On Inequality Decision Problems for Low-Order Holonomic Sequences

George Kenison*, Oleksiy Klurman†, Engel Lefaucheux‡, Florian Luca§¶‖, Pieter Moree**, Joël Ouaknine¶, Markus A. Whiteland††, and James Worrell‡‡

*Institute of Logic and Computation, TU Wien, Vienna, Austria

†School of Mathematics, University of Bristol, Bristol, United Kingdom

‡University of Lorraine, CNRS, Inria, LORIA, Nancy, France

§School of Mathematics, University of the Witwatersrand, South Africa

¶Max Planck Institute for Software Systems, Saarland Informatics Campus, Germany

‖Research Group in Algebraic Structures and Applications, King Abdulaziz University, Jeddah, Saudi Arabia

**Max Planck Institute for Mathematics, Bonn, Germany

††University of Liège, Belgium

‡‡Department of Computer Science, University of Oxford, United Kingdom

***Abstract*—A sequence is holonomic if its terms obey a linear recurrence relation with polynomial coefficients. In this paper we consider decision problems for first- and second-order holonomic sequences involving inequalities. For first-order sequences $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$, we show that the problem of determining whether $u_n \leq v_n$ for each n reduces to the problem of testing equality between periods (in the sense of Kontsevich and Zagier) and their generalisations. For second-order sequences whose coefficients are either constants or linear polynomials, we show that Positivity (i.e., the problem of determining whether the terms of a sequence are all non-negative) also reduces to the problem of testing equality between periods and their generalisations.**

1. Introduction

1.1. Background

The class of holonomic (sometime *P-finite*) sequences, which we will shortly define, underpins a number of computational models with wide-ranging applications such as: reachability questions for planar random walks and the cost of searching in classical data structures such as quadtrees [1]. Further discussions alongside numerous applications in mathematics and the computational sciences are given in a number of sources [2], [3], [4], [5].

A rational-valued sequence $\langle u_n \rangle_n$ is *holonomic* (of *order* k and *degree* d) if there exist polynomials $p_{k+1}, \ldots, p_1 \in \mathbb{Q}[n]$ of degree at most d with $p_1(x), p_{k+1}(x)$ non-zero polynomials such that

$$p_{k+1}(n)u_n = p_k(n)u_{n-1} + \cdots + p_1(n)u_{n-k}.$$

Subject to a standard assumption that $p_{k+1}(n) \neq 0$ for non-negative integer n, the above recurrence uniquely defines an infinite sequence $\langle u_n \rangle_{n=0}^{\infty}$ once the k initial values $u_0, \ldots, u_{k-1}$ are specified.[1]

The class of holonomic sequences contains the familiar class degree-0 sequences or *C-finite sequences*—those sequences that satisfy a linear recurrence relation with *constant* coefficients. Perhaps the best-known example is the second-order Fibonacci sequence $\langle F_n \rangle_n$ satisfying $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = 0$, $F_1 = 1$. The subclass of first-order holonomic sequences are precisely the *hypergeometric* sequences.

1.2. Motivation

Identities for holonomic sequences are a cornerstone of the mathematical literature with much written on the close connections with the analysis of generating functions in fields such as combinatorics and probability [2], [5]. There is *"in contrast,… almost no algorithms are available for inequalities"* as noted by Kauers and Pillwein, [6].

Restricted variants of the inequality problem where at least one of $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ is taken to be constant have garnered considerable attention in the Automated Verification literature. For example, the *Positivity Problem* (i.e., whether every term of a given sequence is non-negative) for C-finite sequences is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [7], [8]. There are even fewer decision procedures for the Positivity Problem concerning holonomic sequences that are not C-finite. However, several partial results and heuristics are known in this setting (cf. [6], [9], [10], [11], [12], [13]).

In the field of numerical analysis, the Minimality Problem for holonomic sequences has received much attention. Briefly, a non-zero holonomic sequence $\langle u_n \rangle_n$ satisfying a given recurrence relation is *minimal* if given an independent solution

1. In the sequel, it will in fact often be convenient to start the sequence at u_{-1} instead of u_0.

$\langle v_n \rangle_n$ one has $u_n/v_n \to 0$ as $n \to \infty$. From numerical calculations to asymptotic analysis, minimal solutions appear in a number of sources [14], [15], [16], [17], [18], [19]—see also the references therein. Kenison et al. [20] showed that Positivity reduces to Minimality for second-order holonomic sequences (a point we return to in our discussion of related work).

On the one hand, decidability of the Positivity Problem for hypergeometric sequences is straightforward to ascertain (see Section 3). On the other hand, the decidability of the *Threshold Problem* (i.e., whether every term of a given sequence is at least a fixed value) is outstanding (we discuss this point further in our discussion of related work).

Herein we focus our attention on low-order sequences (first- and second-order). In the work that follows,

1) We consider the Inequality Problem for hypergeometric sequences.
2) We consider the Positivity Problem for second-order holonomic sequences; that is, sequences that satisfy a recurrence relation of the form
$$p_3(n)u_n = p_2(n)u_{n-1} + p_1(n)u_{n-2}\,, \qquad (1)$$
where $p_1, p_2, p_3 \in \mathbb{Q}[x]$.

1.3. Contributions

Main Results. Our main focus in the present paper is on Turing reductions among inequality problems, and also on reductions to the problem of deciding equality for classes of numbers (known as [univariate] *periods*, *exponential periods*, and *pseudoperiods* that originate from algebraic geometry and number theory). As we develop in the sequel, these quantities also arise as linear combinations of values of hypergeometric functions evaluated at rational or algebraic parameters, or as Beta integrals evaluated at algebraic numbers. Various established conjectures appear in the literature concerning the decidability of equality checking among periods and related expressions, notably those of Kontsevich and Zagier [21].

We summarise our main results as follows:

1) The Hypergeometric Inequality Problem reduces to the problem of deciding equality between pseudoperiods (Theorem 3.3).
2) For degree-1 second-order holonomic sequences, the Positivity and Minimality Problems both reduce to the problem of deciding equality between exponential periods and pseudoperiods (Theorem 4.1).
3) Finally, taking a model-theoretic perspective, we point out that all of the above problems become decidable with the assistance of a classical Gabrielov–Vorobjov oracle, as discussed in Section 5.

Polynomial Continued Fractions. Our approach to inequality decision problems via the analysis of continued fractions will also be of independent interest to researchers further afield.

There is a long history of mathematical research concerning the class of continued fractions whose partial quotients are given by polynomials. For example, the continued fraction expansion for $4/\pi$ given by Lord Brouncker (as reported by Wallis in [22][2])
$$\frac{4}{\pi} = 1 + \operatorname*{K}_{n=1}^{\infty} \frac{(2n-1)^2}{2}$$
has well-behaved partial quotients. Bowman and Mc Laughlin [24] (see also [25]) coined the term *polynomial continued fraction* for such constructions. The evaluation of polynomial continued fractions whose partial quotients have low degrees appears in [26], [27], [28]. For $\deg(a_n) \leq 2$ and $\deg(b_n) \leq 1$, Lorentzen and Waadeland [27, §6.4] express the polynomial continued fraction $\operatorname{K}(a_n/b_n)$ as a ratio of the value of two hypergeometric functions with algebraic parameters evaluated at an algebraic point. However, those authors do not cover all cases at low degrees; for example, the polynomial continued fraction $\operatorname{K}_{n=1}^{\infty} \frac{-n(n+1)}{2n+1}$ corresponding to the recurrence relation $(n+1)u_n = (2n+1)u_{n-1} - (n+1)u_{n-2}$ is not treated. Indeed the presentation in [27] does not handle cases where the corresponding recurrence has a single repeated characteristic root—the above is one such example with its associated characteristic polynomial $x^2 - 2x + 1 = (x-1)^2$. We extend the evaluation of low-degree polynomial continued fractions given by Lorentzen and Waadeland by giving a complete classification in terms of ratios of hypergeometric functions in our proof of Theorem 4.9.

1.4. Related Work

Second-Order Sequences. In recent work, Kenison et al. [20] showed that for second-order holonomic sequences the Positivity Problem reduces to the Minimality Problem. More specifically, those authors gave a semi-algorithm that terminates on all non-minimal solutions by exploiting classical convergence results for continued fractions. The setting herein, degree-1 second-order holonomic sequences, is far stricter. On the other hand, our results are more profound and far-reaching in this setting. Indeed, our reduction establishes new and novel connections to long-standing conjectures on deciding equality between periods (and their generalisations) from algebraic geometry and number theory.

We note, for the avoidance of doubt, that reductions to the aforementioned equality problems is critical on our assumption that the polynomial coefficients in (1) have degree at most one. To the best of our knowledge, there is no such analysis available if we lift this restriction.

First-Order Recurrences. The Threshold and Membership Problems are open even in the setting of hypergeometric sequences—for a hypergeometric sequence $\langle u_n \rangle_n$ and target t, recall that the *Membership Problem* asks to determine whether $u_n = t$ for some n. It is notable that the Membership Problem reduces to the Threshold Problem in this setting due to the asymptotic behaviour of hypergeometric sequences. Since the release of this note as a preprint, further works

2. See the translation by Stedall [23].

by Nosan et al. [29] and Kenison [30] have included discussions on the connection between the Threshold Problem for hypergeometric sequences and the problem of deciding equality between gamma products (see Section 3.1). In this direction, Kenison establishes decidability of the Threshold Problem for classes of hypergeometric sequences with certain restrictive parameter symmetries (such as those sequences whose parameters are all Gaussian integers). We delay formal definitions and terminology to Section 2.

1.5. Structure

The remainder of the paper is structured as follows. In Section 2 we give necessary preliminary material. We prove Theorem 3.3 in Section 3. An overview of the proof of Theorem 4.1 is given in Section 4. A necessary step in the proof of Theorem 4.1, that Positivity reduces to Minimality for degree-1 second-order holonomic sequences, is discussed in Section 4.1. We conclude the paper and point out directions for future research in Section 6. The appendices round out the proof Theorem 4.1.

2. Preliminaries

We briefly recall some terminology for second-order holonomic recurrences [20]. Consider a holonomic recurrence of the form (1). For our purposes, we can safely assume that none of the polynomial coefficients are identically zero (see Section 3). Moreover by considering a shifted recurrence relation we can also assume that each polynomial coefficient has constant sign,[3] and has no roots for $n \geq 0$. Additionally we can assume that $\operatorname{sign}(p_3) = 1$. Thus we define the *signature* of a recurrence relation (1) (or its normalisation (2)) as the ordered pair $(\operatorname{sign}(p_2(n)), \operatorname{sign}(p_1(n)))$.

2.1. Continued Fractions

A *continued fraction*

$$\operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

is defined by an ordered pair of sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ of complex numbers where $a_n \neq 0$ for each $n \in \mathbb{N}$. Comprehensive accounts on the theory of continued fractions are given in [27], [31]. Herein we shall always assume that $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are real-valued rational functions. A continued fraction *converges* to a value $f = \operatorname{K}(a_n/b_n)$ if its *sequence of approximants* $\langle f_n \rangle_{n=1}^{\infty}$ converges to f in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The sequence $\langle f_n \rangle_n$ is recursively defined by the following composition of linear fractional transformations. For $w \in \hat{\mathbb{R}}$, define

$$s_n(w) = \frac{a_n}{b_n + w} \text{ for each } n \in \{1, 2, \ldots\}.$$

We set $f_n := s_1 \circ \cdots \circ s_n(0)$ so that $f_n = \operatorname{K}_{m=1}^{n} \frac{a_m}{b_m}$.

