Robust Positivity Problems for Linear Recurrence Sequences

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Motivation: Linear Dynamical Systems

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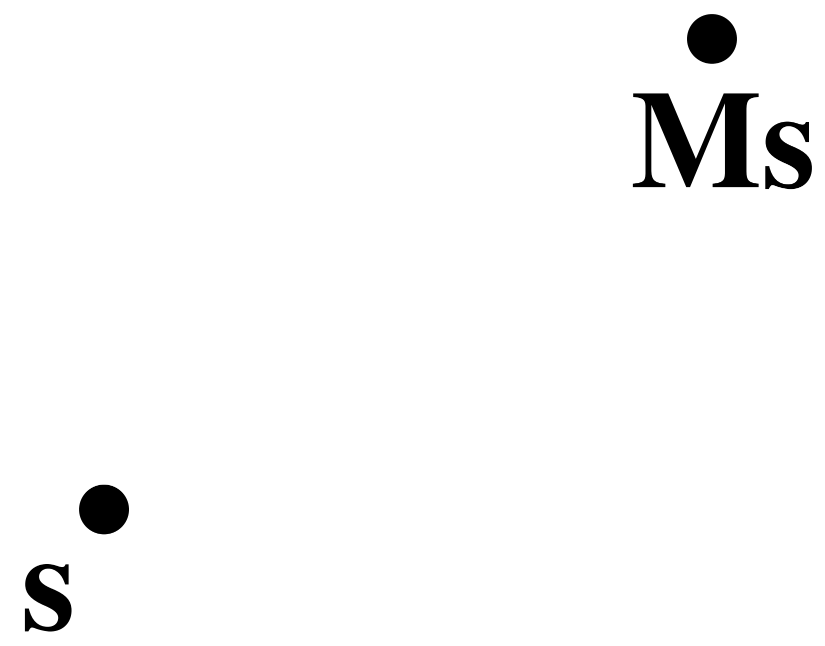

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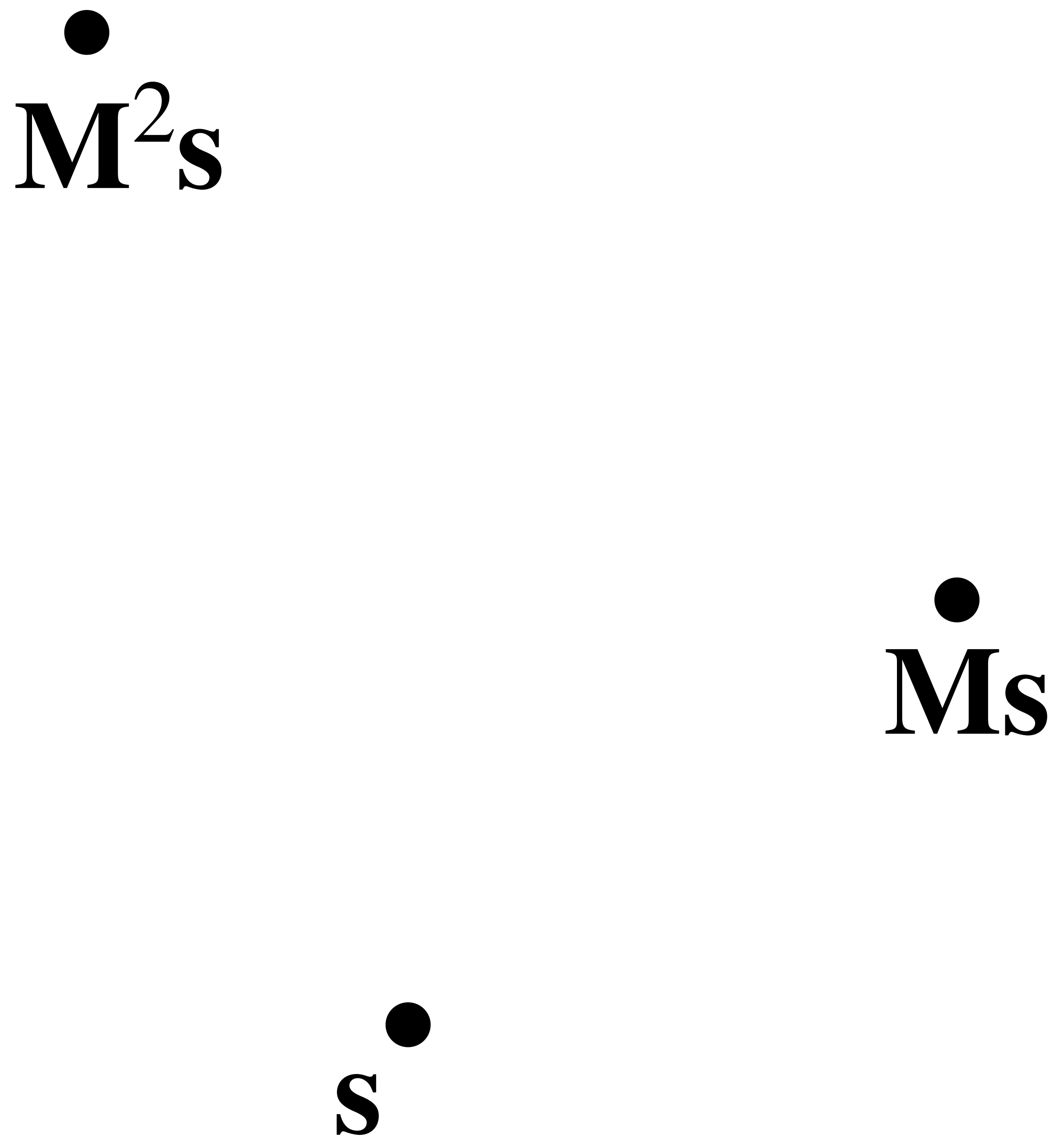

