On Periodic and Aperiodic Optimal Strategies in Solvency Games

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Abstract

Solvency games are a gambling problem on infinite-state MDPs where the investor's fortune $n \in \mathbb{N}$ is the state. In every round, the investor chooses an action from a finite action set, and every action yields a distribution over integer-valued gains in an interval $\{-\ell, \ldots, m\}$. The risk-averse investor wants to minimise the probability of eventual ruin (reaching a fortune ≤ 0).

It was shown in [2] that memoryless deterministic optimal strategies exist, but they do not have a pure tail in general. Even in the special case of gains in $\{-2, \ldots, 1\}$, the optimal strategy may need to make use of two different actions at arbitrarily high fortunes.

We show that optimal strategies in solvency games need to be aperiodic in general (thus disproving a 2012 conjecture of Kučera [10, Sec. 3]). Already in the case of gains in $\{-3, \ldots, 1\}$, it is possible for the optimal strategy to be unique but aperiodic.

However, in the special case of gains in $\{-2, \ldots, 1\}$ there always exists an optimal strategy that either has a pure tail or *strictly alternates* between just two actions.

Finally, we show that the optimal strategy is computable if it is unique. Moreover, (some) optimal strategy can always be computed in the case of gains in $\{-\ell, \ldots, 1\}$ for any $\ell \in \mathbb{N}$.

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1 Introduction

Background. Decision making under uncertainty is a fundamental problem studied in computer science, operations research, and game theory. We study solvency games, a very simple model in which a single player (aka the investor) wants to maximise the probability of perpetual solvency, i.e., to minimise the probability of eventual ruin. Even in this very simple setting, optimal strategies may need to be rather complex.

A solvency game can be described as a gambling problem on an infinite-state MDP. The states of this MDP are numbers $n \in \mathbb{N}$ which correspond to the current fortune of the investor. In every round of the game, the investor chooses an action from a finite action set $\mathcal{A} = \{A, B, \ldots\}$. Each action yields a distribution over integer-valued gains in a bounded interval $\{-\ell, \ldots, m\}$. The new fortune is then the old fortune plus the gain. The risk-averse investor wants to minimise the probability of ruin (eventually reaching a fortune ≤ 0).

There always exist optimal deterministic strategies that depend only on the current fortune (i.e., they are memoryless) [2]. Such strategies can be described by functions $\sigma : \mathbb{N} \to \mathcal{A}$. While having a large fortune can reduce the risk of ruin, maximizing the expected fortune in the long run does **not** coincide with minimizing the risk of eventual ruin. This is easy to see for very small fortunes, e.g., $n = 1$. Suppose action A yields gains $+10$ and -1 with probabilities $1/2$ each, while action B yields gains $+1$ and -1 with probabilities 0.6 and 0.4, resp. While action A has a higher expected gain, the risk-averse investor still needs to play action B at fortune $n = 1$ in order to minimise the probability of ruin.

Still, it seemed plausible that, at least at fortunes n **above** some finite threshold n_0, an optimal strategy σ could always play an action with maximal expected gain, i.e., $\sigma(n) = A$ for all $n \geq n_0$. Such strategies are also called **pure-tail** or **rich-man's** strategies. It is easy to see that there always exist ε-optimal pure-tail strategies for every $\varepsilon > 0$. If at least one action A yields a strictly positive expected gain then some strategy that plays A at all fortunes $n \geq n_0$ (where n_0 depends on ε) can be ε-optimal everywhere, since the risk of eventual ruin from fortune n converges to 0 as $n \to \infty$ [2, Fact 3]. Otherwise, if all actions have an expected gain ≤ 0, then eventual ruin happens almost surely for every strategy (except in the degenerate case where some action has exactly gain 0 only). For a quantitative analysis of these approximation properties see [3].

However, optimal strategies do not have the same properties as ε-optimal strategies. In particular, optimal pure-tail strategies do not always exist [2]. Surprisingly, even in the special case of two actions and gains in $\{-2, \ldots, 1\}$, the optimal strategy may need to use both actions at arbitrarily high fortunes, even though one action has a sub-optimal expected gain. The counterexample from [2] does not show whether the two actions appear in any regular pattern for increasing fortunes n in the optimal strategy. It was conjectured in [10, Sec. 3] that there always exist optimal strategies that are ultimately periodic, i.e., eventually repeat a finite pattern of actions for increasing fortunes n.

Our contributions. Our main contribution is to disprove this conjecture. Already in the case of gains in $\{-3, \ldots, 1\}$ and action sets of size two, optimal strategies may need to be much more complex. In Section 4 we construct an example where the optimal strategy is unique but aperiodic.

However, in the special case of gains in $\{-2, \ldots, 1\}$ there always exists an optimal strategy that either has a pure tail or **strictly alternates** between just two actions (regardless of the size of the action set). The first type is ultimately periodic and the second type is periodic (with period 2) everywhere; cf. Section 3.4.

Finally, we present some results about the computability of optimal strategies, i.e., finding a computable function $\sigma : \mathbb{N} \to \mathcal{A}$ that corresponds to a memoryless deterministic optimal strategy for a given solvency game. In the special case where this optimal strategy σ is unique, $\sigma(n)$ is computable. This is due to the fact that, for every n, the value of the solvency objective at fortune n can be effectively approximated arbitrarily closely [3]. However, this method does not yield any complexity bounds.

Otherwise, even if the memoryless deterministic optimal strategy is not unique, it is possible to find some optimal strategy $\sigma : \mathbb{N} \to \mathcal{A}$ such that $\sigma(n)$ is computable for every $n \in \mathbb{N}$, provided that the gains are in $\{-\ell, \ldots, 1\}$ for some $\ell \in \mathbb{N}$. Here $\sigma(n)$ can be computed in time polynomial in n.

Computability of an optimal strategy remains an open question in the general case with gains in $\{-\ell, \ldots, m\}$ for $m > 1$.

Related work. Many works in gambling theory also consider the problem of minimizing the risk of ruin, e.g., [7, 8, 13, 6, 9]. However, their models differ from ours. Instead of choosing between different actions, their investor chooses which part of his or her fortune to gamble on a fixed action. Additionally, some other models allow borrowing or pay interest on unused capital [4].

More closely related to our work are finite-state Markov decision processes with integer rewards [12, 11]. One can interpret the total reward (the sum of all rewards so far in the run) as the investor's fortune. Alternatively, one can consider the fortune as part of the (now infinite) state space of an MDP. In other words, a state of the MDP is described by a pair (s, n) where s is one of finitely many control states and $n \in \mathbb{N}$ is the fortune. Using terminology from automata theory, this is called a **one-counter MDP** [3]. This model is strictly more general than ours, due to the additional control states, i.e., our model corresponds to the subclass of one-counter MDPs with just one control state.

2 Definition of Solvency Games

▶ **Definition 1.** *Given $\ell, m \in \mathbb{N}$, an (ℓ, m)-solvency game is an infinite-state Markov decision process (MDP) with state space $\mathbb{N}$ and a finite set of actions $\mathcal{A}$ such that each action $A \in \mathcal{A}$ yields a probability distribution $\mathbb{P}_A$ over the finite set $\{-\ell, \ldots, m\}$.*

*The current state $n \in \mathbb{N}$ is called the **fortune** of the player and the values $k \in \{-\ell, \ldots, m\}$ are the possible **gains** in any round of the game (where gains can be negative). $\mathbb{P}_A(k)$ denotes the probability of gain k under action A, hence $\sum_{-\ell \leq k \leq m} \mathbb{P}_A(k) = 1$ for any $A \in \mathcal{A}$.*

Since the state space of solvency games is $\mathbb{N}$, a history can be described by a sequence $h \in \mathbb{N}(\mathcal{A}\mathbb{N})^*$ where the first element in h corresponds to the initial fortune. A **strategy** is a function σ that maps a history h to a mixed action, i.e., a distribution over $\mathcal{A}$. Let $h = h'n$ be a history ending with fortune n. Then the next fortune $n+k$ is determined by adding the realized gain k under the chosen mixed action to n, i.e., $\mathbb{P}(h'nA(n+k) \mid h'n) = \sigma(h'n)(A) * \mathbb{P}_A(k)$.

In solvency games the player has the objective to maximise the probability of perpetual solvency, i.e., keep the fortune > 0. Formally, solvency is the set of runs $\mathbb{N}_{>0}(\mathcal{A}\mathbb{N}_{>0})^\omega$ (a closed set in the Cantor topology and thus Borel measurable). Equivalently, the player aims to minimise the probability of eventual ruin (reaching a fortune ≤ 0). For the solvency objective there always exist optimal **memoryless** and **deterministic** strategies [2], i.e., where $\sigma(h'n)$ depends only on n and $\sigma(h'n)$ is a Dirac distribution. Thus we restrict our attention to memoryless deterministic strategies, which can then be described by simple functions $\sigma : \mathbb{N} \to \mathcal{A}$.

A solvency game, initial fortune $n \in \mathbb{N}$, and memoryless deterministic strategy σ induce an infinite-state Markov chain where the current state corresponds to a random variable X_i, where $X_0 = n$ and $X_{i+1} = X_i + Y_i$, where Y_i has distribution identical to the action $\sigma(X_i)$. The probability of eventual ruin in this Markov chain is

$$p^\sigma(n) := \mathbb{P}(\exists i \in \mathbb{N} \cdot X_i \leq 0)\,.$$

Let $p_x^\sigma(n) := \mathbb{P}(\exists i \in \mathbb{N}\ X_i = x \wedge \forall j < i\ X_j > 0)$, be the probability under strategy σ that the fortune changes from n to x (without hitting zero in between).

A strategy σ is **optimal** if for every other strategy τ we have $p^\sigma(n) \leq p^\tau(n)$ for all $n \in \mathbb{N}$. The existence of optimal strategies in solvency games is shown in [2, Proposition 7]. For an optimal strategy σ we write $p_{\mathrm{opt}}(n) := p^\sigma(n)$ for the probability of eventual ruin given initial fortune n.