Let $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ satisfy the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial values $A_{-1} = 1, A_0 = 0, B_{-1} = 0$, and $B_0 = 1$. As a pair, $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ form a basis for the solution space of the recurrence. We call $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ the sequences of *canonical numerators* and *canonical denominators* of $\operatorname{K}(a_n/b_n)$ because $f_n = A_n/B_n$ for each $n \in \mathbb{N}$.

Two continued fractions are *equivalent* if they share the same sequence of approximants. The following theorem gives a procedure to move between equivalent continued fractions [27], [31].

Theorem 2.1. *The continued fractions* $\operatorname{K}(a_n/b_n)$ *and* $\operatorname{K}(c_n/d_n)$ *are equivalent if and only if there exists a sequence* $\langle \mathfrak{r}_n \rangle_{n=0}^{\infty}$ *with* $\mathfrak{r}_0 = 1$ *and* $\mathfrak{r}_n \neq 0$ *for each* $n \in \mathbb{N}$ *such that* $c_n = \mathfrak{r}_n \mathfrak{r}_{n-1} a_n$ *and* $d_n = \mathfrak{r}_n b_n$ *for each* $n \in \mathbb{N}$.

Second-Order Holonomic Recurrences. Recall that a non-trivial solution $\langle u_n \rangle_{n=-1}^{\infty}$ of the recurrence

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{2}$$

is *minimal* provided that, for all other linearly independent solutions $\langle v_n \rangle_{n=-1}^{\infty}$ of the same recurrence, we have $\lim_{n \to \infty} u_n/v_n = 0$. Since the vector space of solutions has dimension two, it is equivalent for a sequence $\langle u_n \rangle_{n=-1}^{\infty}$ to be minimal for there to exist a linearly independent sequence $\langle v_n \rangle_{n=-1}^{\infty}$ satisfying the above property. In such cases the solution $\langle v_n \rangle_n$ is called *dominant*.

Note that if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent solutions of the above recurrence such that $u_n/v_n \sim C \in \hat{\mathbb{R}}$ then the recurrence relation has a minimal solution [27]. If, in addition, $\langle u_n \rangle_n$ is minimal then all solutions of the form $\langle c u_n \rangle_n$ where $c \neq 0$ are also minimal If $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are respectively minimal and dominant solutions of the recurrence, then together they form a basis of the solution space.

The existence of minimal solutions to a given second-order recurrence has a long history dating back as far as work by Pincherle [32] (cf. [14], [27], [28]).

Theorem 2.2 (Pincherle)**.** *Let* $\langle a_n \rangle_{n=1}^{\infty}$ *and* $\langle b_n \rangle_{n=1}^{\infty}$ *be real-valued sequences such that each of the terms* a_n *is non-zero. First, the recurrence* $u_n = b_n u_{n-1} + a_n u_{n-2}$ *has a minimal solution if and only if the continued fraction* $\operatorname{K}(a_n/b_n)$ *converges. Second, if* $\langle u_n \rangle_n$ *is a minimal solution of this recurrence then the limit of* $\operatorname{K}(a_n/b_n)$ *is* $-u_0/u_{-1}$*. As a consequence, the sequence of canonical denominators* $\langle B_n \rangle_{n=-1}^{\infty}$ *is a minimal solution if and only if the value of* $\operatorname{K}(a_n/b_n)$ *is* $\infty \in \hat{\mathbb{R}}$.

The following determinant lemma below is well-known (see, for example, [27, Lemma 4, §IV]).

3. We denote the sign of a number alternately as belonging to $\{1, 0, -1\}$ or (if zero is excluded) as belonging to $\{+, -\}$.

Lemma 2.3. *Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are both solutions to the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. Then*

$$u_n v_{n-1} - u_{n-1} v_n = (u_0 v_{-1} - u_{-1} v_0) \prod_{k=1}^{n} (-a_k).$$

Convergence of Continued Fractions. The following theorem collects together results from the literature; the first statement follows as a consequence of Worpitzky's Theorem (see [31, Theorem 3.29]) and the convergence results in [33], whilst the second statement follows from the Lane–Wall characterisation of convergence [31, Theorem 3.3].

Theorem 2.4. *Let $\mathrm{K}(\kappa_n/1)$ be a continued fraction with $\langle \kappa_n \rangle_n$ a function in $\mathbb{Q}(n)$. If $\kappa_n < 0$ for all sufficiently large $n \in \mathbb{N}$, then $\mathrm{K}(\kappa_n/1)$ converges to a value in $\hat{\mathbb{R}}$ if and only if, either*

- $\lim_{n\to\infty} \kappa_n$ *exists and is strictly above $-1/4$, or*
- $\lim_{n\to\infty} \kappa_n = -1/4$ *and moreover $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n \log n)^2$ for all sufficiently large n.*

We now turn our attention to positive continued fractions. A continued fraction $\mathrm{K}(a_n/b_n)$ is *positive* if $a_n > 0$ and $b_n \geq 0$ for each $n \in \mathbb{N}$. We will use the following monotonicity result for the odd and even approximants of positive continued fractions [27], [31].

Lemma 2.5. *Suppose that for each $n \in \mathbb{N}$ the sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are positive. Let $\langle f_n \rangle_n$ be the sequence of approximants associated with $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. Then $f_2 \leq f_4 \leq \cdots \leq f_{2m} \leq \cdots \leq f_{2m+1} \leq \cdots \leq f_3 \leq f_1$. If, in addition, $b_1 > 0$ then the subsequences $\langle f_{2n} \rangle_n$ and $\langle f_{2n+1} \rangle_n$ converge to finite, non-negative limits.*

We recall a necessary and sufficient criterion for convergence of a positive continued fraction [31, Theorem 3.14].

Theorem 2.6 (Stern–Stolz Theorem)**.** *A positive continued fraction $\mathrm{K}(a_n/b_n)$ converges if and only if its* Stern–Stolz series $\sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \right|$ *diverges to ∞.*

2.2. Periods and the Kontsevich–Zagier Conjecture

Kontsevich and Zagier [21] define a *period* as a complex number that can be obtained as the value of an integral of an algebraic function over a semialgebraic domain. That is to say, the real and imaginary parts of the number can be written as absolutely convergent integrals of the form

$$\int_{\sigma} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n$$

where f is an algebraic function and the domain $\sigma \subseteq \mathbb{R}^n$ is given by polynomial inequalities with algebraic coefficients. The set of periods forms a countable subring in $\mathbb{C}$ and it is easily seen that this ring contains all the algebraic numbers $\overline{\mathbb{Q}}$. Both $\log \alpha$ (for $\alpha \in \overline{\mathbb{Q}}$) and π are periods since

$$\log \alpha = \int_1^{\alpha} \frac{1}{x} \, dx \quad \text{and} \quad \pi = \int_0^{\infty} \frac{2}{x^2+1} \, dx.$$

Given two algebraic numbers α and β, the problem of determining algorithmically whether $\alpha = \beta$ is known to be decidable. There is no known decision procedure for determining equality between two periods. Subject to the truth of the next conjecture due to Kontsevich and Zagier (cf. [21, Conjecture 1]), the equality of periods is decidable.

Conjecture 2.7. Suppose that a period has two integral representations. One can pass between the representations via a finite sequence of admissible transformations where each transformation preserves the structure that all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$. The admissible transformations are: linearity of the integral, a change of variables, and Stokes's formula.

It is currently not known whether Euler's number e is a period. The following more general notion was introduced in [21] to extend the definition of period to a larger class containing e. An *exponential period* is a complex number that can be written as an absolutely convergent integral of the form

$$\int_{\sigma} \mathrm{e}^{-f(x_1,\ldots,x_n)} g(x_1, \ldots, x_n) \, dx_1 \cdots dx_n$$

where f and g are algebraic functions with algebraic coefficients and the domain $\sigma \subseteq \mathbb{R}^n$ is a semialgebraic set defined by polynomials with algebraic coefficients. Conjecture 2.7 is predicted to generalise to exponential periods in [21].

2.3. Pseudoperiods

In this paper we also encounter integrals that formally resemble univariate periods, and that arise variously as rational linear combinations of values of hypergeometric functions evaluated at real algebraic parameters, and as beta integrals evaluated at algebraic numbers. A (univariate) *pseudoperiod* is a number that can be written as an absolutely convergent integral of the form

$$\int_{\sigma} \sum_{i=1}^{n} \exp\Big(\sum_{j=1}^{k} \alpha_{i,j} \operatorname{Log} f_{i,j}(x) \Big) \, dx \qquad (n, k \in \mathbb{N}),$$

where the domain $\sigma \subseteq \mathbb{R}$ is a semialgebraic set, the functions $f_{i,j}$ are real-valued, non-zero algebraic functions on σ excluding a finite number of points (the poles of f in σ), $\alpha_{i,j} \in \overline{\mathbb{Q}}$, and Log is the principal branch of the complex logarithm.

Integrals of the above form are well-defined since the integrand is measurable[4]. Further, the integrand is (Lebesgue) integrable as the integral is absolutely convergent. Since σ is a finite union of points and open intervals, we lose no generality assuming that σ is an interval with algebraic endpoints, and that moreover it does not contain any poles or zeros of the $f_{i,j}$.

4. For a real-valued algebraic function f, $\operatorname{Log} f(x)$ is defined and is continuous at all but finitely many points (the poles and zeros of f), and is thus measurable. Since $\exp(\sum \alpha_i \operatorname{Log} f_i(x)) = \prod \exp(\alpha_i \operatorname{Log} f_i(x))$, and $\exp(\alpha_i \operatorname{Log} f_i(x))$ is measurable, the claim follows.

The formal resemblance to periods is evidenced by defining $f(x)^\alpha := \exp(\alpha \operatorname{Log} f(x))$, whence the integrand becomes $\sum_{i=1}^{n} \prod_{j=1} f_{i,j}(x)^{\alpha_k}$. When all the $\alpha_{i,j}$ are rational, the pseudoperiod is readily seen to be a *bona fide* univariate period. Conversely, any univariate period is a pseudoperiod.

We remark that there is no particular reason to restrict the definition of pseudoperiods to univariate functions, in which case any period would likewise be a pseudoperiod. In this paper we only encounter univariate entities, so we omit discussion of a more general definition.

Given two finite products of pseudoperiods, the *Pseudoperiod Equality Problem* asks to determine whether the products are equal. The problem is decidable subject to the truth of a stronger version of Conjecture 2.7.

2.4. Infinite Products and the Gamma Function

We say an infinite product $\prod_{k=1}^{\infty} r(k)$ *converges* if the sequence of partial products converges to a finite non-zero limit (otherwise the product is said to diverge). Recall the following classical theorem [34], [35] for the gamma function (see [36]) $\Gamma\colon \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Theorem 2.8. *Consider the rational function*

$$r(k) := \frac{c(k+\varphi_1)\cdots(k+\varphi_m)}{(k+\psi_1)\cdots(k+\psi_{m'})}$$

where we suppose that each $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_{m'}$ is a complex number that is neither zero nor a negative integer. The infinite product $\prod_{k=1}^{\infty} r(k)$ converges to a finite non-zero limit only if $c = 1$, $m = m'$, and $\sum_j \varphi_j = \sum_j \psi_j$. Further, the value of the limit is given by

$$\prod_{k=0}^{\infty} r(k) = \prod_{j=1}^{m} \frac{\Gamma(\psi_j)}{\Gamma(\varphi_j)}.$$

Let $\langle w_n \rangle_{n=-1}^{\infty}$ be a hypergeometric sequence given by $w_n := w_{-1} \prod_{k=0}^{n} r(k)$ with $w_{-1} \neq 0$ and $r(k)$ as above. We deduce that $\langle w_n \rangle_n$ converges to a finite non-zero limit only if $r(k)$, which we term the associated *shift quotient*, satisfies the conditions given in Theorem 2.8.