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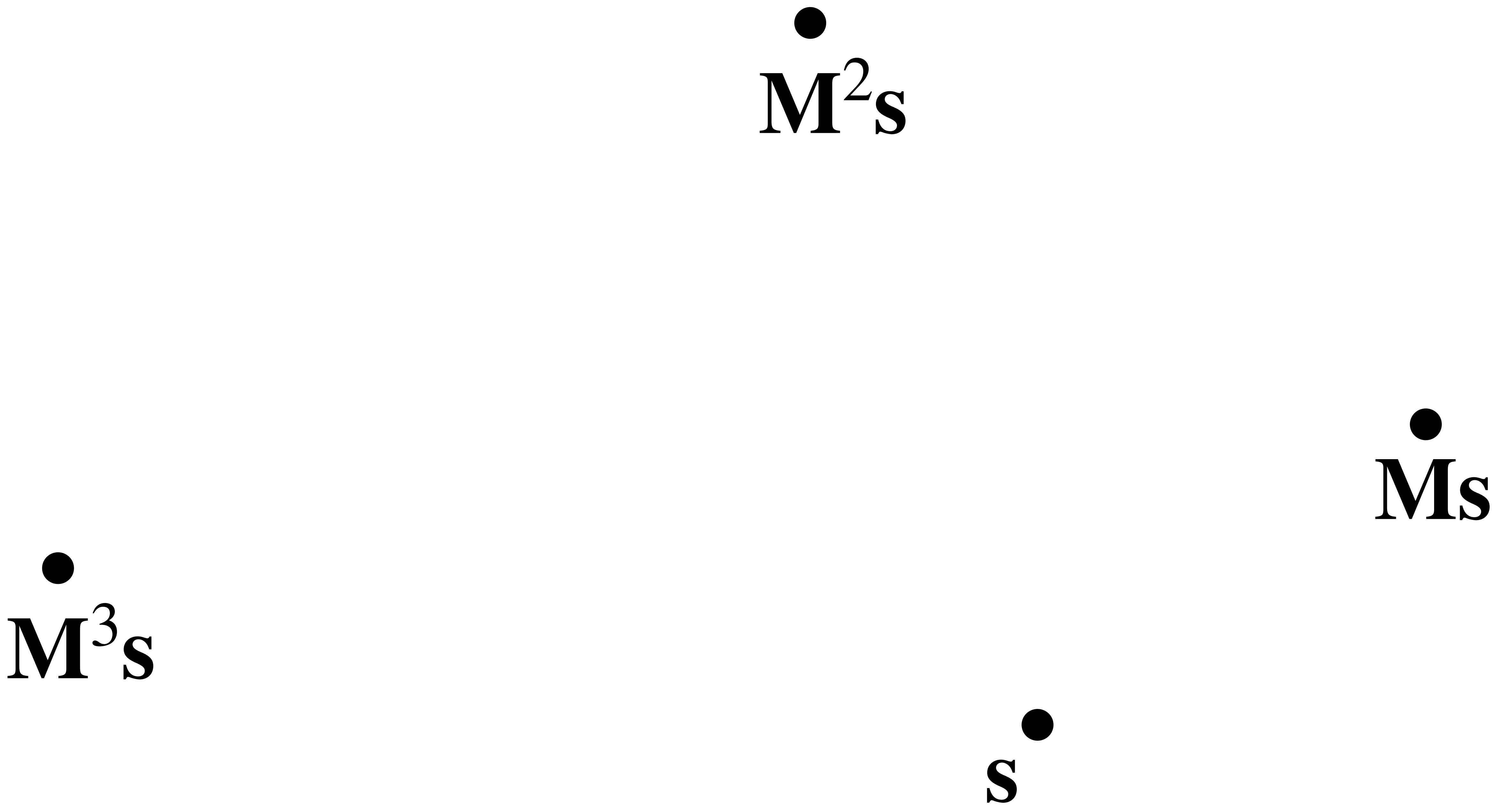

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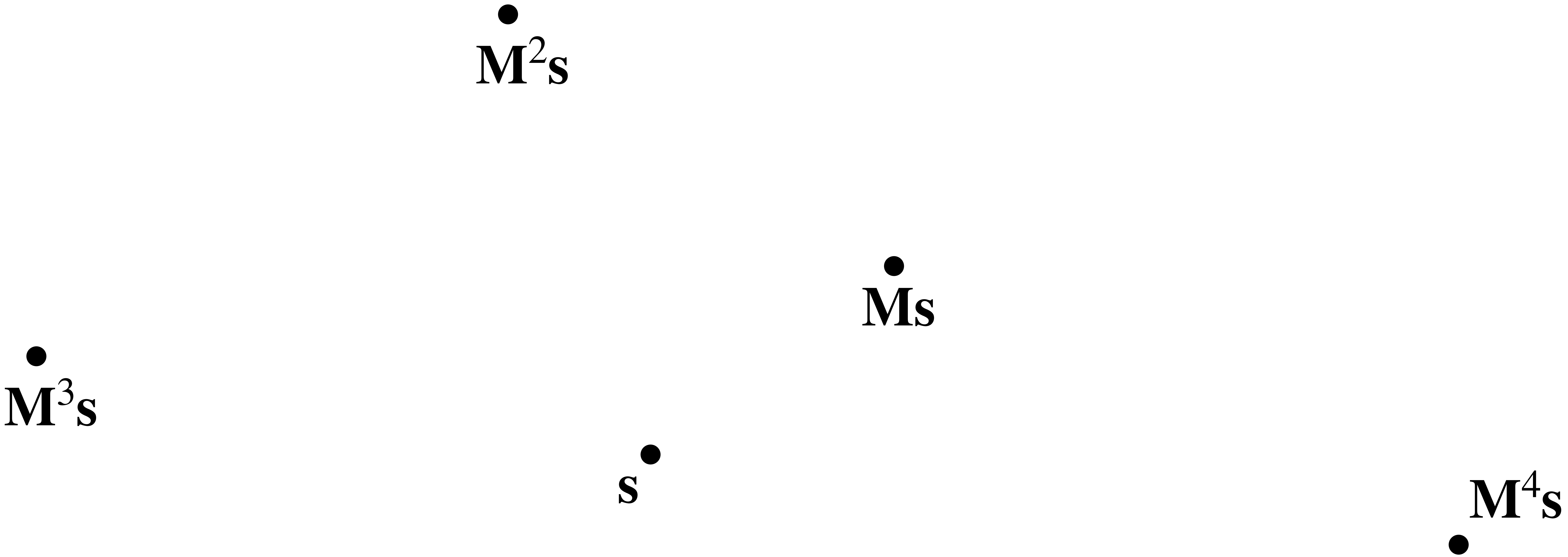