Assumptions. We assume without loss of generality that no two actions A and B have identical distributions $\mathbb{P}_A$ and $\mathbb{P}_B$ over payoffs. Furthermore, if there exists an action A such that $\mathbb{P}$ is supported on a set of nonnegative integers then solvency would be trivial to achieve by always playing A. Thus, we henceforth assume that this is not the case. We also assume that $\mathbb{P}_A(0) = 0$ for every action A. This is also without loss of generality, since otherwise A could be replaced by an 'equivalent' action A' (i.e., without changing the probability of eventual ruin) with $\mathbb{P}_{A'}(0) = 0$ and $\mathbb{P}_{A'}(k) = \mathbb{P}_A(k)/(1 - \mathbb{P}_A(0))$ for all $k \neq 0$.

Finally, we assume that there exists at least one action $A \in \mathcal{A}$ with strictly positive expected gain, i.e., $\sum_{-\ell \leq k \leq m} k\mathbb{P}_A(k) > 0$. Otherwise, ruin would almost surely happen from every fortune under every strategy, and hence all strategies would be optimal. This assumption also implies that $\lim_{n\to\infty} P_{\mathrm{opt}}(n) = 0$ [2, Fact 3].

Following [2, Lemma 1], we define the **characteristic polynomial** for each action. (Our version below uses an additional scaling factor of $1/\mathbb{P}_A(m)$.)

▶ **Definition 2.** *Let $A \in \mathcal{A}$ be an action yielding a distribution over gains in $\{-\ell, \dots, m\}$. Its* ***characteristic polynomial*** *is defined as*

$$\chi_A^{\ell,m}(x) := \frac{1}{\mathbb{P}_A(m)} \left(\sum_{k=-\ell}^{m} \mathbb{P}_A(k) x^{k+\ell} - x^\ell \right)$$

When the context is clear, we may omit to mention ℓ and m and just write $\chi_A(x)$ instead.

By construction, χ_A has leading coefficient 1. Moreover, since $\mathbb{P}_A$ is a distribution, $\chi_A(1) = 0$.

▶ **Proposition 3** ([2, Lemma 1])**.** *For every action A, χ_A has exactly one root $c_A \in (0,1)$, called its* ***primary root****. Moreover, for any complex root r distinct from 1 and c_A, $|r| < c_A$.*

The roots of the characteristic polynomial χ_A of action A which are neither the primary root nor 1 are called **secondary roots**.

The following theorem showed that an optimal pure-tail strategy exists under a certain condition on the primary roots of the characteristic polynomials.

▶ **Theorem 4** ([2, Theorem 8])**.** *Consider a solvency game with action set $\mathcal{A}$ where there exists an action $A \in \mathcal{A}$ such that the primary root c_A of χ_A satisfies $c_A < c_B$ for the primary root c_B of every action $B \neq A$.*

Then there exists an optimal pure-tail strategy $\sigma : \mathbb{N} \to \mathcal{A}$, i.e., there exists an n_0 such that $\sigma(n) = A$ for all $n \geq n_0$.

However, it was also shown in [2] that an optimal pure-tail strategy need not exist in general, even for solvency games with just two actions and gains in $\{-2, \dots, 1\}$.

▶ **Theorem 5** ([2, Theorem 9]). *There exists a* $(2,1)$*-solvency game with two actions* $\mathcal{A} = \{A, B\}$ *such that* $c_A = c_B$ *and if a strategy* $\sigma : \mathbb{N} \to \mathcal{A}$ *is optimal then for every* $W \in \mathbb{N}$ *there exist* $n, n' > W$ *such that* $\sigma(n) = A$ *and* $\sigma(n') = B$*.*

This theorem showed that both actions are needed infinitely often, but did not show whether σ (eventually) strictly alternates between A and B. An example of a $(2,1)$-solvency game where the optimal strategy strictly alternates between actions A and B was described in [5]. In Section 3 we show that all $(2,1)$-solvency games admit optimal strategies that either strictly alternate or are a pure-tail strategy.

Generally, it was open whether there always exists an optimal strategy that is ultimately periodic, i.e., eventually repeats a finite pattern of actions indefinitely for fortune $n \to \infty$; cf. [10, Sec. 3]. We answer this negatively, even for $(3,1)$-solvency games, in Section 4.

3 Computability Results

In this section, we focus on the class of $(\ell, 1)$-solvency games and provide an efficient way to compute optimal strategies for such games. We will assume that all action share the same primary root, c. This is because of Theorem 4: if there were a strategy with a smaller primary root, an optimal strategy would always choose this action for all sufficiently large fortunes.

In the later subsections, we investigate some special cases on which we can say even more about the optimal strategies (namely $(2,1)$-games and $(3,1)$-games).

3.1 Computing the optimal strategy knowing it is unique

It is shown in [3, Theorem 3.1] that the probability of eventual ruin in a solvency game can be effectively approximated—there is a procedure that inputs a solvency game $\mathcal{A}$, initial fortune $n \in \mathbb{N}$, and precision $\varepsilon > 0$, and outputs a rational number q such that $|p_{\mathrm{opt}}(n) - q| < \varepsilon$. (In fact, this approximation works even for the more general model of one-counter MDPs, i.e., with control states.) From this it easily follows that the optimal strategy is computable, provided that it is unique. More precisely, there is a procedure that inputs a solvency game $\mathcal{A}$ and initial fortune $n \in \mathbb{N}$ and that, under the promise that $\mathcal{A}$ has a unique optimal memoryless deterministic strategy $\sigma : \mathbb{N} \to \mathcal{A}$, outputs $\sigma(n)$. Indeed, it is classical [12, Section 4.3] that, being the probability of a finite-horizon objective, p_{opt} satisfies the so-called Bellman optimality equation

$$p_{\mathrm{opt}}(n) = \min_{A \in \mathcal{A}} \sum_k \mathbb{P}_A(k) \cdot p_{\mathrm{opt}}(n+k)\,.$$

Then, by uniqueness of σ, the minimum on the right-hand side of the above equation is achieved for a single action $A \in \mathcal{A}$. To determine this action we find the (unique) minimum summand on the right-hand side. This can be done by computing the values $p_{\mathrm{opt}}(n+k)$, for $k \in \{-\ell, \dots, m\}$, to sufficient precision using the above-mentioned procedure of [3]. Note that the running time of the above procedure depends on the precision required to distinguish the optimal action from given fortune n from the next best action(s), for which we have no *a priori* bound.

3.2 Characteristic polynomials in an $(\ell, 1)$-solvency game

In this first subsection, we make explicit the formulas linking the roots of the characteristic polynomials and the probabilities of the actions. Let $\mathcal{A}$ be an $(\ell, 1)$-solvency game and $A \in \mathcal{A}$ be an action. Recall that all actions of $\mathcal{A}$ share the same primary root $c \in (0, 1)$. Then A has characteristic polynomial

$$\chi_A(x) = (x-1)(x-c)\prod_{i=1}^{\ell-1}(x - r_{A,i})$$

where the $r_{A,i}$s are the remaining complex roots of χ_A. We denote by $r_A = (r_{A,1}, \ldots, r_{A,\ell-1})$ the vector of all these roots. Using Vieta's formulas [1, Chapter 4, Section 4.6] we have:

$$\mathbb{P}_A(1) = \frac{1}{1 + c + s_1(r_A)}$$

and

$$\mathbb{P}_A(-k) = (-1)^{k+1}\frac{s_{k+1}(r_A) - (c+1)s_k(r_A) + cs_{k-1}(r_A)}{1 + c + s_1(r_A)} \quad \text{for } k \in \{1, \ldots, \ell\}$$

where the k-th elementary symmetric polynomial $s_k(r_A)$ is defined by

- $s_0(r_A) := 1$;
- For all $k \in \{1, \ldots, \ell - 1\}$, $s_k(r_A) := \sum_{1 \le i_1 < \cdots < i_k \le \ell-1} r_{A,i_1} \cdots r_{A,i_k}$;
- For $k \geq \ell$, $s_k(r_A) := 0$.

Among all these sums, we give a special name to the sum of all roots $s_A := s_1(r_A)$ and the product of all roots $p_A := s_{\ell-1}(r_A)$.

3.3 Computing optimal strategies for $(\ell, 1)$-games

We first give a simple but useful characterisation of optimal strategies in $(\ell, 1)$-solvency games. Recall that $p_x^\sigma(n)$ denotes the probability under σ that the fortune changes from n to x, while $p^\sigma(n)$ denotes the probability of ruin from fortune n. Finally, $p_{\mathrm{opt}}(n)$ denotes the minimal probability of ruin from n. The following lemma is shown in Section A.1.

▶ **Lemma 6.** *Let $\mathcal{A}$ be an $(\ell, 1)$-solvency game. A memoryless deterministic strategy σ is optimal if and only if it maximises $p_{w+1}^\sigma(n)$ for all $n < w + 1$.*

Lemma 6 does not hold for (ℓ, m)-solvency games with $m > 1$, because the investor might skip over certain fortunes x on the way up. In this case, it matters by how much x is skipped, e.g., $(x - 1) \to (x + 3)$ vs. $(x - 1) \to (x + 7)$, and it does not suffice to just maximise the probability of eventually surpassing x.

We have

$$\forall w \in \{1, \ldots, x-1\} \quad p_x^\sigma(w) = \sum_{k=-\ell}^{1} \mathbb{P}_{\sigma(w)}(k)p_x^\sigma(w+k)$$

Now for a word $W \in \mathcal{A}^*$ we say that a strategy σ **agrees** with W if for all $k \in \{1, \ldots, |W|\}$, $\sigma(k) = W_k$. Note that for any strategies σ, τ that agree with W we have

$$\forall w \in \{1, \ldots, |W|\} \qquad p_{|W|+1}^\sigma(w) = p_{|W|+1}^\tau(w)$$

Therefore, for $w \in \{1, \ldots, |W| + 1\}$, we simply denote $p^W(w) = p_{|W|+1}^\sigma(w)$ where σ is any strategy that agrees with W. Note also that $p^W(w) \neq 0$ since we assume that the action have positive drift hence non-zero probability of going up by 1.

