On Positivity and Minimality for Second-Order Holonomic Sequences

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Abstract

An infinite sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ of real numbers is *holonomic* (also known as *P-recursive* or *P-finite*) if it satisfies a linear recurrence relation with polynomial coefficients. Such a sequence is said to be *positive* if each $u_n \geq 0$, and *minimal* if, given any other linearly independent sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0.

In this paper, we focus on holonomic sequences satisfying a second-order recurrence

$$g_3(n)u_n = g_2(n)u_{n-1} + g_1(n)u_{n-2}\,,$$

where each coefficient $g_3, g_2, g_1 \in \mathbb{Q}[n]$ is a polynomial of degree at most 1. We establish two main results. First, we show that deciding positivity for such sequences reduces to deciding minimality. And second, we prove that deciding minimality is equivalent to determining whether certain numerical expressions (known as *periods*, *exponential periods*, and *period-like integrals*) are equal to zero. Periods and related expressions are classical objects of study in algebraic geometry and number theory, and several established conjectures (notably those of Kontsevich and Zagier) imply that they have a decidable equality problem, which in turn would entail decidability of Positivity and Minimality for a large class of second-order holonomic sequences.

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1 Introduction

Holonomic sequences (also known as *P-recursive* or *P-finite* sequences) are infinite sequences of real (or complex) numbers that satisfy a linear recurrence relation with polynomial coefficients. The earliest and best-known example is the Fibonacci sequence, given by Leonardo of Pisa in the 12th century; more recently, Apéry famously made use of certain holonomic sequences satisfying the recurrence relation

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1} \quad (n \in \mathbb{N})$$

to prove that $\zeta(3) := \sum_{n=1}^{\infty} n^{-3}$ is irrational [4]. Holonomic sequences now form a vast subject in their own right, with numerous applications in mathematics and other sciences; see, for instance, the monographs [36, 12, 13] or the seminal paper [49].

Formally, a holonomic recurrence is a relation of the following form:

$$g_{k+1}(n)u_{n+k} = g_k(n)u_{n+k-1} + \ldots + g_1(n)u_n \,,$$

where $g_{k+1}, \ldots, g_1 \in \mathbb{Q}[n]$ are polynomials with rational coefficients. We define the *order* of the recurrence to be k, and its *degree* to be the maximum degree of the polynomials g_i. Assuming that $g_{k+1}(n) \neq 0$ for non-negative integer n, the above recurrence uniquely defines an infinite sequence $\langle u_n \rangle_{n=0}^{\infty}$ once the k initial values $u_0, \ldots, u_{k-1}$ are specified.[1] Such a sequence is said to be *holonomic*, and—in slight abuse of terminology—will be understood to inherit the *order* and *degree* of its defining recurrence. Degree-0 holonomic sequences—i.e., such that all polynomial coefficients appearing in the recurrence relation are constant—are also known as *C-finite* sequences, and first-order holonomic sequences are known as *hypergeometric* sequences.

Holonomic sequences naturally give rise to *holonomic functions* by considering the associated generating power series $\mathcal{F}(x) = \sum_{n=0}^{\infty} u_n x^n$. As is well-known, the generating functions of C-finite sequences are rational functions, and those of hypergeometric sequences are hypergeometric functions. Properties of holonomic functions—and in particular the differential equations that they obey—will play a central rôle in our analysis of their defining sequences.

There is a voluminous literature devoted to the study of identities for holonomic sequences. However, as noted by Kauers and Pillwein, *"in contrast, [. . .] almost no algorithms are available for inequalities"* [19]. For example, the *Positivity Problem* (i.e., whether every term of a given sequence is non-negative) for C-finite sequences is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [32, 31]; see also [18, 24, 32, 30]. For holonomic sequences that are not C-finite, virtually no decision procedures currently exist for Positivity, although several partial results and heuristics are known (see, for example [25, 19, 29, 48, 37, 38]).

Another extremely important property of holonomic sequences is *minimality*; a sequence $\langle u_n \rangle_n$ is minimal if, given any other linearly independent sequence $\langle v_n \rangle_n$ satisfying the same

[1] In the sequel, it will in fact often be convenient to start the sequence at u_{-1} instead of u_0.

recurrence relation, the ratio u_n/v_n converges to 0. Minimal holonomic sequences play a crucial rôle, among others, in numerical calculations and asymptotics, as noted for example in [15, 16, 14, 10, 2, 11]—see also the references therein. Unfortunately, there is also ample evidence that determining algorithmically whether a given holonomic sequence is minimal is a very challenging task, for which no satisfactory solution is at present known to exist.