For $\xi, \nu \in \mathbb{C}$ with $\operatorname{Re}(\xi), \operatorname{Re}(\nu) > 0$, the *beta function* (see [36]) is given by $B(\xi, \nu) = \int_0^1 t^\xi (1-t)^\nu \, dt$, which is a pseudoperiod when ξ and ν are algebraic. Further, we have the following identity $\Gamma(\xi)\Gamma(\nu) = B(\xi,\nu)\Gamma(\xi+\nu)$, which we shall (repeatedly) apply to products of the form $\Gamma(\psi_1)\cdots\Gamma(\psi_k)$ in the sequel.

2.5. Hypergeometric Functions

A series $\sum c_k x^k$ is called *hypergeometric* if the sequence $\langle c_n \rangle_n$ is hypergeometric. It can be shown (see [36]) that a hypergeometric series can be written as follows

$$\sum_{k=0}^{\infty} c_k x^k = c_0 \sum_{k=0}^{\infty} \frac{(\varphi_1)_k \cdots (\varphi_j)_k}{(\psi_1)_k \cdots (\psi_\ell)_k} \frac{x^k}{k!},$$

where, for $\rho \in \mathbb{C}$, the *(rising) Pochhammer symbol* $(\rho)_n$ is defined as $(\rho)_0 = 1$, and $(\rho)_n = \prod_{j=0}^{n-1}(\rho + j)$ for $n \geq 1$. Here the parameters ψ_m are not negative integers or zero for otherwise the denominator would vanish for some k. The series terminates if any of the parameters φ_m is zero or a negative integer. We let ${}_jF_\ell(\varphi_1, \ldots, \varphi_j; \psi_1, \ldots, \psi_\ell; x)$ denote the series on the right-hand side.

A hypergeometric series ${}_jF_\ell(\varphi_1, \ldots, \varphi_j; \psi_1, \ldots, \psi_\ell; x)$ converges absolutely for all x if $j \leq \ell$ and for $|x| < 1$ if $j = \ell + 1$. It diverges for all $x \neq 0$ if $j > \ell + 1$ and the series does not terminate. Furthermore, in the case that $j = \ell + 1$, the series with $|x| = 1$ converges absolutely if $\operatorname{Re}(\sum \psi_i - \sum \varphi_i) > 0$. The series converges conditionally if $x \neq 1$ and $0 \geq \operatorname{Re}(\sum \psi_i - \sum \varphi_i) > -1$. The series diverges if $\operatorname{Re}(\sum \psi_i - \sum \varphi_i) \leq -1$.

Abusing notation, we let ${}_jF_\ell(\varphi_1, \ldots, \varphi_j; \psi_1, \ldots, \psi_\ell; x)$ denote the analytic function defined by the corresponding series in its radius of convergence (assuming it has a positive radius of convergence), and elsewhere by analytic continuation.

Example 2.9. Of special interest to us is the *Gauss hypergeometric function* ${}_2F_1(a, b; c; x)$, with $a, b, c \in \mathbb{C}$, c neither zero nor a negative integer. It is a single-valued function on the cut complex plane $\mathbb{C} \setminus [1, \infty)$ (cf. [27, §VI.1]).

Another function of interest is the *confluent hypergeometric function (of the first kind)* ${}_1F_1(a; b; x)$, b neither zero nor a negative integer. The series defines an entire function.

For $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, Euler gave the following integral representation: for all $x \in \mathbb{C} \setminus [1, \infty)$, ${}_2F_1(a, b; c; x)$ is equal to

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a} \, dt, \tag{3}$$

where t^b, $(1-t)^{c-b-1}$, and $(1-xt)^{-a}$ have their principal values. Here the path of integration is understood to be along the real line between 0 and 1. We observe that for a, b, c, $x \in \overline{\mathbb{Q}}$, the integral in (3) is a pseudoperiod.

3. The Hypergeometric Inequality Problem

At order one, both minimality and positivity are algorithmically trivial. Indeed, positivity of a hypergeometric sequence is readily determined by inspecting the polynomial coefficients of its defining recurrence relation together with the sign of the first few values of the sequence. For minimality, the solution set of a hypergeometric recurrence is a one-dimensional vector space and so such recurrences cannot possibly admit minimal sequences. Thus we have the following.

Lemma 3.1. *Minimality and Positivity are both decidable for hypergeometric sequences.*

We also invoke the following lemma (considered folklore) in our proof of Theorem 3.3 (below).

Lemma 3.2. *The Skolem Problem for hypergeometric sequences is decidable.*

Recall that the Skolem Problem for holonomic sequences asks to determine whether there is an index n such that $u_n = 0$. Discussions of the Skolem Problem for C-finite sequences at low orders are given in [8].

Let us turn our attention to the subject of this section. Given two hypergeometric sequences $\langle u_n \rangle_{n=-1}^{\infty}$ and $\langle v_n \rangle_{n=-1}^{\infty}$, the *Hypergeometric Inequality Problem* (HIP) asks to determine whether $u_n \leq v_n$ for each $n \in \{-1, 0, \ldots\}$. The main result of this section is the following theorem, which follows from Lemma 3.4 and Proposition 3.6.

Theorem 3.3. *The Hypergeometric Inequality Problem reduces to the problem of checking equality between pseudoperiods.*

3.1. Reduction Argument for Theorem 3.3

An intermediate step in our proof of Theorem 3.3 introduces the *Hypergeometric Threshold Problem* (HTP). Given a hypergeometric sequence $\langle w_n \rangle_n^{\infty}$ and a real-algebraic constant θ (the threshold), the problem is to determine $w_n \leq \theta$ for each n. Clearly the Positivity Problem is a specialisation of the Threshold Problem with threshold zero.

Lemma 3.4. *The Hypergeometric Inequality Problem reduces to the Hypergeometric Threshold Problem.*

Proof. Let $\langle \hat{u}_n \rangle_n$ and $\langle \tilde{u}_n \rangle_n$ be hypergeometric sequences with associated shift quotients $\hat{r}(n)$ and $\tilde{r}(n)$, respectively. We can assume that $\hat{r}(n)$ and $\tilde{r}(n)$ are constant in sign by shifting the sequence as appropriate. By Lemma 3.2, one can decide whether there exists an $n \in \{-1, 0, \ldots\}$ such that $\tilde{u}_n = 0$. If such an n exists, the HIP reduces to determining the Positivity Problem for $\langle \hat{u}_n \rangle_n$ (which is trivial by Lemma 3.1) with a number of comparisons between initial values determined by the shift. Thus we can assume there is no $n \in \{-1, 0, \ldots\}$ for which $\tilde{u}_n = 0$.

Let $\langle w_n \rangle_n$ be the sequence with terms given by $w_n := \operatorname{sign}(\tilde{u}_n)\hat{u}_n/\tilde{u}_n$. The sequence $\langle w_n \rangle_n$ is well-defined and satisfies the recurrence $w_n = r(n)w_{n-1}$ where $r(n) := \operatorname{sign}(\tilde{r})\hat{r}(n)/\tilde{r}(n)$. Thus $\langle w_n \rangle_n$ is a hypergeometric sequence. It is clear that $\hat{u}_n \leq \tilde{u}_n$ for each n if and only if $w_n \leq \operatorname{sign}(\tilde{r})$. There are two cases to consider: either $\operatorname{sign}(\tilde{r})$ is alternating or constant. If $\operatorname{sign}(\tilde{r})$ is alternating then one considers two instances of the HTP as it is sufficient to determine whether $w_{2n} \leq -\operatorname{sign}(\tilde{u}_{-1})$ and $w_{2n-1} \leq \operatorname{sign}(\tilde{u}_{-1})$ for each n. Thus in both cases the HIP reduces to instances of the HTP. $\square$

Let us list our ongoing assumptions (which we make without loss of generality):

1) Given an instance of the HTP, we shall assume that the threshold is non-zero (Lemma 3.1).
2) Given a hypergeometric sequence $\langle w_n \rangle_n$ we shall assume that $\operatorname{sign}(w_n)$ is constant (Lemma 3.4) and that there is no such $n \in \mathbb{N}$ for which $w_n = 0$ (Lemma 3.2).

We add two further assumptions to our list:

3) Given a hypergeometric sequence $\langle w_n \rangle_n$, we shall assume that the associated shift quotient takes the form
$$r(n) = c\frac{(n + \varphi_1) \cdots (n + \varphi_m)}{(n + \psi_1) \cdots (n + \psi_{m'})}$$
such that $c > 0$ and each parameter $\varphi_j, \psi_j \in \overline{\mathbb{Q}}$. By taking a suitable shift, we can assume that $\operatorname{Re}(\varphi_j), \operatorname{Re}(\psi_j) > 0$ for each j.
4) Subject to a suitable shift, we assume that both $\langle r(n) \rangle_n$ and $\langle r(n) - 1 \rangle_n$ are monotone and have constant sign.

Lemma 3.5. *The Hypergeometric Threshold Problem reduces to the problem of determining equality between a real-algebraic number and the limit of a hypergeometric sequence.*

Proof. We have $w_n - w_{n-1} = (r(n) - 1)w_{n-1}$. Sequences $\langle r(n) - 1 \rangle_n$ and $\langle w_n \rangle_n$ have constant sign (by assumption), so we deduce that $\langle w_n \rangle_n$ is monotone. For the class of sequences that diverge to $\pm\infty$, the HTP is easily decidable. Indeed, if $\langle w_n \rangle_n$ diverges to $-\infty$ then it suffices to check whether $w_{-1} \leq \theta$. Thus we can safely assume that the limit of $\langle w_n \rangle_n$ exists and is finite.

Assume that $\lim_{n\to\infty} w_n = \ell \in \mathbb{R}$. Let θ, the threshold, be a real-algebraic constant. Then one needs to determine whether: $\ell > \theta$, in which case one can detect that eventually $w_n > \theta$; $\ell < \theta$, in which case it suffices to check whether $w_{-1} \leq \theta$ as $\langle w_n \rangle_n$ is monotone; or $\ell = \theta$. Thus the HIP is recursively enumerable and, in addition, the problem is decidable subject to an oracle that can determine whether $\ell = \theta$. $\square$

Proposition 3.6. *The Hypergeometric Threshold Problem reduces to the Pseudoperiod Equality Problem.*

Proof. By Lemma 3.5, the HTP is decidable subject to determining whether $\lim_{n\to\infty} w_n = \ell$ is equal to a given algebraic threshold θ. As previously noted, we can freely assume that $\theta \neq 0$. This in turn means we can assume that $\ell \neq 0$. Thus the associated shift quotient $r(n)$ satisfies the necessary assumptions in Theorem 2.8 and, additionally, $\ell = \theta$ if and only if $\Gamma(\psi_1) \cdots \Gamma(\psi_k) = \theta \cdot \Gamma(\varphi_1) \cdots \Gamma(\varphi_k)$.