▶ **Notation.** *For $W \in \mathcal{A}^*$ and $k \leq |W|$, we denote by $W[1:k]$ the prefix of length k of W. In particular, the zero-length prefix $W[1:0]$ is the empty word ε. To handle "negative-length" prefixes we define, for $k < 0$, $W[1:k] := \bot$ for some symbol $\bot \notin \mathcal{A}^*$.*

▶ **Proposition 7.** *(cf. Section A.2) For all word $W \in \mathcal{A}^+$ and $i \in \{1, \dots, |W|\}$, $p^W(i) = \frac{Q_{W[1:i-1]}}{Q_W}$ where Q_W is defined as follows:*

- *$Q_\bot := 0$ and $Q_\varepsilon := 1$.*
- *For $A \in \mathcal{A}$, $Q_A := \frac{1}{\mathbb{P}_A(1)}$.*
- *For $A \in \mathcal{A}$ and $W \in \mathcal{A}^+$,*

$$Q_{WA} = \frac{1}{\mathbb{P}_A(1)} \left(Q_W - \sum_{k=-\ell}^{-1} \mathbb{P}_A(k) Q_{W[1:|W|+k]} \right)$$

This proposition shows that the probability of reaching $|W| + 1$ playing the word W and starting from i is a quotient of polynomials (in the probabilities of the actions) whose numerator has a regular form and whose denominator does not depend on i.

▶ **Lemma 8.** *For all $W \in \mathcal{A}^*$, Q_W is positive.*

Proof. This can be proven by induction. $Q_\varepsilon = 1 > 0$. Since, for $W \in \mathcal{A}^*$ and $A \in \mathcal{A}$, $p^{WA}(i)$ is a probability, by induction, $Q_{W[1:i]}$ being positive, it forces Q_W to be positive too. ◀

▶ **Proposition 9.** *Let $\mathcal{A}$ be an $(\ell, 1)$-solvency game. Then there is a computable optimal strategy σ for $\mathcal{A}$ and for all $n \in \mathbb{N}^*$, $\sigma(n)$ is computable in time linear with n in terms of operations over $\mathbb{Q}$.*

Proof. Assume we have computed σ for $1, \dots, n-1$ and that it agrees with the word $W = W_1 \dots W_{n-1}$. Using Lemma 6, to compute $\sigma(n)$, we just have to choose an action $A \in \mathcal{A}$ that maximises the probability $p^{WA}(n)$. Using Proposition 7, we have

$$p^{WA}(n) = \frac{Q_W}{Q_{WA}}$$

Having already computed $Q_W, \cdots, Q_{W[1:|W|-\ell]}$ we just need to compute all the Q_{WA} for $A \in \mathcal{A}$, which is possible by the definition of Q_{WA} and just pick the action corresponding to the lowest Q_{WA}. This leads to the computation of $|\mathcal{A}|n$ quantities to compute $\sigma(n)$. ◀

This computation method leads to the introduction of the following quantity:

▶ **Notation.** $\Delta_W^{A,B} := Q_{WA} - Q_{WB}$

The point of $\Delta_W^{A,B}$ is to say that assuming we play a strategy that agrees with W, at wealth $|W| + 1$ we should play A instead of B if and only if $\Delta_W^{A,B} < 0$. If $\Delta_W^{A,B} > 0$ then we should prefer B over A. Unfolding the probabilities in the quantities Q_W, the shared primary root c appear in their expressions. It is actually possible to consider a modified version of it to get rid of all the c's.

▶ **Notation.** *For $W \in \mathcal{A}^*$, we denote $R_W = Q_W - (c+1)Q_{W[1:|W|-1]} + cQ_{W[1:|W|-2]}$ where c is the primary root shared by all the actions of $\mathcal{A}$.*

An immediate consequence is that $R_{WA} - R_{WB} = Q_{WA} - Q_{WB} = \Delta_W^{A,B}$. Also we can note that $R_\varepsilon = 1$ and $R_\bot = 0$.

▶ **Lemma 10.** *(cf. Section A.3) For all $W \in \mathcal{A}^*$ and for all action $X \in \mathcal{A}$,*

$$R_{WX} = \sum_{k=1}^{\ell-1} (-1)^{k+1} s_k(r_X) R_{W[1:|W|-k+1]}$$

3.4 $(2,1)$-solvency games

Let us consider a $(2,1)$-solvency game $\mathcal{A}$. For any $A \in \mathcal{A}$, χ_A has degree 3. Since, by Proposition 3, it has two roots in $(0,1]$, all its roots must be real. Moreover, the last root must be negative. We know by Theorem 4 that if one action A has a smaller primary root than all the other actions, there is an optimal pure-tail strategy with an infinite tail that selects action A. In this section, we consider the complementary case where all the actions of $\mathcal{A}$ share the same primary root: c. We will show that in this situation, the optimal strategy is exactly an alternation between two actions.

Note that in this case, the vector of remaining roots, r_A, consists in a single entry, $r_{A,1}$. Let $A_{\min}, A_{\max} \in \mathcal{A}$ be such that $r_{A_{\min},1} := \min\{\, r_{A,1} \mid A \in \mathcal{A}\}$ and $r_{A_{\max},1} := \max\{\, r_{A,1} \mid A \in \mathcal{A}\}$ respectively. We show the following:

▶ **Theorem 11.** *The strategy* $\sigma(n) := \begin{cases} A_{\min} & n \in 2\mathbb{N}+1 \\ A_{\max} & n \in 2\mathbb{N} \end{cases}$ *is the unique optimal strategy for the solvency game* $\mathcal{A}$.

Proof. Using Lemma 10, we have

$$R_{WX} = \sum_{k=1}^{1} (-1)^{k+1} s_k(r_X) R_{W[1:|W|-k+1]} = s_1(r_X) R_W$$

This means that

$$\Delta_W^{A,B} = (s_1(r_A) - s_1(r_B)) R_W = (r_A - r_B) R_W$$

We now prove by induction that the announced strategy is the unique optimal strategy.

- At $n = 1$, we need to take the action A that minimises $Q_A = 1 + c + r_A$. This action must be $A_{\min}$. Therefore, $\sigma(1) = A_{\min}$.
- Assume that we have shown that an optimal strategy σ agrees with the word $W = A_{\min} A_{\max} A_{\min} \dots$ of length $n-1$. By Proposition 9, $\sigma(n)$ must be the action that minimises $Q_{W\sigma(n)}$. Let $A, B \in \mathcal{A}$ any two distinct actions such that $r_A < r_B < 0$. Using the equation satisfied by R_W, we have

  $$R_W = r_{A_{\min}}^{\lceil \frac{n-1}{2} \rceil} r_{A_{\min}}^{\lfloor \frac{n-1}{2} \rfloor}$$

  which is negative if $n \in 2\mathbb{N}$ and positive if $n \in 2\mathbb{N}+1$. In particular if $n \in 2\mathbb{N}$, for all actions A distinct from $A_{\max}$, $\Delta_W^{A,A_{\max}} > 0$ meaning that $Q_{WA_{\max}} < Q_{WA}$ and thus we must have $\sigma(n) = A_{\max}$. Similarly, if $n \in 2\mathbb{N}+1$, for all actions A distinct from $A_{\min}$, $\Delta_W^{A_{\min},A} < 0$ meaning that $Q_{WA_{\min}} < Q_{WA}$ and thus we must have $\sigma(n) = A_{\min}$. ◀

3.5 $(3,1)$-solvency games

Let us consider a $(3,1)$-solvency game $\mathcal{A}$. Using Lemma 10, we have for all $W \in \mathcal{A}^*$ and $X \in \mathcal{A}$,

$$R_{WX} = \sum_{k=1}^{2} (-1)^{k+1} s_k(r_X) R_{W[1:|W|-k+1]} = s_1(r_X) R_W - s_2(r_X) R_{W[1:|W|-1]} \tag{R3}$$

This means that

$$\Delta_W^{A,B} = (s_1(r_A) - s_1(r_B)) R_W - (s_2(r_A) - s_2(r_B)) R_{W[1:|W|-1]}$$

This in particular shows that R_W can be computed by iterating a 2×2 matrix and thus $\Delta_W^{A,B}$ by iterating 3×3 matrix. In this section we will show that we can actually bring this down to a 2×2 matrix. In a later section, we will actually use this to find example of optimal strategies that must be aperiodic.

For brevity we denote for any action X, $s_X = s_1(r_X)$ the sum of its secondary roots and $p_X = s_2(r_X)$ their product. Note that for any two distinct actions A, B we must have $s_A \neq s_B$ or $p_A \neq p_B$. If both were equal, the payoff distributions $\mathbb{P}_A$ and $\mathbb{P}_B$ would be identical, which is assumed not to be the case. We now consider two distinct actions A and B.

Case $s_A \neq s_B$.

▶ **Lemma 12.** *For all actions $A, B, X \in \mathcal{A}$ and for all word $W \in \mathcal{A}^*$,*

$$\Delta_{WX}^{A,B} = \left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} - \left(\left(\frac{p_A - p_B}{s_A - s_B}\right)^2 - s_X\frac{p_A - p_B}{s_A - s_B} + p_X\right)\sum_{k=1}^{|W|}\left(\frac{p_A - p_B}{s_A - s_B}\right)^{k-1}\Delta_{W[1:|W|-k]}^{A,B}$$

See Section A.4 for the proof.

▶ **Corollary 13.** *Denoting $E_W^{A,B} = \sum_{k=1}^{|W|}\left(\frac{p_A - p_B}{s_A - s_B}\right)^{k-1}\Delta_{W[1:|W|-k]}^{A,B}$, we can efficiently compute $\Delta_W^{A,B}$ for all $W \in \mathcal{A}^*$ iterating the following linear expression:*

$$\begin{pmatrix}\Delta_{WX}^{A,B} \\ E_{WX}^{A,B}\end{pmatrix} = \begin{pmatrix} s_X - \frac{p_A - p_B}{s_A - s_B} & -\left(\frac{p_A - p_B}{s_A - s_B}\right)^2 + s_X\frac{p_A - p_B}{s_A - s_B} - p_X \\ 1 & \frac{p_A - p_B}{s_A - s_B}\end{pmatrix}\begin{pmatrix}\Delta_W^{A,B} \\ E_W^{A,B}\end{pmatrix}$$

with $\begin{pmatrix}\Delta_\varepsilon^{A,B} \\ E_\varepsilon^{A,B}\end{pmatrix} = \begin{pmatrix} s_A - s_B \\ 0\end{pmatrix}$.

