On the Positivity Problem for Second-Order Holonomic Sequences

George Kenison
Department of Computer Science, KU Leuven, Belgium

Oleksiy Klurman
School of Mathematics, University of Bristol, UK

Engel Lefaucheux
INRIA Nancy, France

Florian Luca
Mathematics Division, Stellenbosch University, South Africa
Max Planck Institute for Software Systems, Saarland Informatics Campus, Germany

Pieter Moree
Max Planck Institute for Mathematics, Bonn, Germany

Joël Ouaknine
Max Planck Institute for Software Systems, Saarland Informatics Campus, Germany

Emre Can Sertöz
Mathematical Institute, Leiden University, Netherlands

Markus A. Whiteland
Department of Computer Science, Loughborough University, UK

James Worrell
Department of Computer Science, Oxford University, UK

Abstract

A sequence is holonomic if its terms obey a linear recurrence relation with polynomial coefficients. In this paper we consider the Positivity Problem for second-order holonomic sequences with linear coefficients, i.e., the question of determining, for a given sequence $\langle u_n \rangle_n$ obeying the recurrence $(a_1 n + a_0)u_n = (b_1 n + b_0)u_{n-1} + (c_1 n + c_0)u_{n-2}$, whether all terms of $\langle u_n \rangle_n$ are non-negative. Our main result establishes decidability in the case of two distinct rational characteristic roots. We achieve this by leveraging recent results on effective transcendence of values of E-functions and 1-periods, which are integrals playing a central role in the theory of algebraic curves.

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1 Introduction

1.1 Background

Holonomic (or P-recursive) sequences are those whose terms satisfy a linear recurrence with polynomial coefficients; equivalently, their ordinary generating functions are D-finite, meaning they satisfy a linear differential equation with polynomial coefficients. Formally, a rational-valued sequence $\langle u_n \rangle_n$ is *holonomic* (of *order* k and *degree* d) if there exist polynomials $p_{k+1}, \ldots, p_1 \in \mathbb{Q}[x]$ of degree at most d with $p_1(x), p_{k+1}(x)$ non-zero polynomials such that

$$p_{k+1}(n)u_n = p_k(n)u_{n-1} + \cdots + p_1(n)u_{n-k} \, .$$

Subject to the standard assumption that $p_{k+1}(n) \neq 0$ for non-negative integers n, the above recurrence uniquely defines an infinite sequence $\langle u_n \rangle_{n=0}^{\infty}$ once the k initial values $u_0, \ldots, u_{k-1}$ are specified.[1]

The class of holonomic sequences is very general; e.g., C-finite sequences (such as the Fibonacci sequence) are holonomic of degree 0, while hypergeometric sequences are holonomic of order 1. At the same time, the class is closed under sums, products, and many other standard transformations, and it supports algorithms for term computation, asymptotics, and identities (for example via the technique of so-called creative telescoping). Holonomic sequences find applications across computer science and mathematics, particularly in analysis of algorithms and data structures, formal languages and power series, combinatorics, number theory, and special functions [40, 11, 14, 28, 47].

The Positivity Problem for holonomic sequences has as input a recurrence and initial values, and asks to determine whether all terms of the resulting sequence are non-negative. Motivation to study this problem comes from a wide variety of areas, including number theory [45], combinatorics [42], and special-function theory [19]. Nevertheless, in contrast to the rich literature around automatic procedures for proving identities among holonomic sequences [40, 47], Kauers and Pillwein remark that *"... there are almost no algorithms available for inequalities"* [29].

There have been a number of works on the Positivity Problem in the intervening years since [29]. Ouaknine and Worrell [38, 39] show decidability of the problem for C-finite recurrences of order at most 5. Hagihara and Kawamura [21], generalising earlier work of [36], give an algorithm to decide positivity of a class of second-order holonomic sequences $u_n = p(n)u_{n-1} + q(n)u_{n-2}$ for arbitrary $p, q \in \mathbb{Q}[x]$, and all initial values u_0, u_1 outside an *exceptional line*. The references [30, 37] contain conditional decidability results for positivity of inhomogeneous first-order recurrences. Ibrahim and Salvy [26] give a method to certify the positivity of holonomic sequences of arbitrary order and degree, subject to the restriction that there be a single dominant positive real characteristic root and that the initial conditions are assumed to lie outside of an exceptional hyperplane; see [25] for a generalisation of this work.

The present paper gives a decision procedure for the Positivity Problem for a subclass of second-order degree-one recurrences, *with no restrictions on the initial conditions.* The exceptional initial values in the work of [21] are those that lead to so-called *minimal sequences*, that is, sequences $\langle u_n \rangle_n$ such that for any other independent solution $\langle v_n \rangle_n$ of the recurrence, one has $u_n/v_n \to 0$ as $n \to \infty$. Kenison et al. [31] showed that Positivity reduces to Minimality for second-order holonomic sequences. The central contribution of the present

[1] In the sequel, it will in fact often be convenient to start the sequence at u_{-1} instead of u_0, in keeping with standard conventions from the literature.

paper is an algorithm to solve the Minimality Problem on a sub-class of second-order holonomic recurrences. Efficient numerical methods to *approximate* minimal solutions to second-order recurrences appear in a number of sources [17, 18, 16, 9, 2, 10] — see also the references therein. However these methods do not suffice to solve the Minimality Problem. Instead, as we explain below, our results rely on recent advances on transcendence of period integrals and values of E-functions.

1.2 Main Results

The Positivity Problem for first-order holonomic sequences is straightforward to decide. Indeed, for such a sequence $\langle u_n \rangle_n$, we have that $\frac{u_n}{u_{n-1}} = \frac{p_1(n)}{p_2(n)}$ is a rational function and hence has eventually constant sign. In this paper we focus on second-order sequences, for which the Positivity Problem already appears to be very challenging. Our results concern second-order sequences whose coefficient polynomials have degree at most one; that is, we consider recurrences of the form

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2} \tag{1}$$

in which the numerical constants α_i, β_i, and γ_i are rational. The case that $\alpha_1 = \beta_1 = \gamma_1 = 0$ corresponds to order-two C-finite sequences, where the Positivity Problem is known to be decidable [22], and we henceforth assume without loss of generality that at least one of $\alpha_1, \beta_1, \gamma_1$ is non-zero. In this general case the *characteristic polynomial* $p(x) := \alpha_1 x^2 - \beta_1 x - \gamma_1$ of the recurrence (1) is not identically zero.

Our main result is as follows:

▶ **Theorem 1.** *The Positivity and Minimality Problems are decidable for the class of degree-one second-order holonomic recurrences with two distinct rational characteristic roots.*

The technical development combines a number of significant elements, as we now briefly discuss. First, we rely on the theory of polynomial continued fractions. An example of such, given in the so-called *Kettenbruch notation*, is the expression

$$\frac{4}{\pi} = 1 + \operatorname*{K}_{n=1}^{\infty} \frac{(2n-1)^2}{2}\,.$$

Bowman and Mc Laughlin [7] (see also [35]) coined the term *polynomial continued fraction* for such constructions. For $\deg(a_n) \leq 2$ and $\deg(b_n) \leq 1$, Lorentzen and Waadeland [33, §6.4] express the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ in terms of values of hypergeometric functions evaluated at algebraic points.