For the former product we have
$$\Gamma(\psi_1) \cdots \Gamma(\psi_k) = \Gamma(\textstyle\sum_{i=1}^{k} \psi_i) \displaystyle\prod_{i=1}^{k-1} B(\textstyle\sum_{j\leq i} \psi_j, \psi_{i+1}),$$
and we obtain an analogous expression (also in terms of the beta function) for the latter product. Since $\sum_{i=1}^{k} \psi_i = \sum_{i=1}^{k} \varphi_i$, the problem of determining whether $\Gamma(\psi_1) \cdots \Gamma(\psi_k)$ and $\theta \cdot \Gamma(\varphi_1) \cdots \Gamma(\varphi_k)$ are equal reduces to determining whether
$$\prod_{i=1}^{k-1} B(\textstyle\sum_{j\leq i} \psi_j, \psi_{i+1}) = \theta \displaystyle\prod_{i=1}^{k-1} B(\textstyle\sum_{j\leq i} \varphi_j, \varphi_{i+1}).$$
Consequently, determining whether $\ell = \theta$ reduces to an instance of the Pseudoperiod Equality Problem. $\square$

Remark 3.7. If the sequence parameters φ_ℓ and ψ_m in the preceding displayed equation are rational, then the act of determining whether the equality holds is an instance of testing equality between periods. Hence in this restricted setting, the HIP is decidable subject to the truth of Conjecture 2.7.

As can be seen from the preceding discussion, the HIP reduces to determining the multiplicative relations of the Gamma function for given inputs. The latter problem is the subject of an older conjecture due to Rohrlich [37, Conjecture 21]. The Rohrlich Conjecture predicts that any multiplicative relation of the form $\prod_{j=1}^{n}(2\pi)^{-1/2}\Gamma(\xi) = \theta$ for appropriate $\xi_j \in \mathbb{Q}$ and real-algebraic θ can be derived from the standard functional identities for the gamma function (the translation, reflection, and multiplicative properties). We remark that, similar to Conjecture 2.7, Rohrlich's conjecture implies there is a semi-algorithm that can decide the HIP with rational inputs by enumerating all applications of the functional relations.

4. The Positivity and Minimality Problems for Degree-1 Second-Order Holonomic Sequences

Let us state the main result of this section.

Theorem 4.1. *For degree-*1 *second-order holonomic sequences, the Positivity and Minimality Problems reduce to deciding equalities between pseudoperiods and exponential periods.*

Proof Strategy. We give a high-level overview of our proof strategy. First, we will show that Positivity reduces to Minimality for degree-1 second-order holonomic sequences (Section 4.1). We thus deduce the desired corollary.

Corollary 4.2. *For degree-1 second-order holonomic sequences the Positivity Problem reduces to the Minimality Problem.*

In light of Corollary 4.2, in order to establish Theorem 4.1 it remains to prove that Minimality for degree-1 second-order sequences reduces to the aforementioned equality checking problems. For reference, Theorem 4.1 will follow as a consequence of Corollary 4.7 and Theorem 4.9 (below).

4.1. Positivity reduces to Minimality at Low Degrees

For second-order holonomic sequences, Kenison et al. [20] proved that Positivity reduces to Minimality for second-order holonomic sequences. There is a single crucial step in their reduction argument that does not preserve the degrees of the polynomial coefficients—which is necessary to establish Theorem 4.1 above. More specifically, this step transforms a $(-,+)$ relation into a $(+,-)$ relation. It is therefore necessary to closer examine the case $(-,+)$ under our running assumption that the recurrence in question is degree-1. We perform this examination in the work that follows.

We shall require the following technical lemma.

Lemma 4.3. *Suppose that $\langle u_n\rangle_n$ is a solution sequence for recurrence* (4) *with signature $(-,+)$. Assume that $u_{-1} > 0$. For even $n \in \mathbb{N}$, $u_n > 0$ if and only if $f_n > -u_0/u_{-1}$. For odd $n \in \mathbb{N}$, $u_n > 0$ if and only if $f_n < -u_0/u_{-1}$.*

Proof. Recall the canonical solution sequences $\langle A_n\rangle_{n=-1}^{\infty}$ and $\langle B_n\rangle_{n=-1}^{\infty}$. Then $u_n = A_n u_{-1} + B_n u_0$ for each $n \in \{-1, 0, \dots\}$. For recurrences with signature $(-,+)$, it is easy to show by induction that $B_n < 0$ for each odd $n \in \mathbb{N}$, and $B_n > 0$ for each even $n \in \mathbb{N}$. Thus for even $n \in \mathbb{N}$ we have that $u_n > 0$ if and only if $A_n/B_n + u_0/u_{-1} = f_n + u_0/u_{-1} > 0$. The case for odd $n \in \mathbb{N}$ is handled in a similar fashion. □

Another consequence of the work by Kenison et al. shows that, for our purposes, it is sufficient to consider only those recurrence relations whose associated continued fraction $\mathrm{K}(a_n/b_n)$ converges to a finite limit. This claim is explained by the following observation (due to [20]): the positive solutions of a second-order holonomic recurrence relation with signature $(-,+)$ are precisely those solutions that are minimal. Without loss of generality we can assume the limit of the associated continued fraction is finite due to Pincherle's Theorem. Thus all that remains is to prove the following proposition.

Proposition 4.4. *Suppose that $\langle u_n\rangle_n$ with initial values $u_{-1}, u_0 > 0$ is a solution sequence for a degree-1 recurrence* (4) *with signature $(-,+)$ whose associated continued fraction $\mathrm{K}(a_n/b_n)$ converges to a finite limit f. The following statements are equivalent:*

1) *the sequence $\langle u_n\rangle_n$ is positive,*
2) *the sequence $\langle u_n\rangle_n$ is minimal, and*
3) *$-u_0/u_{-1} = f$.*

Proof. The sequence $\langle -f_n\rangle_{n=1}^{\infty}$ is the sequence of approximants associated with $\mathrm{K}(a_n/-b_n)$. This is a positive continued fraction and so, by Lemma 2.5, the subsequences $\langle -f_{2n}\rangle_{n=1}^{\infty}$ and $\langle -f_{2n-1}\rangle_{n=1}^{\infty}$ converge to finite limits $-\ell_1$ and $-\ell_2$, respectively. By Lemma 4.3, a solution sequence $\langle u_n\rangle_n$ is positive if and only if $\ell_2 \le -u_0/u_{-1} \le \ell_1$. The Stern–Stolz series (Theorem 2.6) associated with $\mathrm{K}(a_n/-b_n)$ diverges due to our assumption that each of the coefficients in (4) is a polynomial with degree in $\{0,1\}$. We conclude that $\ell_1 = \ell_2$. Thus $\langle u_n\rangle_n$ is positive if and only if $-u_0/u_{-1}$ is equal to $f = \ell_1 = \ell_2$. From Theorem 2.2, a solution sequence $\langle u_n\rangle_n$ is minimal if and only if $-u_0/u_{-1}$ is the value of the continued fraction $\mathrm{K}(a_n/b_n)$. □

We thus deduce the desired result, Corollary 4.2.

4.2. Parameterising the Minimality Problem

In the sequel it is helpful to parameterise the Minimality Problem as follows.

Problem 4.5 (Minimal(j,k,ℓ))**.** Given a solution $\langle v_n\rangle_{n=-1}^{\infty}$ to (1) with $\deg(p_3) = j, \deg(p_2) = k$, and $\deg(p_1) = \ell$, determine whether $\langle v_n\rangle_n$ is minimal.

Cases where any of the polynomial coefficients are identically 0 are straightforward consequences of the work in Section 3. Thus we shall focus on $j, k, \ell \in \{0, 1\}$.

4.3. Interreductions for $\mathsf{Minimal}(j,k,\ell)$

Consider problem $\mathsf{Minimal}(0,0,0)$: determine whether a holonomic sequence that solves a second-order C-finite recurrence is a minimal solution. Since this problem is a special case of $\mathsf{Minimal}(1,1,1)$ (multiply each of the coefficients by $(n+1)$) we make no further mention of $\mathsf{Minimal}(0,0,0)$ in the sequel. Thus we need only consider the eight remaining cases associated with $\mathsf{Minimal}(j,k,\ell)$. We first reduce the number of problems to five by establishing interreductions between instances of $\mathsf{Minimal}(j,k,\ell)$. Next we employ minimality-preserving transformations to obtain canonical instances of each of the remaining problems (Corollary 4.7). In the sequel we show that these canonical instances reduce to checking whether a pseudoperiod or an exponential period vanishes. The different cases are listed in Theorem 4.9.

For the remaining problem instances it is useful to establish the following conventions. In each instance $\mathsf{Minimal}(j,k,\ell)$ we consider a recurrence relation of the form

$$(\alpha_1 n+\alpha_0)u_n = (\beta_1 n+\beta_0)u_{n-1} + (\gamma_1 n+\gamma_0)u_{n-2}. \quad (4)$$

In the case that $\deg(\alpha_1 n+\alpha_0)=0$ we understand that $\alpha_1=0$ (and adopt a similar convention for the other coefficients). We shall always assume that the values α_0,β_0, and γ_0 are non-zero in accordance with the assumption that the polynomial coefficients do not vanish on non-negative integers. We also introduce the notation $\alpha := \alpha_0/\alpha_1$.

For recurrence (4) with $\alpha_1\neq 0$, we define the associated *characteristic polynomial* as $\alpha_1 x^2-\beta_1 x-\gamma_1$, and refer to its roots as the associated *characteristic roots*.

Proposition 4.6 presents the cases we need to consider to establish the Minimality Problem at low degrees. The proof of Proposition 4.6, a necessary (but tedious) exercise in accounting, is relegated to Appendix A.

Proposition 4.6.

1) $\mathsf{Minimal}(0,k,1)$ *and* $\mathsf{Minimal}(1,k,0)$ *are interreducible.*
2) $\mathsf{Minimal}(1,0,0)$ *reduces to* $\mathsf{Minimal}(1,1,1)$.
3) $\mathsf{Minimal}(1,1,1)$ *reduces to the Minimality Problem for a recurrence of the form*

$$u_n = \frac{\beta_1 n+\beta_0}{n+\alpha}u_{n-1} + \frac{\gamma_1 n+\gamma_0}{n+\alpha}u_{n-2}, \quad (5)$$

where $\alpha,\beta_0,\beta_1,\gamma_0,\gamma_1\in\mathbb{Q}$, $\beta_1>0$, *and* $|\gamma_1|=\beta_1$. *The characteristic roots associated to the recurrence are* $(\beta_1\pm\sqrt{\beta_1^2+4\gamma_1})/2$. *If* $\beta_1^2+4\gamma_1=0$, *then we have a further reduction to the Minimality Problem for solutions to*

$$u_n = \frac{2n+\beta_0}{n+\alpha}u_{n-1} - \frac{n+\gamma_0}{n+\alpha}u_{n-2}, \quad (6)$$

where $\alpha,\beta_0,\gamma_0\in\mathbb{Q}$ *and the recurrence has a single repeated characteristic root* 1.

4) $\mathsf{Minimal}(1,0,1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*[5]

$$u_n = \frac{\beta_0}{n+\alpha}u_{n-1} + \frac{n+\gamma_0}{n+\alpha}u_{n-2}, \quad (7)$$

where $\alpha,\gamma_0\in\mathbb{Q}$ *and* $\beta_0\in\overline{\mathbb{Q}}\cap\mathbb{R}_{>0}$. *The characteristic roots associated to the recurrence are* ± 1.