Motivation: Linear Dynamical Systems

Positivity: Trajectory never goes underwater

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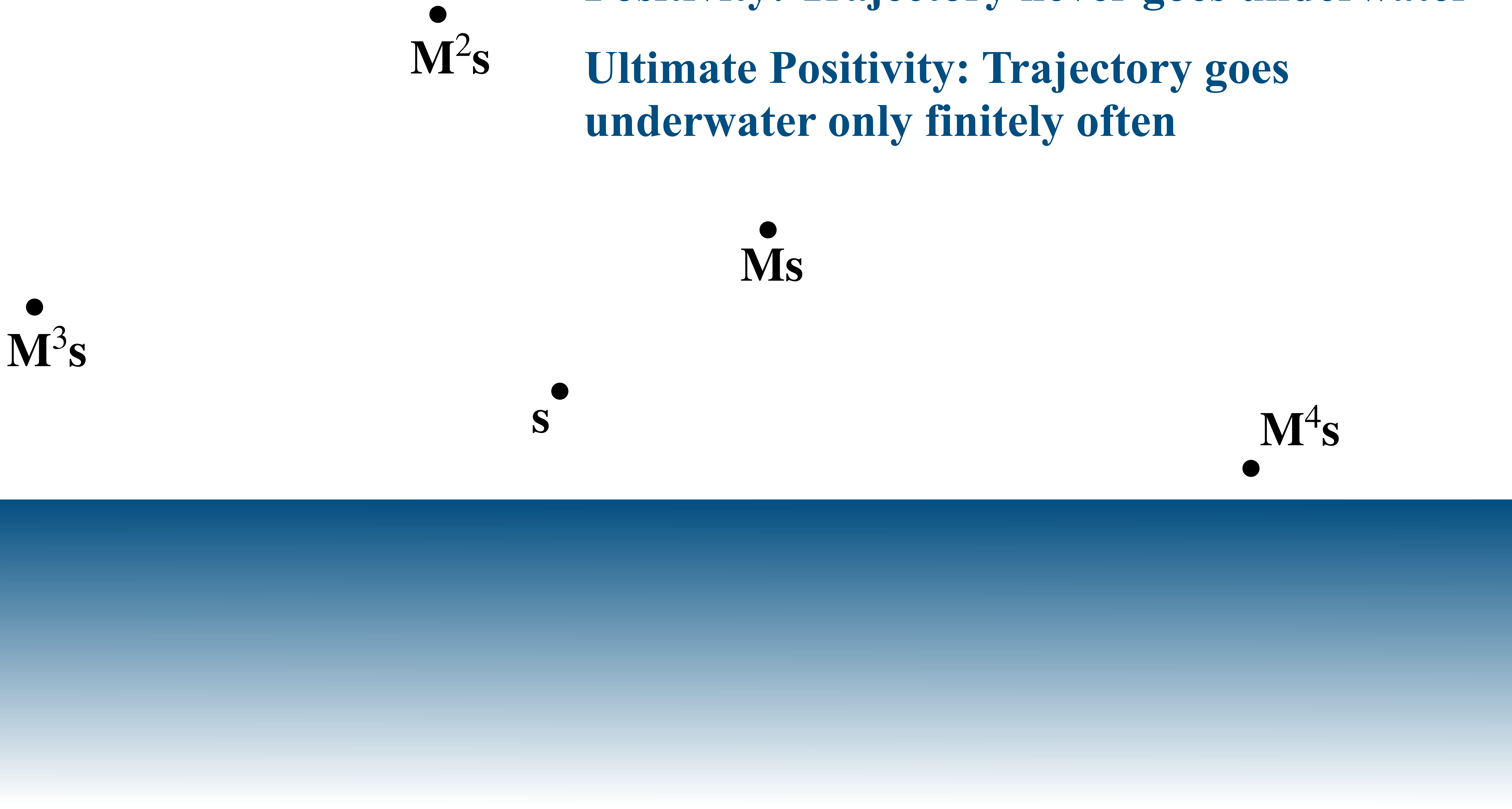
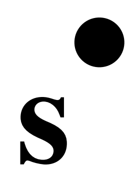
Positivity: Trajectory never goes underwater

Ultimate Positivity: Trajectory goes underwater only finitely often



Practical Concerns



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Is **s** known precisely?

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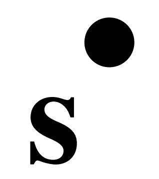

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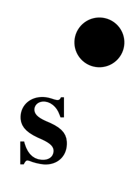
We need to reason about neighbourhoods of **s**



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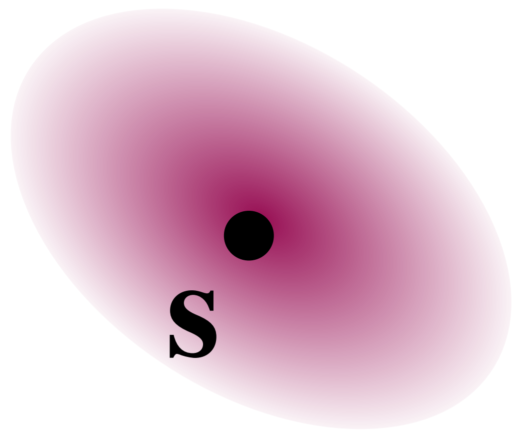
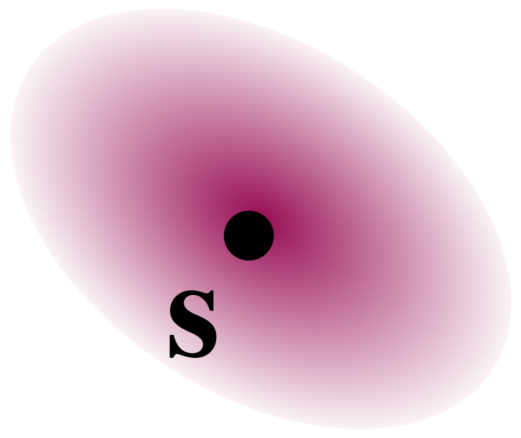


Robust Positivity Problems

Our neighbourhoods are specified by positive definite $\boldsymbol{\Sigma}$

$$(\mathbf{s}' - \mathbf{s})^T \boldsymbol{\Sigma}^{-1} (\mathbf{s}' - \mathbf{s}) \leq 1$$



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Do all points in the neighbourhood initialise trajectories that avoid the water?

The neighbourhood goes underwater only finitely often

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Uniform

There is a threshold step beyond which the entire neighbourhood stays above

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Uniform

There is a threshold step beyond which the entire neighbourhood stays above

Non-uniform

Each of the individual points have a different threshold

An equivalent encoding of the setting

A Linear Recurrence Sequence, indexed by n

A Linear Recurrence Sequence (LRS) of order k is

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A Linear Recurrence Sequence, indexed by n

A Linear Recurrence Sequence (LRS) of order k is

An infinite sequence of numbers $(u_0, u_1, u_2, \ldots)$ satisfying

$$\forall n \,.\; u_{n+k} = a_{k-1}u_{n+k-1} + \ldots + a_0 u_n$$

For some constants $a_0, \ldots, a_{k-1}$

An example

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$$u_n = 5^n \cos n\theta, \text{ where } \theta = \arccos(4/5)$$

$$\mathbf{h}^T\mathbf{M}^n\mathbf{s} = \mathbf{p}^T\mathbf{q_n} = \langle \mathbf{p}, \mathbf{q_n} \rangle = [p_1 \quad p_2] \begin{bmatrix} 5^n \cos n\theta \\ 5^n \sin n\theta \end{bmatrix}$$

Scenery ⟷ LRS

Scenery	LRS
Never underwater?	Positivity Problem for LRS Are all terms non-negative?
Underwater only finitely often?	Ultimate Positivity Problem Are there finitely many negative terms?
$\mathbf{h}^T\mathbf{M}^n\mathbf{s} \geq 0$	$u_n = \langle \mathbf{p}, \mathbf{q_n} \rangle \geq 0$