Related to the reduction of the Positivity Problem to the Minimality Problem, we reduce the question of determining positivity of second-order holonomic sequences to that of determining whether the value of a given polynomial continued fraction is greater than a given rational threshold. By the above-mentioned work of Lorentzen and Waadeland, this problem in turn reduces to the question of determining the value of a hypergeometric function with rational parameters at a rational argument. Such values are, in general, not algebraic numbers, but they respectively fall in two broader classes of numbers that still admit effective equality-testing procedures. Specifically, we rely on an algorithm of Fischler and Rivoal [12] for determining all polynomial relations between the values of given family of so-called *E-functions* evaluated at a common algebraic point. Secondly, we use a new algorithm of Sertöz, Ouaknine, and Worrell [43] for determining linear relations between values of 1-period integrals. Very briefly, E-functions are a generalisation of the exponential function (see Sec. 2.3), introduced by Siegel [44], while 1-periods are integrals of algebraic functions of a single variable (see Sec. 2.4). We also highlight some cases, beyond the scope of Thm. 1, in which the Minimality Problem reduces to the task of determining zeroness of *exponential 1-periods*, an important open problem in algebraic geometry (see Sec. 2.4 for more on this).

2 Mathematical Background

2.1 Continued Fractions and Minimal Solutions

A *continued fraction*

$$\operatorname*{K}_{n=1}^{\infty}\frac{a_n}{b_n} := \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots}}}$$

is defined by an ordered pair of sequences $\langle a_n\rangle_n$ and $\langle b_n\rangle_n$ of real numbers where $a_n \neq 0$ for each $n \in \mathbb{N}$. Comprehensive accounts on the theory of continued fractions are given in [33, 34]. A continued fraction *converges* to a value $f = \operatorname{K}(a_n/b_n)$ if its *sequence of approximants* $\langle f_n\rangle_{n=1}^{\infty}$ converges to f in $\hat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$. The sequence $\langle f_n\rangle_n$ is recursively defined by the following composition of linear fractional transformations. For $w \in \hat{\mathbb{R}}$, define

$$s_n(w) = \frac{a_n}{b_n+w} \text{ for each } n \in \{1,2,\dots\}\,.$$

We set $f_n := s_1\circ\cdots\circ s_n(0)$ so that $f_n = \operatorname{K}_{m=1}^{n}\frac{a_m}{b_m}$. In App. A we collect together a number of results on the convergence of continued fractions that will be used in the sequel.

The following observation establishes a connection between continued fractions and second-order recurrences. Let $\langle A_n\rangle_{n=-1}^{\infty}$ and $\langle B_n\rangle_{n=-1}^{\infty}$ satisfy the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial values $A_{-1} = 1, A_0 = 0, B_{-1} = 0$, and $B_0 = 1$. As a pair, $\langle A_n\rangle_{n=-1}^{\infty}$ and $\langle B_n\rangle_{n=-1}^{\infty}$ form a basis for the solution space of the recurrence. We call $\langle A_n\rangle_n$ and $\langle B_n\rangle_n$ the sequences of *canonical numerators* and *canonical denominators* of $\operatorname{K}(a_n/b_n)$ because $f_n = A_n/B_n$ for each $n \in \mathbb{N}$.

A non-trivial solution $\langle u_n\rangle_{n=-1}^{\infty}$ of the recurrence

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{2}$$

is *minimal* provided that, for all other linearly independent solutions $\langle v_n\rangle_{n=-1}^{\infty}$ of the same recurrence, we have $\lim_{n\to\infty} u_n/v_n = 0$. Since the vector space of solutions has dimension two, it is equivalent for a sequence $\langle u_n\rangle_{n=-1}^{\infty}$ to be minimal for there to exist a linearly independent sequence $\langle v_n\rangle_{n=-1}^{\infty}$ satisfying the above property. In such cases the solution $\langle v_n\rangle_n$ is called *dominant*.

Note that if $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are linearly independent solutions of the above recurrence such that $u_n/v_n \sim C \in \hat{\mathbb{R}}$ then the recurrence relation has a minimal solution [33]. If, in addition, $\langle u_n\rangle_n$ is minimal then all solutions of the form $\langle cu_n\rangle_n$ where $c \neq 0$ are also minimal. If $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are respectively minimal and dominant solutions of the recurrence, then together they form a basis of the solution space.

The following result of Pincherle [41] gives a sufficient and necessary condition for the existence of a minimal solution of a given second-order recurrence.

▶ **Theorem 2** (Pincherle). *Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be real-valued sequences such that each of the terms a_n is non-zero. First, the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ has a minimal solution if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges in $\hat{\mathbb{R}}$. Second, if $\langle u_n \rangle_n$ is a minimal solution of this recurrence then the limit of $\mathrm{K}(a_n/b_n)$ is $-u_0/u_{-1}$.*[2]

2.2 Hypergeometric Functions

In our applications, the values of the continued fractions arising from Pincherle's Theorem will admit expressions in terms of values of hypergeometric functions.

A series $\sum c_k x^k$ is called *hypergeometric* if it has the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_j)_k}{(b_1)_k \cdots (b_\ell)_k} \frac{x^k}{k!} ,$$

where, for $\rho \in \mathbb{C}$, the *(rising) Pochhammer symbol* $(\rho)_n$ is defined as $(\rho)_0 = 1$, and $(\rho)_n = \prod_{j=0}^{n-1}(\rho + j)$ for $n \geq 1$. Here the parameters b_m do not lie in $\{0, -1, -2, \ldots\}$ for otherwise the denominator would vanish for some k. The series terminates if any of the parameters a_m is zero or a negative integer. We let ${}_jF_\ell(a_1, \ldots, a_j; b_1, \ldots, b_\ell; x)$ denote the series on the right-hand side. This converges absolutely for all x if $j \leq \ell$, and for $|x| < 1$ if $j = \ell + 1$. By abuse of notation, we let ${}_jF_\ell(a_1, \ldots, a_j; b_1, \ldots, b_\ell; x)$ denote the analytic function defined by the corresponding series in its radius of convergence.

Of special interest to us are the functions ${}_1F_1(a; b; x)$ and ${}_2F_1(a, b; c; x)$ for rational parameters a, b, c. In order to algorithmically treat values of these functions, we will use recent results about E-functions and 1-periods.

2.3 *E*-Functions

In extending the Lindemann-Weierstrass theorem, Siegel [44] introduced the class of E-functions as generalisations of the exponential function.

▶ **Definition 3.** *A power series $g(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$, where $c_n \in \mathbb{Q}$, is an E-*function *if the following conditions hold:*

1. *The series $g(z)$ is the solution of a linear differential equation with coefficients in $\mathbb{Q}(z)$;*
2. *There exists $C > 0$ such that $|c_n| \leq C^{n+1}$ for all n;*
3. *There exists $D > 0$ and integers $1 \leq d_n \leq D^{n+1}$ such that $d_n c_m$ is an integer for all $0 \leq m \leq n$.*

The power series in the above defines an entire function on the complex numbers.

As shown by Siegel [44] (see also [13, Sec. 1]), the hypergeometric function ${}_1F_1(a, b; z)$ is an E-function for $a, b \in \mathbb{Q}$ with $b \notin \{0, -1, -2, \ldots\}$.

We will use the following result of Fischler and Rivoal [12] that computes all polynomial relations among the values of a given collection of E-functions at a rational point. For algorithmic purposes, an E-function g is specified by (i) a linear differential equation with coefficients in $\mathbb{Q}(z)$ that is satisfied by g; (ii) sufficiently many coefficients of the Taylor expansion of f to uniquely determine g among solutions of the above equation; (iii) a certificate that g satisfies the growth conditions in Items 2 and 3.

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[2] In particular, the sequence of canonical denominators $\langle B_n \rangle_{n=-1}^{\infty}$ is a minimal solution if and only if the value of $\mathrm{K}(a_n/b_n)$ is $\infty \in \hat{\mathbb{R}}$.

