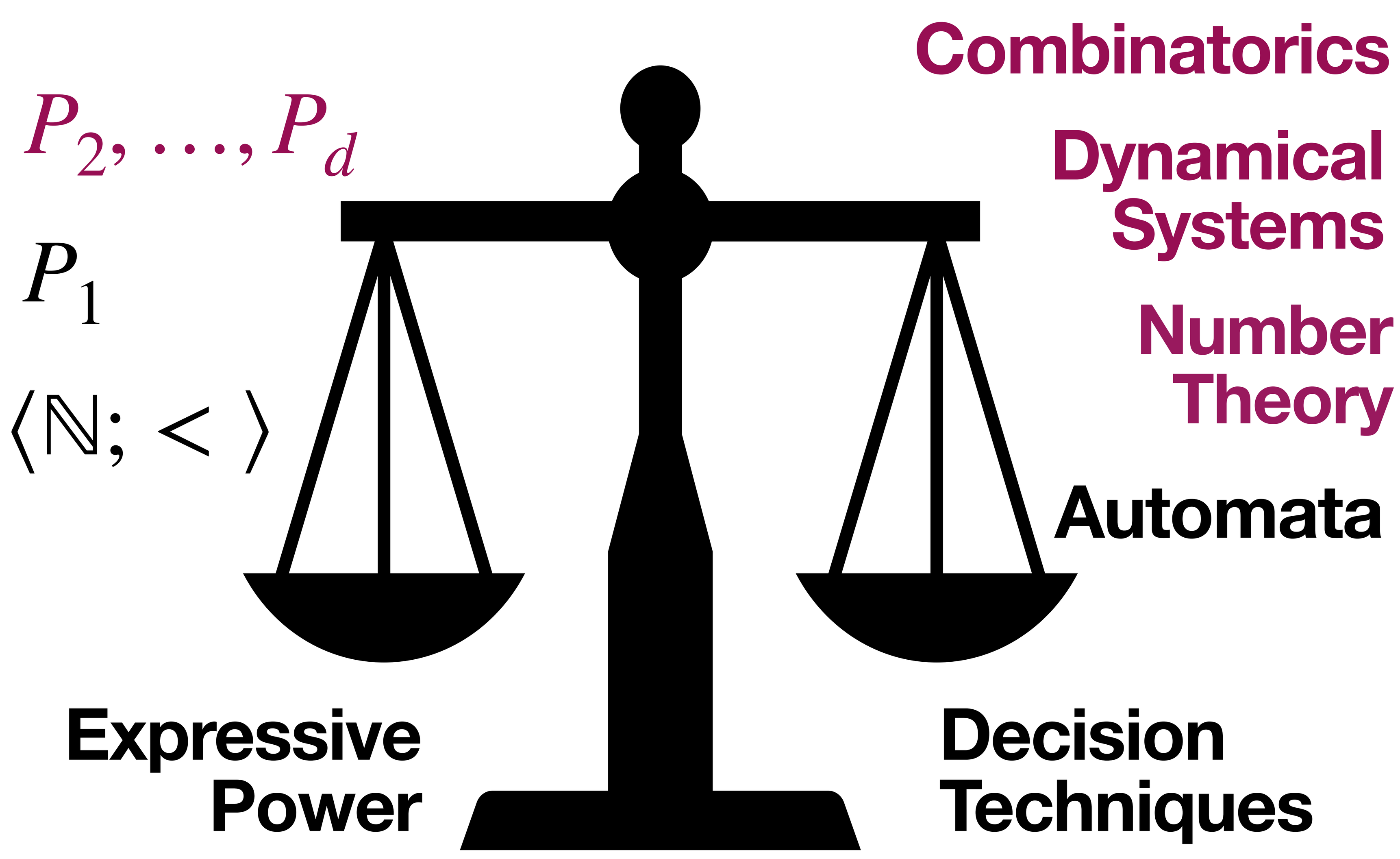
Are there infinitely many m, n such that:

1. m is a power of 2; n is a power of 3
2. The units digits of m, n are $8, 9$ respectively
3. m is the smallest power of 2 larger than n, and their difference is at least 100

…is a playful way of asking a research question…

Which sets of unary predicates can be added to the monadic second-order logic of order MSO while retaining decidability?