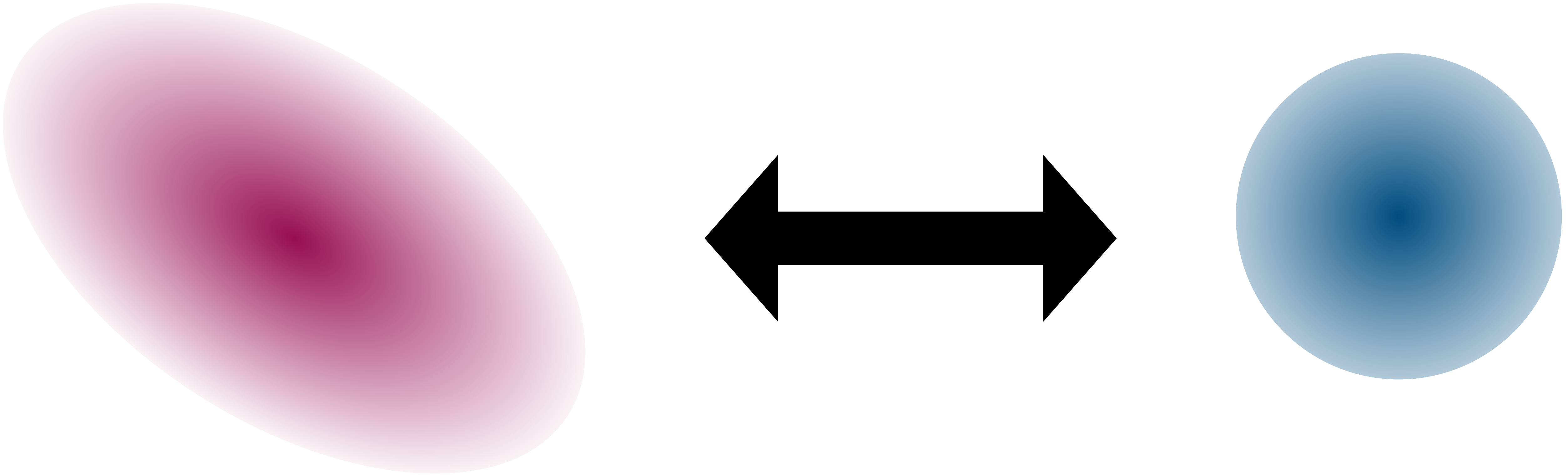
For all $\mathbf{r}$ such that $\mathbf{r}^T\boldsymbol{\Sigma}^{-1}\mathbf{r} \leq 1$

$$\langle \mathbf{p} + \mathbf{r}, \mathbf{q_n} \rangle \geq 0$$

$$\Updownarrow$$

$$\langle \mathbf{p}, \mathbf{q_n} \rangle \geq \max_{\mathbf{r} \in \mathcal{N}} \langle \mathbf{r}, \mathbf{q_n} \rangle$$

Idea: Consider the invertible linear map from the neighbourhood to the Euclidean unit ball



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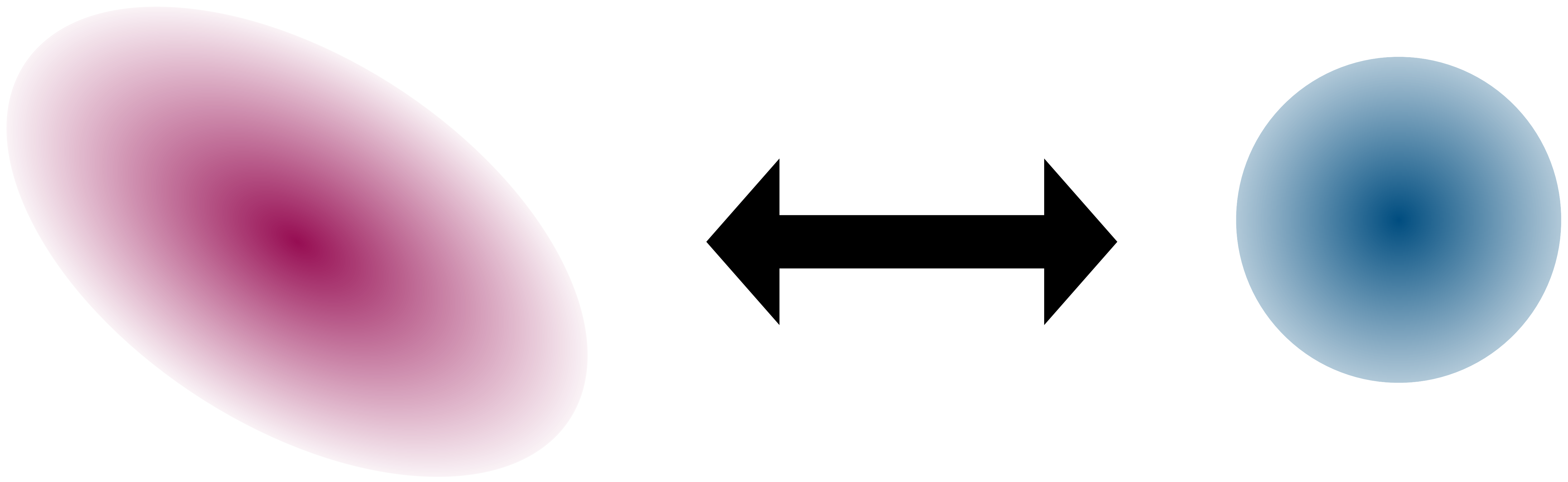
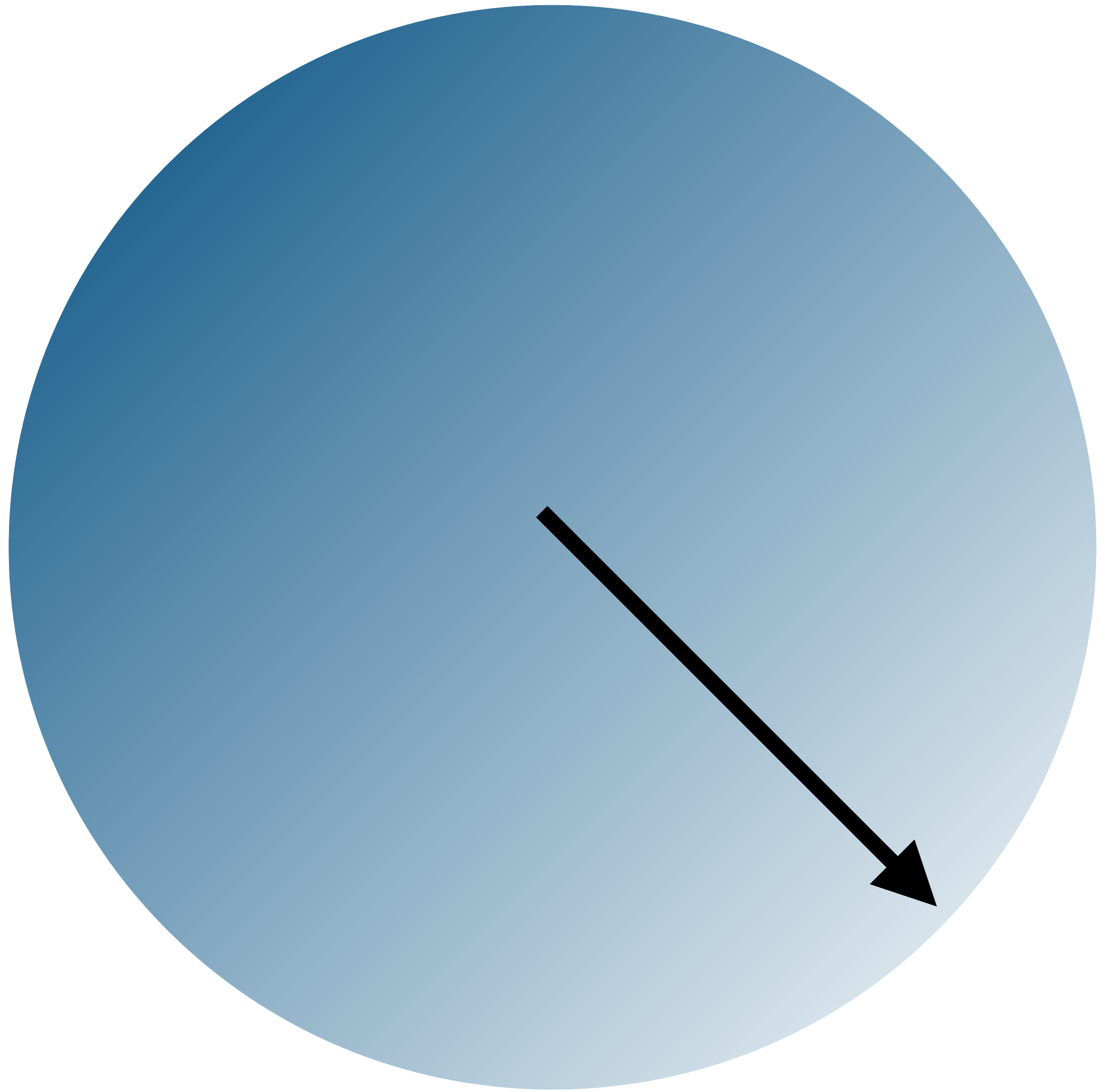