The systematic study of Positivity and Minimality for holonomic sequences of order two and above is a vast undertaking.[2] Accordingly, our focus in the present paper is on second-order, degree-1 sequences.[3] The generating functions of such sequences satisfy certain linear differential equations, whose solutions involve integrals of a particular shape; depending on the original sequence, the definite forms of these integrals are known either as *periods*, *exponential periods*, or *period-like integrals*. Periods and related expressions are classical objects of study in algebraic geometry and number theory, and several established conjectures—notably those of Kontsevich and Zagier [20]—imply that they have a decidable equality problem (see Appendix A for a more detailed account of these facts and considerations). At a high level, whether a given holonomic sequence is minimal or not is related to the radius of convergence of its associated generating function, which in turn hinges on the precise value of these definite integrals. Consequently, we reduce the problem of determining minimality of a given sequence to whether the corresponding integral is zero. Unfortunately, for holonomic sequences of order greater than two, or of degree higher than one, solving the attendant differential equations no longer yields integrals of the appropriate shape.

Main results. We summarise our main results as follows. Consider the class of real-algebraic, second-order, degree-1 holonomic sequences. For this class:

1. The Positivity Problem reduces to the Minimality Problem (Theorem 3.1).
2. The Minimality Problem reduces to determining whether a period, an exponential period, or a period-like integral is equal to zero (Theorem 4.2).

2 Preliminaries

2.1 Second-order linear recurrences

We study the behaviour of solutions $\langle v_n \rangle_{n=-1}^{\infty}$ to second-order recurrence relations of the form

$$g_3(n)v_n = g_2(n)v_{n-1} + g_1(n)v_{n-2}, \quad \text{or} \tag{2.1a}$$

$$v_n = \frac{g_2(n)}{g_3(n)}v_{n-1} + \frac{g_1(n)}{g_3(n)}v_{n-2}, \quad n \in \mathbb{N}. \tag{2.1b}$$

where $g_1, g_2, g_3 \in \mathbb{Q}[x]$. Solutions to such recurrences are called *holonomic sequences*. In the sequel it is useful to transform recurrence (2.1) as follows. For $g_3 \in \mathbb{Q}[x]$, let $\langle u_n \rangle_{n=-1}^{\infty}$ and $\langle v_n \rangle_n$ be real-valued sequences such that $u_{-1} = v_{-1}$ and $u_n = g_3(n) \cdots g_3(0) v_n$ for $n \in \mathbb{N}_0$. Then it is easily seen that $\langle v_n \rangle_n$ is a solution to (2.1) if and only if $\langle u_n \rangle_n$ is a solution to the recurrence

$$u_n = g_2(n)u_{n-1} + g_1(n)g_3(n-1)u_{n-2}. \tag{2.2}$$

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[2] At order one, both problems are algorithmically trivial: indeed, the positivity of a hypergeometric sequence is readily determined by inspecting the polynomial coefficients of its defining recurrence, together with the sign of the first few values of the sequence; and since the solution set of a hypergeometric recurrence is a one-dimensional vector space, such recurrences cannot possibly admit minimal sequences.

[3] Positivity and minimality for second-order C-finite sequences can straightforwardly be determined from their closed-form solutions; see [18].

With this transformation we can translate statements about minimality and positivity of solutions to (2.2), subject to the condition that $g_3(n) > 0$ for each $n \in \mathbb{N}_0$. If $g_3(n) > 0$ for each $n \in \mathbb{N}_0$ then $\langle u_n \rangle_n$ is minimal (positive) if and only if $\langle v_n \rangle_n$ is minimal (positive).

Let $\langle u_n \rangle_n$ be a sequence satisfying the second-order relation (2.1). Note that if g_2 is identically 0, then $\langle u_n \rangle_n$ consists of the interleaving of two hypergeometric sequences, in which case positivity of $\langle u_n \rangle_n$ is simply equivalent to the positivity of both individual hypergeometric sequences, something which can readily be determined as noted in the Introduction. Moreover, such a recurrence admits no minimal solutions: this follows straightforwardly from the observation that the limit $\lim_{n\to\infty} A_n/B_n$ does not exist for the linearly independent solutions $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ defined by $A_{-1} = 1$, $A_0 = 0$, $B_{-1} = 0$, $B_0 = 1$. Similarly, if g_1 is identially 0, then positivity and minimality of $\langle u_n \rangle_n$ likewise become trivial. In what follows, we will therefore assume that none of g_1, g_2, g_3 are identically 0.