…that pushes the limits of decidability.



We show the following MSO theories to be decidable:

$\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3, \mathsf{Pow}_6 \rangle$

$\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Fibonacci} \rangle$

$\langle \mathbb{N}; <, \mathsf{Pow}_4, \mathsf{Squares} \rangle$

$\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3, \mathsf{Pow}_5 \rangle^*$

$\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Squares} \rangle^{**}$

---

*Subject to Schanuel's Conjecture

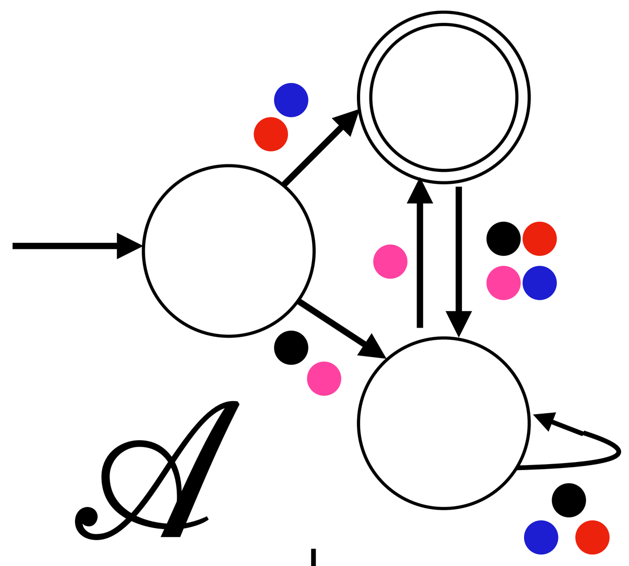
**Provided the binary expansion of $\sqrt{2}$ is weakly normal

Statement — Logical theory of a structure

φ —holds in?→ $\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3 \rangle$

construct ↓ **Automata Theory** ↓ express as

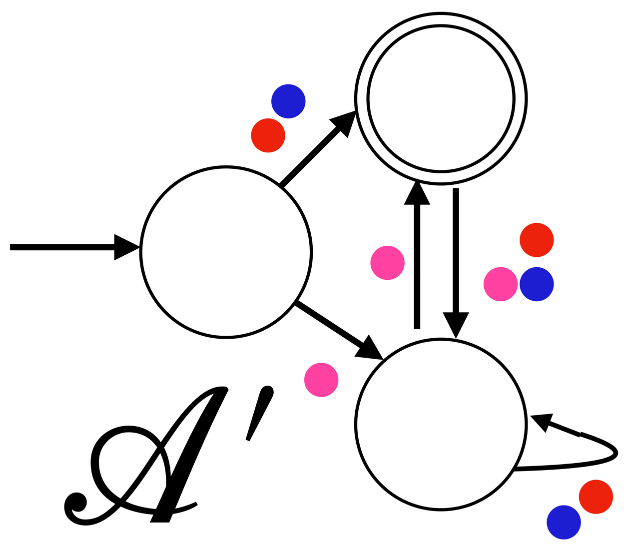
Automaton —accepts?→ characteristic word



$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \binom{0}{0} & \binom{1}{1} & \binom{1}{0} & \binom{0}{1} & \binom{1}{0} & \binom{0}{0} \end{matrix} \cdots$