▶ **Theorem 4** (Fischler and Rivoal [12]). *There exists an algorithm to perform the following task. Given as input E-functions $g_1(z), \ldots, g_k(z)$ and $\alpha \in \mathbb{Q}$, it outputs generators of the ideal of all polynomials in $\mathbb{Q}[x_1, \ldots, x_k]$ that vanish on $(g_1(\alpha), \ldots, g_k(\alpha))$.*

2.4 1-Periods

A *1-period* is the value of an integral $\int_a^b f(x)\,\mathrm{d}x$, where a, b are real algebraic numbers and the integrand $f(x)\colon [a, b] \to \mathbb{C}$ is a complex-valued algebraic function.[3]

▶ **Theorem 5** (Sertöz, Ouaknine, and Worrell [43]). *For any finite tuple of 1-periods, one can effectively compute a basis for the vector space of linear relations between them with complex algebraic coefficients.*

The theorem above makes explicit the result of Huber and Wüstholz [24, Thm. 13.3] which states that there are no linear relations between 1-periods beyond those that have an algebro-geometric origin.

For $c > b > 0$ and $|x| < 1$, the value of the hypergeometric function ${}_2F_1(a, b; c; x)$ has the form

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}\, dt\,, \tag{3}$$

where t^b, $(1-t)^{c-b-1}$, and $(1-xt)^{-a}$ have their principal values. When a, b, c are rational parameters, the fact that the integral appearing in (3) is a 1-period will be critical for our purposes.

All of this is part of a much larger conjectural framework regarding periods. Kontsevich and Zagier [32] define a *period* as a complex number whose real and imaginary parts are the values of absolutely convergent integrals of the form

$$\int_\sigma f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \tag{4}$$

where the domain $\sigma \subseteq \mathbb{R}^n$ is semi-algebraic and the integrand $f(x_1, \ldots, x_n)$ is algebraic, both defined using polynomials with algebraic coefficients. Periods appear in many seemingly different but ultimately equivalent formulations, see [23, §12].

Just like the periods themselves, the expectation is that *identities* between periods should correspond to algebro-geometric constructions and should, therefore, be decidable. This is expressed by the Kontsevich-Zagier Period Conjecture (cf. [32, Conj. 1]) which, in different guises [20, 4, 5], is one of the central questions of the field.

In this paper we will also refer to *exponential 1-periods*, see [32, §4.3] and [6, 15]. Briefly, a real *exponential period* is an integral as in (4) but where the integrand is allowed to be of the form $e^{-f(x_1,\ldots,x_n)}g(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$ with f, g algebraic over real algebraic numbers.

The Period Conjecture can be extended to a decidability statement on identities over exponential periods, generalising other well-known conjectures in transcendental number theory which have stood the test of time [15, §1.3]. Note that, unlike the situation with 1-periods, the decidability of zeroness for exponential 1-periods remains an important open problem in algebraic geometry.

[3] A function $f\colon [a, b] \to \mathbb{C}$ is *algebraic* if it is continuous over $[a, b]$ and moreover solves some polynomial equation $P(x, f(x)) = 0$ over $[a, b]$, where $P \in \overline{\mathbb{Q}}[x, y]$ is a non-zero polynomial with complex algebraic coefficients.

3 The Positivity and Minimality Problems for Degree-One Second-Order Holonomic Sequences

In this section we prove the main result of the paper:

▶ **Theorem 6.** *The Positivity and Minimality Problems are both decidable for the class of degree-one second-order holonomic recurrences with two distinct rational characteristic roots. In the case of a single characteristic root, the Positivity and Minimality Problems reduce to the task of determining zeroness of exponential 1-periods.*

The proof proceeds in three steps:

1. We reduce the Positivity Problem to the Minimality Problem for the class of degree-one second-order recurrences.
2. Using Pincherle's Theorem, we reduce the Minimality Problem to the problem of verifying certain rational linear relations between values of hypergeometric functions.
3. We use a procedure for verifying the zeroness of 1-periods and values of E-functions to solve instances in Step 2 that arise from recurrences with distinct rational characteristic roots. The remaining cases (either a repeated characteristic root, or the characteristic polynomial has degree 1) are handled by recourse to an oracle for deciding zeroness of exponential 1-periods.

Throughout this section we work with a second-order degree-one recurrence

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2} \tag{5}$$

with rational constants α_i, β_i and γ_i, such that $\alpha_1, \beta_1, \gamma_1$ are not all zero. Note that we can assume, without loss of generality, that none of the polynomial coefficients is identically zero (for otherwise we are essentially dealing with order-1 recurrences, for which the Positivity Problem is straightforwardly handled, as discussed in Sec. 1). By considering a shifted recurrence relation, we can also assume that each polynomial coefficient has constant (non-zero) sign. In particular, we have that $\alpha_0, \beta_0, \gamma_0$ are non-zero. In this case we can equivalently write the recurrence in the form

$$u_n = b_n u_{n-1} + a_n u_{n-2} \tag{6}$$

for rational functions $b_n := (\beta_1 n + \beta_0)/(\alpha_1 n + \alpha_0)$ and $a_n := (\gamma_1 n + \gamma_0)/(\alpha_1 n + \alpha_0)$, that are defined for all $n \geq 0$. We define the *signature* of the recurrence (6) to be the ordered pair $(\text{sign}(b_n), \text{sign}(a_n))$, which is independent of n.

3.1 Positivity Reduces to Minimality at Low Degrees

The results of this section make no assumption about the rationality of the characteristic roots of (5).

▶ **Lemma 7.** *For the class of second-order degree-one recurrences, the Positivity Problem reduces to the Minimality Problem.*

For second-order holonomic sequences, Kenison et al. [31] proved that Positivity reduces to Minimality for second-order holonomic recurrences. However their reduction does not preserve the degrees of the polynomial coefficients in the case of recurrences of signature $(-,+)$. We therefore treat this case separately below.

▶ **Proposition 8.** *Suppose that $\langle u_n \rangle_n$ satisfies recurrence* (6) *with signature* $(-,+)$*. Assume that* $u_{-1} > 0$ *and write* $f_n = \mathrm{K}_{m=1}^{n} \frac{a_m}{b_m}$*. Then for even* $n \in \mathbb{N}$*, we have* $u_n > 0$ *if and only if* $f_n > -u_0/u_{-1}$*. For odd* $n \in \mathbb{N}$*, we have* $u_n > 0$ *if and only if* $f_n < -u_0/u_{-1}$*.*

Proof. Recall the canonical solution sequences $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$. Then $u_n = A_n u_{-1} + B_n u_0$ for each $n \in \{-1, 0, \ldots\}$. For recurrences with signature $(-,+)$, it is easy to show by induction that $B_n < 0$ for each odd $n \in \mathbb{N}$, and $B_n > 0$ for each even $n \in \mathbb{N}$. Thus for even $n \in \mathbb{N}$ we have that $u_n > 0$ if and only if $A_n/B_n + u_0/u_{-1} = f_n + u_0/u_{-1} > 0$. The case for odd $n \in \mathbb{N}$ is handled in a similar fashion. ◀

Another consequence of the work by Kenison et al. shows that, for our purposes, it is sufficient to consider only those recurrence relations whose associated continued fraction $\mathrm{K}(a_n/b_n)$ converges to a finite limit. This claim is justified by the following observation (due to [31]): the positive solutions of a second-order holonomic recurrence relation with signature $(-,+)$ are precisely those solutions that are minimal. Without loss of generality we can assume the limit of the associated continued fraction is finite due to Pincherle's Theorem. Thus all that remains is to prove the following proposition.