Let λ and μ be the roots of the associated characteristic polynomial such that $|\lambda|\leq|\mu|$. In recurrences (5)–(7) γ_1 is not zero so we have $\lambda,\mu\neq 0$. Further, by setting $\mu=1$ for (7), we have that $\mu>0$ for the associated recurrences (5)–(7), as the coefficient $\beta_1>0$ in the first two.

We shall treat recurrences (5) and (6) as distinct cases in the sequel. That is to say, in the former we shall always assume that $\beta_1^2+4\gamma_1\neq 0$ (so that the characteristic roots are distinct).

The following corollary is an immediate application of Proposition 4.6.

Corollary 4.7. *For* $j,k,\ell\in\{0,1\}$, *decidability of problem* $\mathsf{Minimal}(j,k,\ell)$ *reduces to proving decidability of* $\mathsf{Minimal}(0,1,0)$, $\mathsf{Minimal}(0,1,1)$, *and decidability of the Minimality Problem for solutions to recurrences* (5)–(7).

Recall that the problem of determining whether a degree-1 second-order recurrence relation admits minimal solutions is decidable. The next lemma gives necessary and sufficient conditions for the relevant recurrences to admit minimal solutions. The straightforward proof, which uses Pincherle's Theorem and standard results on the convergence of continued fractions stated in the Preliminaries, is given in Appendix B. In particular, in the sequel we shall assume that the characteristic roots of a recurrence relation are real-valued.

Lemma 4.8.

1) *A recurrence* $u_n=(\beta_1 n+\beta_0)u_{n-1}+(\gamma_1 n+\gamma_0)u_{n-2}$ *with* $\beta_0,\beta_1,\gamma_0,\gamma_1\in\mathbb{Q}$, *with* $\beta_0,\beta_1,\gamma_0\neq 0$, *admits minimal solutions.*
2) *A recurrence of the form* (5) *or* (7) *admits minimal solutions if and only if the associated characteristic roots are real.*
3) *A recurrence of the form* (6) *admits minimal solutions if and only if* $\beta_0-\alpha-\gamma_0\geq 0$.

4.4. Reduction to Equality Checking

We now turn our attention to the final step in the proof of Theorem 4.1. For degree-1 second-order holonomic sequences, Theorem 4.1 claims that the Positivity and Minimality Problems reduce to deciding equalities between pseudoperiods and exponential periods. In light of the preceding discussion, it suffices to prove the following result.

Theorem 4.9.

1) $\mathsf{Minimal}(0,1,0)$ *reduces to checking whether an exponential period vanishes.*

5. Notice that sequences satisfying (7) are not necessarily holonomic as β_0 can be a real algebraic number.

2) $\mathsf{Minimal}(0,1,1)$ *reduces to checking whether an exponential period vanishes.*
3) *Minimality for recurrences* (5) *and* (7) *reduce to the Pseudoperiod Equality Problem.*
4) *Minimality for recurrences* (6) *reduces to checking whether an exponential period vanishes.*

We thus devote the remainder of the section to the proof of Theorem 4.9. The lynchpin in deciding minimality is Pincherle's Theorem (Theorem 2.2), which, as the reader will recall, transforms such problems into questions about the convergence of polynomial continued fractions. The remainder of our approach to the proof of Theorem 4.1 is via techniques and tools given by Lorentzen and Waadeland in [27, §VI.4.1]. Those authors consider low-degree polynomial continued fractions and express their limits as quotients of *hypergeometric functions*. These functions admit, after suitable transformations, integral representations that involve pseudoperiods or exponential periods.

We include the proof of Theorem 4.9(3) in the main text. In order to avoid repetition amongst similar arguments, the proofs of the remaining items are relegated to Appendix B.

As a point of independent interest, it is notable that the evaluation of low-degree polynomial continued fractions by Lorentzen and Waadeland does not extend to the continued fractions associated with Theorem 4.9(4) (and, by extension, recurrences of the form (6)). As far as we are aware, our analysis is the first to complete this gap in the literature.

Proof of Theorem 4.9(3). Consider recurrence (5). Recall μ and λ are the roots of the characteristic polynomial of recurrence with $\mu > |\lambda| > 0$. Let $\langle u_n \rangle_n$ be a minimal solution to the recurrence. By Theorem 2.2, $-u_0/u_{-1} = \mathrm{K}(a_n/b_n)$. Then $\mathrm{K}(a_n/b_n)$ is equivalent to the continued fraction

$$\frac{1}{\alpha\delta^2} \operatorname*{K}_{n=1}^{\infty} \frac{(\gamma_1 n + \gamma_0)(n+\alpha-1)\delta^2}{(\beta_1 n + \beta_0)\delta}$$

by applying Theorem 2.1 with $\mathfrak{r}_0 = 1$, $\mathfrak{r}_n = \delta(n+\alpha)$ for $n \in \mathbb{N}$, and δ a non-zero real number. Setting $\delta = 1/\mu$, $x = \lambda/\mu$, and either

$$a = \frac{\lambda^2\alpha - \lambda\beta_0 - \gamma_0}{\lambda(\lambda-\mu)},\ b = \alpha - 1,\ c = a + \frac{\gamma_0}{\gamma_1};\quad \text{or}$$

$$a = \frac{\mu^2\alpha - \mu\beta_0 - \gamma_0}{\mu(\lambda-\mu)} + 1,\ b = \frac{\gamma_0}{\gamma_1},\ c = a + \alpha - 1$$

we obtain, by substitution, the continued fraction

$$\frac{\mu^2}{\alpha} \operatorname*{K}_{n=1}^{\infty} \frac{-(c-a+n)(b+n)x}{c+n+(b-a+1+n)x}. \tag{8}$$

Here we note that $\gamma_1 = -\lambda\mu$ and $\beta_1 = \lambda + \mu$. Notice here that $|x| < 1$ by assumption. Furthermore, by shifting the sequence appropriately, the parameter a does not change, but b and c are shifted by an integer value. Hence we may assume that c is a positive number. Finally, $a, b, c, x \in \mathbb{Q}(\mu)$ and so are real algebraic numbers. By Lorentzen and Waadeland's work [27, §VI, Theorem 4(A)],

$$-\frac{\alpha}{\mu^2}\frac{u_0}{u_{-1}} = \frac{c\,{}_2F_1(a,b;c;x)}{{}_2F_1(a,b+1;c+1;x)} - \frac{\beta_0}{\mu}. \tag{9}$$

Let us first show that the equality problem is decidable if a, b, $c-a$, or $c-b$ is a negative integer, or if either $a=0$, or $c-b=0$. If either $a=0$, or that either a or b is a negative integer, then both the hypergeometric series terminate. Thus equality testing reduces to checking equality between algebraic numbers, which is decidable. Moreover, we have Euler's transformation (cf. [36, Eq. 2.2.7]):

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x),$$

which implies that if $c-b=0$ or one of $c-a$ and $c-b$ is a negative integer then again the series terminates, and the problem reduces to checking equality between algebraic numbers. Thus for the remainder of the proof we assume that $a, b, c-a, c-b$ are not negative integers and, in addition, that both a and $c-b$ are non-zero.

Consider the sequence $\langle P_n(x) \rangle_n$ defined by

$$P_{2n}(x) = {}_2F_1(a+n, b+n; c+2n; x) \text{ and}$$
$$P_{2n+1}(x) = {}_2F_1(a+n, b+n+1; c+2n+1; x).$$

Let $\langle s_n \rangle_{n=1}^{\infty}$ be the sequence of linear fractional transformations given by by $s_n(w) := a_n x/(1+w)$ such that a_{2n+1} and a_{2n} are given by

$$\frac{-(a+n)(c-b+n)}{(c+2n)(c+2n+1)} \text{ and } \frac{-(b+n)(c-a+n)}{(c+2n-1)(c+2n)},$$

respectively. It can be shown that

$$\frac{P_0(x)}{P_1(x)} = s_1 \circ \cdots \circ s_n\left(\frac{P_n(x)}{P_{n+1}(x)}\right)$$

(see [27, § VI.1]). Under the aforementioned assumptions we have $a_i \neq 0$ for each $i = 1, \ldots, n$, and so the composition of the s_i is an invertible linear fractional transformation. It follows that

$$\frac{P_0(x)}{P_1(x)} = \frac{X_n P_{n+1}(x) + Y_n P_n(x)}{Z_n P_{n+1}(x) + W_n P_n(x)}, \tag{10}$$

where the sequences $\langle W_n \rangle_n, \langle X_n \rangle_n, \langle Y_n \rangle_n, \langle Z_n \rangle_n$ are over $\mathbb{Q}(a,b,c,x)$ and satisfy $X_n W_n - Y_n Z_n \neq 0$. Take $N \in \mathbb{N}$ to be both even and sufficiently large such that $c + 2N > b + N$. Substituting (10) into (9) with $n = N$ and rearranging, we obtain the equation $\mathfrak{a}_N P_{N+1}(x) = \mathfrak{b}_N P_N(x)$, where $\mathfrak{a}_N$ and $\mathfrak{b}_N$ are defined by

$$X_N - Z_N \frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2} \text{ and } \frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2} W_N - Y_N,$$

respectively. Observe now that Euler's integral representation (3) holds for both P_{N+1} and P_N. By linearity of the integral, we see that deciding Minimality for recurrences of the form (5) reduces to checking whether the integral

$$\int_0^1 \frac{t^{b+N-1}(1-t)^{c-b+N-1}}{(1-xt)^{a+N}} \left(\mathfrak{a}_N \tfrac{(c+2N)}{b+N} t - \mathfrak{b}_N\right) dt$$

vanishes. This integral is a pseudoperiod, so we are done for this part.

Now consider recurrence (6). In a similar manner to previous arguments, a solution $\langle u_n \rangle_n$ of (6) is minimal if

$-\alpha u_0/u_{-1}$ is equal to the continued fraction in (8) when one sets $x = -1$, and either

$$a = \frac{1}{2}(\alpha + \beta_0 - \gamma_0),\ b = \alpha - 1,\ c = \frac{1}{2}(\alpha + \beta_0 + \gamma_0); \text{ or}$$
$$a = \frac{1}{2}(\beta_0 + \gamma_0 - \alpha) + 1,\ b = \gamma_0,\ c = \frac{1}{2}(\beta_0 + \gamma_0 + \alpha).$$

As $c+a-b-1 = \beta_0 > 0$ and the parameters are real-valued, by [27, §VI, Theorem 4(A)] the continued fraction converges to (9). The remainder of the proof follows similarly. □

We briefly summarise the connection between the outstanding parts of Theorem 4.9 and properties of various hypergeometric functions. For Theorem 4.9(1) we show that the equality problem reduces to checking whether ${}_0F_1(;a;x)/{}_0F_1(;a+1;x)$, where $a, x \in \mathbb{Q}$, is equal to an algebraic number. Similarly, for Theorem 4.9(2) the problem reduces to checking whether ${}_1F_1(a;b;x)/{}_1F_1(a+1;b+1;x)$ is equal to an algebraic number. Here again $a, x \in \mathbb{Q}$. Finally for Theorem 4.9(4) we show that the problem reduces to checking whether $U(a,b,x)/U(a+1,b+1,x)$ is equal to an algebraic number. Here again $a, b, x \in \mathbb{Q}$, and U is defined for all $a, b, x \in \mathbb{C}$, $\mathrm{Re}(a), \mathrm{Re}(x) > 0$ as

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zt} t^{a-1} (1+t)^{b-a-1}\, dt.$$

The definition of U immediately gives the desired result after rearranging (and shifting the sequence appropriately).