Moreover (considering a shifted recurrence relation if necessary), we can assume without loss of generality that each polynomial coefficient has constant sign, and has no roots for $n \geq 0$. Additionally we can assume that $\operatorname{sign}(g_3) = 1$. The *signature* of this relation is defined as the ordered tuple $(\operatorname{sign}(g_2(n)), \operatorname{sign}(g_1(n)))$.

2.2 Asymptotic equalities for second-order linear recurrences

Here we state asymptotic results established by Poincaré and Perron in the restricted setting of second-order recurrence relations. Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be real-valued sequences. We say that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a *Poincaré recurrence* if the limits $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist and are finite. The next result, initially considered by Poincaré [40] and expanded upon by Perron [33], considers Poincaré recurrences as perturbations of C-finite recurrences.

▶ **Theorem 2.1** (Poincaré–Perron Theorem). *Suppose that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a Poincaré recurrence and $a_n, b_n \neq 0$ for each $n \in \mathbb{N}$. Let λ and Λ be the roots of the associated characteristic polynomial $x^2 - bx - a$ and suppose that $|\lambda| \neq |\Lambda|$. Then the above recurrence has two linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ such that $u_{n+1}^{(1)}/u_n^{(1)} \sim \lambda$ and $u_{n+1}^{(2)}/u_n^{(2)} \sim \Lambda$.*

Later work by Perron [34] considered the case that the two roots are equal in modulus, as follows.

▶ **Theorem 2.2.** *Suppose that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a Poincaré recurrence and $a_n, b_n \neq 0$ for each $n \in \mathbb{N}$. Let λ and Λ be the roots of the associated characteristic polynomial $x^2 - bx - a$. Then the above recurrence has two linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ such that $\limsup_{n\to\infty} \sqrt[n]{|u_n^{(1)}|} = |\lambda|$ and $\limsup_{n\to\infty} \sqrt[n]{|u_n^{(2)}|} = |\Lambda|$.*

We note that one cannot obtain the neat asymptotic equalities of the form given in the Poincaré–Perron Theorem when the moduli of the roots coincide (consider, for example the Poincaré recurrence $u_n = u_{n-2}$ whose characteristic roots are ± 1). However, later work by Kooman gives a complete characterisation of the asymptotic behaviour of linearly independent solutions for a family of second-order Poincaré recurrence relations. We give two illustrating examples illustrating when the characteristic roots have equal modulus. The proof is a straightforward application of results in [23]. We shall make use of these particular forms in the sequal.

▶ **Example 2.3** (Appendix H).

1. The recurrence relation $(n+\alpha)u_n = \beta u_{n-1} + (n+\gamma)u_{n-2}$ with α β, $\gamma \in \mathbb{R}$ and $\beta > 0$ admits linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ with the following asymptotic equalities: $u_n^{(1)} \sim n^{\frac{1}{2}(\beta+\gamma-\alpha)}$ and $u_n^{(2)} \sim (-1)^n n^{\frac{1}{2}(-\beta+\gamma-\alpha)}$.
2. The recurrence relation $(n+\alpha)u_n = (2n+\beta)u_{n-1} - (n+\gamma)u_{n-2}$, with α, β, $\gamma \in \mathbb{R}$ with $\beta > \alpha+\gamma$ admits linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ with the following asymptotic equalities:

$$u_n^{(1)} \sim n^{(1-2\alpha+2\gamma)/4} \exp(2\sqrt{(\beta-\alpha-\gamma)n}), \quad \text{and}$$
$$u_n^{(2)} \sim n^{(1-2\alpha+2\gamma)/4} \exp(-2\sqrt{(\beta-\alpha-\gamma)n}).$$

2.3 Continued fractions

A *continued fraction* is an ordered pair $((\langle a_n \rangle_{n=1}^{\infty}, \langle b_n \rangle_{n=0}^{\infty}), \langle f_n \rangle_{n=0}^{\infty})$ where $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are sequences of complex numbers such that for each $n \in \mathbb{N}$, $a_n \neq 0$ and $\langle f_n \rangle_n$ is a sequence in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ recursively defined by the following composition of linear fractional transformations. For $w \in \hat{\mathbb{C}}$, define

$$s_0(w) = b_0 + w \text{ and } s_n(w) = \frac{a_n}{b_n + w} \text{ for each } n \in \{1, 2, \dots\}.$$

We set $f_n := s_0 \circ \cdots \circ s_n(0)$ so that

$$f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n}}}}.$$

It is convenient to introduce concise notation for continued fractions and their convergents. We shall make use of Gauss's Kettenbruch notation $f_n =: b_0 + \mathrm{K}_{k=1}^{n}(a_k/b_k)$, and abuse the infinite form of this notation to refer both to the continued fractions and to their limits (if they converge).