A particular example of the case $s_A \neq s_B$ is when $p_A = p_B$. If so, Corollary 13 becomes:

$$\begin{pmatrix}\Delta_{WX}^{A,B} \\ \Delta_W^{A,B}\end{pmatrix} = \begin{pmatrix} s_X & -p_X \\ 1 & 0\end{pmatrix}\begin{pmatrix}\Delta_W^{A,B} \\ \Delta_{W[1:|W|]}^{A,B}\end{pmatrix}$$

This is especially the case when there is some $r > 0$ such that for all action $X \in \mathcal{A}$,

$$r_{X,1} = r\exp(\mathbf{i}\theta_X) \qquad \text{and} \qquad r_{X,2} = r\exp(-\mathbf{i}\theta_X)$$

for some $\theta_X \in \mathbb{R}$. Note that, if so, for all action X, $s_2(r_X) = r^2$ and $s_1(r_X) = 2r\cos(\theta_X)$. In particular the relation on $\Delta^{A,B}$ simplifies to

$$\Delta_{WX}^{A,B} = 2r\cos(\theta_X)\Delta_W^{A,B} - r^2\Delta_{W[1:|W|-1]}^{A,B}$$

We can consider the new quantity

$$\Delta'^{A,B}_W = \frac{\Delta_W^{A,B}}{2\left(\cos(\theta_A) - \cos(\theta_B)\right)r^{|W|+1}}$$

This is well defined because $\mathbb{P}_A(1) > \mathbb{P}_B(1)$ forces $s_1(r_A) < s_1(r_2)$ hence $\cos(\theta_A) - \cos(\theta_B) < 0$. One can check that the newly defined quantity satisfies for all $W \in \{A, B\}^*$ and $X \in \{A, B\}$

$$\Delta'^{A,B}_{WX} = 2\cos(\theta_X)\Delta'^{A,B}_W - \Delta'^{A,B}_{W[1:|W|-1]} \qquad (\Delta 3\text{r})$$

and also $\Delta'^{A,B}_\varepsilon = 1$. This is the situation we will consider in Section 4 to exhibit a solvency game that has only aperiodic optimal strategies. Note that this relation depends neither on A nor B.

Case $s_A = s_B$. In this case, we must have $p_A \neq p_B$ and

$$\Delta_W^{A,B} = (p_B - p_A) R_{W[1:|W|-1]}$$

Using $(R3)$, this leads to the relation standing for $W \in \mathcal{A}^*$ and $X, Y \in \mathcal{A}$:

$$\begin{pmatrix} \Delta_{WXY}^{A,B} \\ \Delta_{WX}^{A,B} \end{pmatrix} = \begin{pmatrix} s_X & -p_X \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_{WX}^{A,B} \\ \Delta_{W}^{A,B} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \Delta_{X}^{A,B} \\ \Delta_{\varepsilon}^{A,B} \end{pmatrix} = \begin{pmatrix} p_B - p_A \\ 0 \end{pmatrix}$$

4 A Necessarily Aperiodic Optimal Strategy

In this section we show that there are games that need the optimal strategies to be aperiodic. We provide an explicit example of a $(3,1)$-solvency game with exactly two actions A and B sharing the same primary root $c > 0$, and with the secondary roots being complex conjugate to each other with modulus shared by both actions. To make things clearer A and B have the following characteristic polynomials:

$$\chi_A(x) = (x-1)(x-c)\,(x - r\exp(\mathbf{i}\theta_A))\,(x - r\exp(-\mathbf{i}\theta_A))$$

and

$$\chi_B(x) = (x-1)(x-c)\,(x - r\exp(\mathbf{i}\theta_B))\,(x - r\exp(-\mathbf{i}\theta_B))$$

We denote, for short, $x_A := \cos(\theta_A)$ and $x_B := \cos(\theta_B)$. This places us in the case $s_A \neq s_B$ of Section 3.5. Without loss of generality, we assume that $x_A < x_B$, which means that $\mathbb{P}_A(1) > \mathbb{P}_B(1)$.

▶ **Lemma 14.** *For any action A associated with a polynomial of the above form we have*

$$-\frac{r^2 + c}{r(c+1)} \leq 2x_A \leq -r\frac{c+1}{c} < 0$$

Conversely, any $r < c \in (0,1)$ and $x_A \in [-1, 0)$ satisfying these conditions give a valid action.

See Section B.2 for the proof.

Note that x_A can be chosen freely in $[-1, 0)$ when choosing r to be sufficiently small.

As explained in the proof of Proposition 9, we have:

▶ **Fact 15.** *Having computed an optimal strategy up to $n-1$ agreeing with the word W, A is optimal at n if and only if $\Delta_W^{A,B} > 0$. If instead $\Delta_W^{A,B} < 0$ then B is optimal. Finally, if $\Delta_W^{A,B} = 0$ then they are equally as good.*

The quantity $\Delta'^{A,B}_W$ is actually simpler to use. Since $x_A < x_B$, it as the opposite sign to $\Delta_W^{A,B}$. Also since $\mathbb{P}_A(1) > \mathbb{P}_B(1)$, we have

$$\Delta'^{A,B}_{\epsilon} = 1 \qquad \text{and} \qquad \Delta'^{A,B}_{A} = 2x_A$$

Finally, using Equation Δ3r, we get

▶ **Fact 16.** *Let σ an optimal strategy and $W_n = \sigma(1)\ldots\sigma(n) \in \mathcal{A}^n$ the word of length n on which it agrees. Then*

$$\Delta'^{A,B}_{W_{n+2}} = 2x_{X_n}\Delta'^{A,B}_{W_{n+1}} - \Delta'^{A,B}_{W_n}$$

where $X_n = A$ *if* $\Delta'^{A,B}_{W_{n+1}} > 0$, $X_n = B$ *if* $\Delta'^{A,B}_{W_{n+1}} < 0$ *and either* A *or* B *if* $\Delta'^{A,B}_{W_{n+1}} = 0$. *In all cases, defining the sequence*

$$\begin{cases} u_0 &= 1 \\ u_1 &= 2x_A \\ u_{n+2} &= (x_A + x_B)u_{n+1} - u_n - (x_A - x_B)|u_{n+1}| \end{cases}$$

we have for all $n \in \mathbb{N}$, $\Delta'^{A,B}_{W_n} = u_n$. *Also, optimal strategies have forced value wherever* $u_n \neq 0$ *and are free when* $u_n \neq 0$. *In particular, the optimal strategy is unique if and only if, the sequence* $(u_n)_{n\in\mathbb{N}}$ *never hits* 0.

We are now ready to give an explicit example of a solvency game that has a single optimal strategy which is aperiodic. We take the $(3,1)$-solvency game $\mathcal{A} = \{A, B\}$ with A, B as just explained and satisfying $x_A = -\dfrac{3}{2}$ and $x_B = -\dfrac{11}{10}$. Hence, $u_0 = 1,\ u_1 = -3/2$ and

$$u_{n+2} = c_{\mathrm{sign}(u_{n+1})}u_{n+1} - u_n, \qquad \text{where} \qquad c_+ = -3/2,\ c_- = -11/10.$$

Thus,

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \end{pmatrix} = L_{c_{\mathrm{sign}(u_{n+1})}} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \quad \text{with} \quad L_{c_+} = \begin{pmatrix} 0 & 1 \\ -1 & c_+ \end{pmatrix}, \quad L_{c_-} = \begin{pmatrix} 0 & 1 \\ -1 & c_- \end{pmatrix}.$$

It suffices to prove (cf. Section B.1):

▶ **Theorem 17.** *For all* $n \in \mathbb{N}$, $u_n \neq 0$ *and the sequence* $(\mathrm{sign}(u_n))_{n\in\mathbb{N}}$ *is not eventually periodic.*

5 Outlook and Related Problems

$(2,1)$-solvency games allow relatively simple optimal strategies: either a pure-tail strategy or strictly alternating between just two actions. In contrast, already in $(3,1)$-solvency games the optimal strategy may have to be aperiodic. However, this aperiodic strategy $\sigma : \mathbb{N} \to \mathcal{A}$ is still computable, and our computability result even holds for all $(\ell, 1)$-solvency games for every $\ell \in \mathbb{N}$.

For general (ℓ, m)-solvency games with $m > 1$ our characterization of optimal strategies in Lemma 6 does not hold, and computability of optimal strategies remains open.

In the more general model of MDPs with multiple control states, it would be easy to reduce the (ℓ, m) case to the $(\ell, 1)$ case, by dividing the fortune by m and encoding the bounded remainder in the control state. However, the introduction of control states in the model makes it more complex. Given a control state s, fortune n and constant $c \in (0, 1)$, deciding whether the minimal ruin probability from (s, n) is $< c$ is at least as hard as the Positivity problem [11, Corollary 4.5] (which in turn is at least as hard as the Skolem problem). In particular, this implies that checking whether two different actions A, B are equally good at a given configuration (s, n) is also Positivity-hard. On the other hand, if the optimal strategy is unique, then it can still be computed in this general model, via the approximation technique described in Section 3.1.