The remainder of the proof in each of the outstanding cases follows by inspection of integral representations of the aforementioned hypergeometric functions. The integrals for ${}_0F_1(;s+1;z)$ and ${}_1F_1(a;b;x)$ are

$$\frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} \int_{-1}^{1} \mathrm{e}^{-2\sqrt{z}t} (1-t^2)^{s-1/2}\, dt, \text{ and}$$
$$\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} \mathrm{e}^{xt}\, dt,$$

respectively (cf. [38, Eq. 9.1.69], [36, Eq. 4.7.5], and [38, Eq. 13.2.1]). These representations hold for $\mathrm{Re}(a) > -1/2$ and $\mathrm{Re}(b) > \mathrm{Re}(a) > 0$, respectively. The desired results are then easily deduced by a simple rearrangement argument.

5. Pfaffian Functions and Gabrielov–Vorobjov Oracles

We now turn to the final theme in this paper. In this section we give a brief model-theoretic discussion of periods and related expressions, along with the attendant decision problems. We note that the classes of periods, exponential periods, and pseudoperiods discussed herein can also be expressed as values of univariate Pfaffian functions evaluated at rational arguments, and in turn all the decision problems considered in this paper become decidable subject to classical Gabrielov–Vorobjov oracles.

Khovanskii introduced Pfaffian functions in [39], [40]. Let U be an open domain in $\mathbb{R}$. A *Pfaffian chain* in U is a sequence of complex analytic functions

$$\phi_1, \phi_2, \ldots, \phi_r \colon U \to \mathbb{C}$$

satisfying Pfaffian differential equations

$$\phi_j'(x) = P_j(x, \phi_1, \ldots, \phi_j) \quad \text{for } j = 1, \ldots, r$$

where each $P_j \in \mathbb{Z}[i][x, y_1, \ldots, y_j]$ is a polynomial with coefficients among the Gaussian integers. A function $\phi \colon U \to \mathbb{C}$ is called a (univariate) *Pfaffian function* if there exists a Pfaffian chain $\phi_1, \phi_2, \ldots, \phi_r$ and a polynomial $P \in \mathbb{Z}[i][x, y_1, \ldots, y_r]$ such that $\phi(x) = P(x, \phi_1, \ldots, \phi_r)$.

Gabrielov and Vorobjov [41] introduce the concept of an oracle for deciding the consistency of a system of Pfaffian constraints.[6] Subject to such an oracle, they provide an algorithm for the smooth stratification of semi-Pfaffian sets. This notion of a *Gabrielov–Vorobjov oracle* has since been reused in a wide range of contexts (see, e.g., [42], [43], [44], [45], [46], [47]).

It is straightforward to show that any univariate period, exponential period, or pseudoperiod α can be written as $\alpha = \lim_{x\to 1^-} f(x)$, where $f : (0,1) \to \mathbb{C}$ is a univariate Pfaffian function. In fact, one can write $f(x) = \int_0^x g(y)\, dy$, where g itself is also Pfaffian on $(0,1)$.

Suppose one wishes to determine whether two given products of pseudoperiods (or [exponential] periods) $\alpha_1 \cdots \alpha_j$ and $\beta_1 \cdots \beta_j$ are equal, in other words solving a particular instance of the Pseudoperiod Equality Problem. Write each α_i as $\lim_{x\to 1^-} f_i(x)$, with

$$f_i(x) = \int_0^x g_i(y)\, dy,$$

and each β_i as $\lim_{x\to 1^-} p_i(x)$, with

$$p_i(x) = \int_0^x q_i(y)\, dy,$$

where all the functions in play are Pfaffian. Observe that each $F_i(x) := f_i(x) + \int_x^1 g_i(y)\, dy$ is Pfaffian with domain $(0,1)$, and of constant value $F_i(x) = \alpha_i$. Likewise one can write $P_i(x) = \beta_j$ for Pfaffian P_i defined over $(0,1)$. The question of whether $\alpha_1 \cdots \alpha_j = \beta_1 \cdots \beta_j$ therefore boils down to asking whether there is some $x \in (0,1)$ such that $F_1(x) \cdots F_j(x) = P_1(x) \cdots P_k(x)$, for which a Gabrielov–Vorobjov oracle readily provides an answer, since Pfaffian functions are closed under products and we are therefore indeed dealing with a (rather simple) Pfaffian constraint system.

6. Conclusion

Summary of Results. The main results in this paper, Theorem 3.3) and Theorem 4.1, draw new and novel connections between inequality decision problems for low-order holonomic sequences and the problem of checking equality

6. Gabrielov and Vorobjov consider both real Pfaffian and complex Pfaffian functions in their paper. For real Pfaffian functions, a constraint system consists of a Boolean combination of inequalities among Pfaffian functions; the system is *consistent* if there exist values of the variables within the relevant functions' domains for which the constraint system is satisfied. In the case of complex Pfaffian functions, only Boolean combinations of equalities and disequalities are allowed.

between the class of periods (and their generalisations). We also give interesting connections between inequality decision problems in this setting and the theory of Pfaffian functions (Section 5); in particular, these decision problems are decidable subject to a standard Gabrielov–Vorobjov oracle.

Directions for Future Research. The problem of deciding equality between general periods is a large open problem. Nevertheless, we present compelling arguments and further motivation to investigate the specific class of periods we identify in previous sections: those that admit representations that are single-dimensional (in the sense that the integrand is univariate) and moreover arise exclusively as rational linear combinations of the values of hypergeometric functions with rational parameters. A clear direction for future research is to establish unconditional decidability

Another possible direction for future research is suggested by recent work due to Nosan et al. [29] and Kenison [30]. In both of these works, the authors restrict the polynomial coefficients of the hypergeometric sequences. For instance, Nosan et al. establish decidability of the Membership Problem for hypergeometric sequences whose polynomial coefficients have rational roots. An interesting pursuit for future work for inequality decision problems for second-order sequences might consider pursuing decidability for second-order sequences under similar restrictions.

References

[1] M. Kauers and P. Paule, *The Concrete Tetrahedron*. Springer Vienna, 2011. [Online]. Available: https://doi.org/10.1007/978-3-7091-0445-3

[2] M. Petkovšek, H. Wilf, and D. Zeilberger, *A=B*. A. K. Peters, 1997.

[3] G. Everest, A. J. van der Poorten, I. E. Shparlinski, and T. Ward, *Recurrence Sequences*, ser. Mathematical surveys and monographs. American Mathematical Society, 2003, vol. 104.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*. Cambridge University Press, 2009.

[5] D. Zeilberger, "A holonomic systems approach to special functions identities," *Journal of Computational and Applied Mathematics*, vol. 32, no. 3, pp. 321–368, 1990.

[6] M. Kauers and V. Pillwein, "When can we detect that a P-finite sequence is positive?" in *Symbolic and Algebraic Computation, International Symposium, ISSAC 2010, Munich, Germany, July 25-28, 2010, Proceedings*. ACM, 2010, pp. 195–201.

[7] J. Ouaknine and J. Worrell, "Positivity problems for low-order linear recurrence sequences," in *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*. ACM, New York, 2014, pp. 366–379.

[8] ——, "On linear recurrence sequences and loop termination," *SIGLOG News*, vol. 2, no. 2, pp. 4–13, 2015.

[9] L. Liu, "Positivity of three-term recurrence sequences," *Electron. J. Combin.*, vol. 17, no. 1, pp. Research Paper 57, 10, 2010.

[10] M. Mezzarobba and B. Salvy, "Effective bounds for P-recursive sequences," *J. Symbolic Comput.*, vol. 45, no. 10, pp. 1075–1096, 2010.

[11] E. X. W. Xia and X. M. Yao, "The signs of three-term recurrence sequences," *Discrete Applied Mathematics*, vol. 159, no. 18, pp. 2290–2296, 2011.

[12] V. Pillwein, "Termination conditions for positivity proving procedures," in *International Symposium on Symbolic and Algebraic Computation, ISSAC'13, Boston, MA, USA, June 26-29, 2013*. ACM, 2013, pp. 315–322.

[13] V. Pillwein and M. Schussler, "An efficient procedure deciding positivity for a class of holonomic functions," *ACM Comm. Computer Algebra*, vol. 49, no. 3, pp. 90–93, 2015.

[14] W. Gautschi, "Computational aspects of three-term recurrence relations," *SIAM Rev.*, vol. 9, pp. 24–82, 1967.

[15] ——, "Anomalous Convergence of a Continued Fraction for Ratios of Kummer Functions," *Mathematics of Computation*, vol. 31, no. 140, pp. 994–999, 1977.

[16] ——, "Minimal solutions of three-term recurrence relations and orthogonal polynomials," *Mathematics of Computation*, vol. 36, no. 154, 1981.

[17] A. Deaño and J. Segura, "Transitory minimal solutions of hypergeometric recursions and pseudoconvergence of associated continued fractions," *Mathematics of Computation*, vol. 76, no. 258, pp. 879–901, 2007.

[18] G. Amparo, J. Segura, and N. M. Temme, *Numerical Methods for Special Functions*. SIAM, 2007, ch. 4.

[19] A. Deaño, J. Segura, and N. M. Temme, "Computational properties of three-term recurrence relations for Kummer functions," *J. Computational Applied Mathematics*, vol. 233, no. 6, pp. 1505–1510, 2010.

[20] G. Kenison, O. Klurman, E. Lefaucheux, F. Luca, P. Moree, J. Ouaknine, M. A. Whiteland, and J. Worrell, "On Positivity and Minimality for Second-Order Holonomic Sequences," in *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, vol. 202, 2021, pp. 67:1–67:15.

[21] M. Kontsevich and D. Zagier, "Periods," in *Mathematics unlimited—2001 and beyond*. Springer, Berlin, 2001, pp. 771–808.

[22] J. Wallis, "Arithmetica infinitorum, sive nova methodus inquirendi in curvilineorum quadraturam, aliaque difficiliori matheseos problemata," *Oxford*, pp. 1–199, 1655.

[23] ——, *The arithmetic of infinitesimals*, ser. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag, New York, 2004, translated from the Latin and with an introduction by Jaequeline A. Stedall.

[24] D. Bowman and J. Mc Laughlin, "Polynomial continued fractions," *Acta Arith.*, vol. 103, no. 4, pp. 329–342, 2002.

[25] J. Mc Laughlin and N. J. Wyshinski, "Real numbers with polynomial continued fraction expansions," *Acta Arith.*, vol. 116, no. 1, pp. 63–79, 2005.

[26] O. Perron, *Die Lehre von den Kettenbrüchen. Dritte, verbesserte und erweiterte Aufl. Bd. II. Analytisch-funktionentheoretische Kettenbrüche*. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1957.

[27] L. Lorentzen and H. Waadeland, *Continued fractions with applications*, ser. Studies in Computational Mathematics. North-Holland Publishing Co., Amsterdam, 1992, vol. 3.

[28] A. Cuyt, V. B. Petersen, B. Verdonk, H. Waadeland, and W. B. Jones, *Handbook of continued fractions for special functions*. Springer, New York, 2008.

[29] K. Nosan, A. Pouly, M. Shirmohammadi, and J. Worrell, "The membership problem for hypergeometric sequences with rational parameters," in *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*. New York, NY, USA: Association for Computing Machinery, 2022, p. 381–389.

