

Transcendence for Pisot morphic words over an algebraic base

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Abstract. It is known that for a uniform morphic sequence $\mathbf{u} = \langle u_n \rangle_{n=0}^\infty$ and an algebraic number β such that $|\beta| > 1$, the number $\llbracket \mathbf{u} \rrbracket_\beta := \sum_{n=0}^\infty (u_n / \beta^n)$ either lies in $\mathbb{Q}(\beta)$ or is transcendental. In this paper, we show a similar rational–transcendental dichotomy for sequences defined by irreducible Pisot morphisms on binary alphabets. Subject to the Pisot conjecture (an irreducible Pisot morphism has pure discrete spectrum), we generalise the latter result to arbitrary finite alphabets. In certain cases, we are able to show transcendence of $\llbracket \mathbf{u} \rrbracket_\beta$ outright. In particular, for $k \geq 2$, if $\mathbf{u}$ is the k -Bonacci word, then $\llbracket \mathbf{u} \rrbracket_\beta$ is transcendental.

Key words: balanced-pair algorithm, Cobham’s conjecture, k -Bonacci words, Pisot conjecture, subspace theorem

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1. Introduction

In 1968, Cobham [11] conjectured that every number whose expansion in an integer base is an automatic sequence is either rational or transcendental [11]. Recall that a sequence $\mathbf{u} = \langle u_n \rangle_{n=0}^\infty$ is k -automatic for $k \geq 2$ if there is an automaton that outputs u_n when given a base- k representation of n as input. A k -automatic sequence can equivalently be characterised as the fixed point of a k -uniform morphism, that is, a morphism on a finite alphabet that maps each letter to a string of length k . The transcendence of irrational

automatic numbers over an integer base was proven in 2004 by Adamczewski, Bugeaud and Luca [3], using Schlickewei's p -adic subspace theorem.

In [11], Cobham also stated a more general version of his conjecture, in which the rational–transcendental dichotomy is posited for numbers whose expansion in an integer base is given by a morphism of exponential growth. Recall here that a primitive morphism has exponential growth if its expansion factor (the Perron–Frobenius eigenvalue of its characteristic matrix) is strictly greater than one. The expansion factor of a k -uniform morphism is k and, hence, such morphisms have exponential growth for $k \geq 2$. This second form of Cobham's conjecture was confirmed in [4] using a construction similar to that used in the case of uniform morphisms. It is noted in [1] that the further extension to words generated by morphisms of merely polynomial growth encompasses several other longstanding open problems in transcendence theory.

Another natural generalisation of Cobham's conjecture involves a formulation for algebraic-number bases as opposed to rational integer bases: for $b \geq 2$, $\mathbf{u} \in \{0, 1, \dots, b-1\}^\omega$ a fixed point of a morphism of exponential growth and β an algebraic number, $|\beta| > 1$, the number $\llbracket \mathbf{u} \rrbracket_\beta := \sum_{n=0}^{\infty} (u_n / \beta^n)$ either lies in $\mathbb{Q}(\beta)$ or is transcendental. Such a result was recently obtained in the case of uniform morphisms by Adamczewski and Faverjon [5] using Mahler's method. Beyond the case of uniform morphisms, transcendence results for morphic words over algebraic bases are more restricted. However, the techniques of [1] can be extended to prove that for a word $\mathbf{u}$ and algebraic number β , subject to a non-trivial inequality between the height of β and a measure of the periodicity of $\mathbf{u}$, called the *Diophantine exponent*, the number $\llbracket \mathbf{u} \rrbracket_\beta$ is either transcendental or lies in $\mathbb{Q}(\beta)$; see [2, Theorem 1]. This result can be applied in the case where $\mathbf{u}$ is the fixed point of a morphism of exponential growth and β has height bounded above by a quantity determined by $\mathbf{u}$.

The object of this paper is to make further progress on the extension of Cobham's conjecture to algebraic-number bases. We focus on morphisms whose expansion factor is a *Pisot number*, that is, an algebraic integer whose Galois conjugates all have absolute value strictly less than one. Pisot morphisms are of particular interest in view of the *Pisot substitution conjecture*. This says that the shift dynamical system associated with an irreducible Pisot morphism has pure point spectrum (see [6, 17] and [26, Ch. 6.3]) or, equivalently, is measurably isomorphic to a translation on a compact abelian group equipped with the Haar measure [15, Theorem 9.3]. Notable positive examples of this conjecture include the Fibonacci morphism, whose associated shift dynamical system arises as the coding of a rotation on the unit circle, and the Tribonacci morphism, whose associated dynamical system arises as the coding, via the Rauzy fractal, of a translation on the two-torus [7, 27].

One of our main contributions is to exhibit a link between the respective conjectures of Pisot and Cobham. This is established via a combinatorial criterion on an infinite word called *echoing*, given in Definition 3.1. In Theorem 5.1, we show that if $\mathbf{u} \in \{0, \dots, b-1\}^\omega$ is echoing, then for all algebraic numbers β , either $\llbracket \mathbf{u} \rrbracket_\beta$ is transcendental or it lies in the field $\mathbb{Q}(\beta)$. We furthermore give a sufficient condition for a word to be echoing in terms of Livshits' balanced-pair algorithm: namely, we show that a morphic word on which this algorithm terminates with coincidence is echoing. It is known

that the algorithm terminates with coincidence for any irreducible Pisot morphism on a binary alphabet. Moreover, for an irreducible Pisot morphism on an arbitrary alphabet, the algorithm terminates with coincidence if and only if the morphism has pure discrete spectrum [6, Theorem 5.3]. We conclude that all words defined by irreducible Pisot morphisms on a binary alphabet are echoing and hence obey (the algebraic-number-base extension of) Cobham's conjecture. Assuming the Pisot conjecture, words defined by irreducible Pisot morphisms on arbitrary finite alphabets are also echoing and thereby obey Cobham's conjecture.

We further propose in Definition 6.1 a strengthening of the echoing condition, called *strongly echoing*, and show that if $\mathbf{u} \in \{0, \dots, b-1\}^\omega$ is strongly echoing, then $\llbracket \mathbf{u} \rrbracket_\beta$ is transcendental, that is, we eliminate the eventuality that $\llbracket \mathbf{u} \rrbracket_\beta$ lies in $\mathbb{Q}(\beta)$. By a combinatorial analysis of balanced pairs, we show that for all $k \geq 2$, the k -Bonacci word is strongly echoing and hence that for any such word $\mathbf{u}$, the number $\llbracket \mathbf{u} \rrbracket_\beta$ is transcendental.

The present paper is a development of the conference paper [19]. The combinatorial transcendence conditions presented below are based on a condition introduced therein and in the MSc thesis [18]. These conditions in turn build on ideas introduced in [24] to prove transcendence results for Sturmian sequences. In the present paper, we have divided the notion of echoing sequence into weak and strong variants, highlighting the importance of the non-vanishing condition in the formulation of the latter. The applications to Pisot morphisms in §4, including the connection with the Pisot conjecture, and k -Bonacci sequences in §7 did not appear in [19].

2. Preliminaries

2.1. Morphic sequences. Consider an alphabet $\Sigma = \{0, \dots, k-1\}$. We endow the set Σ^ω of infinite words over Σ with the product topology, where Σ has the discrete topology. The shift map $\sigma : \Sigma^\omega \rightarrow \Sigma^\omega$ is defined by $\sigma(u_0u_1u_2 \dots) = u_1u_2u_3 \dots$.

A *morphism* is a homomorphism $\varphi : \Sigma^+ \rightarrow \Sigma^+$ of the free semigroup Σ^+ . The *incidence matrix* $M_\varphi \in \mathbb{N}^{k \times k}$ of φ is defined by taking $(M_\varphi)_{i,j}$ to be the number of occurrences of the symbol j in $\varphi(i)$. We say that φ is *primitive* if some power of M_φ is positive and we say that φ has *exponential growth* if the spectral radius $\rho(M_\varphi)$ of M_φ is strictly greater than one. If M_φ is primitive then, by the Perron–Frobenius theorem, its spectral radius is a strictly positive real eigenvalue and all other eigenvalues have absolute value strictly less than $\rho(M_\varphi)$. We say that φ is *irreducible* if its characteristic polynomial is irreducible over $\mathbb{Q}$. We define φ to be of *Pisot type* (or simply *Pisot*) if $\rho(M_\varphi)$ is a Pisot number, that is, all its Galois conjugates have absolute value strictly less than one.

Let φ be a primitive morphism over alphabet Σ . Assume that φ is *prolongable*, that is, $\varphi(0) = 0u$ for some word $u \in \Sigma^+$. Then, the sequence $(\varphi^n(0))_{n \geq 0}$ of finite words converges to an infinite word $\mathbf{u}$ that is a fixed point of φ . The word $\mathbf{u}$ is *uniformly recurrent*, that is, each finite factor occurs infinitely often and there is an upper bound on the length of the gaps between every pair of successive occurrences of the factor.

We refer to the shift dynamical system (X_φ, σ) , where X_φ is the topological closure of the orbit of $\mathbf{u}$ under the shift map and, by a slight abuse of notation, σ denotes the restriction of the shift map to X_φ . For a primitive morphism, this dynamical system has a

unique ergodic measure μ and we have a unitary operator $U_\varphi : L^2(X_\varphi, \mu) \rightarrow L^2(X_\varphi, \mu)$, defined by $U_\varphi(f) = f \circ \sigma$ for f a square-integrable function on the measure space (X_φ, μ) . The morphism φ is said to have *pure discrete spectrum* if the space $L^2(X_\varphi, \mu)$ has a basis of eigenvectors of U_φ .

2.2. Number theory. Let K be a number field of degree d over $\mathbb{Q}$ and let $M(K)$ be the set of *places* of K . We divide $M(K)$ into the collection of *Archimedean places*, which are determined either by an embedding of K in $\mathbb{R}$ or a complex-conjugate pair of embeddings of K in $\mathbb{C}$, and the set of *non-Archimedean places*, which are determined by prime ideals in the ring $\mathcal{O}_K$ of integers of K .