▶ **Proposition 9.** *Suppose that $\langle u_n \rangle_n$ with initial values $u_{-1}, u_0 > 0$ is a solution sequence for a degree-1 recurrence* (6) *with signature $(-,+)$ whose associated continued fraction $\mathrm{K}(a_n/b_n)$ converges to a finite limit f. The following statements are equivalent:*

1. *the sequence $\langle u_n \rangle_n$ is positive,*
2. *the sequence $\langle u_n \rangle_n$ is minimal, and*
3. $-u_0/u_{-1} = f$*.*

Proof. The sequence $\langle -f_n \rangle_{n=1}^{\infty}$ is the sequence of approximants associated with $\mathrm{K}(a_n/-b_n)$. This is a positive continued fraction and so, by Lem. 16, the subsequences $\langle -f_{2n} \rangle_{n=1}^{\infty}$ and $\langle -f_{2n-1} \rangle_{n=1}^{\infty}$ converge to finite limits $-\ell_1$ and $-\ell_2$, respectively. By Prop. 8, a solution sequence $\langle u_n \rangle_n$ is positive if and only if $\ell_2 \le -u_0/u_{-1} \le \ell_1$. The Stern-Stolz series (Thm. 18) associated with $\mathrm{K}(a_n/-b_n)$ diverges due to our assumption that each of the coefficients in (6) is a polynomial with degree in $\{0,1\}$. We conclude that $\ell_1 = \ell_2$. Thus $\langle u_n \rangle_n$ is positive if and only if $-u_0/u_{-1}$ is equal to $f = \ell_1 = \ell_2$. From Thm. 2, a solution sequence $\langle u_n \rangle_n$ is minimal if and only if $-u_0/u_{-1}$ is the value of the continued fraction $\mathrm{K}(a_n/b_n)$. ◀

We thus obtain the desired result, Lem. 7.

3.2 Parameterising the Minimality Problem

In the sequel it is helpful to parameterise the Minimality Problem as follows:

▶ **Problem 1** (Minimal(j,k,ℓ))**.** *Given a solution $\langle v_n \rangle_{n=-1}^{\infty}$ to the recurrence* (5) *where* $\deg(\alpha_1 n + \alpha_0) = j, \deg(\beta_1 n + \beta_0) = k$*, and* $\deg(\gamma_1 n + \gamma_0) = \ell$*, determine whether $\langle v_n \rangle_n$ is minimal.*

Given our assumption that none of the polynomial coefficients in (5) is identically zero, we may suppose that $j, k, \ell \in \{0,1\}$.

3.3 Interreductions for Minimal(j,k,ℓ)

Consider problem Minimal$(0,0,0)$: determine whether a sequence that satisfies a second-order C-finite recurrence is a minimal solution. Since this problem is a special case of Minimal$(1,1,1)$

(multiply each of the coefficients by $(n+1)$) we make no further mention of $\mathsf{Minimal}(0,0,0)$ in the sequel. Thus we need only consider the seven remaining cases associated with $\mathsf{Minimal}(j,k,\ell)$. We first reduce the number of problems to five by establishing interreductions between instances of $\mathsf{Minimal}(j,k,\ell)$. Next we employ minimality-preserving transformations to obtain canonical instances of each of the remaining problems (Cor. 11).

Proposition 10 presents the cases we need to consider to establish the Minimality Problem at low degrees. The proposition uses the notation $\alpha := \alpha_0/\alpha_1$ in case $\alpha_1 \neq 0$. The (routine) proof of Prop. 10 is relegated to App. B.

▶ **Proposition 10.**

1. $\mathsf{Minimal}(0,k,1)$ *and* $\mathsf{Minimal}(1,k,0)$ *are interreducible.*
2. $\mathsf{Minimal}(1,0,0)$ *reduces to* $\mathsf{Minimal}(1,1,1)$.
3. $\mathsf{Minimal}(1,1,1)$ *reduces to the Minimality Problem for a recurrence of the form*

$$u_n = \frac{\beta_1 n + \beta_0}{n+\alpha} u_{n-1} + \frac{\gamma_1 n + \gamma_0}{n+\alpha} u_{n-2}\,, \tag{7}$$

where $\alpha, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{Q}$, $\beta_1 > 0$, *and* $|\gamma_1| = \beta_1$. *The characteristic roots associated to the recurrence are* $(\beta_1 \pm \sqrt{\beta_1^2 + 4\gamma_1})/2$. *If* $\beta_1^2 + 4\gamma_1 = 0$, *then we have a further reduction to the Minimality Problem for solutions to*

$$u_n = \frac{2n + \beta_0}{n+\alpha} u_{n-1} - \frac{n + \gamma_0}{n+\alpha} u_{n-2}\,, \tag{8}$$

where $\alpha, \beta_0, \gamma_0 \in \mathbb{Q}$ *and the recurrence has a single repeated characteristic root* 1.

4. $\mathsf{Minimal}(1,0,1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*[4]

$$u_n = \frac{\beta_0}{n+\alpha} u_{n-1} + \frac{n + \gamma_0}{n+\alpha} u_{n-2}\,, \tag{9}$$

where $\alpha, \gamma_0 \in \mathbb{Q}$ *and* $\beta_0 \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$. *The characteristic roots associated to the recurrence are* ± 1.

Let λ and μ be the roots of the associated characteristic polynomial such that $|\lambda| \le |\mu|$. In recurrences (7)–(9), γ_1 is not zero so we have $\lambda, \mu \neq 0$. Further, by setting $\mu = 1$ for (9), we have that $\mu > 0$ for the associated recurrences (7)–(9), as the coefficient $\beta_1 > 0$ in the first two.

We shall treat recurrences (7) and (8) as distinct cases in the sequel. That is to say, in the former we shall always assume that $\beta_1^2 + 4\gamma_1 \neq 0$ (so that the characteristic roots are distinct).

The following corollary is an immediate application of Prop. 10.

▶ **Corollary 11.** *For* $j,k,\ell \in \{0,1\}$, *decidability of problem* $\mathsf{Minimal}(j,k,\ell)$ *reduces to proving decidability of* $\mathsf{Minimal}(0,1,0)$, $\mathsf{Minimal}(0,1,1)$, *and decidability of the Minimality Problem for solutions to recurrences* (7)*–*(9).

The next lemma gives necessary and sufficient conditions for the relevant recurrences to admit minimal solutions. The straightforward proof, which uses Pincherle's Theorem and standard results on the convergence of continued fractions stated in the Preliminaries, is given in App. C. In particular, in the sequel we shall assume that the characteristic roots of a recurrence relation are real-valued.

[4] Notice that sequences satisfying (9) are not necessarily holonomic as β_0 can be a real algebraic number.

▶ **Lemma 12.**

1. *A recurrence* $u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$ *with* $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{Q}$*, with* $\beta_0, \beta_1, \gamma_0 \neq 0$*, admits minimal solutions.*
2. *A recurrence of the form* (7) *or* (9) *admits minimal solutions if and only if the associated characteristic roots are real.*
3. *A recurrence of the form* (8) *admits minimal solutions if and only if* $\beta_0 - \alpha - \gamma_0 \geq 0$*.*

3.4 Reduction to Equality Checking

We next reduce the Minimality Problem to the respective problems of verifying zeroness of values of E-functions, 1-periods, and exponential 1-periods.

▶ **Theorem 13.** *For the class of recurrences with rational characteristic roots, we have:*

1. $\mathsf{Minimal}(0,1,1)$ *reduces to the problem of verifying that an* E*-function vanishes at a rational argument.*
2. *Minimality for recurrences* (7) *and* (9) *reduces to the problem of verifying the zeroness of a 1-period.*
3. $\mathsf{Minimal}(0,1,0)$ *reduces to the problem of verifying the zeroness of an exponential 1-period.*
4. *Minimality for recurrence* (8) *reduces to the problem of verifying the zeroness of an exponential 1-period.*

We devote the remainder of the section to the proof of Thm. 13(2). The remaining items appear in App. C. The lynchpin in deciding minimality is Pincherle's Theorem (Thm. 2), which, as explained in Sec. 2, transforms such problems into questions about the convergence of polynomial continued fractions. The remainder of our approach to the proof of Thm. 13 is via techniques of Lorentzen and Waadeland in [33, §VI.4.1], which express the limits of certain continued fractions as quotients of *hypergeometric functions*. These functions admit, after suitable transformations, either representations as E-functions or integral representations that involve periods or exponential periods.