We respectively call $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ the sequences of *partial numerators* and *partial denominators* (together the *partial quotients*) of the continued fraction $\mathrm{K}(a_n/b_n)$. We call $\langle f_n \rangle_n$ the sequence of *convergents*. Let $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ satisfy the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial values $A_{-1} = 1, A_0 = 0, B_{-1} = 0$, and $B_0 = 1$. Then $\langle A_n \rangle_n$ and $\langle B_n \rangle_n$ are respectively called the sequences of *canonical numerators* and *canonical denominators* of $\mathrm{K}(a_n/b_n)$ because $f_n = A_n/B_n$ for each $n \in \mathbb{N}_0$. We call $f \in \hat{\mathbb{C}}$ the *limit* of the continued fraction $\mathrm{K}(a_n/b_n)$ if $f_n \to f$ as $n \to \infty$ and say that $\mathrm{K}(a_n/b_n)$ *converges* if such a limit exists. The results presented herein consider continued fractions whose partial quotients are real-valued. Nevertheless it is often useful to adopt the standard notion of convergence in $\hat{\mathbb{C}}$ in order to exploit the algebraic properties of $\hat{\mathbb{C}}$.

Two continued fractions are said to be *equivalent* if they have the same sequence of convergents. From the standard equivalence transformation (as described in [9, §1.4] or [26, Chapter II, Cor. 10]), We have the following equivalences for the continued fractions associated to the respective recurrences (2.1b) and (2.2):

$$\operatorname*{K}_{n=1}^{\infty} \frac{g_1(n)/g_3(n)}{g_2(n)/g_3(n)} \equiv \frac{g_1(1)/g_2(1)}{1 + \mathrm{K}_{n=2}^{\infty}(d_n/1)}, \quad \text{and}$$
$$\operatorname*{K}_{n=1}^{\infty} \frac{g_1(n)g_3(n-1)}{g_2(n)} \equiv \frac{g_1(1)g_3(0)/g_2(1)}{1 + \mathrm{K}_{n=2}^{\infty}(d_n/1)}.$$

Here $d_n = \frac{g_1(n)g_3(n-1)}{g_2(n-1)g_2(n)}$ for each $n \in \mathbb{N}$. Note that we are permitted to make these transformations under the assumption that $g_2(n) \neq 0$ for each $n \in \mathbb{N}$. It is clear from the tails of the above continued fractions and Pincherle's Theorem (Theorem 2.8) that the transformations described between the recurrence forms preserves the existence of minimal solutions.

A *simple continued fraction* takes the form $b_0 + \mathrm{K}_{n=1}^{\infty}(1/b_n)$ where each partial denominator is a positive integer. The number π has an erratic simple continued fraction expansion whose sequence of partial denominators begins $(3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots)$. However, Lord Brouncker (as reported by Wallis in [45][4]) gave a continued fraction expansion for $4/\pi$ whose partial quotients are polynomials as follows:

$$\frac{4}{\pi} = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2}.$$

Apéry's constant $\zeta(3)$ has a continued fraction expansion (see [43])

$$\zeta(3) = \frac{6}{5 + \mathrm{K}_{n=1}^{\infty}(-n^6/(34n^3 + 51n^2 + 27n + 5))}$$

whose partial quotients are ultimately polynomials. We refer the reader to [9] for further examples of continued expansions of famous constants. Motivated by such constructions, Bowman and Mc Laughlin [7] (see also [28]) coined the term *polynomial continued fraction*. A polynomial continued fraction $\mathrm{K}(a_n/b_n)$ has integer partial quotients such that for sufficiently large $n \in \mathbb{N}$ we have $a_n = p(n)$ and $b_n = q(n)$ for $p, q \in \mathbb{Z}[x]$. The evaluation of polynomial continued fractions whose partial quotients have low degrees appears in the accounts [35, 26, 9]. For $\deg(a_n) \leq 2$ and $\deg(b_n) \leq 1$, Lorentzen and Waadeland [26, §6.4] express the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ as a ratio of two hypergeometric functions with algebraic parameters. However, their methods do not cover all cases at low degrees; for example, the polynomial continued fraction $\mathrm{K}_{n=1}^{\infty} \frac{-n(n+1)}{2n+1}$ corresponding to the recurrence relation $(n+1)u_n = (2n+1)u_{n-1} - (n+1)u_{n-2}$ cannot be so treated. Indeed the method presented in [26] cannot handle cases where the corresponding recurrence has a single repeated characteristic root—the above is one such example with its associated characteristic polynomial $x^2 - 2x + 1 = (x-1)^2$.

