Equivalence of Continuous-Time Markov Chains and Linear Dynamical Systems

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Abstract

The purpose of this short note is to record that an analogue of the following result, which is known for discrete-time linear dynamical systems, also holds in the continuous-time setting. The dynamics of a d-state Markov chain is governed by that of a linear dynamical system of dimension at most $d-1$; conversely, a linear dynamical system of dimension $d-1$ can be "embedded" into a Markov chain with d states.

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1 Statements

A discrete-time linear dynamical system of dimension d is given by an update matrix $M \in \mathbb{Q}^{d \times d}$ and an initial vector $u_0 = u \in \mathbb{Q}^d$. The system evolves as $u_{n+1} = Mu_n$, and hence $u_n = M^n u_0$. We say that this system is a (discrete) Markov chain if (i) the initial vector u_0 is a distribution, i.e., all entries are non-negative, and the sum of entries is 1, and (ii) each column of M is a distribution, making it a stochastic matrix. We immediately observe that the orbit is thus a sequence of distributions.

Analogously, a continuous-time linear dynamical system of dimension d is given by an infinitesimal generator (or simply generator) matrix $A \in \mathbb{Q}^{d \times d}$ and an initial vector $v(0) \in \mathbb{Q}^d$. The system evolves as $\frac{d}{dt}v(t) = Av(t)$, and hence $v(t) = \exp(At)v(0)$. We say that this system is a continuous Markov chain if (i) the initial vector $v(0)$ is a distribution as before, and (ii) the entries in each column of the generator matrix A add up to 0, and the off-diagonal entries are non-negative. We refer to such a matrix A as a stochastic generator matrix.

We prove analogues of the core results in [1] (see also [2]). For notational convenience, we use $\mathbf{0}_d$ to denote the d-dimensional vector with all entries 0, $\mathbf{1}_d$ to denote the d-dimensional vector with all entries 1, and I_d to denote the $d \times d$ identity matrix. We may omit the subscripts when the dimensions are clear from the context. We use O to denote a block of appropriate dimensions, all of whose entries are 0. The symbol $\top$, when in superscript, denotes matrix transposition. Linear algebraic notions that we assume familiarity with can be found in [1, Sec. 2.3].

▶ **Theorem 1.** *Given a distribution $v \in \mathbb{Q}^d$ and a stochastic generator matrix $A \in \mathbb{Q}^{d \times d}$, let $\ell \geq 1$ be the multiplicity of the eigenvalue 0 of A. We can compute a generator matrix $B \in \mathbb{Q}^{(d-\ell)\times(d-\ell)}$, an embedding matrix $Q \in \mathbb{Q}^{d \times (d-\ell)}$, a distribution $s \in \mathbb{Q}^d$, and a vector $u \in \mathbb{Q}^{d-\ell}$ such that for all $t \geq 0$, we have that*

$$\exp(At)v = s + Q\exp(Bt)u.$$

In the above, the choice of Q is independent of v.

▶ **Theorem 2.** *Given a vector $u \in \mathbb{Q}^{d-1}$ and a generator matrix $C \in \mathbb{Q}^{(d-1)\times(d-1)}$, we can compute $\rho, \eta \in \mathbb{Q}$, a distribution $v \in \mathbb{Q}^d$, and a stochastic generator matrix $A \in \mathbb{Q}^{d \times d}$ such* *that for all $t \geq 0$ we have that*

$$\exp(At)v = s + \eta e^{-\rho t} \cdot Q \exp(Ct)u,$$

where the pair s, Q is arbitrarily chosen as $s = \mathbf{1}_d/d$ and $Q = \begin{bmatrix} I_{d-1} \\ -\mathbf{1}_{d-1}^\top \end{bmatrix}$.

2 Proofs

We begin by noting a basic property of generator matrices, which resolves a potential ambiguity in the statement of Thm. 1 regarding the multiplicity of the eigenvalue 0.

▶ **Lemma 3.** *Let A be a stochastic generator matrix. We have that A has 0 as an eigenvalue, and its algebraic multiplicity is equal to its geometric multiplicity. Furthermore, all other eigenvalues of A have negative real part.*

Proof. It follows by definition that $\mathbf{1}^\top A = \mathbf{0}^\top$, and we thus have found a left eigenvector for the eigenvalue 0. Observe that we can choose $a > 0$ small enough so that $M = I + aA$ is a stochastic matrix. It is easy to check that λ is an eigenvalue of A if and only if $\lambda' = 1 + a\lambda$ is an eigenvalue of M, and that their corresponding eigenvectors have the same order, and span the same spaces. Perron-Frobenius theory (see, e.g., [1, Lem. 3]) gives us the desired properties of the spectrum of M: it necessarily has 1 as an eigenvalue with equal algebraic and geometric multiplicities, and all other eigenvalues have absolute value at most 1. The former corresponds to the eigenvalue 0 of A, the latter can only correspond to eigenvalues of A that are strictly to the left of the imaginary axis. ◀

The proofs of Thms. 1 and 2 mirror the proofs in [1, Sec. 3].