We note that the evaluation of low-degree polynomial continued fractions by Lorentzen and Waadeland does not extend to the continued fractions associated with recurrences of the form (8). As far as we are aware, our analysis is the first to complete this gap in the literature.

Proof of Thm. 13(2). Consider recurrence (7). Recall that μ and λ are the (rational) roots of the characteristic polynomial of recurrence, with $\mu > |\lambda| > 0$. Let $\langle u_n \rangle_n$ be a minimal solution to the recurrence. By Thm. 2, $-u_0/u_{-1} = \mathrm{K}(a_n/b_n)$. Now $\mathrm{K}(a_n/b_n)$ is equivalent to the continued fraction

$$\frac{1}{\alpha\delta^2} \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(\gamma_1 n + \gamma_0)(n + \alpha - 1)\delta^2}{(\beta_1 n + \beta_0)\delta}$$

by applying Thm. 15 with $\mathfrak{r}_0 = 1$, $\mathfrak{r}_n = \delta(n + \alpha)$ for $n \in \mathbb{N}$, and δ a non-zero real number. Setting $\delta = 1/\mu$, $x = \lambda/\mu$, and either

$$a = \frac{\lambda^2\alpha - \lambda\beta_0 - \gamma_0}{\lambda(\lambda - \mu)},\ b = \alpha - 1,\ c = a + \frac{\gamma_0}{\gamma_1}; \quad \text{or}$$

$$a = \frac{\mu^2\alpha - \mu\beta_0 - \gamma_0}{\mu(\lambda - \mu)} + 1,\ b = \frac{\gamma_0}{\gamma_1},\ c = a + \alpha - 1$$

we obtain, by substitution, the continued fraction

$$\frac{\mu^2}{\alpha} \operatorname*{K}_{n=1}^{\infty} \frac{-(c-a+n)(b+n)x}{c+n+(b-a+1+n)x} \,. \tag{10}$$

Here we note that $\gamma_1 = -\lambda\mu$ and $\beta_1 = \lambda + \mu$. Notice here that $|x| < 1$ by construction. Furthermore, shifting the sequence appropriately, the parameter a does not change, but b and c are shifted by an integer value. Hence we may assume that c is a positive number. Finally, a, b, c, x are all rational. By Lorentzen and Waadeland's work [33, §VI, Thm. 4(A)],

$$-\frac{\alpha}{\mu^2}\frac{u_0}{u_{-1}} = \frac{c\,{}_2F_1(a,b;c;x)}{{}_2F_1(a,b+1;c+1;x)} - \frac{\beta_0}{\mu} \,. \tag{11}$$

Let us first show that the equality problem is decidable if a, b, $c-a$, or $c-b$ is a negative integer, or if either $a = 0$, or $c - b = 0$. If either $a = 0$, or that either a or b is a negative integer, then both the hypergeometric series terminate. Thus equality testing reduces to checking equality of rational numbers. Moreover, we have Euler's transformation (cf. [3, Eq. 2.2.7]):

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)\,,$$

which implies that if $c - b = 0$ or one of $c - a$ and $c - b$ is a negative integer then again the series terminates, and the problem reduces to checking equality between rational numbers. Thus for the remainder of the proof we assume that $a, b, c-a, c-b$ are not negative integers and, in addition, that both a and $c - b$ are non-zero.

Consider the sequence $\langle P_n(x)\rangle_n$ defined by

$$\begin{aligned} P_{2n}(x) &= {}_2F_1(a+n, b+n; c+2n; x) \text{ and} \\ P_{2n+1}(x) &= {}_2F_1(a+n, b+n+1; c+2n+1; x)\,. \end{aligned}$$

Let $\langle s_n\rangle_{n=1}^{\infty}$ be the sequence of linear fractional transformations given by $s_n(w) := a_n x/(1+w)$ such that a_{2n+1} and a_{2n} are given by

$$\frac{-(a+n)(c-b+n)}{(c+2n)(c+2n+1)} \text{ and } \frac{-(b+n)(c-a+n)}{(c+2n-1)(c+2n)}\,,$$

respectively. It can be shown that

$$\frac{P_0(x)}{P_1(x)} = s_1 \circ \cdots \circ s_n\left(\frac{P_n(x)}{P_{n+1}(x)}\right)$$

(see [33, § VI.1]). Under the aforementioned assumptions we have $a_i \neq 0$ for each $i = 1, \ldots, n$, and so the composition of the s_i is an invertible linear fractional transformation. It follows that

$$\frac{P_0(x)}{P_1(x)} = \frac{X_n P_{n+1}(x) + Y_n P_n(x)}{Z_n P_{n+1}(x) + W_n P_n(x)}\,, \tag{12}$$

where the sequences $\langle W_n\rangle_n, \langle X_n\rangle_n, \langle Y_n\rangle_n, \langle Z_n\rangle_n$ are over $\mathbb{Q}$ and satisfy $X_nW_n - Y_nZ_n \neq 0$. Take $N \in \mathbb{N}$ to be both even and sufficiently large such that $c + 2N > b + N$. Substituting (12) into (11) with $n = N$ and rearranging, we obtain the equation $\mathfrak{a}_N P_{N+1}(x) = \mathfrak{b}_N P_N(x)$, where $\mathfrak{a}_N$ and $\mathfrak{b}_N$ are defined by

$$X_N - Z_N\frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2} \text{ and } \frac{\beta_0\mu - \alpha u_0/u_{-1}}{c\mu^2}W_N - Y_N\,,$$

respectively. Observe now that Euler's integral representation (3) holds for both P_{N+1} and P_N. By linearity of the integral, we see that deciding Minimality for recurrences of the form (7) reduces to checking whether the integral

$$\int_0^1 \frac{t^{b+N-1}(1-t)^{c-b+N-1}}{(1-xt)^{a+N}} \left(\mathfrak{a}_N \tfrac{(c+2N)}{b+N} t - \mathfrak{b}_N\right) dt \tag{13}$$

vanishes. This integral is a period, so we are done for this part.

Now consider recurrence (8). In a similar manner to previous arguments, a solution $\langle u_n \rangle_n$ of (8) is minimal if $-\alpha u_0/u_{-1}$ is equal to the continued fraction in (10) when one sets $x = -1$, and either

$$a = \frac{1}{2}(\alpha + \beta_0 - \gamma_0),\ b = \alpha - 1,\ c = \frac{1}{2}(\alpha + \beta_0 + \gamma_0);\ \text{or}$$
$$a = \frac{1}{2}(\beta_0 + \gamma_0 - \alpha) + 1,\ b = \gamma_0,\ c = \frac{1}{2}(\beta_0 + \gamma_0 + \alpha)\,.$$

As $c + a - b - 1 = \beta_0 > 0$ and the parameters are real-valued, by [33, §VI, Thm. 4(A)] the continued fraction converges to (11). The remainder of the proof follows similarly. ◀

Theorem 6 follows by combining the reduction of the different variants of the Minimality Problem, given in Prop. 10, with the procedure for determining zeroness of 1-periods in Thm. 5 and the algorithm for determining polynomial relations between values of E-functions in Thm. 4. In particular, the case of two distinct rational characteristic roots falls under either $\mathsf{Minimal}(1,1,0)$ (which is interreducible with $\mathsf{Minimal}(0,1,1)$ by Prop. 10) or the cases of $\mathsf{Minimal}(1,0,1)$ and $\mathsf{Minimal}(1,1,1)$ corresponding to recurrences (9) and (7). These are all covered by Items 1 and 2 of Thm. 13, and thus can be solved unconditionally. Not only is there an algorithm for the decision version of the Minimality Problem, but we also have a procedure to solve the following function version of the problem:

▶ **Theorem 14.** *There is a procedure that inputs a second-order degree-one recurrence* (5) *with distinct rational characteristic roots, and either outputs algebraic initial values* u_{-1}, u_0 *giving rise to a minimal solution, or reports that no such algebraic values exist.*

Proof. First, we can determine whether the recurrence admits a minimal solution using Pincherle's Theorem (Thm. 2) and the Stern-Stolz convergence criterion (Thm. 17). If there is a minimal solution, it remains to see that there exists one with algebraic elements.