[30] G. Kenison, "Applications of transcendental number theory to decision problems for hypergeometric sequences," *CoRR*, vol. abs/2211.02447, 2022. [Online]. Available: https://doi.org/10.48550/arXiv.2211.02447

[31] L. Lorentzen and H. Waadeland, *Continued fractions. Vol. 1*, 2nd ed., ser. Atlantis Studies in Mathematics for Engineering and Science. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008, vol. 1.

[32] S. Pincherle, "Delle Funzioni ipergeometriche, e di varie questioni ad esse attinenti," *Giorn. Mat. Battaglini*, vol. 32, pp. 209–291, 1894.

[33] L. Jacobsen and D. R. Masson, "On the convergence of limit periodic continued fractions $K(a_n/1)$, where $a_n \to -\frac{1}{4}$. III," *Constr. Approx.*, vol. 6, no. 4, pp. 363–374, 1990.

[34] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge University Press, Sep. 1996.

[35] M. Chamberland and A. Straub, "On gamma quotients and infinite products," *Advances in Applied Mathematics*, vol. 51, no. 5, pp. 546 – 562, 2013.

[36] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999, vol. 71.

[37] M. Waldschmidt, "Transcendence of periods: the state of the art," *Pure Appl. Math. Q.*, vol. 2, no. 2, Special Issue: In honor of John H. Coates. Part 2, pp. 435–463, 2006.

[38] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ser. National Bureau of Standards Applied Mathematics Series 55. Tenth Printing. ERIC, 1972.

[39] A. Khovanskii, "A class of systems of transcendental equations," *Dokl. Akad. Nauk SSSR*, vol. 255, no. 4, pp. 804–807, 1980.

[40] ——, *Fewnomials*. American Mathematical Soc., 1991, vol. 88.

[41] A. Gabrielov and N. Vorobjov, "Complexity of stratification of semi-Pfaffian sets," *Discrete & Computational Geometry*, vol. 14, no. 1, pp. 71–91, 1995.

[42] S. Pericleous and N. Vorobjov, "New complexity bounds for cylindrical decompositions of sub-Pfaffian sets," in *Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation, ISSAC 2001, ORCCA & University of Western Ontario, London, Ontario, Canada, July 22-25, 2001*, E. Kaltofen and G. Villard, Eds. ACM, 2001, pp. 268–275.

[43] A. Gabrielov and N. Vorobjov, "Complexity of computations with Pfaffian and Noetherian functions," in *Normal Forms, Bifurcations and Finiteness Problems in Differential Equations, volume 137 of NATO Science Series II*. Kluwer, 2004, pp. 211–250.

[44] M. V. Korovina and N. Vorobjov, "Pfaffian hybrid systems," in *Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpacz, Poland, September 20-24, 2004, Proceedings*, ser. Lecture Notes in Computer Science, J. Marcinkowski and A. Tarlecki, Eds., vol. 3210. Springer, 2004, pp. 430–441.

[45] ——, "Bounds on sizes of finite bisimulations of Pfaffian dynamical systems," *Theory Comput. Syst.*, vol. 43, no. 3-4, pp. 498–515, 2008.

[46] ——, "Computing combinatorial types of trajectories in Pfaffian dynamics," *J. Log. Algebraic Methods Program.*, vol. 79, no. 1, pp. 32–37, 2010.

[47] M. L. Barbagallo, G. Jeronimo, and J. Sabia, "Zero counting for a class of univariate pfaffian functions," *Journal of Algebra*, vol. 452, pp. 549–573, 2016.

[48] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions. Vols. I, II*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[49] N. M. Temme, "The numerical computation of the confluent hypergeometric function u(a,b,z)," *Numeriche Mathematik*, vol. 41, no. 1, pp. 63–82, 1983.

Appendix A.
Proof of Proposition 4.6

In the work that follows it will be useful to normalise recurrence (2). Our normalisation is an equivalence transformation (in the sense of Theorem 2.1). Set $b_0 = 1$, $\kappa_n := a_n/(b_n b_{n-1})$, and consider

$$w_n = w_{n-1} + \kappa_n w_{n-2}. \tag{11}$$

Then $\langle w_n \rangle_n$ with $w_{-1} = u_{-1}$ and $w_n := u_n/(\prod_{k=1}^n b_k)$ is a solution to (11) if and only if $\langle u_n \rangle_n$ is a solution to (2).

Proof of Proposition 4.6.

1) The result follows immediately from the equivalence transformation between (2) and (11).
2) Division by α_1 normalises the recurrence in the following way:

$$(n+\alpha)u_n = \beta u_{n-1} + \gamma u_{n-2}. \tag{12}$$

We shall assume that $\alpha := \alpha_0/\alpha_1 > 1$ (this can be achieved by shifting as appropriate). Suppose that $\langle u_n \rangle_n$ is a solution of the normalised recurrence. We use the updated reduction argument in Section 4 to obtain a second recurrence. We have

$$\begin{aligned}(2n+\alpha)&(2n+\alpha-1)u_{2n} \\ &= (\beta^2 + \gamma(4n+2\alpha-3))u_{2n-2} - \gamma^2 u_{2n-4}. \end{aligned} \tag{13}$$

This defines a second-order recurrence with solutions $\langle v_n \rangle_{n=0}^\infty$ given by $v_n := u_{2n}$. The mapping $n \mapsto 2n$ establishes a one-to-one correspondence between the solutions of recurrences (12) and (13) and we claim this correspondence preserves minimality. In order to prove this claim we show that linear independence and the asymptotic equalities are preserved. For linear independence one direction is trivial: if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly dependent solutions of (12), then $\langle u_{2n} \rangle_n$ and $\langle v_{2n} \rangle_n$ are linearly dependent solutions of (13). For the converse, suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent. Assume, for a contradiction, that there exists an $\ell \in \mathbb{R}$ such that $u_{2n} = \ell v_{2n}$ for each n. We study the sequence $\langle u_n - \ell v_n \rangle_{n=-1}^\infty$. By assumption $0 \neq u_{-1} - \ell v_{-1}$ and $0 = u_0 - \ell v_0$. Using (12) we then compute

$$u_1 - \ell v_1 = \frac{\gamma}{1+\alpha}(u_{-1} - \ell v_{-1}) \quad \text{and}$$
$$u_2 - \ell v_2 = \frac{\beta}{2+\alpha}(u_1 - \ell v_1) \neq 0,$$

a contradiction to our assumption.
We turn our attention to minimality. Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are minimal and dominant solutions of (12), respectively. Then $\lim_{n\to\infty} u_n/v_n = \lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since $\langle u_{2n} \rangle_n$ and $\langle v_{2n} \rangle_n$ are linearly independent by the above, $\langle u_{2n} \rangle_n$ is necessarily a minimal solution of (13). Conversely, assume that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent solutions such that $\langle u_{2n} \rangle_n$ is a minimal solution of (13) (recall that the existence of minimal solutions is decidable for each recurrence). Since $\langle v_{2n} \rangle_n$ is linearly independent of $\langle u_{2n} \rangle_n$, $\lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since recurrence (12) must also admit minimal solutions, one easily deduces that $\langle u_n \rangle_n$ is likewise minimal.
Notice that the Minimality Problem for $\langle v_n \rangle_n$ is an instance of $\mathsf{Minimal}(2,1,0)$ where the polynomial g_3 has rational roots. The equivalence transformations between (2) and (11) give the reduction to $\mathsf{Minimal}(1,1,1)$ under this assumption.
3) A solution sequence $\langle u_n \rangle_n$ satisfies a normalised recurrence of the form

$$u_n = \frac{\beta_1 n + \beta_0}{n+\alpha} u_{n-1} + \frac{\gamma_1 n + \gamma_0}{n+\alpha} u_{n-2}.$$

If $\beta_1 = |\gamma_1|$ then we are done. If not, consider the sequence $\langle v_n \rangle_n$ with terms given by $v_n := (\text{sign}(\gamma_1)\beta_1/\gamma_1)^n u_n$. Not only is it evident that $\langle v_n \rangle_n$ satisfies a recurrence of the desired form, but the sequence $\langle v_n \rangle_n$ is also a minimal solution if and only if $\langle u_n \rangle_n$ is a minimal solution.
Assume now that $\beta_1^2 + 4\gamma_1 = 0$ in (5). As $|\gamma_1| = |\beta_1|$, it follows immediately that $\beta_1 = 4 = -\gamma_1$. Now the sequence $\langle (1/2)^n u_n \rangle_n$ satisfies a recurrence of the form (6) and minimality is clearly preserved by this transformation.

4) In this case the recurrence admits minimal solutions if and only if $\gamma_1\alpha_1 > 0$ (compare to Lemma 4.8). This follows by an application of Theorem 2.4 as the standard normalisation (11) has $\kappa_n = (\gamma_1 n + \gamma_0)(\alpha_1(n-1) + \alpha_0)/\beta_0^2$. The reduction to (7) follows by considering the sequence $\langle (\text{sign}(\beta_0)\sqrt{\alpha_1/\gamma_1})^n u_n \rangle_n$. □

Appendix B. Proof of Theorem 4.9 Continued

Let us first prove Lemma 4.8.

Proof of Lemma 4.8. First, it is useful to normalise the recurrences using the normalisation in (11) so that one can determine whether a minimal solution exists using the criteria in Theorem 2.4 and Theorem 2.2. Under our assumptions, this normalisation does not change the signature of the recurrence. Second, recall that the characteristic roots of a recurrence are real if and only if $\beta_1^2 + 4\gamma_1 \geq 0$. Further, the characteristic roots are distinct if and only if $\beta_1^2 + 4\gamma_1 \neq 0$.

1) Regardless of whether $\gamma_1 = 0$ or not, when the recurrence has signature $(-,+)$ it is clear that $\lim_{n\to\infty} \kappa_n = 0$. When the recurrence has signature $(+,+)$ it is clear that $\sum_{n=2}^{\infty} 1/(n\kappa_n) = \infty$ and so the associated Stern–Stolz series (Theorem 2.6) diverges to ∞. These conditions are sufficient to prove the statement.
2) First, let us consider recurrence (5) under the assumption that $\beta_1^2 + 4\gamma \neq 0$. When (5) has signature $(-,+)$ there are minimal solutions if and only if $1 + 4\gamma_1/\beta_1^2 > 0$ if and only if λ, $\mu \in \mathbb{R}$. When (5) has signature $(+,+)$ it is clear that $\lim_{n\to\infty} \kappa_n = \gamma_1/\beta_1^2 > 0$ and so always admits minimal solutions. Finally, recurrence (7) also always admits minimal solutions: here the recurrence has signature $(+,+)$ and the Stern–Stolz series (Theorem 2.6) diverges to ∞ since $\sum_{n=2}^{\infty} 1/\sqrt{\kappa_n} = \infty$.
3) The normalisation of (6) is of the form $w_n = w_{n-1} + \kappa_n w_{n-2}$ with
$$\kappa_n = -\frac{1}{4} - \frac{\alpha - \beta_0 + \gamma_0}{4n} - \frac{\varepsilon}{16n^2} + \mathscr{O}(1/n^3)$$
and $\varepsilon = 4\beta_0(\beta_0 - \alpha - \gamma_0) + 4\alpha(1 + \gamma_0) - \beta_0(\beta_0 + 2)$. There are two cases to consider. If $\beta_0 - \alpha - \gamma_0 \neq 0$ then the recurrence admits minimal solutions if and only if $\beta_0 - \alpha - \gamma_0 > 0$. Otherwise $\beta_0 = \alpha + \gamma_0$, in which case κ_n simplifies as follows:
$$\kappa_n := -\frac{1}{4} - \frac{-(\alpha-\gamma)(\alpha-\gamma-2)}{16n^2} + \mathscr{O}(1/n^3).$$
Since $-x(x-2) \leq 1$ for each $x \in \mathbb{R}$, by Theorem 2.4, we deduce that this subcase always admits minimal solutions. □

We move on to complete the proof of Theorem 4.9(1)&(2)&(4) in order.