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A Proofs of Section 3

A.1 Proof of Lemma 6

▶ **Lemma 6.** *Let $\mathcal{A}$ be an $(\ell,1)$-solvency game. A memoryless deterministic strategy σ is optimal if and only if it maximises $p^{\sigma}_{w+1}(n)$ for all $n < w+1$.*

Proof. First we show that a sub-optimal strategy σ cannot maximise $p^{\sigma}_{w+1}(n)$ for all $n < w+1$. Consider a sub-optimal memoryless deterministic strategy σ and an optimal memoryless deterministic strategy σ'. Then there exists a fortune $n \in \mathbb{N}$ and actions $A, B \in \mathcal{A}$ such that $\sigma(n) = A$ and $\sigma'(n) = B$ such that $\delta := \sum_{\ell \le k \le m} \mathbb{P}_A(k) p_{\text{opt}}(n+k) - p_{\text{opt}}(n) > 0$. Since $\lim_{w \to \infty} P_{\text{opt}}(w) = 0$, we can pick a sufficiently large fortune $w+1 > n$ such that $p_{\text{opt}}(w+1) \le \delta/2$. Since almost all runs from n that forever stay below fortune $w+1$ end in ruin and σ' is optimal, we obtain $p^{\sigma'}_{w+1}(n) \ge 1 - p_{\text{opt}}(n)$. Towards a contradiction, suppose that σ maximises $p^{\sigma}_{w+1}(n)$ for all $n < w+1$. Then $p^{\sigma}_{w+1}(n) \ge p^{\sigma'}_{w+1}(n) \ge 1 - p_{\text{opt}}(n)$. Let σ'' be the strategy that first plays like σ from n, but upon the first visit to $w+1$ it switches to the optimal strategy σ'. (Here we use the property of $(\ell,1)$-solvency games that the fortune can increase by only $+1$, and hence fortune $w+1$ is not skipped.) Then $1 - p^{\sigma''}(n) \ge p^{\sigma}_{w+1}(n) \cdot (1 - p_{\text{opt}}(w+1)) \ge (1 - p_{\text{opt}}(n)) \cdot (1 - \delta/2)$. However, since $\sigma''(n) = A$ we have $1 - p^{\sigma''}(n) \le 1 - \sum_{\ell \le k \le m} \mathbb{P}_A(k) p_{\text{opt}}(n+k) = 1 - (p_{\text{opt}}(n) + \delta)$ and thus $(1 - p_{\text{opt}}(n)) \cdot (1 - \delta/2) \le 1 - (p_{\text{opt}}(n) + \delta)$ and therefore $(1 - p_{\text{opt}}(n))/2 \ge 1$, a contradiction.

For the other direction we show that a strategy σ that does not maximise $p^{\sigma}_{w+1}(n)$ for all $n < w+1$ cannot be optimal for solvency. Consider such a σ and fortunes $n < w+1$ such that $p^{\sigma}_{w+1}(n)$ is not maximal. Then there exists a different strategy σ' such that $p^{\sigma'}_{w+1}(n) > p^{\sigma}_{w+1}(n)$. Let σ'' be the strategy that first plays like σ' from n, but upon the first visit to $w+1$ it switches to some optimal strategy σ^*. (Here again we use the property of $(\ell,1)$-solvency games that the fortune can increase by only $+1$ and $w+1$ is not skipped.) Then $1 - p^{\sigma''}(n) \ge p^{\sigma'}_{w+1}(n)(1 - p_{\text{opt}}(w+1)) > p^{\sigma}_{w+1}(n)(1 - p_{\text{opt}}(w+1)) \ge 1 - p^{\sigma}(n)$, since almost all runs that forever stay below fortune $w+1$ end in ruin. Therefore σ is not optimal from n. ◀

A.2 Proof of Proposition 7

▶ **Proposition 7.** *(cf. Section A.2) For all word $W \in \mathcal{A}^+$ and $i \in \{1, \dots, |W|\}$, $p^W(i) = \dfrac{Q_{W[1:i-1]}}{Q_W}$ where Q_W is defined as follows:*

- $Q_\perp := 0$ *and* $Q_\varepsilon := 1$.
- *For* $A \in \mathcal{A}$, $Q_A := \dfrac{1}{\mathbb{P}_A(1)}$.
- *For* $A \in \mathcal{A}$ *and* $W \in \mathcal{A}^+$,

$$Q_{WA} = \frac{1}{\mathbb{P}_A(1)} \left(Q_W - \sum_{k=-\ell}^{-1} \mathbb{P}_A(k) Q_{W[1:|W|+k]} \right)$$

Proof.
- By definition, for $A \in \mathcal{A}$, $p^A(1) = \mathbb{P}_A(1) = \dfrac{Q_\varepsilon}{Q_A} = \dfrac{Q_{A[1:0]}}{Q_A}$.
- Assume the property for some $W \in \mathcal{A}^+$. Let $A \in \mathcal{A}$. Let $i \le |W|+1$. Since we have to first reach wealth $|W|+1$ before reaching wealth $|WA|+1$, we have

$$p^{WA}(i) = p^W(i) p^{WA}(|W|+1)$$

Let $Q_{WA} := \dfrac{Q_W}{p^{WA}(|W|+1)}$. Recall that this is well defined. By induction, this is also not zero. Using the induction hypothesis we then get

$$\forall i \in \{1, \dots, |W|+1\} \qquad p^{WA}(i) = \frac{Q_{WA[1:i-1]}}{Q_{WA}}$$

It remains to prove that Q_{WA} satisfies the announced relation. Note that the probability of going back to wealth $|W|+1$ starting at $|W|+1$ and without ever reaching $|W|+2$ is

$$\sum_{k=1}^{\ell} \mathbb{P}_A(-k) p^W(|W|+1-k)$$

Therefore to reach $|W|+2$ we may lose some wealth and come back to $|W|+1$ n times then finally getting to $|W|+2$. Therefore

$$p^{WA}(|W|+1) = \sum_{n=0}^{\infty} \mathbb{P}_A(1) \left(\sum_{k=1}^{\ell} \mathbb{P}_A(-k) p^W(|W|+1-k) \right)^n = \frac{\mathbb{P}_A(1)}{1 - \sum_{k=1}^{\ell} \mathbb{P}_A(-k) p^W(|W|+1-k)}$$

Using the induction hypothesis,

$$\begin{aligned}
\frac{Q_W}{Q_{WA}} &= p^{WA}(|W|+1) \\
&= \frac{\mathbb{P}_A(1)}{1 - \sum_{k=1}^{\ell} \mathbb{P}_A(-k) \frac{Q_{W[1:|W|-k]}}{Q_W}} \\
\frac{1}{Q_{WA}} &= \frac{\mathbb{P}_A(1)}{Q_W - \sum_{k=1}^{\ell} \mathbb{P}_A(-k) Q_{W[1:|W|-k]}} \\
Q_{WA} &= \frac{1}{\mathbb{P}_A(1)} \left(Q_W - \sum_{k=-\ell}^{-1} \mathbb{P}_A(k) Q_{W[1:|W|+k]} \right)
\end{aligned}$$

◀

A.3 Proof of Lemma 10

We show that for all $p \in \{1, \dots, \ell-1\}$, the following property $\mathcal{P}(p)$ holds: for $W \in \mathcal{A}^*$ and $X \in \mathcal{A}$,

$$R_{WX} = \sum_{k=1}^{p} (-1)^{k+1} s_k(r_X) R_{W[1:|W|-k+1]} + (-1)^p \left(s_{p+1}(r_X) Q_{W[1:|W|-p]} + c s_p(r_X) Q_{W[1:|W|-p-1]} \right)$$
$$+ \sum_{k=p+1}^{\ell} (-1)^k \left(s_{k+1}(r_X) + (c+1) s_k(r_X) + c s_{k-1}(r_X) \right) Q_{W[1:|W|-k]}$$

- Recall that $R_{WX} = Q_{WX} - (c+1)Q_W + cQ_{W[1:|W|-1]}$. Using the definitions of Q_{WX}

and the formulas in Section 3.2,

$$\begin{aligned}
R_{WX} &= (1+c+s_1(r_X))Q_W - \sum_{k=1}^{\ell}(-1)^{k+1}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]} \\
&\qquad - (c+1)Q_W + cQ_{W[1:|W|-1]} \\
&= s_1(r_X)R_W + \left(c+s_1(r_X)(c+1)\right)Q_{W[1:|W|-1]} - s_1(r_X)cQ_{W[1:|W|-2]} \\
&\qquad + \sum_{k=1}^{\ell}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]} \\
&= s_1(r_X)R_W - s_2(r_X)Q_{W[1:|W|-1]} - s_1(r_X)cQ_{W[1:|W|-2]} \\
&\qquad + \sum_{k=2}^{\ell}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]}
\end{aligned}$$

which is exactly $\mathcal{P}(1)$.

- Assume that we have shown $\mathcal{P}(p)$ for some $p \in \{1,\dots,\ell-2\}$. Then,

$$\begin{aligned}
R_{WX} &= \sum_{k=1}^{p}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} + (-1)^p\left(s_{p+1}(r_X)Q_{W[1:|W|-p]} + cs_p(r_X)Q_{W[1:|W|-p-1]}\right) \\
&\qquad + \sum_{k=p+1}^{\ell-1}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]} \\
&= \sum_{k=1}^{p+1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} + (-1)^p cs_p(r_X)Q_{W[1:|W|-p-1]} \\
&\qquad + (-1)^{p+2}s_{p+1}(r_X)\left((c+1)Q_{W[1:|W|-p-1]} - cQ_{W[1:|W|-p-2]}\right) \\
&\qquad + \sum_{k=p+1}^{\ell-1}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]} \\
&= \sum_{k=1}^{p+1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} + (-1)^p cs_p(r_X)Q_{W[1:|W|-p-1]} \\
&\qquad + (-1)^{p+2}s_{p+1}(r_X)\left((c+1)Q_{W[1:|W|-p-1]} - cQ_{W[1:|W|-p-2]}\right) \\
&\qquad + \sum_{k=p+2}^{\ell-1}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]} \\
&\qquad + (-1)^{p+1}\left(s_{p+2}(r_X)+(c+1)s_{p+1}(r_X)+cs_p(r_X)\right)Q_{W[1:|W|-p-1]} \\
&= \sum_{k=1}^{p+1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} \\
&\qquad - (-1)^{p+2}s_{p+1}(r_X)cQ_{W[1:|W|-p-2]} + (-1)^{p+1}s_{p+2}(r_X)Q_{W[1:|W|-p-1]} \\
&\qquad + \sum_{k=p+2}^{\ell-1}(-1)^{k}\left(s_{k+1}(r_X)+(c+1)s_k(r_X)+cs_{k-1}(r_X)\right)Q_{W[1:|W|-k]}
\end{aligned}$$

which is exactly $\mathcal{P}(p+1)$.