For $a \in K$ and $v \in M(K)$, define the absolute value $|a|_v$ as follows: $|a|_v := |\sigma(a)|^{1/d}$ if v corresponds to a real embedding $\sigma : K \rightarrow \mathbb{R}$; $|a|_v := |\sigma(a)|^{2/d}$ if v corresponds to a complex-conjugate pair of embeddings $\sigma, \bar{\sigma} : K \rightarrow \mathbb{C}$; $|a|_v := N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(a)/d}$ if v corresponds to a prime ideal $\mathfrak{p}$ in $\mathcal{O}$ and $\text{ord}_{\mathfrak{p}}(a)$ is the order to which $\mathfrak{p}$ divides the ideal $a\mathcal{O}$. With the above definitions, we have the *product formula*: $\prod_{v \in M(K)} |a|_v = 1$ for all $a \in K^\times$. Given a set of places $S \subseteq M(K)$, the ring $\mathcal{O}_S$ of *S-integers* is the subring comprising all $a \in K$ such that $|a|_v \leq 1$ for all non-Archimedean places $v \notin S$.

For $m \geq 1$, the *Weil height* of the projective point $\mathbf{a} = [a_0 : a_1 : \dots : a_m] \in \mathbb{P}^m(K)$ is

$$H(\mathbf{a}) := \prod_{v \in M(K)} \max(|a_0|_v, \dots, |a_m|_v).$$

This definition is independent of the choice of the field K containing $a_0, \dots, a_m$. We define the height $H(a)$ of $a \in K$ to be the height $H([1 : a])$ of the corresponding point in $\mathbb{P}^1(K)$. For a non-zero Laurent polynomial $f = x^n \sum_{i=0}^m a_i x^i \in K[x, x^{-1}]$, where $m \geq 1$ and $n \in \mathbb{Z}$, following [20], we define its height $H(f)$ to be the height $H([a_0 : \dots : a_m])$ of the vector of coefficients.

The following special case of the p -adic subspace theorem of Schlickewei [28] is one of the main ingredients of our approach. (We formulate the special case of the subspace theorem in which all but one of the linear forms are coordinate variables.)

THEOREM 2.1. *Let $S \subseteq M(K)$ be a finite set of places of K that contains all Archimedean places. Let $v_0 \in S$ be a distinguished place and choose a continuation of $|\cdot|_{v_0}$ to $\overline{\mathbb{Q}}$, also denoted $|\cdot|_{v_0}$. Given $m \geq 2$, let $L(x_1, \dots, x_m)$ be a linear form with algebraic coefficients and let $i_0 \in \{1, \dots, m\}$ be a distinguished index such that x_{i_0} has non-zero coefficient in $\overline{\mathbb{Q}}$. Then, for any $\varepsilon > 0$, the set of solutions $\mathbf{a} = (a_1, \dots, a_m) \in (\mathcal{O}_S)^m$ of the inequality*

$$|L(\mathbf{a})|_{v_0} \cdot \left(\prod_{\substack{(i,v) \in \{1, \dots, m\} \times S \\ (i,v) \neq (i_0, v_0)}} |a_i|_v \right) \leq H(\mathbf{a})^{-\varepsilon}$$

is contained in a finite union of proper linear subspaces of K^m .

We need the following proposition about roots of univariate polynomials.

PROPOSITION 2.1. [20, Proposition 2.3] *Let $f \in K[x, x^{-1}]$ be a Laurent polynomial with at most $k+1$ terms. Assume that f can be written as the sum of two polynomials*

g and h , where every monomial of g has degree at most d_0 and every monomial of h has degree at least d_1 . Let β be a root of f that is not a root of unity. If $d_1 - d_0 > (\log(k H(f)))/(\log H(\beta))$, then β is a common root of g and h .

We also need the following separation bound [30, Ch. 3].

PROPOSITION 2.2. *Let $f \in \mathbb{Z}[x]$ have degree d and let L be the sum of the absolute value of its coefficients. If $\beta \in \overline{\mathbb{Q}}$ is not a root of f , then*

$$|f(\beta)| > \frac{1}{c^d L^{\deg(\beta)}},$$

where c is a constant that depends only on the height of β .

3. Echoing words

In this section, we present the main definition of the paper—the notion of echoing word. To motivate this, we first present an informal analysis of periodicity properties of the Fibonacci and Tribonacci words, which are perhaps the two best-known examples of Pisot morphic words.

3.1. The Fibonacci word. Let $\Sigma = \{0, 1\}$ and consider the morphism $\varphi : \Sigma^+ \rightarrow \Sigma^+$ given by $\varphi(0) = 01$ and $\varphi(1) = 0$. The *Fibonacci word* $\mathbf{u}_{\text{Fib}} \in \Sigma^\omega$ is the morphic word

$$\mathbf{u}_{\text{Fib}} := \lim_{n \rightarrow \infty} \varphi^n(0) = 01001010010010100 \dots$$

The Fibonacci word is not periodic and, hence, $\mathbf{u}_{\text{Fib}}$ is not equal to any of its tails $\sigma^n(\mathbf{u}_{\text{Fib}})$ for $n > 0$. However, if the shift n is well chosen, then the mismatches between $\mathbf{u}_{\text{Fib}}$ and $\sigma^n(\mathbf{u}_{\text{Fib}})$ are sparse. This intuition will be formalised in the definition of echoing word. A good choice of shifts are the elements of the sequence $\langle 1, 2, 3, 5, 8, \dots \rangle$ of Fibonacci numbers: for example, vertically aligning $\mathbf{u}_{\text{Fib}}$ and $\sigma^5(\mathbf{u}_{\text{Fib}})$, and shading mismatches, we have

$$\begin{aligned} \mathbf{u}_{\text{Fib}} &:= 010010\mathbf{10}01001010010\mathbf{10}010010\mathbf{10}0100101001\dots, \\ \sigma^5(\mathbf{u}_{\text{Fib}}) &:= 010010\mathbf{01}01001010010\mathbf{01}010010\mathbf{01}0100101001\dots \end{aligned}$$

Observe that each mismatch involves a factor 10 of $\mathbf{u}_{\text{Fib}}$ for which the corresponding factor in $\sigma^5(\mathbf{u}_{\text{Fib}})$ is the reverse, 01. In fact, we see the same phenomenon for all shifts of $\mathbf{u}_{\text{Fib}}$ by an element of the Fibonacci sequence. Furthermore, it turns out that for each successive shift, the distance between the mismatched factors increases. This is formalised below as the *expanding gaps property*.

3.2. The Tribonacci word. The following example illustrates some more complicated phenomena than the Fibonacci case. Let $\Sigma = \{0, 1, 2\}$ and consider the morphism $\varphi : \Sigma^+ \rightarrow \Sigma^+$ given by $\varphi(0) = 01$, $\varphi(1) = 02$ and $\varphi(2) = 0$. The *Tribonacci word* $\mathbf{u}_{\text{Tri}} \in \Sigma^\omega$ is the morphic word

$$\mathbf{u}_{\text{Tri}} := \lim_{n \rightarrow \infty} \varphi^n(0) = 0102010010201\dots$$

Associated with the Tribonacci word, we have the sequence $\langle t_n \rangle_{n=0}^\infty$ of Tribonacci numbers, defined by the recurrence $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ and initial conditions $t_0 = 1$, $t_1 = 2$, $t_2 = 4$. It is easy to see that the word $\varphi^n(0)$ has length t_n for all $n \in \mathbb{N}$.

In the spirit of our analysis of the Fibonacci word, we match the Tribonacci word against shifts of itself by elements of the Tribonacci sequence $\langle 1, 2, 4, 7, 13, 24, \dots \rangle$. For example, comparing $\mathbf{u}_{\text{Tri}}$ and $\sigma^{13}(\mathbf{u}_{\text{Tri}})$, we have

$$\begin{aligned}\mathbf{u}_{\text{Tri}} &:= 01020100102010\mathbf{1020100102010201001020100102010102010010} \dots, \\ \sigma^{13}(\mathbf{u}_{\text{Tri}}) &:= 01020100102010\mathbf{2010010201010201001020100102010010201010} \dots.\end{aligned}$$

Notice that the shaded mismatches appear as a fixed set of factors (either 10, 20 or 102) in $\mathbf{u}_{\text{Tri}}$ that get reversed in $\sigma^{13}(\mathbf{u}_{\text{Tri}})$. Unlike with the Fibonacci word, this time, the factors may appear close to each other. Nevertheless, by suitably grouping neighbouring mismatch factors, we recover a form of the expanding gaps property, and we are moreover able to show that the mismatches between $\mathbf{u}_{\text{Tri}}$ and its shifts are relatively sparse.

3.3. Definition of echoing words. Drawing on the respective examples of the Fibonacci and Tribonacci words, we give in this section the formal definition of echoing word. To this end, we introduce the following notation and terminology. Given two non-empty intervals $I, J \subseteq \mathbb{N}$, write $I < J$ if $a < b$ for all $a \in I$ and $b \in J$, and define the distance of I and J to be $d(I, J) := \min\{|a - b| : a \in I, b \in J\}$. Furthermore, define the density of a non-empty finite set $S \subseteq \mathbb{N}$ to be $\text{den}(S) := |S|/\max(S)$. For two functions $f(n)$ and $g(n)$, we write $f \ll g$ if there exist positive constants n_0, c_1 such that for all $n \geq n_0$, we have $f(n) \leq c_1 g(n)$. We also write $f \asymp g$ if both $f \ll g$ and $g \ll f$.

Definition 3.1. Let $\Sigma := \{0, \dots, b-1\}$ be a finite alphabet. An infinite word $\mathbf{u} = u_0 u_1 u_2 \dots \in \Sigma^\omega$ is said to be *echoing* if for all $\varepsilon > 0$, there exist sequences of integers r_n, s_n and non-empty intervals $I_{n,j}$ indexed by $n \in \mathbb{N}$, such that $0 \leq r_n < s_n$ and $\{0\} = I_{n,0} < I_{n,1} < I_{n,2} < \dots$, satisfying the following properties:

- (1) (covering) $\{i \in \mathbb{N} : u_{i+s_n} \neq u_{i+r_n}\} \subseteq \bigcup_{j=1}^\infty I_{n,j}$;
- (2) (density) $\text{den}(\bigcup_{j=1}^m I_{n,j}) \leq \varepsilon$ for $\min(m, n)$ sufficiently large;
- (3) (expanding gaps) $\lim_{n \rightarrow \infty} (s_n - r_n) = \infty$ and for all $j \in \mathbb{N}$, we have $d(I_{n,j}, I_{n,j+1}) \gg s_n$, with the implied constant being independent of j .