Proof of Theorem 4.9(1). If $\langle u_n \rangle_n$ is a minimal solution to recurrence $u_n = (\beta_1 n + \beta_0)u_{n-1} + \gamma_0 u_{n-2}$ then, by Theorem 2.2, $-u_0/u_{-1}$ is equal to
$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\gamma_0}{\beta_1 n + \beta_0} = \beta_0 \frac{{}_0F_1(;\beta_0/\beta_1;\gamma_0/\beta_1^2)}{{}_0F_1(;\beta_0/\beta_1+1;\gamma_0/\beta_1^2)} - \beta_0$$
(see [27, §VI.4]). Hence the Minimality Problem for the above recurrence reduces to checking the equality
$$(\beta_0 u_{-1} - u_0)\, {}_0F_1\left(;\frac{\beta_0}{\beta_1}+1;\frac{\gamma_0}{\beta_1^2}\right) = u_{-1}\beta_0\, {}_0F_1\left(;\frac{\beta_0}{\beta_1};\frac{\gamma_0}{\beta_1^2}\right).$$
The *Bessel functions of the first kind*, sometimes called *cylinder functions*, $J_s(z)$ are a family of functions that solve Bessel's differential equation [28], [36], [38]. For $z, s \in \mathbb{C}$ the function $J_s(z)$ is defined by the hypergeometric series [38, Equation 9.1.69]
$$J_s(z) := \frac{1}{\Gamma(s+1)}\left(\frac{z}{2}\right)^s {}_0F_1(;s+1;-z^2/4).$$
We obtain the principal branch of $J_s(z)$ by assigning $(z/2)^s$ its principal value. When $\mathrm{Re}(s) > -1/2$ we have the following integral representation [36, Equation 4.7.5],
$$J_s(z) = \frac{1}{\sqrt{\pi}\Gamma(s+1/2)}\left(\frac{z}{2}\right)^s \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{s-1/2}\,dt.$$
Hence for $\mathrm{Re}(s) > -1/2$, we have the following integral representation
$${}_0F_1(;s+1;z) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} \int_{-1}^{1} \mathrm{e}^{-2\sqrt{z}t}(1-t^2)^{s-1/2}\,dt.$$
Let us return to minimal solutions of the aforementioned recurrence relation. By substitution and linearity of the integral, we see that $\mathsf{Minimal}(0,1,0)$ reduces to checking whether the following integral
$$\int_{-1}^{1} \mathrm{e}^{\frac{-2\sqrt{\gamma_0}}{\beta_1}t}(1-t^2)^{\beta_0/\beta_1-3/2}\left(\tfrac{2(\beta_0 u_{-1}-u_0)}{2\beta_0-1}(1-t^2)-1\right)dt$$
vanishes. The integral in question is an exponential period, as the values β_i and γ_0 are rational numbers. To ensure that the integral converges absolutely note that we can shift the recurrence so that $\beta_0/\beta_1 > 3/2$. □

Proof of Theorem 4.9(2). Consider an instance of $\mathsf{Minimal}(0,1,1)$: without loss of generality it is of the form $u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$. By shifting the sequence appropriately, we may assume that $\frac{\beta_0}{\beta_1} + \frac{\gamma_1}{\beta_1^2}$ is positive. Let $\langle u_n \rangle_n$ be a minimal solution. Then $-u_0/u_{-1}$ is equal to
$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\gamma_1 n + \gamma_0}{\beta_1 n + \beta_0} = \frac{(\beta_0 + \frac{\gamma_1}{\beta_1})\, {}_1F_1(\frac{\gamma_0}{\gamma_1};\frac{\beta_0}{\beta_1}+\frac{\gamma_1}{\beta_1^2};\frac{\gamma_1}{\beta_1^2})}{{}_1F_1(\frac{\gamma_0}{\gamma_1}+1;\frac{\beta_0}{\beta_1}+\frac{\gamma_1}{\beta_1^2}+1;\frac{\gamma_1}{\beta_1^2})} - \beta_0.$$

Here the value on the right-hand side is given in [27, §VI.4].

Let $a = \gamma_0/\gamma_1$, $b = \beta_0/\beta_1 + \gamma_1/\beta_1^2$, and $x = \gamma_1/\beta_1^2$, then $\mathsf{Minimal}(0,1,1)$ reduces to checking the equality

$$\frac{{}_1F_1(a;b;x)}{{}_1F_1(a+1;b+1;x)} = \frac{(u_{-1}\beta_0 - u_0)}{u_{-1}(\beta_0 + \frac{\gamma_1}{\beta_1})}.$$

Notice here that $a > 0$ and $b > 0$ by assumption. Applying the transformation ${}_1F_1(a;b;x) = \mathrm{e}^x\,{}_1F_1(b-a;b;-x)$ (cf. [38, 13.1.27]), we have that if $b-a$ is zero or a negative integer, the values of the associated finite series are elements of $\mathbb{Q}(a,b,x)$. As these parameters are rational, the equality is plain to check. We thus assume that $b-a$ is not zero or a negative integer.

Proceeding as in the proof of Theorem 4.9(3), it can be shown that the sequence $\langle P_n(x)\rangle_{n=0}^\infty$ with $P_n = {}_1F_1(a+n;b+2n;x)$ if n is even, $P_n = {}_1F_1(a+n+1;c+2n+1;x)$ if n is odd, satisfies

$$\frac{P_0(x)}{P_1(x)} = s_1\circ\cdots\circ s_n\left(\frac{P_n(x)}{P_{n+1}(x)}\right),$$

where $\langle s_n\rangle_{n=1}^\infty$ is a sequence of linear fractional transformations given by $s_n(w) = d_n x/(1+w)$, where

$$d_n = \begin{cases} -\dfrac{b-a+n}{(b+2n)(b+2n+1)} & \text{if } n \text{ is odd, and} \\ \dfrac{a+n}{(b+2n-1)(b+2n)} & \text{if } n \text{ is even} \end{cases}$$

(see [27, §VI.2.2]). As $d_n \neq 0$ under our assumptions on a and b, the composition of the linear fractional transformations is an invertible. Thus

$$\frac{P_0(x)}{P_1(x)} = \frac{X_nP_{n+1}(x) + Y_nP_n(x)}{Z_nP_{n+1}(x) + W_nP_n(x)},$$

with $X_nW_n - Y_nZ_n \neq 0$ for all $n \geq 1$. We recall that ${}_1F_1(a;b;x)$ admits the integral representation

$${}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\mathrm{e}^{xt}\,dt$$

whenever $\mathrm{Re}(b) > \mathrm{Re}(a) > 0$ [27, Appendix 4.4] or [48, §6.5]. One can then complete the proof in a manner analogous to the proof of Theorem 4.9(3). ☐

It remains to prove Theorem 4.9(4). We first deal with a simple case that turns out to be decidable.

Lemma B.1. *Let $\langle u_n\rangle_n$ be a non-trivial solution to* (6) *with $\beta_0 = \alpha + \gamma_0$. If $\alpha \leq \gamma_0 + 1$ then $\langle u_n\rangle_n$ is minimal if and only if $u_0/u_{-1} = 1$. If $\alpha > \gamma_0 + 1$ then $\langle u_n\rangle_n$ is minimal if and only if $u_0/u_{-1} = (\gamma_0+1)/\alpha$.*

Proof. If $\beta_0 = \alpha + \gamma_0$, then the constant sequence $\langle 1\rangle_n$ is a solution to the recurrence by inspection. Hence $\langle 1\rangle_n$ and $\langle B_n\rangle_n$ defined by $B_{-1} = 0$, $B_0 = 1$ are linearly independent solutions and, by Lemma 2.3,

$$B_n = \sum_{k=0}^n\prod_{m=1}^k \frac{m+\gamma_0}{m+\alpha} = \sum_{k=0}^n \frac{(\gamma_0+1)_k}{(\alpha+1)_k}.$$

By a straightforward application of Stirling's approximation, $(\gamma_0+1)_n/(\alpha+1)_n \sim n^{\gamma_0-\alpha}$ as $n\to\infty$. Hence if $\gamma_0 - \alpha \geq -1$ the series diverges (by comparison to the harmonic series) from which we deduce that $\langle 1\rangle_n$ is minimal. If $\gamma_0 - \alpha < -1$, then $\lim_{n\to\infty} B_n$ converges to the value

$$\begin{aligned}\sum_{k=0}^\infty \frac{(\gamma_0+1)_k(1)_k}{(\alpha+1)_k}\frac{1}{k!} &= {}_2F_1(\gamma_0+1,1;\alpha+1;1) \\ &= \frac{\Gamma(\alpha+1)\Gamma(\alpha-\gamma_0-1)}{\Gamma(\alpha-\gamma_0)\Gamma(\alpha)} = \frac{\alpha}{\alpha-\gamma_0-1}.\end{aligned}$$

In the second equality we use [36, Thm. 2.2.2]. It follows that $\langle u_n\rangle_n = \frac{\alpha}{\alpha-\gamma_0-1}\langle 1\rangle_n - \langle B_n\rangle_n$ is a minimal solution, and we may compute $u_0/u_{-1} = (\gamma_0+1)/\alpha$. ☐

Proof of Theorem 4.9(4). The case when $\beta_0 = \alpha + \gamma_0$ is decidable by the above lemma, so we consider the case $\beta_0 > \alpha + \gamma_0$; otherwise the recurrence admits no minimal solutions by Lemma 4.8.

The function $U(a,b,x)$, the *confluent hypergeometric function of the second kind*, is defined for all $a,b,x \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(x) > 0$ as

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-xt}t^{a-1}(1+t)^{b-a-1}\,dt.$$

As noted by Temme in [49], the sequence $\langle u_n\rangle_{n=-1}^\infty$ given by $u_{-1} = \frac{1}{a-1}U(a-1,b,x)$, $u_n := (a)_nU(a+n,b,x)$ is a minimal solution of the recurrence

$$u_n = \frac{2n+x+2a-b-2}{n+a-b}u_{n-1} - \frac{n+a-2}{n+a-b}u_{n-2} \quad (14)$$

(assuming that $a \neq 1$ and $a-b$ is not a negative integer). Notice that the recurrence holds also for $n = 1$ because

$$(2a+x-b)U(a,b,z) - U(a-1,b,z) = a(1+a-b)U(a+1,b,c).$$

When one substitutes the values $a = \gamma_0+2$, $b = \gamma_0+2-\alpha$, and $x = \beta_0 - \gamma_0 - \alpha$ into (6) one obtains recurrence (14). Subject to an initial shift of the sequence, we may assume that have $a > 2$. We also have $x > 0$ by assumption (shifting the sequence has no effect on x). Hence, a minimal solution to (6) satisfies $u_0/u_{-1} = (a-1)U(a,b,x)/U(a-1,b,x)$. We may apply the integral representation for U immediately. Since the parameters involved are rational numbers, the integrals obtained are exponential periods, and the claim follows. ☐