Consider Case 2 in the proof of Thm. 13. Here, a sequence is minimal if and only if its initial values are such that the integral (13) vanishes. Observe that the constants $\mathfrak{a}_N$ and $\mathfrak{b}_N$ in this integral depend linearly on u_0/u_{-1}. Treating u_0 and u_{-1} as indeterminates, multiply the expression (13) by u_{-1} to make it homogenous linear on u_0 and u_{-1} we deduce that minimality is equivalent to the equation

$$u_{-1}\mathfrak{p}_{-1} + u_0\mathfrak{p}_0 = 0 \tag{14}$$

where $\mathfrak{p}$'s are 1-periods. We can now use the main result of [43], see Thm. 5, to compute the space of solutions to (14) over algebraic u_{-1}, u_0.

Next, consider Case 1 of the proof of Thm. 13. Here, minimality reduces to the vanishing of a linear expression in u_{-1} and u_0 whose coefficients are the values of two E-functions at a common rational point. The algorithm of Thm. 4 allows one to compute the set of all algebraic such solutions u_{-1}, u_0. ◀

Note, as an immediate corollary, that we can also decide whether there exist *rational* initial values u_{-1}, u_0 giving rise to a minimal solution: indeed, the latter will hold if and only if there exist algebraic u_{-1}, u_0 giving rise to a minimal solution whose ratio is moreover rational.

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A Continued Fractions

Two continued fractions are *equivalent* if they share the same sequence of approximants. The following theorem gives a procedure to move between equivalent continued fractions [33, 34].

▶ **Theorem 15.** *The continued fractions* $\mathrm{K}(a_n/b_n)$ *and* $\mathrm{K}(c_n/d_n)$ *are equivalent if and only if there exists a sequence* $\langle \mathfrak{r}_n\rangle_{n=0}^{\infty}$ *with* $\mathfrak{r}_0 = 1$ *and* $\mathfrak{r}_n \neq 0$ *for each* $n \in \mathbb{N}$ *such that* $c_n = \mathfrak{r}_n\mathfrak{r}_{n-1}a_n$ *and* $d_n = \mathfrak{r}_n b_n$ *for each* $n \in \mathbb{N}$.

A continued fraction $\mathrm{K}(a_n/b_n)$ is *positive* if $a_n > 0$ and $b_n \geq 0$ for each $n \in \mathbb{N}$. We will use the following monotonicity result for the odd and even approximants of positive continued fractions [33, 34].

▶ **Lemma 16.** *Suppose that for each* $n \in \mathbb{N}$ *the sequences* $\langle a_n\rangle_n$ *and* $\langle b_n\rangle_n$ *are positive. Let* $\langle f_n\rangle_n$ *be the sequence of approximants associated with* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. *Then* $f_2 \leq f_4 \leq \cdots \leq f_{2m} \leq \cdots \leq f_{2m+1} \leq \cdots \leq f_3 \leq f_1$. *If, in addition,* $b_1 > 0$ *then the subsequences* $\langle f_{2n}\rangle_n$ *and* $\langle f_{2n+1}\rangle_n$ *converge to finite, non-negative limits.*

The following theorem collects together results from the literature concerning the convergence of continued fractions; the first statement follows as a consequence of Worpitzky's Theorem (see [34, Thm. 3.29]) and the convergence results in [27], whilst the second statement follows from the Lane–Wall characterisation of convergence [34, Thm. 3.3].

▶ **Theorem 17.** *Let $\mathrm{K}(\kappa_n/1)$ be a continued fraction with $\langle \kappa_n \rangle_n$ a function in $\mathbb{Q}(n)$. If $\kappa_n < 0$ for all sufficiently large $n \in \mathbb{N}$, then $\mathrm{K}(\kappa_n/1)$ converges to a value in $\hat{\mathbb{R}}$ if and only if, either*

- *$\lim_{n\to\infty} \kappa_n$ exists and is strictly above $-1/4$, or*
- *$\lim_{n\to\infty} \kappa_n = -1/4$ and moreover $\kappa_n \geq -1/4 - 1/(4n)^2 - 1/(4n\log n)^2$ for all sufficiently large n.*

We recall a necessary and sufficient criterion for convergence of a positive continued fraction [34, Thm. 3.14].

▶ **Theorem 18** (Stern-Stolz Theorem)**.** *A positive continued fraction $\mathrm{K}(a_n/b_n)$ converges if and only if its* Stern-Stolz series $\sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \right|$ *diverges to ∞.*

B Proof of Proposition 10

In the work that follows it will be useful to normalise recurrence (2). Our normalisation is an equivalence transformation (in the sense of Thm. 15). Set $b_0 = 1$, $\kappa_n := a_n/(b_n b_{n-1})$, and consider

$$w_n = w_{n-1} + \kappa_n w_{n-2}\,. \tag{15}$$

Then $\langle w_n \rangle_n$ with $w_{-1} = u_{-1}$ and $w_n := u_n/(\prod_{k=1}^{n} b_k)$ is a solution to (15) if and only if $\langle u_n \rangle_n$ is a solution to (2).

Proof of Proposition 10.

1. The result follows immediately from the equivalence transformation between (2) and (15).
2. Division by α_1 normalises the recurrence in the following way:

$$(n+\alpha)u_n = \beta u_{n-1} + \gamma u_{n-2}\,. \tag{16}$$

We shall assume that $\alpha := \alpha_0/\alpha_1 > 1$ (this can be achieved by shifting as appropriate). Suppose that $\langle u_n \rangle_n$ is a solution of the normalised recurrence. We use the updated reduction argument in Sec. 3 to obtain a second recurrence. We have

$$(2n+\alpha)(2n+\alpha-1)u_{2n} = (\beta^2 + \gamma(4n+2\alpha-3))u_{2n-2} - \gamma^2 u_{2n-4}\,. \tag{17}$$

This defines a second-order recurrence with solutions $\langle v_n \rangle_{n=0}^{\infty}$ given by $v_n := u_{2n}$. The mapping $n \mapsto 2n$ establishes a one-to-one correspondence between the solutions of recurrences (16) and (17) and we claim this correspondence preserves minimality. In order to prove this claim we show that linear independence and the asymptotic equalities are preserved. For linear independence one direction is trivial: if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly dependent solutions of (16), then $\langle u_{2n} \rangle_n$ and $\langle v_{2n} \rangle_n$ are linearly dependent solutions of (17). For the converse, suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are linearly independent. Assume, for a contradiction, that there exists an $\ell \in \mathbb{R}$ such that $u_{2n} = \ell v_{2n}$ for each n. We study the sequence $\langle u_n - \ell v_n\rangle_{n=-1}^{\infty}$. By assumption $0 \neq u_{-1} - \ell v_{-1}$ and $0 = u_0 - \ell v_0$. Using (16) we then compute

$$u_1 - \ell v_1 = \frac{\gamma}{1+\alpha}(u_{-1} - \ell v_{-1}) \quad \text{and}$$
$$u_2 - \ell v_2 = \frac{\beta}{2+\alpha}(u_1 - \ell v_1) \neq 0\,,$$

a contradiction to our assumption.