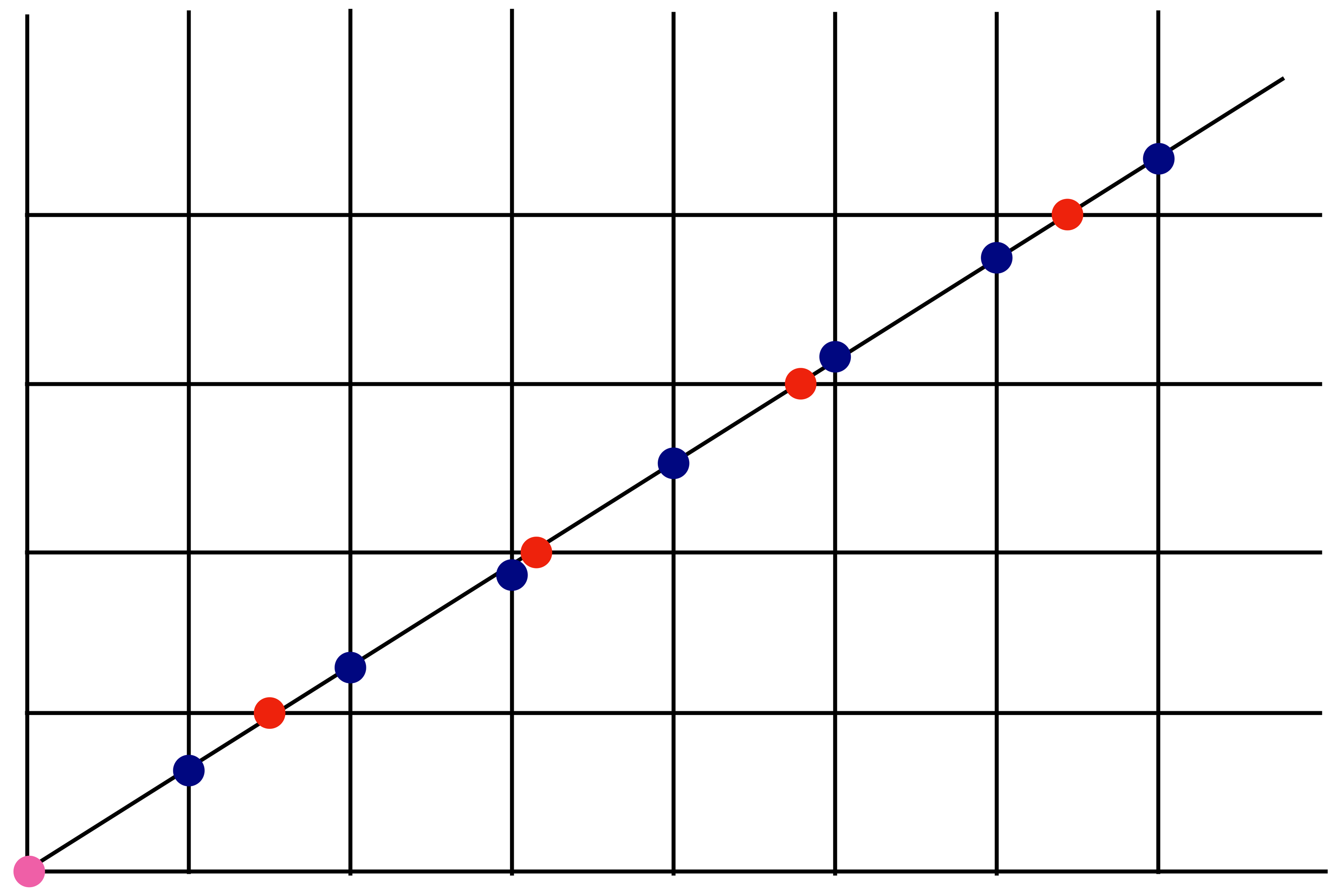
simulate with ↓ **Number Theory** ↓ compress to

Automaton —accepts?→ order word



$\begin{matrix} 1 & 2 & 3 & 4 & 8 & 9 \\ \binom{1}{1} & \binom{1}{0} & \binom{0}{1} & \binom{1}{0} & \binom{1}{0} & \binom{0}{1} \end{matrix} \cdots$

Order words are often traces of
Dynamical Systems



We can use their **Combinatorics** to solve the above Automaton Acceptance Problem