In the end we get that $\mathcal{P}(\ell-1)$ holds hence

$$R_{WX} = \sum_{k=1}^{\ell-1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} + (-1)^{\ell-1}\left(s_\ell(r_X)Q_{W[1:|W|-\ell+1]} + cs_{\ell-1}(r_X)Q_{W[1:|W|-\ell]}\right) + (-1)^{\ell}\left(s_{\ell+1}(r_X) + (c+1)s_\ell(r_X) + cs_{\ell-1}(r_X)\right)Q_{W[1:|W|-\ell]}$$

Using that $s_\ell(r_X) = s_{\ell+1}(r_X) = 0$ this simplifies to

$$\begin{aligned} R_{WX} &= \sum_{k=1}^{\ell-1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} + (-1)^{\ell-1}cs_{\ell-1}(r_X)Q_{W[1:|W|-\ell]} \\ &\quad + (-1)^{\ell}cs_{\ell-1}(r_X)Q_{W[1:|W|-\ell]} \\ &= \sum_{k=1}^{\ell-1}(-1)^{k+1}s_k(r_X)R_{W[1:|W|-k+1]} \end{aligned}$$

This concludes the proof of the lemma.

A.4 Proof of Lemma 12

We show that for all $p \in \{0, \ldots, |W|\}$, the following property $\mathcal{P}(p)$ holds: for $W \in \mathcal{A}^*$ and $X \in \mathcal{A}$,

$$\begin{aligned} \Delta_{WX}^{A,B} &= \left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} \\ &\quad - \left(\left(\frac{p_A - p_B}{s_A - s_B}\right)^2 - s_X\frac{p_A - p_B}{s_A - s_B} + p_X\right)\sum_{k=1}^{p}\left(\frac{p_A - p_B}{s_A - s_B}\right)^{k-1}\Delta_{W[1:|W|-k-1]}^{A,B} \\ &\quad - \left(\left(\frac{p_A - p_B}{s_A - s_B}\right)^2 - s_X\frac{p_A - p_B}{s_A - s_B} + p_X\right)\left(\frac{p_A - p_B}{s_A - s_B}\right)^{p}(s_A - s_B)R_{W[1:|W|-p-1]} \end{aligned}$$

- By definition,

$$\begin{aligned} \Delta_{WX}^{A,B} &= (s_A - s_B)R_{WX} - (p_A - p_B)R_W \\ &= (s_A - s_B)\left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)R_W + p_X(s_A - s_B)R_{W[1:|W|-1]} \\ &= \left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} + \left(\left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)(p_A - p_B) - p_X(s_A - s_B)\right)R_{W[1:|W|-1]} \\ &= \left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} - \left(\left(\frac{p_A - p_B}{s_A - s_B}\right)^2 - s_X\frac{p_A - p_B}{s_A - s_B} + p_X\right)(s_A - s_B)R_{W[1:|W|-1]} \end{aligned}$$

  which is exactly $\mathcal{P}(0)$.
- Assume we have shown $\mathcal{P}(p)$ for some $p \in \{0, \ldots, |W|-1\}$. Recall that

$$(s_A - s_B)R_{W[1:|W|-p-1]} = \Delta_{W[1:|W|-p-1]} + (p_A - p_B)R_{W[1:|W|-p-2]}$$

Hence

$$\begin{aligned}
\Delta_{WX}^{A,B} = &\left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\sum_{k=1}^{p}\left(\frac{p_A-p_B}{s_A-s_B}\right)^{k-1}\Delta_{W[1:|W|-k-1]}^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\left(\frac{p_A-p_B}{s_A-s_B}\right)^{p}(s_A-s_B)R_{W[1:|W|-p-1]} \\
= &\left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\sum_{k=1}^{p+1}\left(\frac{p_A-p_B}{s_A-s_B}\right)^{k-1}\Delta_{W[1:|W|-k-1]}^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\left(\frac{p_A-p_B}{s_A-s_B}\right)^{p}(p_A-p_B)R_{W[1:|W|-p-2]} \\
= &\left(s_X - \frac{p_A - p_B}{s_A - s_B}\right)\Delta_W^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\sum_{k=0}^{p+1}\left(\frac{p_A-p_B}{s_A-s_B}\right)^{k-1}\Delta_{W[1:|W|-k-1]}^{A,B} \\
&- \left(\left(\frac{p_A-p_B}{s_A-s_B}\right)^2 - s_X\frac{p_A-p_B}{s_A-s_B} + p_X\right)\left(\frac{p_A-p_B}{s_A-s_B}\right)^{p+1}(s_A-s_B)R_{W[1:|W|-p-2]}
\end{aligned}$$

which is exactly $\mathcal{P}(p+1)$.
We then get that $\mathcal{P}(|W|)$ holds. Now using that $R_{W[1:-1]} = R_\perp = 0$, $\mathcal{P}(|W|)$ simplifies to the announced statement.

B Proofs of Section 4

B.1 Proof of Theorem 17

▶ **Theorem 17.** *For all $n \in \mathbb{N}$, $u_n \neq 0$ and the sequence $(\operatorname{sign}(u_n))_{n\in\mathbb{N}}$ is not eventually periodic.*

Going in this way from 0 to infinity we create an infinite word on an alphabet with two letters $\{L_{c_+}, L_{c_-}\}$. Here is the beginning of the word:

$$L_{c_-}, L_{c_+}, L_{c_+}, L_{c_-}, L_{c_+}, L_{c_-}, L_{c_-}, L_{c_+}, L_{c_-}, L_{c_-}, \ldots$$

▶ **Lemma 18.** *For all natural number $n \in \mathbb{N}$, there are $p_n \in \mathbb{N}$ coprime with 2 and 5 and $q_n \in \mathbb{N}$ such that*

$$u_n = \frac{p_n}{2^n 5^{q_n}}$$

Proof. By induction. We verify it immediately for $n = 0, 1$. Assume now that we have found p_n, p_{n+1}, q_n and q_{n+1} for some $n \geq 0$. Then

$$u_{n+2} = c_\pm u_{n+1} - u_n = \frac{\alpha p_{n+1} 5^{q_n} - 4 p_n 5^{q_{n+1}+\beta}}{2^{n+2} 5^{q_n + q_{n+1} + \beta}}, \quad (\alpha, \beta) = (-3, 0),\ (-11, 1),$$

according to whether $u_{n+1} < 0$ or $u_{n+1} > 0$, respectively. ◀

▶ **Corollary 19.** *For all natural number n, $u_n \neq 0$ and all the values of the sequence are distinct from each other.*

▶ **Lemma 20.** *Writing $u_n = \frac{p_n}{2^n 5^{q_n}}$ as above, for all $n \in \mathbb{N}$, $\gcd(p_n, p_{n+1}) = 1$.*

Proof. By induction. We verify it immediately for $n = 0, 1$. Assume now that we have shown this for some n, then for some $(\alpha, \beta) \in \{(-3, 1), (-11, 5)\}$,

$$u_{n+2} = c_{\pm} u_{n+1} - u_n = \frac{\alpha p_{n+1} 5^{q_n} - 4p_n 5^{q_{n+1}+\beta}}{2^{n+2} 5^{q_n + q_{n+1}+\beta}}$$

Let d a common divisor of p_{n+1} and p_{n+2}. In particular, d is divisible by neither 2 nor 5. p_{n+2} is $\alpha p_{n+1} 5^{q_n} - 4p_n 5^{q_{n+1}+\beta}$ divided by some power of 5. Then d divides $\alpha p_{n+1} 5^{q_n} - 4p_n 5^{q_{n+1}+\beta}$ hence it also divides $4p_n 5^{q_{n+1}+\beta}$. Therefore d must divide p_n. By induction hypothesis $d = 1$. ◀

Next we record the indices i where $u_i > 0, u_{i+1} > 0$. Let $(b_k)_{k \in \mathbb{N}}$ the increasing sequence of all such indices. Let also $\mathcal{B} = \{\, q_K \mid k \in \mathbb{N}\}$. Here are the first elements of this set:

$$\mathcal{B} = \{3, 14, 17, 25, 28, 39, 50, 53, 61, 64, \ldots\}. \tag{B}$$

It seems that for all $k \in \mathbb{N}$, $b_{k+1} - q_K \in \{3, 8, 11\}$. This is the objective of our next lemma.

▶ **Lemma 21.** *Assume that $i \in \mathcal{B}$ and let $u_i = a,\ u_{i+1} = b$. Then the next $j > i$ such that $j \in \mathcal{B}$ is one of*

1. *If $\frac{b}{a} > \frac{26}{21} \approx 1.2381$ then $j = i + 3$ and*

$$u_{i+3} = \frac{11}{10}a + \frac{13}{20}b \qquad \text{and} \qquad u_{i+4} = -\frac{13}{20}a + \frac{21}{40}b.$$

2. *If $\frac{b}{a} < \frac{21942}{65107} \approx 0.3370$ then $j = i + 8$ and*

$$u_{i+8} = \frac{10971}{40000}a - \frac{65107}{80000}b \qquad \text{and} \qquad u_{i+9} \frac{65107}{80000}a + \frac{196981}{160000}b.$$

3. *Otherwise $j = i + 11$ and*

$$u_{i+11} = \frac{7216067}{8000000}a - \frac{4905339}{16000000}b \qquad \text{and} \qquad u_{i+12} = \frac{4905339}{16000000}a + \frac{32141837}{32000000}b$$

This follows from a routine calculation. Note that the "cut-off" values $26/13,\ 21942/65107$ are never achieved with $b/a = u_{n+1}/u_n$ because by Lemma 18: u_{n+1}/u_n is a rational number with an even denominator, whereas the three numbers above have even numerators. Let now

$$M_3 = L_{c_+} L_{c_-} L_{c_+} = \begin{pmatrix} 11/10 & 13/20 \\ -13/20 & 21/40 \end{pmatrix},$$

$$M_8 = L_{c_+} L_{c_-} L_{c_+} L_{c_-} L_{c_-} L_{c_+} L_{c_-} L_{c_+} = \begin{pmatrix} \frac{10971}{40000} & -\frac{65107}{80000} \\ \frac{65107}{80000} & \frac{196981}{160000} \end{pmatrix},$$

$$M_{11} = L_{c_+} L_{c_-} L_{c_+} L_{c_-} L_{c_-} L_{c_+} L_{c_-} L_{c_-} L_{c_+} L_{c_-} L_{c_+} = \begin{pmatrix} \frac{7216067}{8000000} & -\frac{4905339}{16000000} \\ \frac{4905339}{16000000} & \frac{32141837}{32000000} \end{pmatrix}.$$