Proof of Thm. 1. Let V_0 be the ℓ-dimensional subspace spanned by the eigenvectors of the eigenvalue 0, and let W be the $(d - \ell)$-dimensional subspace (of $\mathbb{C}^d$) spanned by the (generalised) eigenvectors of all the other eigenvalues. Recall the basic properties that V_0, W together span $\mathbb{C}^d$ [1, Thm. 6], and that W is perpendicular to the space spanned by the left eigenvectors of the eigenvalue 0.

We can thus compute matrices $P \in \mathbb{Q}^{d \times \ell}, Q \in \mathbb{Q}^{d \times (d-\ell)}$ whose columns respectively form bases of V_0, W. Observe that v can be uniquely expressed as $Pw + Qu$, where $w \in \mathbb{Q}^\ell$, and $u \in \mathbb{Q}^{d-\ell}$. We choose $s = Pw$.

It remains to define B, and then show that $\exp(At)v = s + Q\exp(Bt)u$. To that end, define the invertible $R = \begin{bmatrix} P & Q \end{bmatrix}$, and express R^{-1} as $\begin{bmatrix} P' \\ Q' \end{bmatrix}$. Consider the matrix

$$\begin{aligned} D &= R^{-1}AR \\ &= \begin{bmatrix} P' \\ Q' \end{bmatrix} A \begin{bmatrix} P & Q \end{bmatrix} \\ &= \begin{bmatrix} P' \\ Q' \end{bmatrix} \begin{bmatrix} O & AQ \end{bmatrix} \\ &= \begin{bmatrix} O & O \\ O & Q'AQ. \end{bmatrix} \end{aligned}$$

In the above, we used the property that all of the columns of AQ are in W, and that the rows of P' span the space perpendicular to W. We define $B = Q'AQ$, and observe that by similar reasoning as above, for all $n \geq 1$

$$D^n = \begin{bmatrix} O & O \\ O & B^n \end{bmatrix} = \begin{bmatrix} O & O \\ O & Q'A^nQ \end{bmatrix}.$$

Dually, we also deduce that for all $n \geq 1$, we have $A^n = QB^nQ'$.

Let us now express the matrix exponential using the formal power series. We get

$$\begin{aligned}
\exp(At)v &= R\left(\exp(Dt)\right)R^{-1}v \\
&= \begin{bmatrix} P & Q \end{bmatrix} \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} D^i\right) \begin{bmatrix} w \\ u \end{bmatrix} \\
&= s + Qu + \sum_{i=1}^{\infty} \frac{t^i}{i!} \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} O & O \\ O & B^i \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \\
&= s + Qu + \sum_{i=1}^{\infty} \frac{t^i}{i!} QB^i u \\
&= s + Q\left(\sum_{i=0}^{\infty} \frac{t^i}{i!} B^i\right) u \\
&= s + Q\exp(Bt)u,
\end{aligned}$$

as desired. ◀

We remark that [1, Sec. 3] also discusses how the spectral decompositions of B and A are related. The same discussion applies in the present setting too. In particular, B and A have the same nonzero eigenvalues, and hence $Q\exp(Bt)u$ converges to $\mathbf{0}$.

Proof of Thm. 2. We prove the theorem by using the calculations from the previous proof. We have fixed Q, and P is the matrix whose single column is s. This determines P', Q'; in particular $P' = \mathbf{1}^\top$. We choose the scalar η to ensure that $v = s + \eta Qu$ is a distribution. The scalar ρ is chosen to define a matrix $B = C - \rho I$ such that QBQ' is a stochastic generator matrix. It would then follow that

$$\exp(At)v = s + \eta Q\exp(Bt)u = s + \eta e^{-\rho t}\exp(Ct)u.$$

Let us show how to choose ρ. We already have that $\mathbf{1}^\top QCQ' = \mathbf{0}^\top$ because $\mathbf{1}^\top Q = \mathbf{0}^\top$. We only need to show that we can make the off-diagonal entries non-negative by subtracting ρI from C.

We have

$$Q(C - \rho I)Q' = QCQ' - \rho QQ'.$$

Since

$$\begin{bmatrix} s & Q \end{bmatrix} \begin{bmatrix} \mathbf{1}^\top \\ Q' \end{bmatrix} = I,$$

we have that $QQ' = I - S$, where S is the matrix with all entries equal to $1/d$. Thus, $Q(C - \rho I)Q' = QCQ' + \rho S - \rho I$. Choosing ρ to be large enough thus ensures that all the off-diagonal entries of QBQ' are non-negative, as desired. ◀

References

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