We turn our attention to minimality. Suppose that $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are minimal and dominant solutions of (16), respectively. Then $\lim_{n\to\infty} u_n/v_n = \lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since $\langle u_{2n}\rangle_n$ and $\langle v_{2n}\rangle_n$ are linearly independent by the above, $\langle u_{2n}\rangle_n$ is necessarily a minimal solution of (17). Conversely, assume that $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are linearly independent solutions such that $\langle u_{2n}\rangle_n$ is a minimal solution of (17) (recall that the existence of minimal solutions is decidable for each recurrence). Since $\langle v_{2n}\rangle_n$ is linearly independent of $\langle u_{2n}\rangle_n$, $\lim_{n\to\infty} u_{2n}/v_{2n} = 0$. Since recurrence (16) must also admit minimal solutions, one easily deduces that $\langle u_n\rangle_n$ is likewise minimal.

Notice that the Minimality Problem for $\langle v_n\rangle_n$ is an instance of $\mathsf{Minimal}(2,1,0)$ where the polynomial g_3 has rational roots. The equivalence transformations between (2) and (15) give the reduction to $\mathsf{Minimal}(1,1,1)$ under this assumption.

3. A solution sequence $\langle u_n\rangle_n$ satisfies a normalised recurrence of the form

$$u_n = \frac{\beta_1 n + \beta_0}{n+\alpha} u_{n-1} + \frac{\gamma_1 n + \gamma_0}{n+\alpha} u_{n-2}\,.$$

If $\beta_1 = |\gamma_1|$ then we are done. If not, consider the sequence $\langle v_n\rangle_n$ with terms given by $v_n := (\operatorname{sign}(\gamma_1)\beta_1/\gamma_1)^n u_n$. Not only is it evident that $\langle v_n\rangle_n$ satisfies a recurrence of the desired form, but the sequence $\langle v_n\rangle_n$ is also a minimal solution if and only if $\langle u_n\rangle_n$ is a minimal solution.

Assume now that $\beta_1^2 + 4\gamma_1 = 0$ in (7). As $|\gamma_1| = |\beta_1|$, it follows immediately that $\beta_1 = 4 = -\gamma_1$. Now the sequence $\langle (1/2)^n u_n\rangle_n$ satisfies a recurrence of the form (8) and minimality is clearly preserved by this transformation.

4. In this case the recurrence admits minimal solutions if and only if $\gamma_1\alpha_1 > 0$ (compare to Lem. 12). This follows by an application of Theorem 17 as the standard normalisation (15) has $\kappa_n = (\gamma_1 n + \gamma_0)(\alpha_1(n-1)+\alpha_0)/\beta_0^2$. The reduction to (9) follows by considering the sequence $\langle (\operatorname{sign}(\beta_0)\sqrt{\alpha_1/\gamma_1})^n u_n\rangle_n$. ◀

C Proof of Theorem 13 Continued

Let us first prove Lem. 12.

Proof of Lem. 12. First, it is useful to normalise the recurrences using the normalisation in (15) so that one can determine whether a minimal solution exists using the criteria in Thm. 17 and Thm. 2. Under our assumptions, this normalisation does not change the signature of the recurrence. Second, recall that the characteristic roots of a recurrence are real if and only if $\beta_1^2 + 4\gamma_1 \geq 0$. Further, the characteristic roots are distinct if and only if $\beta_1^2 + 4\gamma_1 \neq 0$.

1. Regardless of whether $\gamma_1 = 0$ or not, when the recurrence has signature $(-,+)$ it is clear that $\lim_{n\to\infty}\kappa_n = 0$. When the recurrence has signature $(+,+)$ it is clear that $\sum_{n=2}^{\infty} 1/(n\kappa_n) = \infty$ and so the associated Stern-Stolz series (Theorem 18) diverges to ∞. These conditions are sufficient to prove the statement.

2. First, let us consider recurrence (7) under the assumption that $\beta_1^2 + 4\gamma \neq 0$. When (7) has signature $(-,+)$ there are minimal solutions if and only if $1 + 4\gamma_1/\beta_1^2 > 0$ if and only if $\lambda,\, \mu \in \mathbb{R}$. When (7) has signature $(+,+)$ it is clear that $\lim_{n\to\infty} \kappa_n = \gamma_1/\beta_1^2 > 0$ and so always admits minimal solutions. Finally, recurrence (9) also always admits minimal solutions: here the recurrence has signature $(+,+)$ and the Stern-Stolz series (Theorem 18) diverges to ∞ since $\sum_{n=2}^{\infty} 1/\sqrt{\kappa_n} = \infty$.
3. The normalisation of (8) is of the form $w_n = w_{n-1} + \kappa_n w_{n-2}$ with

$$\kappa_n = -\frac{1}{4} - \frac{\alpha - \beta_0 + \gamma_0}{4n} - \frac{\varepsilon}{16n^2} + \mathscr{O}(1/n^3)$$

and $\varepsilon = 4\beta_0(\beta_0 - \alpha - \gamma_0) + 4\alpha(1+\gamma_0) - \beta_0(\beta_0 + 2)$. There are two cases to consider. If $\beta_0 - \alpha - \gamma_0 \neq 0$ then the recurrence admits minimal solutions if and only if $\beta_0 - \alpha - \gamma_0 > 0$. Otherwise $\beta_0 = \alpha + \gamma_0$, in which case κ_n simplifies as follows:

$$\kappa_n := -\frac{1}{4} - \frac{-(\alpha-\gamma)(\alpha-\gamma-2)}{16n^2} + \mathscr{O}(1/n^3)\,.$$

Since $-x(x-2) \leq 1$ for each $x \in \mathbb{R}$, by Theorem 17, we deduce that this subcase always admits minimal solutions. ◀

We move on to complete the proof of Theorem 13.

Proof of Theorem 13(1). Consider an instance of Minimal$(0,1,1)$. Without loss of generality it is of the form $u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}$. By shifting the sequence appropriately, we may assume that $\frac{\beta_0}{\beta_1} + \frac{\gamma_1}{\beta_1^2}$ is positive. Let $\langle u_n \rangle_n$ be a minimal solution. Then $-u_0/u_{-1}$ is equal to

$$\operatorname*{K}_{n=1}^{\infty} \frac{\gamma_1 n + \gamma_0}{\beta_1 n + \beta_0} = \frac{(\beta_0 + \frac{\gamma_1}{\beta_1})\, {}_1F_1(\frac{\gamma_0}{\gamma_1}; \frac{\beta_0}{\beta_1} + \frac{\gamma_1}{\beta_1^2}; \frac{\gamma_1}{\beta_1^2})}{{}_1F_1(\frac{\gamma_0}{\gamma_1} + 1; \frac{\beta_0}{\beta_1} + \frac{\gamma_1}{\beta_1^2} + 1; \frac{\gamma_1}{\beta_1^2})} - \beta_0\,.$$

Here the value on the right-hand side is given in [33, §VI.4].