This suggests that we have an automaton such at each instance $(u_i, u_{i+1}) = (a, b)$ with $a > 0,\ b > 0$, the next time that both $x_j,\ x_{j+1}$ are positive is obtained by applying one of

$M_3,\ M_8,\ M_{11}$ to $(a,b)^T$. So, we study this automaton. We let $u, v \in \{3, 8, 11\}$ and write $M_u \to M_v$ to indicate if after having applied M_u to $(a,b)^T$, the next possible move is M_v. Just to get a feeling of what we might expect we calculated the first few terms of the sequence $b_{j+1} - b_j$ listed in (B) getting

$$\mathcal{C} = \{11, 3, 8, 3, 11, 11, 3, 8, 3, 11, 3, 8, 3, 8, 3, 11, \ldots\}.$$

▶ **Lemma 22.**
1. *There are no two consecutive* 3*'s;*
2. *There may be two consecutive* 11*'s in* C *but not three of them;*
3. *An 8 is never followed by an* 8 *or* 11*;*
4. *An* 11 *is never followed by an* 8*.*
5. *Any group* $11,\ 3$ *is followed by an* 8*.*

See Section B.3 for the proof. Now we prove the following lemma:

▶ **Lemma 23.** *Each of the states* $3, 8, 11$ *is being visited infinitely often.*

Proof. Since 8's are only followed by 3's and 11's are followed by 3 or 11, 3, it follows that 3 is being visited infinitely often. Since there is no group 11, 3, 11, it follows that if 11 is visited infinitely often, so is 8. So, we only have to rule out the case when there are only finitely many 11's. In this case our sequence after a while looks like

$$M_3 M_8 M_3 M_8, \ldots.$$

Let $L := M_3 M_8$. One checks that

$$L^{1000} = \begin{pmatrix} -, - \\ +, + \end{pmatrix},$$

showing that with

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = L^{1000} \begin{pmatrix} a \\ b \end{pmatrix},$$

we have $b' < 0$, which is wrong. This shows that in fact $M_3 M_8$ cannot repeat itself more than 1000 times without an M_{11} appearing. This finishes the proof of this lemma. ◀

Now we study the growth of $(u_n)_{n\in\mathbb{N}}$. Although it is likely to be bounded, we could not prove it. Nonetheless we can show the following:

▶ **Proposition 24.** *We have* $u_n = \underset{n\to+\infty}{\mathrm{O}}(1.06^n)$*.*

See Section B.4 for the proof.

We now give a bound on the quantity q_n seen in Lemma 18.

▶ **Lemma 25.** *We have* $q_n < 0.32n$ *for all large enough* n*.*

See Section B.5 for the proof. We are now ready to conclude.

▶ **Theorem 17.** *For all* $n \in \mathbb{N}$*,* $u_n \neq 0$ *and the sequence* $(\mathrm{sign}(u_n))_{n\in\mathbb{N}}$ *is not eventually periodic.*

Proof. Assume it is. In particular, the part $\mathcal{B}$ namely of those n such that $u_n > 0$ and $u_{n+1} > 0$ is also periodic. Assume it is of period k. It then follows that for some t_0 and k and some matrix M we have that

$$\begin{pmatrix} u_{kt+r} \\ u_{kt+r+1} \end{pmatrix} = M \begin{pmatrix} u_{k(t-1)+r} \\ u_{k(t-1)+r+1} \end{pmatrix}$$

for all $r \in \{0, 1, \ldots, k-1\}$ and all $t \geq t_0$. In particular,

$$\begin{pmatrix} u_{kt+r} \\ u_{kt+r+1} \end{pmatrix} = M^{t-t_0} \begin{pmatrix} u_{n_0} \\ u_{n_0+1} \end{pmatrix},$$

with $n_0 := k(t - t_0) + r$. This shows that $(u_{kt+r})_{t \geq t_0}$ is binary recurrent. The characteristic equation is given by M^k, a matrix of determinant 1. So, there are various scenarios.

1. The matrix M has complex conjugate eigenvalues of absolute value 1. It then follows that

$$u_{kt+r} = c_r \cos(\theta_r + t\phi),$$

where M^k has eigenvalues $e^{\pm i\phi}$. If ϕ is a rational multiple of π then such sequence has a finite range as t varies which contradicts Corollary 19. If ϕ is an irrational multiple of π then $c_r \cos(\theta_r + t\phi)$ will change signs infinitely often which is false since u_{kt+r} are positive. Thus, this case is not possible.
2. The matrix M has roots in $\{\pm 1\}$. Then $(u_{kt+r})_{t \geq t_0}$ either has a finite range or is an arithmetic progression. The former contradicts Corollary 19 and the latter contradicts Lemma 18 which states that the denominators of u_{kt+r} are divisible by arbitrarily large powers of 2 while the denominators of numbers in an arithmetic progression remain bounded.
3. M has real eigenvalues whose product is 1. Assume that these eigenvalues are irrational and let them be ζ and η, with $|\zeta| > 1 > |\eta|$. In particular, the pairs (u_n, u_{n+1}) over these n such that $u_n,\ u_{n+1}$ are both positive range over finitely many pairs binary recurrent sequences all with the same characteristic roots namely ζ and η. Namely, each one of these sequences has a Binet formula of the form

$$c_r \zeta^{\lfloor n/k \rfloor} + d_r \eta^{\lfloor n/k \rfloor},$$

where the pairs (c_r, d_r) are drawn from a finite collection of pairs of conjugated numbers in $\mathbb{K} = \mathbb{Q}(\zeta)$. Note that since u_n is rational, none of $c_r,\ d_r$ is zero. Thus, u_n/u_{n+1} as n goes to infinity approach finitely many numbers in $\mathbb{K} = \mathbb{Q}(\zeta)$. Let us show that none of them is rational. Assume that u_n/u_{n+1} has a rational limit. Writing

$$u_n = c_r \zeta^{\lfloor n/k \rfloor} + d_r \eta^{\lfloor n/k \rfloor}, \qquad u_{n+1} = c'_r \zeta^{\lfloor n/k \rfloor} + d_r \eta^{\lfloor n/k \rfloor},$$

we get that $c_r/c'_r = \lambda$, where $\lambda \in \mathbb{Q}$. By Galois conjugation $d_r/d'_r = \lambda$ which implies $u_n/u_{n+1} = \lambda$. Writing

$$u_n = \frac{p_n}{2^n 5^{q_n}}, \quad u_{n+1} = \frac{p_{n+1}}{2^{n+1} 5^{q_{n+1}}},$$

where $\gcd(p_n, 5) = \gcd(p_{n+1}, 5) = 1$. By Lemma 20, $\gcd(p_n, p_{n+1}) = 1$. Since λ is rational, this shows that the numerator of u_n (and u_{n+1} as well) is a bounded integer times a power of 5. By the Binet formula for u_n and linear forms in 5-adic logarithms, the power of 5 in the numerator of u_n, if present at all, is of size $n^{O(1)}$. In particular u_n should tend to zero which is inconsistent with the Binet formula (the numbers $c_r,\ d_r$ are nonzero).

Since each of the three moves 3, 8, 11 is executed infinitely often, it follows that u_n/u_{n+1} has at least three cluster points, namely one in the region $u_{n+1}/u_n = b/a > 26/21$, another in $u_{n+1}/u_n < 0,33704$ and finally one in the intermediary region $0.33704 < b/a < 26/21$. The matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

maps the finitely many cluster points of $\{u_{n+1}/u_n\}_{n\in\mathcal{B}}$ let them be $\{y_1,\ldots,y_k\}$ to the points

$$\left\{\frac{m_{21}+m_{22}y_1}{m_{11}+m_{12}y_1},\ldots,\frac{m_{21}+m_{22}y_k}{m_{11}+m_{12}y_k}\right\}.$$

So, M induces a function from $\{y_1,\ldots,y_k\}$ into itself via Möbius transformations. Note that ∞ and 0 do not participate since $y_1,\ldots,y_k$ are in $\mathbb{K}$ and are irrational and $k \geq 3$ so the tops or bottom of the above fractions are not close to zero. Since M is a Möbius transformation, it it injective on this finite set, so also surjective. Letting g be the order of M as permutation on this set of k elements, we get that M^g invaries all $y_1,\ldots,y_k$. Since $k \geq 3$, M^g must be the identity (the identity is the only Möbius transformation with three fixed points), so the characteristic roots of M are roots of unity, a case already treated.

4. M has real rational eigenvalues $\zeta,\ \eta$. We show that this cannot be the case using a size argument. Assume that M is replaced by a suitable power of itself so it is a concatenation of matrices of the form $D_{i_1}\cdots D_{i_{10}}$, where $i_1,\ldots,i_{10} \in \{1,2,3\}$, where these matrices have the same meaning as in Lemma 25. Letting K be the total number of matrices in M, we have that

$$M = \begin{pmatrix} \frac{a(K)}{2^{K-2}} & \frac{b(K)}{2^{K-1}} \\ \frac{c(K)}{2^{K-1}} & \frac{d(K)}{2^{K}} \end{pmatrix}.$$

Further

$$\mathrm{Tr}(M) = \frac{4a(K)+d(K)}{2^K} = \frac{u(K)}{2^K 5^{q_K}},$$

where by Lemma 25 we have that $q_K < 0.32K$. We want that

$$\lambda^2 - \mathrm{Tr}(M)\lambda + 1 = (\lambda-\zeta)(\lambda-\eta),$$

where $\zeta,\ \eta$ are rational. Then,

$$\zeta,\ \eta = \frac{u(K)/(2^K 5^{q_K}) \pm \sqrt{(u(K)^2/(2^{2K}5^{2q_K})-4}}{2}.$$

