Reachability in VASS Extended with Integer Counters

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Plan

1 Introduction

Plan

1. Introduction
2. Motivation

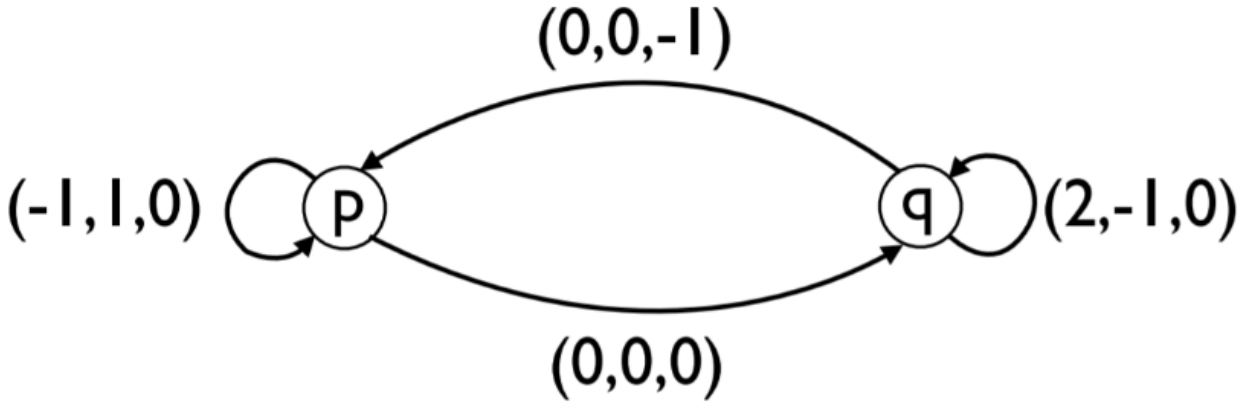
Plan

1. Introduction
2. Motivation
3. Results

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4. Future research

Vector Addition Systems with States



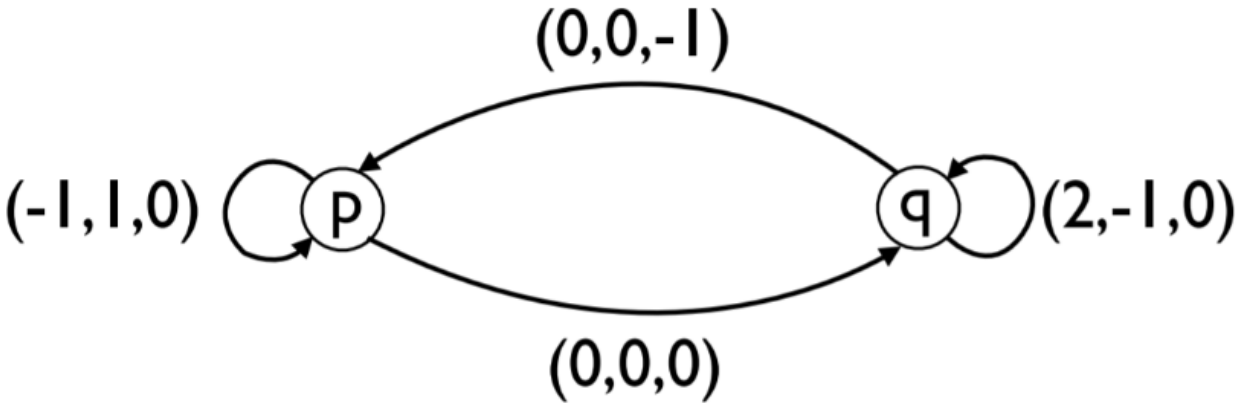
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Given: A Vector Addition System with States (VASS) V, two configurations s and t.

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Short history

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- Useful as a subprocedure in algorithms for reachability in VASSs

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  - Additional k $\mathbb{Z}$-counters can be simulated by $2k$ $\mathbb{N}$-counters.
- Question: What about a fixed number of $\mathbb{N}$-counters?