$$\max_{\mathbf{r}\in\mathcal{N}}\langle \mathbf{r}, \mathbf{q_n}\rangle = \max_{\mathbf{d}\in\mathcal{B}}\langle \mathbf{d}, \mathbf{B}\mathbf{q_n}\rangle$$

Maximising a linear function over the Euclidean unit ball



$$\max_{\mathbf{d}\in\mathscr{B}} \langle \mathbf{d}, \mathbf{f} \rangle = \|\mathbf{f}\| = \sqrt{f_1^2 + \ldots + f_k^2}$$

For all $\mathbf{r}$ such that $\mathbf{r}^T\mathbf{\Sigma}^{-1}\mathbf{r} \leq 1$

$$\langle \mathbf{p} + \mathbf{r}, \mathbf{q_n} \rangle \geq 0$$

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$$\langle \mathbf{p}, \mathbf{q_n} \rangle \geq \|\mathbf{B}\mathbf{q_n}\| \geq 0$$

$$\|\mathbf{B}\mathbf{q_n}\| = \sqrt{\langle \mathbf{b_1}, \mathbf{q_n} \rangle^2 + \ldots + \langle \mathbf{b_k}, \mathbf{q_n} \rangle^2}$$

To Solve Robust Uniform Positivity, we need to check the (Ultimate) Positivity of two LRS

$$u_n = \langle \mathbf{p}, \mathbf{q_n} \rangle$$

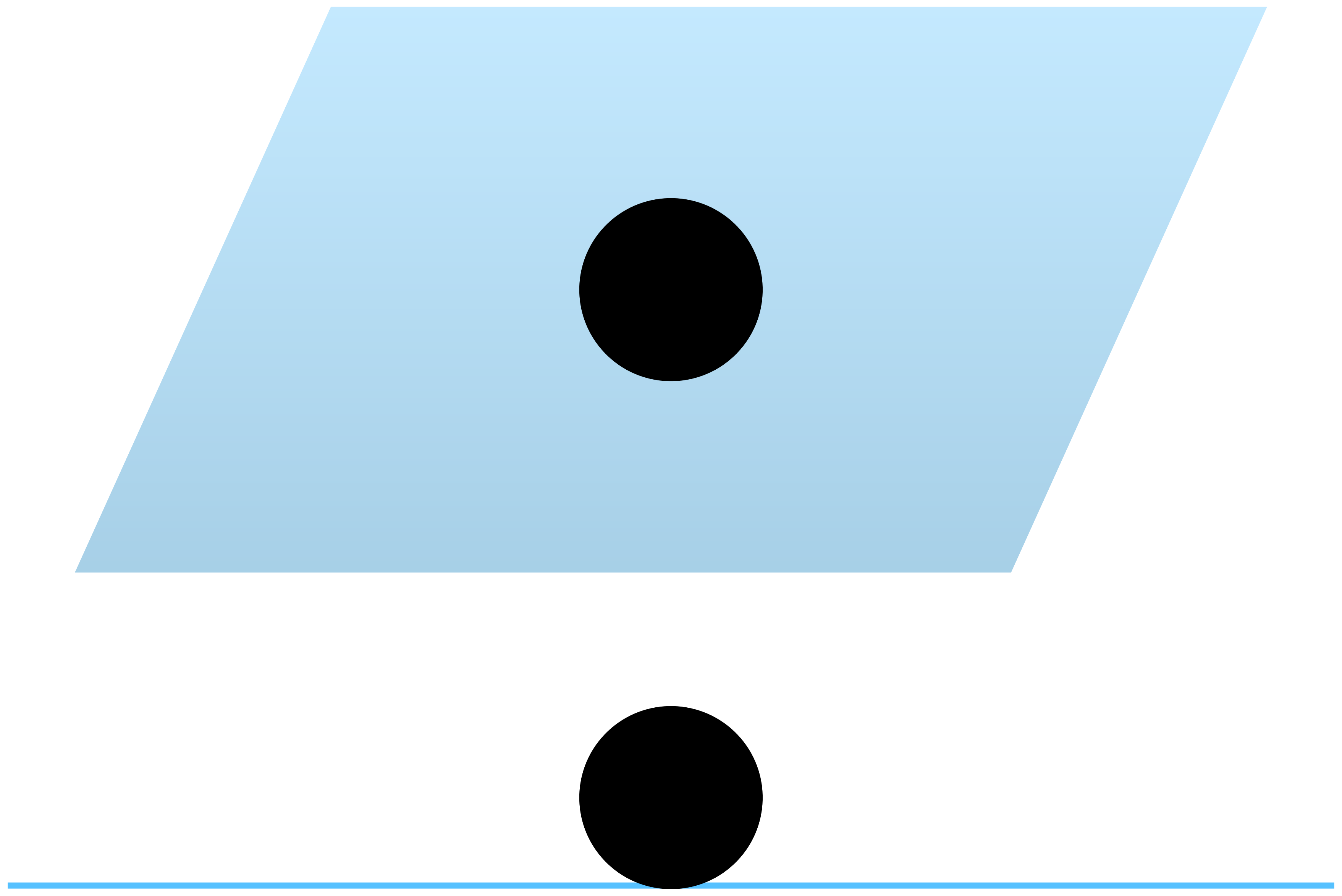
$$v_n = \langle \mathbf{p}, \mathbf{q_n} \rangle^2 - \langle \mathbf{b_1}, \mathbf{q_n} \rangle^2 - \ldots - \langle \mathbf{b_k}, \mathbf{q_n} \rangle^2$$