▶ Remark 2.4. Let $\mathbb{P} \subset \mathbb{R}$ be the set of real numbers that have a polynomial continued fraction expansion. We have that $\mathbb{Q} \subset \mathbb{P}$ and, as can be seen from the literature, there is a plethora of examples of both algebraic and transcendental numbers in $\mathbb{P}$. We shall be interested in the problem of determining whether a polynomial continued fraction expansion and an algebraic number are equal.

2.4 Convergence criteria for continued fractions

A continued fraction $\mathrm{K}(a_n/b_n)$ is said to be *positive* if $a_n > 0$ and $b_n \geq 0$ for each $n \in \mathbb{N}$.

▶ **Lemma 2.5.** *Suppose that for each $n \in \mathbb{N}$ the sequences $\langle a_n \rangle_n$ and $\langle b_n \rangle_n$ are positive. Let $\langle f_n \rangle_n$ be the sequence of convergents associated to the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. Then*

$$f_2 \leq f_4 \leq \cdots \leq f_{2m} \leq \cdots \leq f_{2m+1} \leq \cdots \leq f_3 \leq f_1. \tag{2.3}$$

[4] See the translation by Stedall [46].

If, in addition, $b_1 > 0$ then the subsequences $\langle f_{2n} \rangle_n$ and $\langle f_{2n+1} \rangle_n$ converge to finite, non-negative limits. If $b_n > 0$ for each $n \in \mathbb{N}$ then (2.3) *above holds with strict inequalities.*

We recall a necessary and sufficient criterion for convergence of a positive continued fraction [27, Theorem 3.14].

▶ **Theorem 2.6** (Seidel–Stern Theorem). *A positive continued fraction $\mathrm{K}(a_n/b_n)$ converges if and only if its* Stern–Stolz series

$$\mathcal{S} := \sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n-k+1}} \right|$$

diverges to ∞.

2.5 Second-order linear recurrences and continued fractions

A non-trivial solution $\langle u_n \rangle_{n=-1}^{\infty}$ of the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ is called *minimal* if there exists another linearly independent solution $\langle v_n \rangle_{n=-1}^{\infty}$ such that $\lim_{n\to\infty} u_n/v_n = 0$. (In such cases the solution $\langle v_n \rangle_n$ is called *dominant*). If $\langle u_n \rangle_n$ is minimal then all solutions of the form $\langle cu_n \rangle_n$ where $c \neq 0$ are also minimal. Note that if $\langle y_n \rangle_n$ and $\langle z_n \rangle_n$ are linearly independent solutions of the above recurrence such that $y_n/z_n \sim C \in \hat{\mathbb{C}}$ then the recurrence relation has a minimal solution [26]. In general, a system of recurrences may not have a minimal solution. Nevertheless, if $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ are respectively minimal and dominant solutions of the recurrence, then together they form a basis of the solution space.

▶ Remark 2.7. When a second-order recurrence relation has minimal solutions, it is often beneficial from a numerical standpoint to provide a basis of solutions where one of the elements is a minimal solution. Such a basis can be used to approximate any element of the vector space of solutions: taking $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ as above, a general solution $\langle w_n \rangle_n$ is given by $w_n = \alpha_1 u_n + \alpha_2 v_n$ and is therefore dominant unless $\alpha_2 = 0$.

Let $\langle u_n \rangle_{n=-1}^{\infty}$ be a non-trivial solution of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. If $u_{n-1} \neq 0$ then we can rearrange the relation to obtain

$$-\frac{u_{n-1}}{u_{n-2}} = \frac{a_n}{b_n - \frac{u_n}{u_{n-1}}}$$

for $n \in \mathbb{N}$. In the event that $u_{n-2} = 0$ we take the usual interpretation in $\hat{\mathbb{C}}$. Since $\langle u_n \rangle_n$ is non-trivial and $a_n \neq 0$ for each $n \in \mathbb{N}$, the sequence $\langle u_n \rangle_n$ does not vanish at two consecutive indices. Thus if $u_{n-1} = 0$ then $u_{n-2}, u_n \neq 0$ and so both the left-hand the right-hand sides of the last equation are well-defined in $\hat{\mathbb{C}}$ and are equal to 0. Thus the sequence with terms $-u_n/u_{n-1}$ is well-defined in $\hat{\mathbb{C}}$ for each $n \in \mathbb{N}_0$.