Let $a = \gamma_0/\gamma_1$, $b = \beta_0/\beta_1 + \gamma_1/\beta_1^2$, and $x = \gamma_1/\beta_1^2$, then Minimal$(0,1,1)$ reduces to checking the equality

$$\frac{{}_1F_1(a;b;x)}{{}_1F_1(a+1;b+1;x)} = \frac{(u_{-1}\beta_0 - u_0)}{u_{-1}(\beta_0 + \frac{\gamma_1}{\beta_1})}\,.$$

Notice here that $a > 0$ and $b > 0$ by assumption. This equation expresses a linear relation between the values of two E-functions, as required by the proposition. ◀

Proof of Theorem 13(3). Our objective is to show that Minimal$(0,1,0)$ reduces to the task of determining whether an exponential period vanishes. If $\langle u_n \rangle_n$ is a minimal solution of the recurrence $u_n = (\beta_1 n + \beta_0)u_{n-1} + \gamma_0 u_{n-2}$ then, by Theorem 2, $-u_0/u_{-1}$ is equal to

$$\operatorname*{K}_{n=1}^{\infty} \left(\frac{\gamma_0}{\beta_1 n + \beta_0} \right) = \beta_0 \frac{{}_0F_1(;\beta_0/\beta_1;\gamma_0/\beta_1^2)}{{}_0F_1(;\beta_0/\beta_1 + 1;\gamma_0/\beta_1^2)} - \beta_0$$

(see [33, §VI.4.1]). Hence the Minimality Problem for the above recurrence reduces to checking the equality

$$(\beta_0 u_{-1} - u_0)\, {}_0F_1\left(;\frac{\beta_0}{\beta_1} + 1; \frac{\gamma_0}{\beta_1^2}\right) = \beta_0\, {}_0F_1\left(;\frac{\beta_0}{\beta_1}; \frac{\gamma_0}{\beta_1^2}\right).$$

The *Bessel functions of the first kind*, sometimes called *cylinder functions*, $J_s(z)$ are a family of functions that solve Bessel's differential equation [8]. For $z, s \in \mathsf{C}$ the function $J_s(z)$ is defined by the hypergeometric series [1, equation 9.1.69]

$$J_s(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+1)} \left(\frac{z}{2}\right)^{2k+s} = \frac{1}{\Gamma(s+1)} \left(\frac{z}{2}\right)^s {}_0F_1(;s+1;-z^2/4).$$

We obtain the principal branch of $J_s(z)$ by assigning $(z/2)^s$ its principal value. When $\mathrm{Re}(s) > -1/2$ we have the following integral representation [3, equation 4.7.5],

$$J_s(z) = \frac{1}{\sqrt{\pi}\Gamma(s+1/2)} \left(\frac{z}{2}\right)^s \int_{-1}^{1} e^{izt}(1-t^2)^{s-1/2}\, dt.$$

Hence for $\mathrm{Re}(s) > -1/2$, we have the following integral representation

$${}_0F_1(;s+1;z) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} \int_{-1}^{1} e^{-2\sqrt{z}t}(1-t^2)^{s-1/2}\, dt.$$

Let us return to minimal solutions of the aforementioned recurrence relation. By substitution and linearity of the integral, we see that $\mathsf{Minimal}(0,1,0)$ reduces to checking whether the following integral

$$\int_{-1}^{1} e^{\frac{-2\sqrt{\gamma_0}t}{\beta_1}} (1-t^2)^{\frac{\beta_0}{\beta_1}-\frac{3}{2}} \left(\frac{2(\beta_0 u_{-1} - u_0)}{2\beta_0 - 1}(1-t^2) - 1 \right) dt$$

vanishes. To ensure that the integral converges absolutely note that we can shift the recurrence so that $\beta_0/\beta_1 > 3/2$. The integral is an exponential period, which completes the proof. ◀

Before presenting the proof of Theorem 13(4), we need some auxilliary lemmas. We first deal with a simple case that turns out to be decidable. To this end, we use the following well-known determinant lemma (see, for example, [33, Lem. 4, §IV]).

▶ **Lemma 19.** *Suppose that $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are both solutions to the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. Then*

$$u_n v_{n-1} - u_{n-1} v_n = (u_0 v_{-1} - u_{-1} v_0) \prod_{k=1}^{n} (-a_k)\,.$$

▶ **Lemma 20.** *Let $\langle u_n \rangle_n$ be a non-trivial solution to (8) with $\beta_0 = \alpha + \gamma_0$. If $\alpha \le \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = 1$. If $\alpha > \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = (\gamma_0+1)/\alpha$.*

Proof. If $\beta_0 = \alpha + \gamma_0$, then the constant sequence $\langle 1 \rangle_n$ is a solution to the recurrence by inspection. Hence $\langle 1 \rangle_n$ and $\langle B_n \rangle_n$ defined by $B_{-1} = 0$, $B_0 = 1$ are linearly independent solutions and, by Lemma 19,

$$B_n = \sum_{k=0}^{n} \prod_{m=1}^{k} \frac{m+\gamma_0}{m+\alpha} = \sum_{k=0}^{n} \frac{(\gamma_0+1)_k}{(\alpha+1)_k}\,.$$

By a straightforward application of Stirling's approximation, $(\gamma_0+1)_n/(\alpha+1)_n \sim n^{\gamma_0-\alpha}$ as $n \to \infty$. Hence if $\gamma_0 - \alpha \ge -1$ the series diverges (by comparison to the harmonic series) from which we deduce that $\langle 1\rangle_n$ is minimal. If $\gamma_0 - \alpha < -1$, then $\lim_{n\to\infty} B_n$ converges to the value

$$\sum_{k=0}^{\infty} \frac{(\gamma_0+1)_k(1)_k}{(\alpha+1)_k}\frac{1}{k!} = {}_2F_1(\gamma_0+1,1;\alpha+1;1) = \frac{\Gamma(\alpha+1)\Gamma(\alpha-\gamma_0-1)}{\Gamma(\alpha-\gamma_0)\Gamma(\alpha)} = \frac{\alpha}{\alpha-\gamma_0-1}\,.$$

In the second equality we use [3, Thm. 2.2.2]. It follows that $\langle u_n\rangle_n = \frac{\alpha}{\alpha-\gamma_0-1}\langle 1\rangle_n - \langle B_n\rangle_n$ is a minimal solution, and we may compute $u_0/u_{-1} = (\gamma_0+1)/\alpha$. ◀

Proof of Theorem 13(4). The case when $\beta_0 = \alpha + \gamma_0$ is decidable by Lemma 20, so we consider the case $\beta_0 > \alpha + \gamma_0$; otherwise the recurrence admits no minimal solutions by Lemma 12.

The function $U(a,b,x)$, the *confluent hypergeometric function of the second kind*, is defined for all $a,b,x \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(x) > 0$ as

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}\,dt\,.$$

As noted by Temme in [46], the sequence $\langle u_n\rangle_{n=-1}^\infty$ given by $u_{-1} = \frac{1}{a-1}U(a-1,b,x)$, $u_n := (a)_n U(a+n,b,x)$ is a minimal solution of the recurrence

$$u_n = \frac{2n+x+2a-b-2}{n+a-b}u_{n-1} - \frac{n+a-2}{n+a-b}u_{n-2} \tag{18}$$

(assuming that $a \neq 1$ and $a-b$ is not a negative integer). Notice that the recurrence holds also for $n = 1$ because

$$(2a+x-b)U(a,b,z) - U(a-1,b,z) = a(1+a-b)U(a+1,b,c)\,.$$

Consider now recurrence (8): $u_n = \frac{2n+\beta_0}{n+\alpha}u_{n-1} - \frac{n+\gamma_0}{n+\alpha}u_{n-2}$. When one substitutes the values $a = \gamma_0 + 2$, $b = \gamma_0 + 2 - \alpha$, and $x = \beta_0 - \gamma_0 - \alpha$ into (8) one obtains recurrence (18). Subject to an initial shift of the sequence, we may assume that have $a > 2$. We also have $x > 0$ by assumption (shifting the sequence has no effect on x). Hence, a minimal solution to (8) satisfies $u_0/u_{-1} = (a-1)U(a,b,x)/U(a-1,b,x)$. We may apply the integral representation for U immediately. Since the parameters involved are rational numbers, the integrals obtained are exponential periods, and the claim follows. ◀