The notion of echoing word concerns the alignment of the suffixes $\sigma^{r_n}(\mathbf{u})$ and $\sigma^{s_n}(\mathbf{u})$ of $\mathbf{u}$, where $s_n - r_n$ tends to infinity. The conditions imply that the set of discrepancies between these two suffixes is contained in a union of intervals of asymptotic density at most ε and such that the distance between successive intervals grows linearly with s_n .

We show that the Fibonacci and Tribonacci words are both echoing. As suggested by the examples above, in the case of the Fibonacci word, a suitable choice of the sequence $\langle s_n \rangle_{n=0}^\infty$ of shifts is the sequence of Fibonacci numbers, while in the case of the Tribonacci word, it is the sequence of Tribonacci numbers (and in both cases, $\langle r_n \rangle_{n=0}^\infty$ is the all-zero sequence). In the case of the Fibonacci word, the intervals $I_{n,j}$ are doubletons, whereas in the case of the Tribonacci word, their total length grows with n , although the asymptotic density of $\bigcup_{j=1}^\infty I_{n,j}$ tends to zero as n tends to infinity.

4. Binary Pisot morphic words are echoing

4.1. *Balanced pairs.* A pair of words $(x, y) \in \Sigma^+ \times \Sigma^+$ is said to be *balanced* if x and y have the same commutative image. A balanced pair is said to be *irreducible* if it cannot be decomposed as a product (in the semigroup $\Sigma^+ \times \Sigma^+$) of two or more balanced pairs. For $x \in \Sigma$, we call the balanced pair (x, x) a *coincidence*. An irreducible balanced pair that is not a coincidence is called a *mismatch*.

Let $\varphi : \Sigma^+ \rightarrow \Sigma^+$ be a morphism. Note that if (x, y) is a balanced pair, then so is $(\varphi(x), \varphi(y))$. A finite set $\Gamma \subseteq \Sigma^+ \times \Sigma^+$ of irreducible balanced pairs is said to be *closed for φ* if for all $(x, y) \in \Gamma$, we can write $(\varphi(x), \varphi(y))$ as a product of balanced pairs in Γ . If Γ is closed, then φ naturally lifts to a morphism $\tilde{\varphi}$ on Γ such that $\tilde{\varphi}((x, y))$ is the decomposition of $(\varphi(x), \varphi(y))$ as a product of elements of Γ . We note that $\tilde{\varphi}$ necessarily maps a coincidence to a string of coincidences. We say that $\tilde{\varphi}$ satisfies the *coincidence condition* if for all $(x, y) \in \Gamma$, there exists $k \geq 0$ such that the string $\tilde{\varphi}^k((x, y))$ contains a coincidence. It is not difficult to see that if $\tilde{\varphi}$ satisfies the *coincidence condition*, then the asymptotic upper density of mismatch symbols in $\tilde{\varphi}^k((x, y))$ decays to zero at an exponential rate in k .

Let $\mathbf{u}$ be a fixed point of a primitive morphism φ . Given a non-empty prefix x of $\mathbf{u}$, since $\mathbf{u}$ is recurrent, we can write $\mathbf{u} = xyxu'$ for some $y \in \Sigma^*$ and suffix $u' \in \Sigma^\omega$. Given as input the prefix x , the *balanced-pair algorithm* [21, 22] successively computes all irreducible factors of the balanced pairs $(\varphi^k(xy), \varphi^k(yx))$ for $k = 1, 2, \dots$. We say that the algorithm *terminates with coincidence* if the resulting set Γ of irreducible factors is finite and the lifting of φ to a morphism $\tilde{\varphi}$ on Γ satisfies the coincidence condition.

The following result is [6, Theorem 5.3]. The fact that (X_φ, σ) has pure discrete spectrum if the balanced-pair algorithm for φ terminates with coincidence on some input prefix is due to Livshits [21, 22]. The converse direction follows by combining [29, Corollary 5.3] and [10, Corollary 3.2].

THEOREM 4.1. *Given an irreducible Pisot substitution φ , the substitutive dynamical system (X_φ, σ) has pure discrete spectrum if and only if the balanced-pair algorithm for φ terminates with coincidence for some input prefix.*

Hollander and Solomyak [16] showed that the balanced-pair algorithm terminates for irreducible Pisot morphisms on two-element alphabets, while Barge and Diamond [8] showed that the output satisfies the coincidence condition. This gives the following theorem.

THEOREM 4.2. *Let φ be an irreducible Pisot morphism on a two-letter alphabet with fixed point $\mathbf{u}$. Then, there exists a non-empty prefix x of $\mathbf{u}$ such that the balanced-pair algorithm terminates with coincidence on input x .*

4.2. *The Tribonacci word is echoing.* This section presents an extended example of an echoing word. The construction is only sketched below, and will be treated more formally and in more generality in Lemma 4.4 and Theorem 7.1.

Consider the alphabet $\Sigma = \{0, 1, 2\}$ and the Tribonacci morphism $\varphi : \Sigma^+ \rightarrow \Sigma^+$, defined by $\varphi(0) = 01$, $\varphi(1) = 02$ and $\varphi(2) = 0$. We introduce the alphabet

$\Gamma := \{a_0, \dots, a_{10}\}$ of irreducible balanced pairs, whose elements (depicted as tiles) are as follows:

$$\begin{aligned} a_0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & a_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & a_2 &= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, & a_3 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, & a_4 &= \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}, & a_5 &= \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \\ a_6 &= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}, & a_7 &= \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, & a_8 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & a_9 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & a_{10} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

We partition Γ into the subset $\Gamma_0 := \{a_0, \dots, a_7\}$ of mismatches and the subset $\Gamma_1 := \{a_8, a_9, a_{10}\}$ of coincidences.

The map φ lifts to a morphism $\tilde{\varphi} : \Gamma^+ \rightarrow \Gamma^+$ that is defined as follows:

$$\begin{aligned} \tilde{\varphi}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}, & \tilde{\varphi}\left(\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \\ \tilde{\varphi}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}, & \tilde{\varphi}\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}, \\ \tilde{\varphi}\left(\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \tilde{\varphi}\left(\begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \tilde{\varphi}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \tilde{\varphi}\left(\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\right) &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and

$$\tilde{\varphi}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \tilde{\varphi}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \tilde{\varphi}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By inspection, $\tilde{\varphi}$ satisfies the coincidence condition.

Let the projection morphisms $\pi_1, \pi_2 : \Gamma^+ \rightarrow \Sigma^+$ be defined by $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$ for all $(x, y) \in \Gamma$. Denote by $\mathbf{u} \in \Sigma^\omega$ the Tribonacci word and by

$$\mathbf{w} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \in \Gamma^\omega$$

the image of $\mathbf{u}$ under the coding $\iota : \Sigma^+ \rightarrow \Gamma^+$ given by

$$\iota(0) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \iota(1) := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad \iota(2) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, $\pi_1(\mathbf{w}) = \mathbf{u}$ and $\pi_2(\mathbf{w}) = \sigma(\mathbf{u})$. Furthermore, for $n_0 \in \mathbb{N}$, we can write

$$\tilde{\varphi}^{n_0}(\mathbf{w}) = w_0 a_{i_1} w_1 a_{i_2} w_2 a_{i_3} \cdots, \quad (4.1)$$

where $w_0, w_1, \dots \in \Gamma^*$ are sequences of coincidence symbols and $a_{i_1}, a_{i_2}, \dots \in \Gamma$ are mismatch symbols. Since $\tilde{\varphi}$ satisfies the coincidence condition, by making $n_0 \in \mathbb{N}$ sufficiently large, the upper asymptotic density of mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$ can be made arbitrarily small.

The data to show the echoing property (see Definition 3.1) are as follows. For all $n \in \mathbb{N}$, consider the image of (4.1) under the n -fold application of $\tilde{\varphi}$:

$$\tilde{\varphi}^{n+n_0}(\mathbf{w}) = \tilde{\varphi}^n(w_0) \underbrace{\tilde{\varphi}^n(a_{i_1})}_{I_{n,1}} \tilde{\varphi}^n(w_1) \underbrace{\tilde{\varphi}^n(a_{i_2})}_{I_{n,2}} \tilde{\varphi}^n(w_2) \underbrace{\tilde{\varphi}^n(a_{i_3})}_{I_{n,3}} \cdots. \quad (4.2)$$

We define $r_n := 0$ and $s_n := |\varphi^{n+n_0}(0)|$. We also have $\pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \mathbf{u}$ and $\pi_2(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \sigma^{s_n}(\mathbf{u})$. For all $j \in \{1, 2, \dots\}$, we define $I_{n,j} \subseteq \mathbb{N}$ to be the smallest interval of positions in $\mathbf{u} = \pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w}))$ that contains all mismatch symbols occurring in the factor $\pi_1(\tilde{\varphi}^n(a_{i_j}))$ (see (4.2) for an illustration). By construction, this choice satisfies the covering property. Intuitively, the density property follows from the fact that, by primitivity of φ , for all $n \in \mathbb{N}$, the upper asymptotic density of $\bigcup_{j=1}^{\infty} I_{n,j}$ is bounded above by a constant multiple of the upper asymptotic density of mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$, which can be made arbitrarily small as previously noted.

It remains to consider the expanding gaps property. Here, we note that, by definition of r_n and s_n , we clearly have $\lim_{n \rightarrow \infty} (s_n - r_n) = \infty$. Moreover, the primitivity of φ can be used to show that $d(I_{n,j}, I_{n,j+1}) \gg s_n$ for all $j \in \mathbb{N}$. Note that this bound holds even when the factor w_j in (4.1) is empty, since $\tilde{\varphi}(a_{i_{j+1}})$ starts with a coincidence symbol.

4.3. Pisot morphic words are echoing. The following is the main result of this section.

THEOREM 4.3. *Let $\mathbf{u}$ be a fixed point of an irreducible Pisot morphism φ over alphabet Σ . Assuming the Pisot conjecture, it holds that $\mathbf{u}$ is echoing. In the case that Σ is a binary alphabet, $\mathbf{u}$ is echoing unconditionally.*

Theorem 4.3 follows from Theorems 4.1 and 4.2 (which respectively give conditions for the balanced-pair algorithm to terminate with coincidence over binary and general alphabets) and the following lemma (showing that termination with coincidence implies the echoing condition). The proof of Lemma 4.4 generalises the construction for the Tribonacci word in §4.2.