To Solve Robust Non-uniform Positivity:

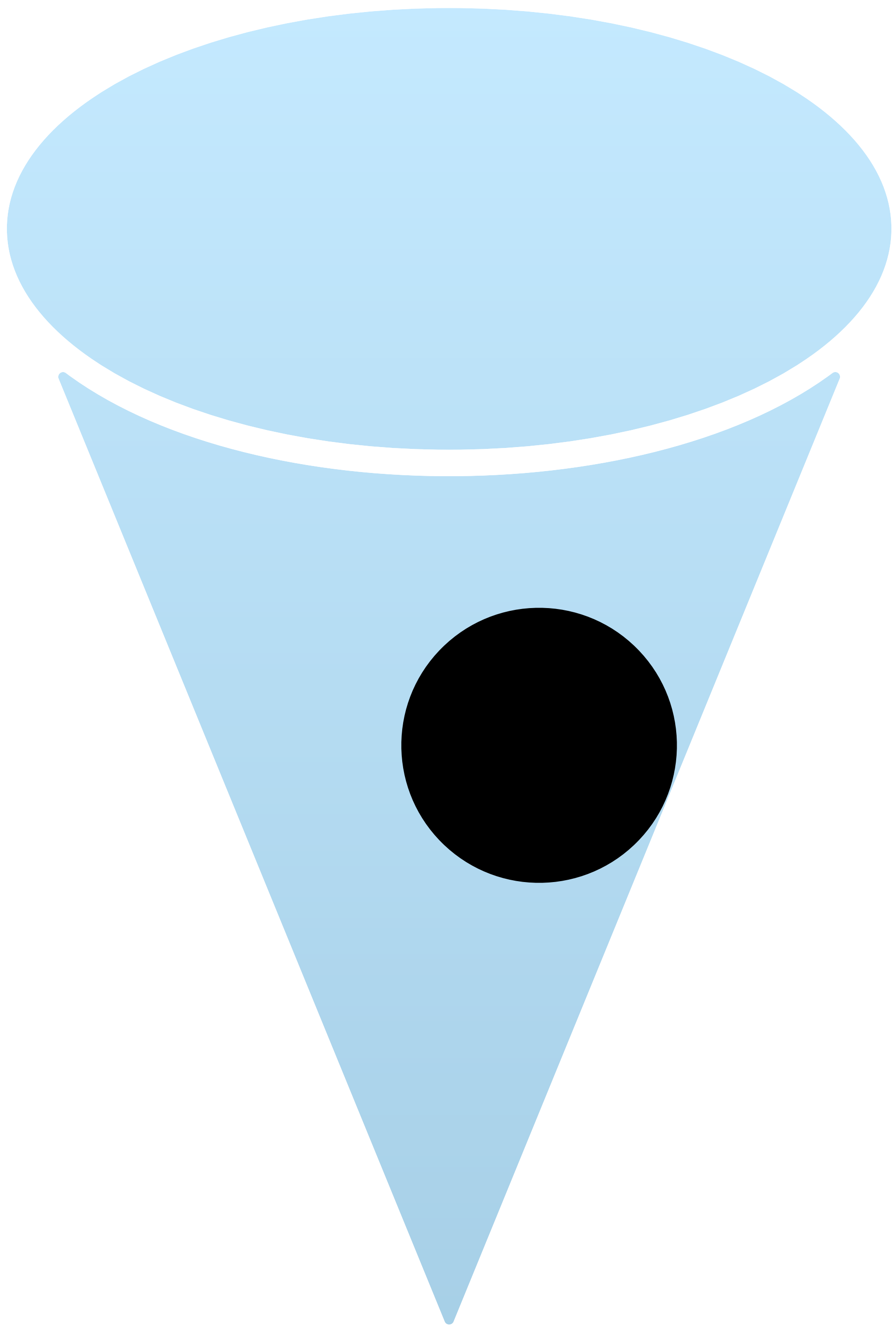
(1) check, using First Order Theory of the Reals, if the neighbourhood is contained in an over-approximation of the region of Ultimately Positive initialisations