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  - KLM decomposition for VASS reachability
- Understanding fixed dimension VASS appears to be highly nontrivial and requires novel approaches

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- ExpSpace-complete for VASS (Rackoff '78)
- Ackermann-complete for VASS+$\mathbb{Z}$

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- Reduction from VASS reachability: one additional $\mathbb{Z}$-counter can be used to store minus the sum of all of the $\mathbb{N}$-counters
- $s \xrightarrow{*} t$ in VASS if and only if from $(s, -\text{sum}(s))$ we can cover $(t, -\text{sum}(t))$ in VASS+$\mathbb{Z}$

Motivation

Given: Two labelled d-VASS V_1 and V_2, decide whether there exist accepted words $w_1 \in L(V_1)$ and $w_2 \in L(V_2)$ such that

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4. Run V_2 while guessing an accepted word w_2.
5. Subtract the Parikh image of w_2 from the same $\mathbb{Z}$-counters.

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- We prove that reachability in 1-VASS+$\mathbb{Z}$ is also NP-complete
- This is obtained by showing that reachability is captured by linear path schemes, i.e., runs of form $\rho_0\sigma_1^*\rho_1 \dots \rho_{l-1}\sigma_l^*\rho_l$

Dimension one

Lemma

Let $V = (Q, T)$ be a 1-VASS+$k\mathbb{Z}$, let s, t be two configurations, with maximal norm of transition M. If $s \xrightarrow{*} t$ in V, then there exists a run $s \xrightarrow{\rho} t$ in V and a decomposition

$$\rho = \rho_0 \sigma_1^{x_1} \rho_1 \dots \rho_{l-1} \sigma_l^{x_l} \rho_l$$

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- Corollary: reachability relation is semilinear

Rearranging cycles in a VASS path

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2. First and last occurrences of state



Rearranging cycles in a VASS path

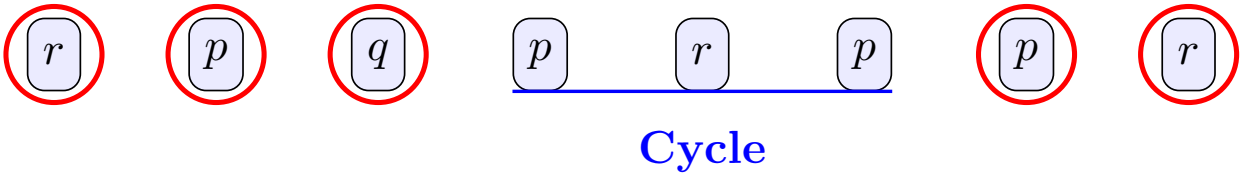
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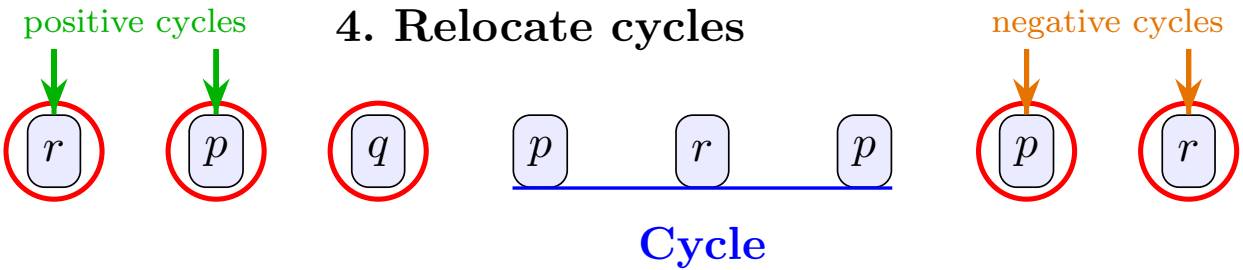
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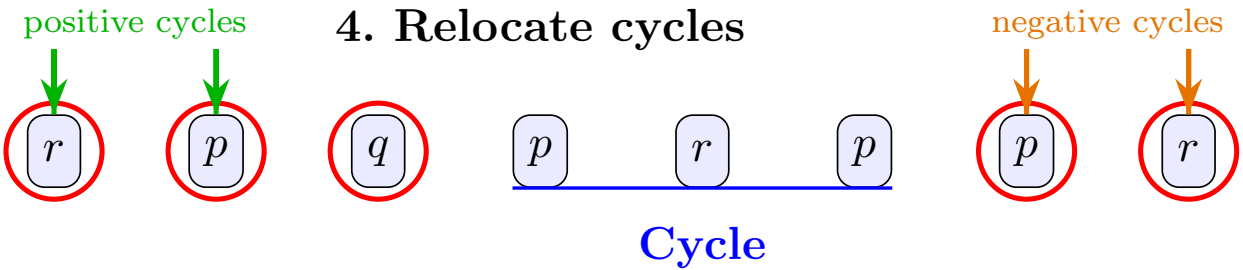
3. Remove cycles not containing first/last occurrences