LEMMA 4.4. *Let $\mathbf{u}$ be a fixed point of a primitive morphism φ over alphabet Σ . If there is a finite set Γ of balanced pairs such that φ lifts to a morphism $\tilde{\varphi} : \Gamma^+ \rightarrow \Gamma^+$ satisfying the coincidence condition, then $\mathbf{u}$ is either eventually periodic or echoing.*

Proof. By assumption, there exists a finite alphabet $\Gamma \subseteq \Sigma^+ \times \Sigma^+$ of irreducible balanced pairs and a lifting of φ to a morphism $\tilde{\varphi} : \Gamma^+ \rightarrow \Gamma^+$ satisfying the coincidence condition. More specifically, we suppose that Γ is the output of the balanced-pair algorithm on input x a non-empty prefix of $\mathbf{u}$. Since x occurs in $\mathbf{u}$ with bounded gaps, we may write $\mathbf{u} = xy_1xy_2x \cdots$ for some $y_1, y_2, \dots \in \Sigma^*$ and obtain an infinite word $\mathbf{w} \in \Gamma^\omega$ by decomposing the infinite sequence of balanced pairs

$$\begin{bmatrix} xy_1 \\ y_1x \end{bmatrix} \begin{bmatrix} xy_2 \\ y_2x \end{bmatrix} \begin{bmatrix} xy_3 \\ y_3x \end{bmatrix} \cdots \quad (4.3)$$

into a sequence of irreducible balanced pairs in Γ . Note that $\pi_1(\mathbf{w}) = \mathbf{u}$ and $\pi_2(\mathbf{w}) = \sigma^\ell(\mathbf{u})$, where $\ell = |x|$. Since φ is primitive, there exists $c_0 > 0$ such that for all $\gamma, \gamma' \in \Gamma$ and all $n \in \mathbb{N}$, we have

$$|\pi_1(\tilde{\varphi}^n(\gamma))| \leq c_0 |\pi_1(\tilde{\varphi}^n(\gamma'))|. \quad (4.4)$$

Assume that $\mathbf{u}$ is not ultimately periodic. Then, given $n_0 \in \mathbb{N}$, the word $\tilde{\varphi}^{n_0}(\mathbf{w})$ contains infinitely many mismatch symbols. Moreover, since $\tilde{\varphi}$ satisfies the coincidence condition,

for n_0 sufficiently large, $\tilde{\varphi}^{n_0}(\mathbf{w})$ also contains infinitely many coincidence symbols. For such n_0 , we thus have either

$$\tilde{\varphi}^{n_0}(\mathbf{w}) = z_0 v_0 z_1 v_1 z_2 \cdots \quad \text{or} \quad \tilde{\varphi}^{n_0}(\mathbf{w}) = v_0 z_0 v_1 z_1 v_2 \cdots, \quad (4.5)$$

where $z_0, z_1, \dots \in \Gamma^+$ are non-null sequences of mismatch symbols and $v_0, v_1, \dots \in \Gamma^+$ are non-null sequences of coincidence symbols. Let $\varepsilon > 0$ be given as in the definition of an echoing sequence. Since $\tilde{\varphi}$ satisfies the coincidence condition, for sufficiently large $n_0 \in \mathbb{N}$, the upper asymptotic density of mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$ is at most ε/c_0 .

The data to show the echoing property (see Definition 3.1) are as follows. Let us assume the left-hand case in (4.5); the reasoning for the right-hand case follows with minor changes. For all $n \in \mathbb{N}$, consider the string

$$\tilde{\varphi}^{n+n_0}(\mathbf{w}) = \tilde{\varphi}^n(z_0) \tilde{\varphi}^n(v_0) \underbrace{\tilde{\varphi}^n(z_1)}_{I_{n,1}} \tilde{\varphi}^n(v_1) \underbrace{\tilde{\varphi}^n(z_2)}_{I_{n,2}} \tilde{\varphi}^n(v_2) \cdots \quad (4.6)$$

Then, for all $n \in \mathbb{N}$, we have $\pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \mathbf{u}$ and $\pi_2(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \sigma^{t_n}(\mathbf{u})$, where $t_n := |\varphi^{n+n_0}(x)|$.

For all $n \in \mathbb{N}$, define $r_n := |\pi_1(\tilde{\varphi}^n(z_0))|$ and $s_n := r_n + t_n$. For all $j \in \{1, 2, \dots\}$, we define $I_{n,j} \subseteq \mathbb{N}$ to be the interval of positions in $\mathbf{u} = \pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w}))$ corresponding to the factor $\pi_1(\tilde{\varphi}^n(z_j))$ (see (4.6)). By construction, this choice satisfies the covering property. Since the mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$ have upper asymptotic density at most ε/c_0 , by (4.4), we have $\text{den}(\bigcup_{j=1}^m I_{n,j}) \leq \varepsilon$ for m sufficiently large; hence, the density property holds. Finally, by primitivity of φ , $\pi_1(\tilde{\varphi}^n(\gamma))$ has the same asymptotic growth rate as a function of n for all $\gamma \in \Gamma$. It follows that $d(I_{n,j}, I_{n,j+1}) \gg s_n$ for all $j \in \mathbb{N}$, with the implied constant being independent of j . Since $s_n - r_n = |\varphi^{n+n_0}(x)|$, we clearly have $\lim_{n \rightarrow \infty} (s_n - r_n) = \infty$. This establishes the expanding gaps property. $\square$

5. Transcendence for echoing words

We have the following transcendence result for echoing words.

THEOREM 5.1. *Let $\Sigma := \{0, 1, \dots, b-1\}$ and suppose that the sequence $\mathbf{u} = \langle u_m \rangle_{m=0}^\infty \in \Sigma^\omega$ is echoing. Then, for any algebraic number β such that $|\beta| > 1$, the sum $\alpha := \sum_{m=0}^\infty (u_m / \beta^m)$ is either transcendental or lies in $\mathbb{Q}(\beta)$.*

Proof. Suppose that α is algebraic. We use the subspace theorem to show that $\alpha \in \mathbb{Q}(\beta)$. Let S comprise all the Archimedean places of $\mathbb{Q}(\beta)$ and all non-Archimedean places corresponding to prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(\beta)}$ such that $\text{ord}_{\mathfrak{p}}(\beta) \neq 0$. Let $v_0 \in S$ be the place corresponding to the inclusion of $\mathbb{Q}(\beta)$ in $\mathbb{C}$. Recall that $|\cdot|_{v_0} = |\cdot|^{e_0}$, where $|\cdot|$ denotes the usual absolute value on $\mathbb{C}$ and $e_0 = 1/(\deg(\beta))$ if β is real and $e_0 = 2/(\deg(\beta))$ otherwise. Let $\kappa \geq b$ be an upper bound of $|\beta|_v$ for all $v \in S$.

Let $\varepsilon := (e_0 \log |\beta|) / (4|S| \log \kappa)$. By the assumption that $\mathbf{u}$ is echoing, for all $n \in \mathbb{N}$, there exist $0 \leq r_n < s_n$ and a sequence of intervals $\{0\} = I_{n,0} < I_{n,1} < I_{n,2} < \dots$ satisfying conditions (1)–(3) of Definition 3.1 for the above choice of ε .

Define $\rho := (4|S| \log \kappa) / (e_0 \log |\beta|)$. The expanding-gaps condition gives $d(I_{n,j}, I_{n,j+1}) \gg s_n$ for $j \in \mathbb{N}$, with the implied constant being independent of j ; thus,

there exists $m \in \mathbb{N}$ such that for all n , the right endpoint t_n of $I_{n,m}$ satisfies $t_n \geq \rho s_n$. For $n \in \mathbb{N}$, define $\mathbf{a}_n = (a_{n,1}, \dots, a_{n,m+3}) \in (\mathcal{O}_S)^{m+3}$ by

$$\begin{aligned} a_{n,1} &:= \beta^{s_n}, & a_{n,2} &:= \beta^{r_n}, & a_{n,3} &:= \sum_{i=0}^{s_n} u_i \beta^{s_n-i} - \sum_{i=0}^{r_n} u_i \beta^{r_n-i}, \\ a_{n,j+3} &:= \sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) \beta^{-i} \quad (j = 1, \dots, m). \end{aligned} \quad (5.1)$$

By passing to a subsequence of $n \in \mathbb{N}$, we may assume that there exists $J \subseteq \{1, \dots, m\}$ such that for all $j \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, we have $a_{n,j+3} \neq 0$ if and only if $j \in J$.

Consider the linear form

$$L(x_1, \dots, x_{m+3}) := \alpha x_1 - \alpha x_2 - x_3 - \sum_{j \in J} x_{j+3}. \quad (5.2)$$

Then, we have

$$\begin{aligned} L(\mathbf{a}_n) &= \left| \alpha \beta^{s_n} - \alpha \beta^{r_n} - \sum_{i=0}^{s_n} u_i \beta^{s_n-i} + \sum_{i=0}^{r_n} u_i \beta^{r_n-i} - \sum_{j \in J} \sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) \beta^i \right| \\ &= \left| \alpha \beta^{s_n} - \alpha \beta^{r_n} - \sum_{i=0}^{s_n} u_i \beta^{s_n-i} + \sum_{i=0}^{r_n} u_i \beta^{r_n-i} - \sum_{j=1}^m \sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) \beta^i \right| \\ &= \left| \sum_{i=t_n+1}^{\infty} (u_{i+s_n} - u_{i+r_n}) \beta^{-i} \right| \\ &\ll |\beta|^{-t_n}, \end{aligned} \quad (5.3)$$

where the second equation holds because the extra summands for $j \notin J$ contribute zero, while the third equation holds because the set $\{i \in \{0, \dots, t_n\} : u_{i+s_n} \neq u_{i+r_n}\}$ is contained in $\bigcup_{j=1}^m I_{n,j}$.