(2) solve the boundary cases

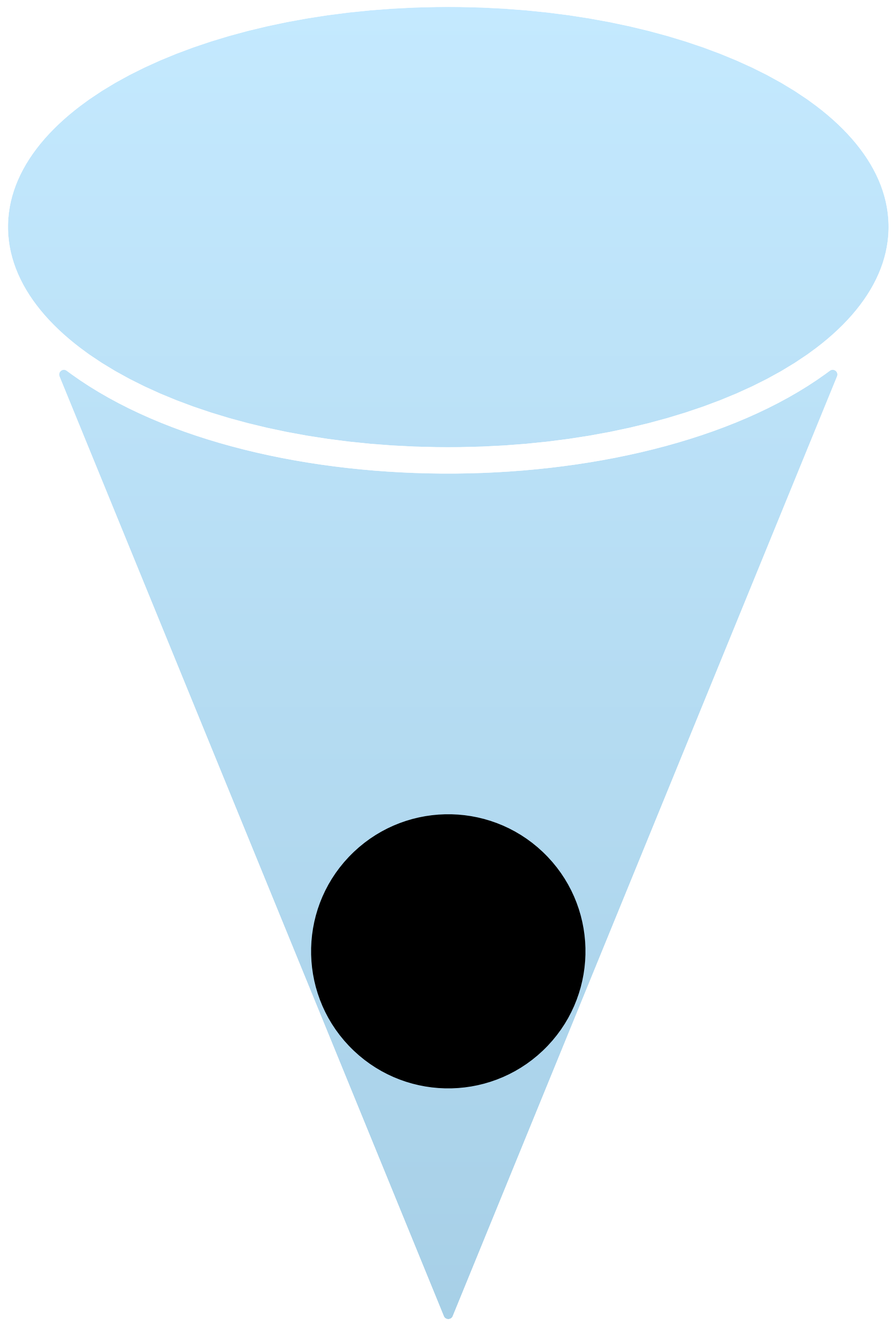
Ball seated atop hyperplane



Ball touching cone



Ball nestled in cone

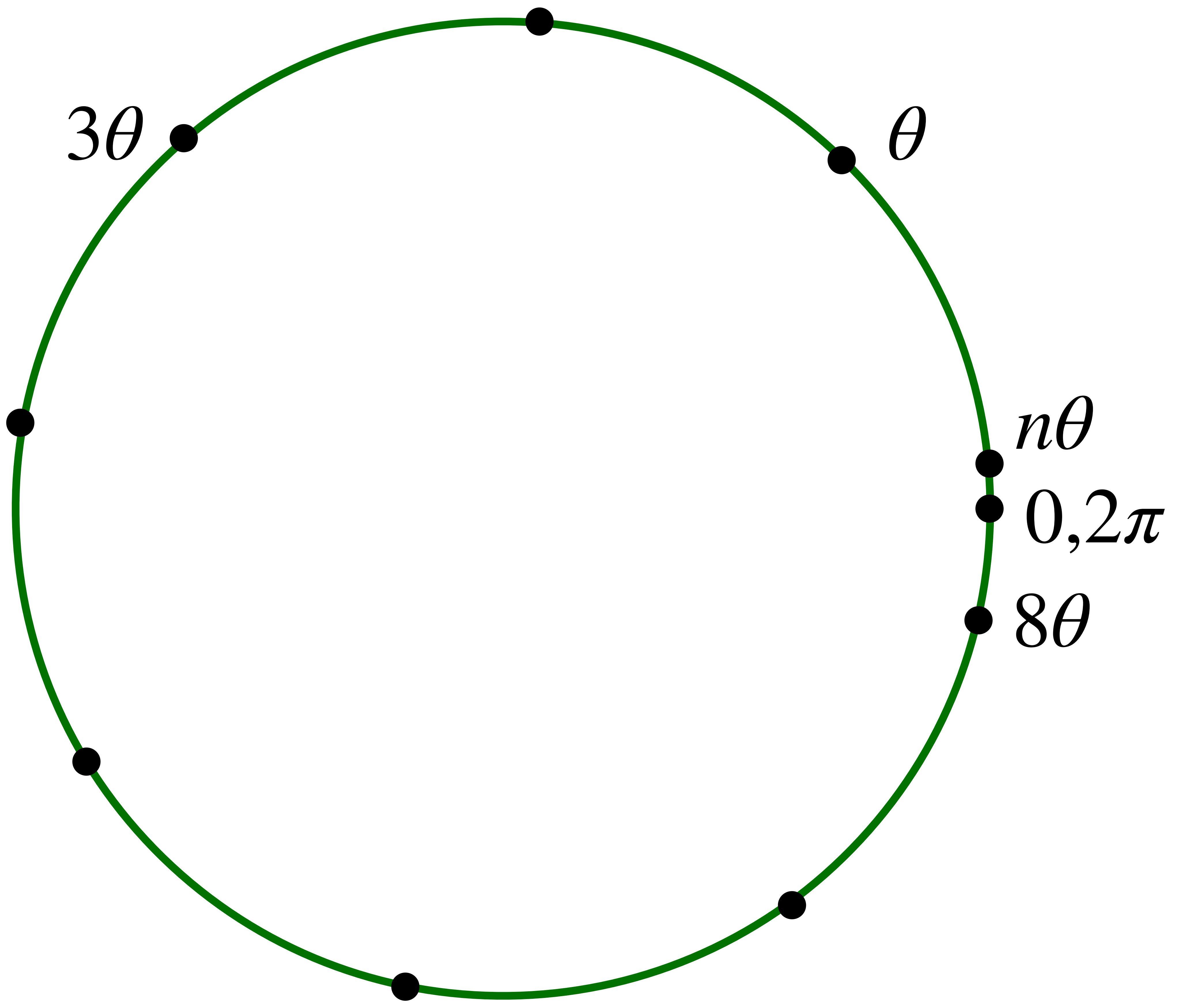


Executing these plans: the critical inequality

$$1 - \cos(n\theta - \varphi) < g(n)$$

g(n) is asymptotically 0.
When does this have a solution? Infinitely many solutions?
Solutions correspond to n for which Positivity is violated

$$1 - \cos n\theta < g(n)$$



How close can nθ get to a multiple of 2π?

Diophantine Approximation

Approximating irrational numbers through continued fractions

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Rational approximation obtained by truncating at the kth level

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Rational approximation obtained by truncating at the kth level

How well do these approximations converge?

$$L(t) = \inf_k q_k \,|\, q_k t - p_k \,|$$

Diophantine Approximation Type

$$L_\infty(t) = \lim_k \inf q_k \,|\, q_k t - p_k \,|$$

Lagrange Constant

What is known about these constants?

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$$0 \leq L(t) \leq L_\infty(t) \leq \frac{1}{\sqrt{5}}$$

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$L_\infty(t) = 0$ for most t **but**

Given t, e.g. as $e^{2\pi \iota t}$, computing $L(t), L_\infty(t)$
is beyond contemporary number theory

