

Skolem and Positivity Completeness of Ergodic Markov Chains

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Consider a regular language $L \subseteq \Sigma^*$

Let $L_n = \{w \in L : |w| = n\}$

Decision problems:

Is $|L_n| \geq 2^n$ for all n ?

Is $|L_n| = 2^n$ for some n ?

Look at a DFA \mathcal{A} accepting L .

$$\mathcal{A} = (\Sigma, \{q_1, \dots, q_k\}, q_{init}, \delta, F)$$

How do we use \mathcal{A} to compute $|L_n|$?

Idea: transition matrix $\mathbf{M} \in \mathbb{N}^{k \times k}$

m_{ij} denotes the number of transitions from q_j to q_i

What about the $(i, j)^{th}$ entry of \mathbf{M}^n ?

The number of words that take q_j to q_i

But what is the $(i, j)^{th}$ entry of \mathbf{M}^n ?

A Linear Recurrence Sequence, indexed by n

A Linear Recurrence Sequence (LRS) of order k is

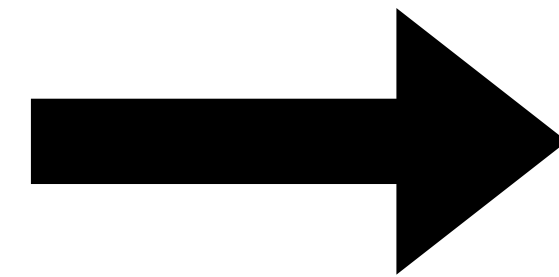
An infinite sequence of numbers (u_0, u_1, u_2, \dots) satisfying

$$\forall n . u_{n+k} = a_{k-1}u_{n+k-1} + \dots + a_0u_n$$

For some constants a_0, \dots, a_{k-1}

$$\mathbf{M}^n \in \mathbb{N}^{k \times k}$$

Automata



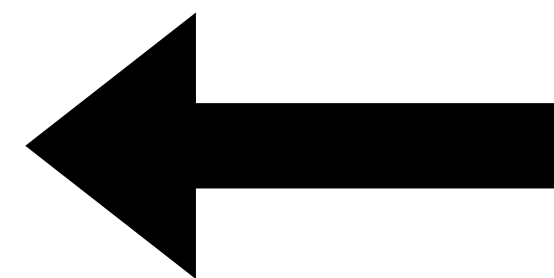
LRS

$$\forall n . |L_n| \geq 2^n?$$

$$\exists n . |L_n| = 2^n?$$

Positivity Problem for LRS
Are all terms non-negative?

Skolem Problem
Is there a zero term?



??

Markov Chains $\xrightarrow{\mathbf{M}^n \in \mathbb{Q}_{\geq 0}^{k \times k}}$ LRS

$\forall n . \Pr[q_\ell] \geq r?$

Positivity Problem for LRS
Are all terms non-negative?

$\exists n . \Pr[q_\ell] = r?$

Skolem Problem
Is there a zero term?

\leftarrow
??

Transition Systems $\xrightarrow{M^n}$ LRS

$\forall n . \Pr[q_\ell] \geq r?$

$\forall n . |L_n| \geq 2^n?$

$\exists n . \Pr[q_\ell] = r?$

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Caveats

LRS of order k reduce to transition systems of order $2k + 4$

These transition systems are periodic

Does this undermine the fundamental nature of LRS?

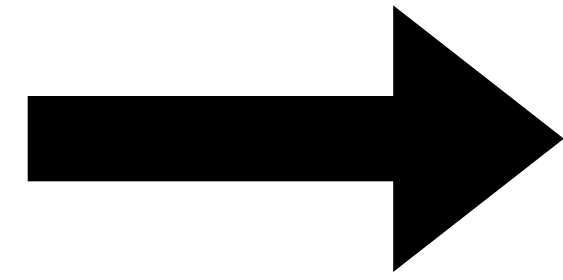
Aperiodic automata: LTL/First Order Definability, additional logical structure

Irreducible, Aperiodic Markov Chains: Ergodic, unique stationary distribution, spectral structure

Ubiquitous!

“No,” we say, “it doesn’t”

LRS of order k



Aperiodic
Transition Systems
of order $k + 1$

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Key Technical Insight

$$\mathbf{M} = \mathbf{S} + \mathbf{D}$$

Transition
Matrix

Stationary
Distribution

“Disturbance”

$$\mathbf{DS} = \mathbf{SD} = \mathbf{0} \quad \lim_{n \rightarrow \infty} \mathbf{D}^n = \mathbf{0}$$

Idea: Encode the LRS in \mathbf{D}

Reduction through Linear Algebra

$$\mathbf{M}_{(i,j)}^n = \mathbf{S}_{(i,j)} + \mathbf{D}_{(i,j)}^n$$

Query this entry Can choose this arbitrarily Encode LRS here

$$\mathbf{D}_{(i,j)}^n = \frac{\eta u_n}{\rho^n}$$

Via changing and change of basis on companion matrix of LRS

Thank you!

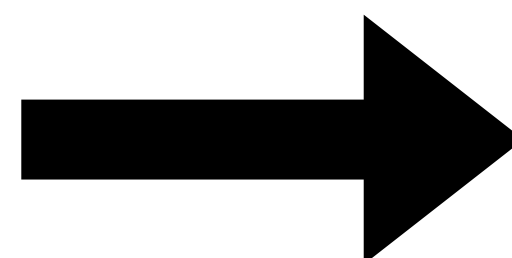
$$\mathbf{M}_{(i,j)}^n = \mathbf{S}_{(i,j)} + \mathbf{D}_{(i,j)}^n$$

Query this entry

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Encode LRS here

LRS of order k



Aperiodic
Transition Systems
of order $k + 1$

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Are all terms non-negative?

$$\forall n . \Pr[q_\ell] \geq r?$$

$$\forall n . |L_n| \geq 2^n?$$

Skolem Problem

Is there a zero term?

$$\exists n . \Pr[q_\ell] = r?$$

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