We apply the subspace theorem (Theorem 2.1) with distinguished index $i_0 = 3$ and distinguished place v_0 as defined at the start of the current proof. To set up the application of the subspace theorem, we estimate the absolute values of the components of $\mathbf{a}_n$. By the product formula, we have $\prod_{v \in S} |a_{n,1}|_v = \prod_{v \in S} |a_{n,2}|_v = 1$. Next, by the facts that $t_n \geq \rho s_n$ and $\rho := (4|S| \log \kappa)/(e_0 \log |\beta|)$, we have

$$\prod_{v \in S \setminus \{v_0\}} |a_{n,3}|_v \leq \prod_{v \in S} \sum_{i=0}^{s_n} \kappa^{i+1} \leq \prod_{v \in S} \kappa^{s_n+2} = \kappa^{(s_n+2)|S|} \ll |\beta|^{e_0 t_n/4}. \quad (5.4)$$

Furthermore, by the density condition, for m sufficiently large, we have $\sum_{j=1}^m |I_{n,j}| \leq \varepsilon t_n$ for all $n \in \mathbb{N}$. By the product formula and the fact that $\varepsilon := (e_0 \log |\beta|)/(4|S| \log \kappa)$, we have

$$\prod_{j \in J} \prod_{v \in S} |a_{n,j+3}|_v \leq \prod_{j \in J} \prod_{v \in S} \sum_{i=0}^{|I_{n,j}|} \kappa^{i+1} \leq \prod_{j \in J} \kappa^{|S|(2+|I_{n,j}|)} \leq \kappa^{2|S|m+|S|\varepsilon t_n} \ll |\beta|^{e_0 t_n/4}. \quad (5.5)$$

Combining (5.4), (5.5) and the bound $|L(\mathbf{a}_n)|_{v_0} \ll |\beta|^{-e_0 t_n}$ from (5.3), we have

$$|L(\mathbf{a}_n)|_{v_0} \cdot \prod_{\substack{(i,v) \in (\{1,2,3\} \cup J) \times S \\ (i,v) \neq (3,v_0)}} |a_{n,i}|_v \ll |\beta|^{e_0 t_n/2} \cdot |\beta|^{-e_0 t_n} = |\beta|^{-e_0 t_n/2}. \quad (5.6)$$

For all but finitely many n , the right-hand side of (5.6) is less than $|\beta|^{-e_0 t_n/3}$. However, there exists a constant $c_1 > 0$ such that the height of $\mathbf{a}_n$ satisfies $H(\mathbf{a}_n) \leq |\beta|^{c_1 t_n}$ for all n . Thus, the right-hand side of (5.6) is at most $H(\mathbf{a}_n)^{-e_0/3c_1}$ for infinitely many n . We can hence apply the subspace theorem (Theorem 2.1) to obtain a non-zero linear form

$$F(x_1, \dots, x_{3+m}) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \sum_{j \in J} \alpha_{3+j} x_{3+j}$$

with coefficients in $\mathbb{Q}(\beta)$ such that $F(\mathbf{a}_n) = 0$ for infinitely many $n \in \mathbb{N}$. In other words,

$$\alpha_1 \beta^{s_n} + \alpha_2 \beta^{r_n} + \alpha_3 \left(\sum_{i=0}^{s_n} u_i \beta^{s_n-i} - \sum_{i=0}^{r_n} u_i \beta^{r_n-i} \right) + \sum_{j \in J} \alpha_{3+j} a_{3+j} = 0 \quad (5.7)$$

for infinitely many $n \in \mathbb{N}$.

We claim that $\alpha_3 \neq 0$ in (5.7). The proof is as follows. Equation (5.7) is equivalent to the assertion that $P_n(\beta) = 0$ for the Laurent polynomial $P_n(x) := P_{n,1}(x) + P_{n,2}(x) + P_{n,3}(x) + \sum_{j \in J} P_{n,3+j}(x)$, where

$$P_{n,1}(x) := \alpha_1 x^{s_n}, \quad P_{n,2}(x) := \alpha_2 x^{r_n}, \quad P_{n,3}(x) := \alpha_3 \left(\sum_{i=0}^{s_n} u_i x^{s_n-i} - \sum_{i=0}^{r_n} u_i x^{r_n-i} \right),$$

$$P_{n,3+j}(x) := \alpha_{3+j} \sum_{i \in I_{n,j}} (u_{s_n+i} - u_{r_n+i}) x^{-i} \quad (j \in J).$$

The height of P_n is bounded independently of n and P_n has at most $s_n + t_n$ terms, that is, the number of terms is bounded by a constant multiple of s_n . However, for $j < j' \in J \cup \{0\}$, the gap between the minimum degree of $P_{n,3+j}(x)$ and the maximum degree of $P_{n,3+j'}(x)$ is $\gg s_n$ by the expanding-gaps condition. Applying Proposition 2.1, we deduce that for n sufficiently large, if $P_n(\beta) = 0$, then for $j \in J$, each summand $P_{n,3+j}(\beta)$ must individually vanish. Since $P_{n,3+j}(\beta) = \alpha_{3+j} a_{n,3+j}$ and $a_{n,3+j} \neq 0$ for all $j \in J$, it follows that $\alpha_{3+j} = 0$ for all $j \in J$. We conclude that $\alpha_3 \neq 0$ since otherwise, we have $F(\mathbf{a}_n) = \alpha_1 \beta^{s_n} + \alpha_2 \beta^{r_n} = 0$ for infinitely many n , which contradicts the fact that $\lim_{n \rightarrow \infty} (s_n - r_n) = \infty$.

Having established that α_3 is not zero, dividing (5.7) by β^{s_n} and letting n tend to infinity, we have $\alpha_1 + \alpha_3 \sum_{i=0}^{\infty} u_i \beta_i = 0$. Hence, $\sum_{i=0}^{\infty} u_i \beta_i$ lies in $\mathbb{Q}(\beta)$, as we wanted to prove. $\square$

6. Strongly echoing sequences

6.1. *The non-vanishing condition.* The following definition presents a strengthening of the echoing condition that allows proving transcendence outright (removing the possibility that $\llbracket \mathbf{u} \rrbracket_\beta$ be algebraic). The change to Definition 3.1 involves strengthening the expanding-gaps condition (to specify both lower and upper bounds on the gaps between

intervals) and adding a fourth condition, called *non-vanishing*. The latter guarantees the existence of at least two intervals of mismatches that are non-zero when evaluated in base β .

Definition 6.1. Let $\Sigma := \{0, \dots, b-1\}$ be a finite alphabet. An infinite word $\mathbf{u} = u_0 u_1 u_2 \dots \in \Sigma^\omega$ is said to be *strongly echoing* if for all $\varepsilon > 0$, there exist sequences of integers r_n, s_n and non-empty intervals $I_{n,j}$ indexed by $n \in \mathbb{N}$ such that $0 \leq r_n < s_n$ and $\{0\} = I_{n,0} < I_{n,1} < I_{n,2} < \dots$, satisfying the following properties:

- (1) (covering) $\{i \in \mathbb{N} : u_{i+s_n} \neq u_{i+r_n}\} \subseteq \bigcup_{j=1}^{\infty} I_{n,j}$;
- (2) (density) $\text{den}(\bigcup_{j=1}^m I_{n,j}) \leq \varepsilon$ for $\min(m, n)$ sufficiently large;
- (3) (expanding gaps) $\lim_{n \rightarrow \infty} (s_n - r_n) = \infty$ and for all $j \in \mathbb{N}$, we have $d(I_{n,j}, I_{n,j+1}) \asymp s_n$, where the implied constant is independent of j ;
- (4) (non-vanishing) for all $\beta \in \overline{\mathbb{Q}}$ such that $|\beta| > 1$, there exist $1 \leq j_0 < j_1$ such that for all $n \in \mathbb{N}$ and $j \in \{j_0, j_1\}$, we have $\sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) \beta^{-i} \neq 0$.

In §7, we show that the substitutive sequence defined by the k -Bonacci morphism is strongly echoing for all $k \geq 2$. The case $k = 2$ (the Fibonacci word) is straightforward. In the following section, we treat the case $k = 3$ (the Tribonacci word), continuing the extended example from §4.2.

6.2. The Tribonacci word is strongly echoing. For $\Sigma := \{0, 1, 2\}$, let $\varphi : \Sigma^+ \rightarrow \Sigma^+$ be the morphism that defines the Tribonacci word $\mathbf{u} \in \Sigma^\omega$ and let $\tilde{\varphi} : \Gamma^+ \rightarrow \Gamma^+$ be its lifting to the closed set of balanced pairs, as defined in §4.2, with $\pi_1, \pi_2 : \Gamma^+ \rightarrow \Sigma^+$ the associated projection morphisms. We recall the construction showing that $\mathbf{u}$ is echoing and argue that the non-vanishing condition is also satisfied. In particular, we refer to the shifts $r_n := 0$ and $s_n := |\varphi^{n+n_0}(0)|$, and intervals $I_{n,j}$ that witness that $\mathbf{u}$ is echoing.

For all $n \in \mathbb{N}$ and $j \in \{1, 2, \dots\}$, define the polynomial $P_{n,j}(x) \in \mathbb{Z}[x]$ by

$$P_{n,j}(x) := \sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) x^i. \quad (6.1)$$

Inspecting condition (4) of Definition 6.1, to show that $\mathbf{u}$ is non-vanishing, we must prove that for all β such that $|\beta| > 1$, there exist indices $j_0 < j_1$ such that $P_{n,j_0}(\beta^{-1}) \neq 0$ and $P_{n,j_1}(\beta^{-1}) \neq 0$ for infinitely many n . To this end, we give a recursive characterisation of the polynomials $P_{n,j}$.

Recall the mismatch symbols $a_0, \dots, a_7$ in Γ . For all $i \in \{0, \dots, 7\}$ and $n \in \mathbb{N}$, define the polynomial $Q_{n,i}(x) \in \mathbb{Z}[x]$ by the equation

$$Q_{n,i}(x) := \sum_{k=0}^{\ell-1} (\pi_1(\tilde{\varphi}^n(a_i))_k - \pi_2(\tilde{\varphi}^n(a_i))_k) x^k, \quad (6.2)$$

where $\ell = |\pi_1(\tilde{\varphi}(a_i))|$. For example, we have $Q_{0,6}(x) = -2 + x - x^2 + 2x^3$ since $a_6 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}$. Referring to the list of symbols $a_{i_1}, a_{i_2}, \dots$ and the corresponding intervals $I_{n,j}$ in (4.2), we observe that for all $n \in \mathbb{N}$ and $j \in \{1, 2, \dots\}$, polynomial

$P_{n,j}$ is the product Q_{n,i_j} and a monomial. Thus, our task is to find $j_0 < j_1$ such that $Q_{n,i_{j_0}}(\beta^{-1}) \neq 0$ and $Q_{n,i_{j_1}}(\beta^{-1}) \neq 0$ for infinitely many n .