Robust Uniform Positivity

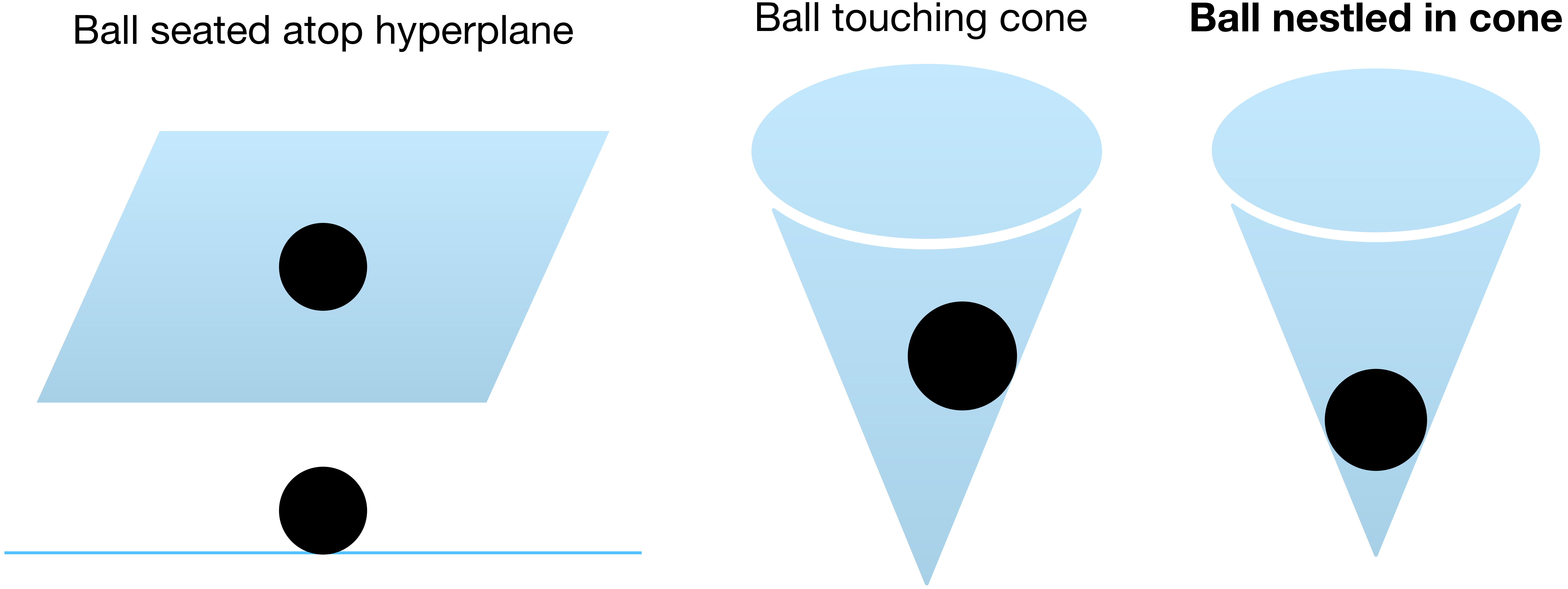
$$1 - \cos n\theta < g(n)$$

$g(n) \in \Theta(1/n)$: infinitely many solutions

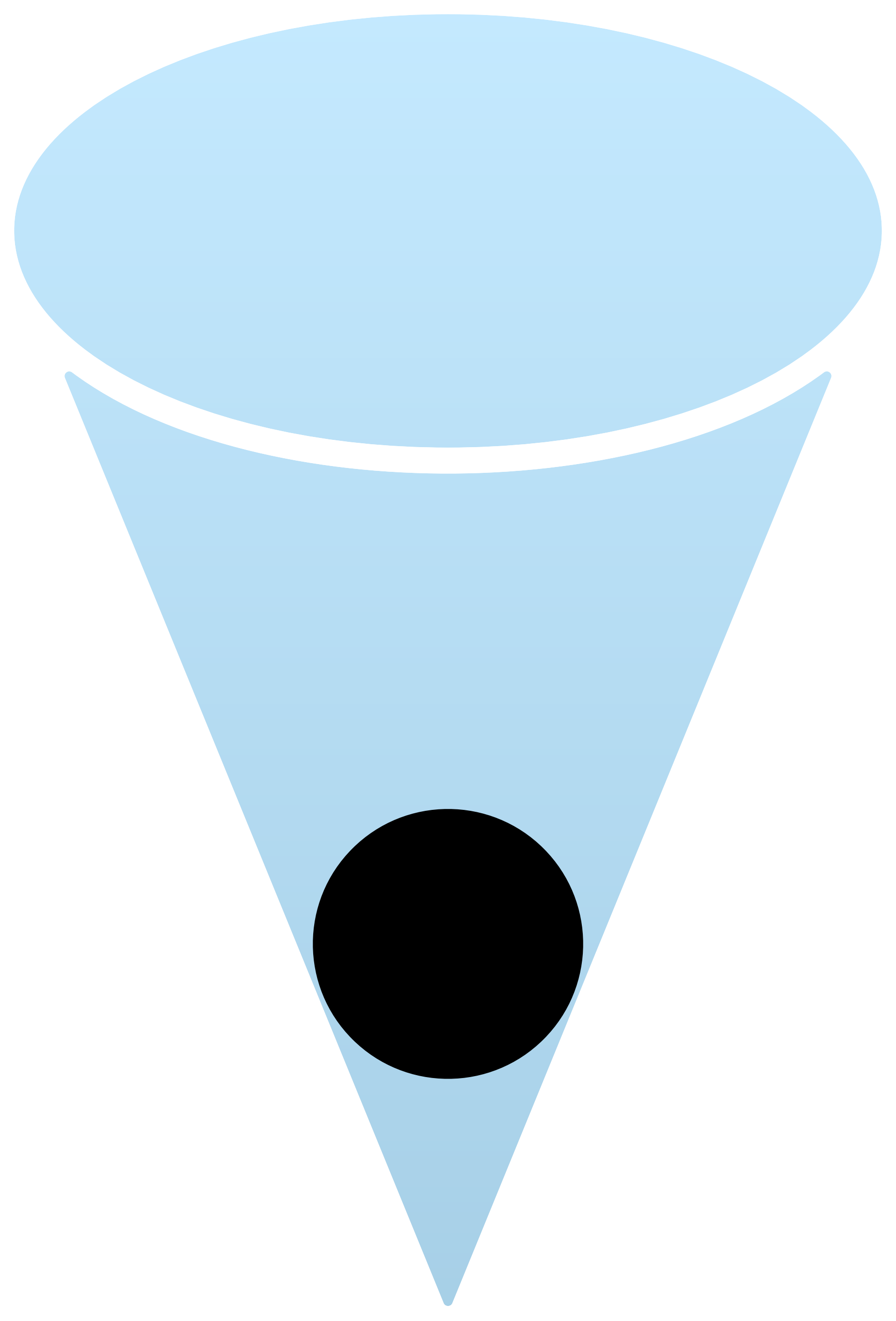
$g(n) \in \Theta(1/n^2)$: depends on $L(t), L_\infty(t)$

$g(n) = (0.99)^n$: finitely many solutions,
effectively enumerable

Robust Non-uniform Positivity



Ball nestled in cone: Always a NO instance



Key technical Lemma

Let g be strictly decreasing.

$$1 - \cos(n\theta - \varphi) < (-1)^n g(n)$$

In any non-empty interval (a, b), there exists φ such that the above inequality holds for infinitely many n.

This φ corresponds to a point that violates Ultimate Positivity

When can we execute these plans?

	Decidability for Simple LRS	Decidability for General LRS	Number-theoretic hardness for General LRS
Robust Positivity	Up to Order 5	Up to Order 4	Order 5 and above
Robust Uniform Ultimate Positivity	**All Orders**	Up to Order 4	Order 5 and above
Robust Non-uniform Ultimate Positivity	Up to Order 4	Up to Order 4	Order 6 and above

Thank You!

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Robust Positivity	Up to Order 5	Up to Order 4	Order 5 and above
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