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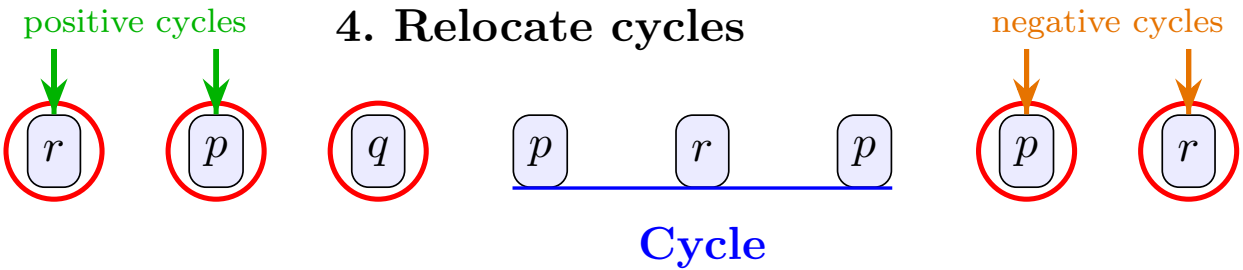


Rearranging cycles in a VASS path



- Positive-effect cycles are moved to the first occurrence of the state where the cycle reaches its minimum.

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Let $X \subset \mathbb{Z}^k$ be a finite set, let $s \in X^*$, and let $M = \max\{\|x\|_\infty : x \in X\}$. There exists a subset $Y \subseteq X$ such that $s \in Y^*$ and $|Y| \leq \text{poly}(k, \log(M))$.

Rearranging cycles in a VASS path

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- Apply Carathéodory's Theorem for Integer Cones
- Use Integer Linear Programming to bound the number of times each cycle is used

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- New technique for simulation of restricted reachability in three counter automata with zero-tests

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This problem is PSPACE-complete.

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- Solution: encode three counters (x, y, z) of counter automaton after run of length L as $2^x \cdot 3^y \cdot 5^z \cdot 7^L$

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- We can compute such number by a 2-VASS+$\mathbb{Z}$

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- Tower-hardness known for 8-VASSs
- We show Tower-hardness for reachability in 3-VASS+$\mathbb{Z}$
- Modification of Tower-hardness for 8-VASSs

Dimension d

- Reachability in d-VASS is in $\mathcal{F}_d$ (KLM decomposition)

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- Reachability in d-VASS is in $\mathcal{F}_d$ (KLM decomposition)
- We show that reachability in d-VASS+$\mathbb{Z}$ is in $\mathcal{F}_{d+2}$
- Almost optimal (for KLM based algorithm) due to existence of finite reachability sets of size F_{d+1}

Future research

- Is reachability in d-VASS+$\mathbb{Z}$ in $\mathcal{F}_{d+1}$?

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Thank You!