We claim that for all n , there exists $i \in \{0, \dots, 7\}$ such that $Q_{n,i}(\beta^{-1}) \neq 0$. From the claim, it follows that there exists $i^* \in \{0, \dots, 7\}$ such that $Q_{n,i^*}(\beta^{-1}) \neq 0$ for infinitely many $n \in \mathbb{N}$. Since $\tilde{\varphi}$ acts primitively on the set of mismatch symbols, for n_0 sufficiently large, i^* occurs infinitely often in the list $i_1, i_2, \dots$ in (4.1). In particular, there exist $j_0 < j_1$ such that $Q_{n,i_{j_0}}(\beta^{-1}) \neq 0$ and $Q_{n,i_{j_1}}(\beta^{-1}) \neq 0$ for infinitely many n . This is what we wanted to prove and it remains to justify the claim.

For all $n \in \mathbb{N}$, we have the vector $\mathbf{Q}_n := (Q_{n,0}, \dots, Q_{n,7}) \in \mathbb{Z}[x]^8$ of polynomials. We want to show that $\mathbf{Q}_n(\beta^{-1}) \neq \mathbf{0}$ for all n . The action of $\tilde{\varphi}$ gives rise to a recurrence for $\mathbf{Q}_n$. For example, we have $\tilde{\varphi}(a_6) = a_8 a_1 a_3 a_1$ and, hence,

$$\tilde{\varphi}^{n+1}(a_6) = \tilde{\varphi}^n(a_8) \tilde{\varphi}^n(a_1) \tilde{\varphi}^n(a_3) \tilde{\varphi}^n(a_1).$$

We deduce that

$$Q_{n+1,6}(x) = x^{\ell_{n,8}} Q_{n,1}(x) + x^{\ell_{n,8} + \ell_{n,1}} Q_{n,3}(x) + x^{\ell_{n,8} + \ell_{n,1} + \ell_{n,2}} Q_{n,1}(x),$$

where $\ell_{n,i} = |\pi_1(\tilde{\varphi}^n(a_i))|$ for all $i \in \{0, \dots, 8\}$. Calculating the corresponding recurrences for each polynomial $Q_{n+1,i}(x)$ yields the vector recurrence $\mathbf{Q}_{n+1} = M_n \mathbf{Q}_n$, where

$$M_n := x^{\ell_{n,8}} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 + x^{\ell_{n,1} + \ell_{n,3}} & 0 & x^{\ell_{n,1}} & 0 & 0 & 0 & 0 \\ 1 + x^{\ell_{n,0} + \ell_{n,2}} & 0 & x^{\ell_{n,0}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.3)$$

However, $Q_{0,0} = (x - 1)$ and $\det(M_n) = x^{\ell_{n,0} + \ell_{n,1} + 8\ell_{n,8}}$. Hence, for all $\beta \neq 1$, we have $\mathbf{Q}_0(\beta^{-1}) \neq \mathbf{0}$ and, by induction on n , $\mathbf{Q}_n(\beta^{-1}) \neq \mathbf{0}$ for all $n \in \mathbb{N}$. The claim is proven and this completes the argument that $\mathbf{u}$ is strongly echoing.

6.3. Transcendence result. For a strongly echoing sequence $\mathbf{u}$ and algebraic β , we can show that $\llbracket \mathbf{u} \rrbracket_\beta$ is outright transcendental. Whereas in the proof of Theorem 5.1 we assume that $\llbracket \mathbf{u} \rrbracket_\beta$ is algebraic and use the subspace theorem to show that it lies in $\mathbb{Q}(\beta)$, in the following, we continue the argument by applying the subspace theorem a second time (employing the non-vanishing condition) to derive a contradiction.

THEOREM 6.1. *Let $\Sigma := \{0, 1, \dots, b-1\}$ and suppose that the sequence $\mathbf{u} = (u_m)_{m=0}^\infty \in \Sigma^\omega$ is strongly echoing. Then, for any algebraic number β such that $|\beta| > 1$, the sum $\alpha := \sum_{m=0}^\infty (u_m / \beta^m)$ is transcendental.*

Proof. We suppose that α is algebraic and obtain a contradiction. We refer to the proof of Theorem 5.1, whose notation we freely use in the following. In particular, we use the distinguished place v_0 on the field $\mathbb{Q}(\beta)$, the tuples $\mathbf{a}_n \in (\mathcal{O})_S^{m+3}$ in (5.1) and the linear

form L in (5.2), which satisfies the inequality $|L(\mathbf{a}_n)| \ll |\beta|^{-t_n}$ shown in (5.3). As before, by passing to a subsequence of $n \in \mathbb{N}$, we assume that there is a set $J \subseteq \{1, \dots, m\}$ such that for all $n \in \mathbb{N}$, we have $a_{n,j} \neq 0$ if and only if $j \in J$. Crucially, J is non-empty in this case since it contains the indices j_0 and j_1 appearing in the non-vanishing condition. The only deviation from the basic set-up of Theorem 5.1 concerns the choice of parameter $m \in \mathbb{N}$ in (5.1), as we now explain.

For the index j_0 from the non-vanishing condition, since $d(I_{n,j}, I_{n,j+1}) \ll s_n$ for all $j < j_0$ by the upper-bound part of the expanding-gaps condition, we can write $a_{n,3+j_0}$ as a polynomial in β of height at most b and degree bounded by a constant multiple of s_n . Applying Proposition 2.2, there exists a constant $c_2 > 0$ such that $|a_{n,3+j_0}| \gg |\beta|^{-c_2 s_n}$. By the expanding gaps condition (this time, the lower bound $d(I_{n,j}, I_{n,j+1}) \gg s_n$), we can choose m sufficiently large that $m \geq j_0, j_1$ and for all n , the right endpoint t_n of interval $I_{n,m}$ satisfies $t_n \geq (\rho + 6c_2)s_n$ for all n .

Exactly as in the proof of Theorem 5.1, from the subspace theorem, we obtain a linear form

$$F(x_1, \dots, x_{3+m}) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \sum_{j \in J} \alpha_{3+j} x_{3+j} \quad (6.4)$$

with algebraic coefficients such that $F(\mathbf{a}_n) = 0$ for infinitely many $n \in \mathbb{N}$, and such that $\alpha_3 \neq 0$ and $\alpha_{3+j} = 0$ for all $j \in J$.

Here is where we deviate from the proof of Theorem 5.1. Subtracting a suitable multiple of F from L , we obtain a new linear form L' whose support contains x_{3+j} for $j \in J$, but not x_3 , and that satisfies $|L'(\mathbf{a}_n)| \ll |\beta|^{-t_n}$ for infinitely many n . Since $j_0, j_1 \in J$ (by the non-vanishing condition) the support of L' contains at least two variables.

Our aim is to apply the subspace theorem to L' , with distinguished index $3 + j_0$ and distinguished place v_0 . To this end, from (5.4) and (5.5), we have

$$\prod_{(i,v) \in (\{1,2,3\} \cup J) \times S} |a_{n,i}|_v \ll |\beta|^{e_0 t_n / 2}. \quad (6.5)$$

Recalling that $|a_{n,3+j_0}| \gg |\beta|^{-c_2 s_n}$ and $|L'(\mathbf{a}_n)| \ll |\beta|^{-t_n}$, since $t_n \geq 6c_2 s_n$, we have

$$|L'(\mathbf{a}_n)|_{v_0} |a_{n,3+j_0}|_{v_0}^{-1} \ll |\beta|^{-e_0 t_n} |\beta|^{e_0 c_2 s_n} \leq |\beta|^{-5e_0 t_n / 6}. \quad (6.6)$$

Multiplying (6.5) and (6.6), we get

$$|L'(\mathbf{a}_n)|_{v_0} \cdot \prod_{\substack{(i,v) \in (\{1,2,3\} \cup J) \times S \\ (i,v) \neq (3+j_0, v_0)}} |a_{n,i}|_v \ll |\beta|^{-e_0 t_n / 3}. \quad (6.7)$$

For all but finitely many n , the right-hand side of (6.7) is less than $|\beta|^{-e_0 t_n / 4}$. However, there exists a constant $c_3 > 0$ such that the height of $\mathbf{a}_n$ satisfies the bound $H(\mathbf{a}_n) \leq |\beta|^{c_3 t_n}$ for all n . Thus, the right-hand side of (6.7) is at most $H(\mathbf{a}_n)^{-e_0 / 4c_3}$ for infinitely many n . Thus, we may apply the subspace theorem to obtain a non-zero linear form

$$G(x_1, \dots, x_{3+m}) := \beta_1 x_1 + \beta_2 x_2 + \sum_{j \in J} \beta_{3+j} x_{3+j}$$

with algebraic coefficients whose support does not include x_3 such that $G(\mathbf{a}_n) = 0$ for infinitely many n . As shown in the proof of the claim at the end of Theorem 5.1, the conditions that $G(\mathbf{a}_n) = 0$ for infinitely many n and that x_3 not appear in the support of G entail that G is identically zero, which yields the desired contradiction. $\square$

7. The k -Bonacci word is strongly echoing

Let $k \geq 2$ and consider the alphabet $\Sigma := \{0, \dots, k-1\}$. The k -Bonacci morphism $\varphi : \Sigma^+ \rightarrow \Sigma^+$ is defined by $\varphi(0) := 01, \varphi(1) := 02, \dots, \varphi(k-2) = 0(k-1)$ and $\varphi(k-1) = 0$. Let $\mathbf{u} := \lim_{n \rightarrow \infty} \varphi^n(0)$ be the k -Bonacci word. The cases $k = 2$ and $k = 3$ respectively yield the Fibonacci and Tribonacci words as considered in §§3.1 and 3.2. The main result of this section shows that $\mathbf{u}$ is strongly echoing.

THEOREM 7.1. *For $k \geq 2$, the k -Bonacci word is strongly echoing.*

The proof of Theorem 7.1 is given below. To prepare the ground, we first describe the output of the balanced-pair algorithm for the k -Bonacci word and the consequent lifting of φ to a morphism $\tilde{\varphi}$ on balanced pairs. As with the example of the Tribonacci word, the key to establishing the non-vanishing property is to show that a certain matrix of polynomials related to $\tilde{\varphi}$ (generalising (6.3)) is non-singular when evaluated at any number $\beta \in \mathbb{C} \setminus \{0, 1\}$.