The expression under the square root must be rational so we write it as $x(K)/(2^K 5^{b(K)})$ where $x(K)$ is a positive integer and we get

$$u(K)^2 - 2^{2K+2}5^{2q_K} = x(K)^2.$$

This gives

$$(|u(K)|+x(K))(|u(K)|-x(K)) = 2^{2k+2}5^{2q(K)}.$$

Here, $q_K \geq 0$, $u(K)$ is odd and coprime to 5 if $q_K > 0$. We see that the only suitable decompositions are

$$|u(K)|+x(K) = 2^{2K+1}, \quad |u(K)|-x(K) = 2\cdot 5^{2q_K};$$

or

$$|u(K)|+x(K) = 2^{2K+1}5^{q_K}, \qquad |u(K)|-x(K) = 2.$$

These give

$$(|u(K)|, x(K)) = (2^{2K} + 5^{2q_K}, 2^{2K} - 5^{2q_K}), \quad \text{or} \quad (2^{2K}5^{2q_K} + 1, 2^{2K}5^{2q_K} - 1),$$

leading to

$$(\eta, \lambda) = \pm\left(\frac{2^K}{5^{q_K}}, \frac{5^{q_K}}{2^K}\right), \qquad \pm\left(2^K 5^{q_K}, \frac{1}{2^K 5^{q_K}}\right).$$

Now

$$u_n = c_r\zeta^m + d_r\eta^m, \quad n \equiv r \pmod{K},\ r \in \{0, \dots, K-1\},\ m = \lfloor n/K \rfloor.$$

Thus, in the first case if $c_r \neq 0$, then

$$|u_n| \gg |\zeta|^{n/K} \gg \frac{2^n}{5^{nq_K/K}}.$$

Since $q_K < 0.32K$, it follows that $a_n \gg (2/5^{0.32})^n$ for large n, which contradicts Proposition 24 since $2/5^{0.32} > 1.19 > 1.06$. The case when $\zeta = 2^K 5^{q_K}$ is even worse since in this case we get $a_n \gg 2^n$, which is again false. This is assuming $c_r \neq 0$. Of course if

$$u_n = c_r\zeta^m + d_r\eta^m, \qquad u_{n+1} = c'_r\zeta^m + d'_r\eta^m,$$

where $c_r = 0,\ c'_r = 0$, then u_{n+1}/u_n when $u_n > 0,\ u_{n+1} > 0$ is a rational number from a finite list. This number is not one of the numbers $26/21,\ 21942/65107,\ 2\cdot 7216067/4905339$ by the argument about u_{n+1}/u_n having even denominator. Now the argument with M^g for some fixed g having three fixed points as a Möbius transformation applies and we are done.

◀

B.2 Proof of Lemma 14

Assume we have an action A for a $(3,1)$-solvency game with characteristic polynomial

$$\chi_A(x) = (x-1)(x-c)\,(x - r\exp(\mathbf{i}\theta_A))\,(x - r\exp(-\mathbf{i}\theta_A))$$

This implies that

- $\mathbb{P}_A(1) = \dfrac{1}{2rx_A + c + 1}$
- $\mathbb{P}_A(-1) = \dfrac{2(c+1)rx_A + r^2 + c}{2rx_A + c + 1}$
- $\mathbb{P}_A(-2) = -\dfrac{2crx_A + (c+1)r^2}{2rx_A + c + 1}$
- $\mathbb{P}_A(-3) = \dfrac{cr^2}{2rx_A + c + 1}$

The above requirement imply the following:

1. $2rx_A + c + 1 > 1$
2. $2(c+1)rx_A + r^2 + c \geq 0$
3. $2crx_A + (c+1)r^2 \leq 0$
4. $cr^2 \geq 0$.

Inequality 4 is actually always satisfied. Rewriting the others we have

5. $2x_A > -\dfrac{c}{r}$
6. $2x_A \geq -\dfrac{r^2 + c}{r(c+1)}$
7. $2x_A \leq -r\dfrac{c+1}{c}$

Since c is the primary root, we have $c > r > 0$ hence $-\frac{r^2+c}{r(c+1)} > -\frac{c^2+c}{r(c+1)} = -\frac{c}{r}$ meaning that Inequality 5 is actually pointless. The two remaining ones form the announced statement.

B.3 Proof of Lemma 22

1. Let us start with 3. We got there because $b/a > 26/13$. After arriving there with

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M_3 \begin{pmatrix} a \\ b \end{pmatrix}$$

we have

$$\frac{b'}{a'} = \frac{(-13/20)a + (21/40)b}{(11/10)a + (13/20)b} < \frac{21/40}{13/20} = \frac{21}{26} < \frac{26}{21},$$

which shows that 3 can only be followed by 8 or 11.

2. Let us turn now to the 11's. In order for (a, b) to end up in M_{11} it is necessary that

$$0.33704 < \frac{b}{a} < 1.2381.$$

Writing

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M_{11} \begin{pmatrix} a \\ b \end{pmatrix},$$

We get

$$\frac{0.306 + 1.004(b/a)}{0.903 - 0.305(b/a)} < 1.239,$$

giving $b/a < 0.59$. Iterating this, we get

$$\frac{0.306 + 1.004(b/a)}{0.903 - 0.305(b/a)} < 0.59,$$

which gives $b/a < 0.2$, but this is false since we must have $b/a > 0.33$.

3. To see that after 8 we cannot have another 8 or 11, note that with

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M_8 \begin{pmatrix} a \\ b \end{pmatrix},$$

we have

$$\frac{b'}{a'} = \frac{65107/80000 + (196981/160000)(b/a)}{10971/40000 - (65107/80000)(b/a)} > \frac{65107/80000}{10971/40000} > 2.96,$$

so it cannot be followed by either 8 or 11.

4. Assuming $(a, b)^T$ ends up being in M_{11}, we then get that with

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M_{11} \begin{pmatrix} a \\ b \end{pmatrix},$$

we must have

$$\frac{b'}{a'} > \frac{1.004(b/a) + 0.306}{0.903 - 0.307(b/a)}.$$

Imposing $b'/a' < 0.338$, and solving for b/a we get $b/a < 0$, a contradiction.

5. We have to rule out a sequence of the form $11,\ 3,\ 11$. Assume that we have M_{11} followed by M_3. We then must have with

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M_{11} \begin{pmatrix} a \\ b \end{pmatrix}$$

that $b'/a' > 1.238$. This gives

$$\frac{0.306 + 1.004(b/a)}{0.902 - 0.306(b/a)} > 1.238,$$

which gives $b/a > 0.58$. But in case $M_3 M_{11}(a,b)^T$ ends up in M_{11}, we must also have with

$$\begin{pmatrix} a'' \\ b'' \end{pmatrix} = M_3 M_{11} \begin{pmatrix} a \\ b \end{pmatrix},$$

that $b''/a'' < 1.239$. This gives

$$\frac{-0.2 + 0.52(b/a)}{1 - 0.19(b/a)} < 1.239,$$

which gives $b/a < 0.48$, which is incompatible with the previous conclusion $b/a > 0.58$.

B.4 Proof of Proposition 24

We compute the 2-norms of $M_3,\ M_8,\ M_{11}$, which is the operator norm associated to the usual Euclidean norm $\|\cdot\|_2$. It is defined for a real square matrix M by $|||M|||_2 = \sqrt{\rho\left(MM^T\right)}$ where ρ is the spectral radius. For M_3, we have that

$$M_3 M_3^T = \begin{pmatrix} \frac{653}{400} & \frac{299}{800} \\ -\frac{299}{800} & \frac{1117}{1600} \end{pmatrix}$$

is of characteristic equation $\lambda^2 - 1.70231\ldots\lambda + 1 = 0$. Since $|1.70| < 2$, this equation has complex roots of modulus 1. Thus, $|||M_3|||_2 = 1$. For M_8 we have that $M_8 M_8^T$ is of characteristic equation

$$\lambda^2 - 2.91577\ldots\lambda + 1 = 0,$$

so $|||M_8|||_2 \leq 1.59 < 1.06^8$. Finally, for M_{11} we have that $M_{11}M_{11}^T$ has characteristic equation

$$\lambda^2 - 2.01049\ldots\lambda + 1 = 0,$$

so $|||M_{11}|||_2 \leq 1.08 < 1.01^{11}$. For all n, we can write

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = L'M' \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

where M' finite product of the matrices M_3, M_8, M_11 whose sum of indices is at most n, L' is the product of at most 10 matrices equal to either L_{c_+} or L_{c_-}. In particular, $|||L'|||_2$ is bounded independently of n and $|||M'|||_2 \leq 1.06^n$. Putting together all the above we get that

$$\left\| \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \right\|_2 \leq |||M'|||_2 |||L'|||_2 \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_2 \leq |||L'|||_2 \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_2 1.06^n$$

hence $u_n = \underset{n\to+\infty}{\mathrm{O}} (1.06^n)$.

B.5 Proof of Lemma 25

Let us consider

$$D_1 = M_3 M_8, \quad D_2 = M_3 M_{11}, \quad D_3 = M_3 M_{11} M_{11} M_3 M_8.$$

By Lemma 22, any acceptable path in the automaton is a concatenation of D_i for $i \in \{1, 2, 3\}$. We generated by computer all matrices of the form

$$D = D_{i_1} \dots D_{i_{10}}, \qquad i_1, \dots, i_{10} \in \{1, 2, 3\},$$

a totality of 3^{10} of them. For each of them we wrote

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

and we calculated the maximum power of 5 that divides the the denominator of one of the $d_{i,j}$'s. This we scaled by the weight of the matrix D (namely how many L_{c_*}'s are in each D) which is given by

$$w(D_{i_1}) + \cdots + w(D_{i_{12}}),$$

where $w(D_1) = 11$, $w(D_2) = 14$ and $w(D_3) = 36$. Denoting μ the maximal value the computer shows that $\mu \approx 0.311111 < 0.32$. Now any, like in the proof of Proposition 24,

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = L' D' \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

where D' is the product of a multiple of 10 of the matrices D_1, D_2, D_3 and L' is the product of at most 360 matrices equal to either L_{c_+} or L_{c_-}. Hence D' is of the form $\frac{1}{2^n 5^{\nu n}} D''$ where D'' is an integer matrix and $\nu \leq \mu \leq 0.32n$. Similarly, L' is of the for form $\frac{1}{K} L''$ with L'' and integer matrix and $K \in \mathbb{N}$ being at most divisible by 5^{360}. Hence $L'D'$ has denominators divisible by at most $5^{\mu n + 360}$ hence $q_n \leq \mu n + 360$ which is less than $0.32n$ for large enough n.