Skolem and Positivity Completeness of Ergodic Markov Chains

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Consider a regular language $L \subseteq \Sigma^*$

Let $L_n = \{w \in L : |w| = n\}$

Decision problems:

Is $|L_n| \geq 2^n$ for all n?

Is $|L_n| = 2^n$ for some n?

Look at a DFA $\mathscr{A}$ accepting L.

$$\mathscr{A} = (\Sigma, \{q_1, \ldots, q_k\}, q_{init}, \delta, F)$$

How do we use $\mathscr{A}$ to compute $|L_n|$?

Idea: transition matrix $\mathbf{M} \in \mathbb{N}^{k \times k}$

m_{ij} denotes the number of transitions from q_j to q_i

What about the $(i, j)^{th}$ entry of $\mathbf{M}^n$?

The number of words that take q_j to q_i

But what is the $(i, j)^{th}$ entry of $\mathbf{M}^n$?

A Linear Recurrence Sequence, indexed by n

A Linear Recurrence Sequence (LRS) of order k is

An infinite sequence of numbers $(u_0, u_1, u_2, \ldots)$ satisfying

$$\forall n \,.\; u_{n+k} = a_{k-1}u_{n+k-1} + \ldots + a_0 u_n$$

For some constants $a_0, \ldots, a_{k-1}$

$$\mathbf{M}^n \in \mathbb{N}^{k \times k}$$

Automata → LRS

$$\forall n \,.\, |L_n| \geq 2^n?$$

Positivity Problem for LRS

Are all terms non-negative?

$$\exists n \,.\, |L_n| = 2^n?$$

Skolem Problem

Is there a zero term?

← ??

$$\mathbf{M}^n \in \mathbb{Q}_{\geq 0}^{k \times k}$$

Markov Chains → LRS

$$\forall n\,.\ \Pr[q_\ell] \geq r?$$

Positivity Problem for LRS

Are all terms non-negative?

$$\exists n\,.\ \Pr[q_\ell] = r?$$

Skolem Problem

Is there a zero term?

← ??

$\mathbf{M}^n$

Transition Systems → LRS

$\forall n \, . \; \Pr[q_\ell] \geq r?$

$\forall n \, . \; |L_n| \geq 2^n?$

Positivity Problem for LRS

Are all terms non-negative?

$\exists n \, . \; \Pr[q_\ell] = r?$

$\exists n \, . \; |L_n| = 2^n?$

Skolem Problem

Is there a zero term?

←

Akshay et al. IPL 2015

Caveats

LRS of order k reduce to transition systems of order $2k + 4$

These transition systems are periodic

Does this undermine the fundamental nature of LRS?

Aperiodic automata: LTL/First Order Definability, additional logical structure

Irreducible, Aperiodic Markov Chains: Ergodic, unique stationary distribution, spectral structure

Ubiquitous!

“No,” we say, “it doesn’t”

LRS of order k → Aperiodic Transition Systems of order $k+1$

Positivity Problem for LRS	$\forall n .\ \Pr[q_\ell] \geq r?$
Are all terms non-negative?	$\forall n .\ \lvert L_n \rvert \geq 2^n?$
Skolem Problem	$\exists n .\ \Pr[q_\ell] = r?$
Is there a zero term?	$\exists n .\ \lvert L_n \rvert = 2^n?$

Key Technical Insight

$$\mathbf{M} = \mathbf{S} + \mathbf{D}$$

Transition Matrix | Stationary Distribution | "Disturbance"

$$\mathbf{D}\mathbf{S} = \mathbf{S}\mathbf{D} = \mathbf{0} \qquad \lim_{n\to\infty} \mathbf{D}^n = \mathbf{0}$$

Idea: Encode the LRS in D

Reduction through Linear Algebra

$$\mathbf{M}^n_{(i,j)} = \mathbf{S}_{(i,j)} + \mathbf{D}^n_{(i,j)}$$

Query this entry

Can choose this arbitrarily

Encode LRS here

$$\mathbf{D}^n_{(i,j)} = \frac{\eta u_n}{\rho^n}$$

Via changing and change of basis on companion matrix of LRS

Thank you!

$$\mathbf{M}^n_{(i,j)} = \mathbf{S}_{(i,j)} + \mathbf{D}^n_{(i,j)}$$

Query this entry — Can choose this arbitrarily — Encode LRS here

LRS of order k → Aperiodic Transition Systems of order $k+1$

Positivity Problem for LRS

Are all terms non-negative?

$\forall n.\ \Pr[q_\ell] \geq r?$

$\forall n.\ |L_n| \geq 2^n?$

Skolem Problem

Is there a zero term?

$\exists n.\ \Pr[q_\ell] = r?$

$\exists n.\ |L_n| = 2^n?$