7.1. Balanced pairs for the k -Bonacci morphism. Write $\Sigma_{\perp} := \Sigma \cup \{-1\}$. For $a \in \Sigma_{\perp}$, the iterated palindromic closure $\text{pal}(a) \in \Sigma^*$ is inductively defined by $\text{pal}(-1) := \varepsilon$ and $\text{pal}(a) := \text{pal}(a-1) a \text{pal}(a-1)$ for $a > -1$. (Note that $\text{pal}(a)$ denotes the iterated palindromic closure of the string $01 \dots a$, as introduced by de Luca [12].) In manipulating balanced pairs, we write expressions in the free group generated by Σ that simplify to elements of the free monoid generated by Σ . For example, for $x, y \in \Sigma^*$, we write $y^{-1}x$ when y is a prefix of x .

- (1) For $a \in \Sigma$, we write $[a]$ for the balanced pair $\begin{bmatrix} a \\ a \end{bmatrix}$.
- (2) For $a > b > c$ in $\Sigma_{\perp}$, $[a, b, c]$ denotes the balanced pair $\begin{bmatrix} a \text{ pal}(b) \text{ pal}(c)^{-1} \\ \text{pal}(c)^{-1} \text{ pal}(b) a \end{bmatrix}$. This pair is irreducible since the only occurrences of a are on the top left and bottom right.
- (3) For $a < b < c$ in $\Sigma_{\perp}$, $[a, b, c]$ denotes the balanced pair $\begin{bmatrix} \text{pal}(a)^{-1} \text{ pal}(b) c \\ c \text{ pal}(b) \text{ pal}(a)^{-1} \end{bmatrix}$. This pair is irreducible since the only occurrences of c are on the top right and bottom left.

We consider the following alphabet of irreducible balanced pairs:

$$\Gamma := \{[a] : a \in \Sigma\} \cup \{[a, b, c] \in (\Sigma_{\perp})^3 : a > b > c \text{ or } a < b < c\}.$$

We extend the definition of iterated palindromic closure to a map $\text{pal} : \Sigma \times \Sigma \rightarrow \Gamma^*$, defined inductively by $\text{pal}(a, a) := \varepsilon$ and

$$\text{pal}(a, b) := \begin{cases} \text{pal}(a-1, b) [a, b, -1] \text{pal}(a-1, b) & (a > b), \\ \text{pal}(a, b-1) [-1, a, b] \text{pal}(a, b-1) & (a < b). \end{cases}$$

We use this map to characterise the lifting of φ to a morphism $\tilde{\varphi} : \Gamma^+ \rightarrow \Gamma^+$ as follows:

$$\begin{aligned}\tilde{\varphi}([a]) &:= \begin{cases} [0] [a+1] & \text{if } a < k-1, \\ [0] & \text{if } a = k-1, \end{cases} \\ \tilde{\varphi}([a, b, c]) &:= \begin{cases} [0] [a+1, b+1, c+1] & \text{if } a < k-1, \\ [0] \text{pal}(c+1, b+1) & \text{if } a = k-1, \end{cases} \quad (a > b > c) \\ \tilde{\varphi}([a, b, c]) &:= \begin{cases} [0] [a+1, b+1, c+1] & \text{if } c < k-1, \\ [0] \text{pal}(c+1, b+1) & \text{if } c = k-1, \end{cases} \quad (a < b < c).\end{aligned}$$

The correctness of the above characterisation can be shown by direct calculation using the facts that $\varphi(\text{pal}(a)) = \text{pal}(a+1)0^{-1}$ for $a \in \{0, \dots, k-2\}$ and $\varphi(\text{pal}(-1)) = \varepsilon = \text{pal}(-1)$. For example, for $k-1 > a > b > c \geq 0$, we have

$$\begin{aligned}\varphi(a \text{pal}(b) \text{pal}(c)^{-1}) &= 0 (a+1) \text{pal}(b+1) 0^{-1} 0 \text{pal}(c+1)^{-1} \\ &= 0 (a+1) \text{pal}(b+1) \text{pal}(c+1)^{-1}\end{aligned}$$

and

$$\begin{aligned}\varphi(\text{pal}(c)^{-1} \text{pal}(b) a) &= 0 \text{pal}(c+1)^{-1} 0 0^{-1} \text{pal}(b+1) (a+1) \\ &= 0 \text{pal}(c+1)^{-1} \text{pal}(b+1)(a+1).\end{aligned}$$

We conclude that $\tilde{\varphi}([a, b, c]) = [0] [a+1, b+1, c+1]$. The other cases follow by similar reasoning.

The construction above shows that the balanced-pair algorithm terminates with coincidence for the k -Bonacci morphism. Using Lemma 4.4, we conclude that the k -Bonacci word is echoing. The rest of the proof is dedicated to establishing the non-vanishing condition.

The incidence graph $\mathcal{G}_{\tilde{\varphi}}$ of $\tilde{\varphi}$ has set of vertices the set $\Gamma_0 := \Gamma \setminus \{[a] : a \in \Sigma\}$ comprising the mismatch symbols $[a, b, c]$ in Γ . There is a directed edge from $[a, b, c]$ to $[a', b', c']$ if and only if $[a', b', c']$ appears in the string $\tilde{\varphi}([a, b, c])$.

PROPOSITION 7.1. *The incidence graph $\mathcal{G}_{\tilde{\varphi}}$ has a unique cycle cover.*

Proof. We specify a cycle cover $\mathcal{C}$ of $\mathcal{G}_{\tilde{\varphi}}$ by defining a function $s : \Gamma_0 \rightarrow \Gamma_0$ that determines an outgoing edge from every vertex. Specifically, for $a > b > c$, we have

$$s([a, b, c]) := \begin{cases} [a+1, b+1, c+1], & a < k-1, \\ [-1, c+1, b+1], & a = k-1, \end{cases}$$

and for $a < b < c$, we have

$$s([a, b, c]) := \begin{cases} [a+1, b+1, c+1], & c < k-1, \\ [b+1, a+1, -1], & c = k-1. \end{cases}$$

Informally, whenever there is a choice of an outgoing edge from a node $[a, b, c]$ in $\mathcal{G}_{\tilde{\varphi}}$, the function s selects the edge leading to the central letter of the string $\tilde{\varphi}([a, b, c])$.

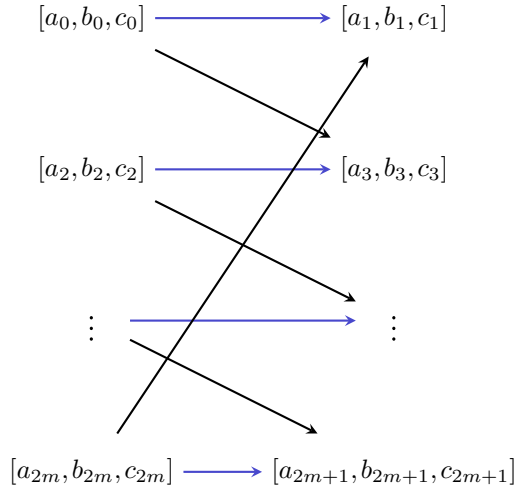


FIGURE 1. An alternating cycle for a cycle cover $\mathcal{C}$. The horizontal edges lie in $\mathcal{C}$ and the non-horizontal edges are not in $\mathcal{C}$. Removing the horizontal edges from $\mathcal{C}$ and replacing them by the non-horizontal edges yields a new cycle cover. However, if $\mathcal{C}'$ is a cycle cover other than $\mathcal{C}$, then the symmetric difference of the respective sets of edges in $\mathcal{C}$ and $\mathcal{C}'$ can be partitioned into alternating paths of the above form.

It can be seen by direct calculation that $\mathcal{C}$ is a cycle cover. Indeed, $\mathcal{C}$ is a union of cycles of the following form, where $a > b > -1$:

$$\begin{aligned}
 [a, b, -1] &\xrightarrow{s^{k-a}} [-1, k-a-1, k+b-a] \xrightarrow{s^{a-b}} [k-b-1, a-b-1, -1] \xrightarrow{s^{b+1}} [-1, b, a] \\
 &\xrightarrow{s^{k-a}} [k+b-a, k-a-1, -1] \xrightarrow{s^{a-b}} [-1, a-b-1, k-b-1] \xrightarrow{s^{b+1}} [a, b, -1].
 \end{aligned}$$

We now show that $\mathcal{C}$ is the unique cycle cover of $\mathcal{G}_{\tilde{\varphi}}$. To this end, recall that a $\mathcal{C}$ -alternating cycle is a bipartite graph $\mathcal{A} = (U, V, E)$, where U and V are disjoint sets of vertices of $\mathcal{G}_{\tilde{\varphi}}$ with $|U| = |V| \geq 2$, $E \subseteq U \times V$ is a set of edges of $\mathcal{G}_{\tilde{\varphi}}$ and each vertex of $\mathcal{A}$ is incident to two edges in E , of which one lies in $\mathcal{C}$. By Berge's theorem [23, Ch. 2], the uniqueness of $\mathcal{C}$ as a cycle cover is equivalent to the non-existence of a $\mathcal{C}$ -alternating cycle; see Figure 1 for a representation of an alternating cycle.

The nodes in $\mathcal{G}_{\tilde{\varphi}}$ that have outdegree at least two have the form $[k-1, a, b]$ or $[a, b, k-1]$. Hence, for any bipartite subgraph of the form shown in Figure 1, there are two cases. The first case is that $a_0 = a_2 = \dots = a_{2m} = k-1$ and $a_1 = a_3 = \dots = a_{2m+1} = -1$. Since the blue edges point to the central factor of an iterated palindromic closure, we have $c_1 > c_3 > \dots > c_{2m+1} > c_1$, which is a contradiction. The second case is that $c_0 = c_2 = \dots = c_{2m} = k-1$ and $c_1 = c_3 = \dots = c_{2m+1} = -1$. Then, we have $a_1 > a_3 > \dots > a_{2m+1} > a_1$, which is again a contradiction. We conclude that an alternating cycle of the form shown in Figure 1 cannot exist and, hence, the cycle cover $\mathcal{C}$ is unique. $\square$

7.2. Strong echoing condition. Write $\Gamma_0 := \{a_0, \dots, a_m\}$ for the set of mismatch symbols in Γ . For $i \in \{0, \dots, m\}$, define $Q_{n,i}(x) \in \mathbb{Z}[x]$ by

$$Q_{n,i}(x) := \sum_{k=0}^{\ell-1} (\pi_1(\tilde{\varphi}^n(a_i))_k - \pi_2(\tilde{\varphi}^n(a_i))_k) x^k.$$

Using Proposition 7.1, we prove the following result.

PROPOSITION 7.2. *The vector $\mathbf{Q}_n := (Q_{n,0}, \dots, Q_{n,m}) \in \mathbb{Z}[x]^{m+1}$ is such that $\mathbf{Q}_n(\beta^{-1}) \neq \mathbf{0}$ for all $\beta \in \mathbb{C} \setminus \{0, 1\}$.*

Proof. Without loss of generality, assume that $a_0 \in \Gamma$ is the matched pair $(01, 10)$. Then, $Q_{0,0}(x)$ is the polynomial $-1 + x$. Hence, $\mathbf{Q}_0(\beta) \neq \mathbf{0}$ for $\beta \neq 1$.

We next describe a recurrence $\mathbf{Q}_{n+1} = M_n \mathbf{Q}_n$, where $M_n \in \mathbb{Z}[x]^{(m+1) \times (m+1)}$. The matrix entries $(M_n)_{i,j}$ are defined as follows. Suppose $\tilde{\varphi}(a_i) = a_{j_1} \cdots a_{j_s}$. Then, $\tilde{\varphi}^{n+1}(a_i) = \tilde{\varphi}^n(a_{j_1}) \cdots \tilde{\varphi}^n(a_{j_s})$. This equation allows us to write the polynomial $Q_{n+1,i}(x)$ as a linear combination of the polynomials $Q_{n,0}(x), \dots, Q_{n,m}(x)$. Specifically, we have

$$(M_n)_{i,j} := \sum_{k:j_k=j} x^{f(k)} \quad \text{where } f(k) = |\pi_1(\tilde{\varphi}^n(a_{j_1}) \cdots \tilde{\varphi}^n(a_{j_{k-1}}))|.$$

Note that $(M_n)_{i,j} \neq 0$ only if there is an edge from a_i to a_j in the incidence graph $\mathcal{G}_{\tilde{\varphi}}$. Thus, we have $\det(M_n) = \prod_{i=0}^m (M_n)_{i,s(i)}$ for the map $s: \{0, \dots, m\} \rightarrow \{0, \dots, m\}$ that specifies the unique cycle cover of $\mathcal{G}_{\tilde{\varphi}}$, as defined in the proof of Proposition 7.1. By definition of s , the symbol $a_{s(i)}$ has a single occurrence in the string $\tilde{\varphi}(a_i)$, namely as the central factor in an iterated palindromic closure. It follows that $M_{i,s(i)}$ is a monomial for all $i \in \{1, \dots, m\}$ and, hence, $\det(M_n)$ is a monomial. Thus, $M_n(\beta^{-1})$ is non-singular for all $\beta \neq 0$.

From recurrence $\mathbf{Q}_{n+1} = M_n \mathbf{Q}_n$ and the facts that $\mathbf{Q}_0(\beta) \neq \mathbf{0}$ for $\beta \neq 1$ and $\det(M_n)(\beta) \neq 0$ for $\beta \neq 0$, we have by induction on $n \in \mathbb{N}$ that $\mathbf{Q}_n(\beta^{-1}) \neq \mathbf{0}$ for all $n \in \mathbb{N}$ and $\beta \in \mathbb{C} \setminus \{0, 1\}$. $\square$

We can now prove the main result of this section. The proof develops the construction in §§4.2 and 6.2 showing that the Tribonacci word is strongly echoing.

Proof of Theorem 7.1. By inspection, the lifting $\tilde{\varphi}$ of the k -Bonacci morphism φ to the set Γ of balanced pairs satisfies the coincidence condition. Furthermore, since φ is primitive, there exists $c_0 > 0$ such that for all $\gamma, \gamma' \in \Gamma$ and all $n \in \mathbb{N}$, we have

$$|\pi_1(\tilde{\varphi}^n(\gamma))| \leq c_0 |\pi_1(\tilde{\varphi}^n(\gamma'))|. \quad (7.1)$$

Let $\mathbf{u} \in \Sigma^\omega$ be the k -Bonacci word and let $\mathbf{w} \in \Gamma^\omega$ be the image of $\mathbf{u}$ under the coding $\iota: \Sigma \rightarrow \Gamma$ given by

$$\iota(0) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \iota(1) := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad \dots, \quad \iota(k-2) := \begin{bmatrix} 0 & k-1 \\ k-1 & 0 \end{bmatrix}, \quad \iota(k-1) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\pi_1 \circ \iota = \varphi$, we have $\pi_1(\mathbf{w}) = \mathbf{u}$. Moreover, since $\sigma \circ \pi_1 \circ \iota$ and $\pi_2 \circ \iota$ coincide on any word on Σ^ω , we have $\pi_2(\mathbf{w}) = \sigma(\mathbf{u})$.

Given $n_0 \in \mathbb{N}$, write

$$\tilde{\varphi}^{n_0}(\mathbf{w}) = w_0 a_{i_1} w_1 a_{i_2} w_2 a_{i_3} \cdots, \quad (7.2)$$

where $w_0, w_1, \dots \in \Gamma^*$ are sequences of coincidence symbols and $a_{i_1}, a_{i_2}, \dots \in \Gamma$ are mismatch symbols. The word $\tilde{\varphi}^{n_0}(\mathbf{w})$ is uniformly recurrent, being the morphic image of the uniformly recurrent word $\mathbf{u}$. Hence, there is a uniform upper bound on the length of the factors w_i , depending only on n_0 . Let $\varepsilon > 0$ be as in the definition of echoing sequence (Definition 6.1). Since $\tilde{\varphi}$ satisfies the coincidence condition, for sufficiently large $n_0 \in \mathbb{N}$, the upper asymptotic density of mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$ is at most ε/c_0 .

The data to show the strong echoing property are as follows. For all $n \in \mathbb{N}$, consider the image of (7.2) under the n -fold application of $\tilde{\varphi}$:

$$\tilde{\varphi}^{n+n_0}(\mathbf{w}) = \tilde{\varphi}^n(w_0) \underbrace{\tilde{\varphi}^n(a_{i_1})}_{I_{n,1}} \tilde{\varphi}^n(w_1) \underbrace{\tilde{\varphi}^n(a_{i_2})}_{I_{n,2}} \tilde{\varphi}^n(w_2) \underbrace{\tilde{\varphi}^n(a_{i_3})}_{I_{n,3}} \cdots \quad (7.3)$$

We define $r_n := 0$ and $s_n := |\varphi^{n+n_0}(0)|$. Then, $\pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \mathbf{u}$ and $\pi_2(\tilde{\varphi}^{n+n_0}(\mathbf{w})) = \sigma^{s_n}(\mathbf{u})$. For all $j \in \{1, 2, \dots\}$, we define $I_{n,j} \subseteq \mathbb{N}$ to be the smallest interval of positions in $\mathbf{u} = \pi_1(\tilde{\varphi}^{n+n_0}(\mathbf{w}))$ that contains all mismatch symbols occurring in the factor $\pi_1(\tilde{\varphi}^n(a_{i_j}))$, as shown in (7.3). By construction, this choice satisfies the covering property. To establish the density property, we note that since the mismatch symbols in $\tilde{\varphi}^{n_0}(\mathbf{w})$ have upper asymptotic density at most ε/c_0 , by (7.1), we have $\text{den}(\bigcup_{j=1}^m I_{n,j}) \leq \varepsilon$ for all sufficiently large m . Finally, the uniform upper bound on the length of the factors w_i and the bound (7.1) imply that $d(I_{n,j}, I_{n,j+1}) \asymp s_n$ for all $j \geq 1$. (Note that this growth bound holds even when the word w_j is empty since $\tilde{\varphi}(a_{i_{j+1}})$ starts with a coincidence.) We have thus established the expanding-gaps property.

It remains to consider the non-vanishing property. To this end, for all $n \in \mathbb{N}$ and $j \in \{1, 2, \dots\}$, define the polynomial $P_{n,j}(x) \in \mathbb{Z}[x]$ by

$$P_{n,j}(x) := \sum_{i \in I_{n,j}} (u_{i+s_n} - u_{i+r_n}) x^i. \quad (7.4)$$

Given $\beta \in \mathbb{C}$ with $|\beta| > 1$, we must show that there exist distinct indices j_0, j_1 such that $P_{n,j_0}(\beta^{-1}) \neq 0$ and $P_{n,j_1}(\beta^{-1}) \neq 0$ for infinitely many $n \in \mathbb{N}$. By Proposition 7.2, for all $n \in \mathbb{N}$, there exists $i \in \{0, \dots, m\}$ such that $Q_{n,i}(\beta^{-1}) \neq 0$. Hence, there exists i^* such that $Q_{n,i^*}(\beta^{-1}) \neq 0$ for infinitely many $n \in \mathbb{N}$. There are arbitrarily large choices of n_0 such that i^* occurs infinitely often in the list $i_1, i_2, i_3, \dots$ in (7.2). For such an n_0 , there exist $j_0 < j_1$ such that $i_{j_0} = i_{j_1} = i^*$. Then, $Q_{n,i_{j_0}}(\beta^{-1}) \neq 0$ and $Q_{n,i_{j_1}}(\beta^{-1}) \neq 0$ for infinitely many n . However, for all $j \in \{1, 2, \dots\}$, $P_{n,j}$ is the product of Q_{n,i_j} and a monomial in x . Hence, $P_{n,j_0}(\beta^{-1}) \neq 0$ and $P_{n,j_1}(\beta^{-1}) \neq 0$ for infinitely many n . $\square$

8. Conclusion

Resolving Pisot's conjecture is an important research objective in dynamical systems [6, 25]. Our main results relate Pisot's conjecture to a version of Cobham's conjecture concerning the transcendence of numbers whose expansions in an algebraic base are given by irreducible Pisot morphisms. A natural direction for further research is to determine classes of morphisms that lead to strongly echoing words, rather than merely echoing

words (for which we would obtain transcendence outright by applying Theorem 6.1, rather than the weaker rational-transcendental dichotomy obtained from Theorem 5.1). In the present work, we have obtained such a result for the class of k -Bonacci morphisms. One can ask whether the strong echoing property holds for the sub-class of Arnoux–Rauzy words whose S -adic expansion is periodic (see [14, Ch. 7] for the notion of S -adic expansion). In this regard, we note that a number whose expansion in an integer base is an Arnoux–Rauzy word is necessarily transcendental [9, 13]. However, [18, §3.1] gives an example of the Episturmian word $\mathbf{u}$ and an algebraic base β such that the non-vanishing condition fails and the number $\llbracket \mathbf{u} \rrbracket_\beta$ belongs to $\mathbb{Q}(\beta)$.

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