The next theorem due to Pincherle [39] connects the existence of minimal solutions for a second-order recurrence to the convergence of the associated continued fraction (see also [15, 26, 9]).

▶ **Theorem 2.8** (Pincherle). *Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be real-valued sequences such that each of the terms a_n is non-zero. First, the recurrence $u_n = b_n u_{n-1} + a_n u_{n-2}$ has a minimal solution if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges. Second, if $\langle u_n \rangle_n$ is a minimal solution of this recurrence then the limit of $\mathrm{K}(a_n/b_n)$ is $-u_0/u_{-1}$. As a consequence, the sequence of canonical denominators $\langle B_n \rangle_{n=-1}^{\infty}$ is a minimal solution if and only if the value of $\mathrm{K}(a_n/b_n)$ is $\infty \in \hat{\mathbb{C}}$.*

We refer to the problem of determining whether the value a given convergent polynomial continued fraction is equal to a particular algebraic number as the *PCF Equality Problem.* We now have:

▶ **Corollary 2.9** (Appendix B). *The PCF Equality Problem and the Minimality Problem are interreducible.*

We denote by $\mathbb{Q}(x)$ the field of rational functions; that is, the field of fractions of the polynomial ring $\mathbb{Q}[x]$. We define the *degree* of $r = p(x)/q(x) \in \mathbb{Q}(x)$ as follows: if $r = 0$ set $\deg(r) = -\infty$, otherwise set $\deg(r) = \deg(p) - \deg(q)$.

The following theorem relates the convergence of the polynomial continued fraction $\mathrm{K}_{n=1}^{\infty}(p(n)/q(n))$ to the behaviour of an associated rational function [22] (see also the version of the theorem presented in [21] for the field of meromorphic fractions).

▶ **Theorem 2.10.** *For $p, q \in \mathbb{Q}[n]$ such that neither p nor q is the zero polynomial, let $r \in \mathbb{Q}(n)$ be the rational function given by $r(n) = 1 + \frac{4q(n)}{p(n-1)p(n)}$ with $\deg(r) = d$. The continued fraction $\mathrm{K}_{n=1}^{\infty}(p(n)/q(n))$ converges if and only if one of the following holds:*
1. $\deg(r) \le -2$ *and* $\lim_{n\to\infty} r(n)n^2 \ge -1/4$, *or*
2. $-1 \le \deg(r) \le 2$ *and* $\lim_{n\to\infty} r(n)n^d > 0$.

We remark the immediate corollary by Theorem 2.8.

▶ **Corollary 2.11.** *Given a recurrence relation of the form* (2.1)*, it is decidable whether the recurrence admits a minimal solution.*

The following technical lemma is well-known (see, for example, [26, Lemma 4, §IV]).

▶ **Lemma 2.12.** *Suppose that $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are both solutions to the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$. Then*

$$u_n v_{n-1} - u_{n-1} v_n = (u_0 v_{-1} - u_{-1} v_0) \prod_{k=1}^{n} (-a_k).$$

▶ Remark 2.13. Given a kth-order recurrence relation R with coefficients in $\mathbb{Q}(x)$, let $Z(R)$ be the set of solution sequences with rational initial values. For $\langle u_n\rangle_n, \langle v_n\rangle_n \in Z(R)$, consider the limit u_n/v_n as $n \to \infty$ if the limit exists and let $L(R)$ be the set of such limits

$$L(R) := \left\{ \alpha \in \mathbb{R} : \alpha = \lim_{n\to\infty} \frac{u_n}{v_n}, \langle u_n\rangle_n, \langle v_n\rangle_n \in Z(R) \right\}.$$

Because $Z(R)$ is a vector space over $\mathbb{Q}$, it follows that $\mathbb{Q} \subset L(R) \subset \mathbb{R}$. Let $\mathbb{L}$ be the union of $L(R)$ over all R. Kooman [21, Chapter 2] makes the following observations: the set $\mathbb{L}$ is a field, is countable, and $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbb{L} \subset \mathbb{R}$. We note the inclusion $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbb{L}$ follows from limits associated to C-finite recurrence relations. The set $\mathbb{L}$ also contains real transcendental numbers. In fact, any real number of the form $\sum_{k=0}^{\infty} \prod_{m=1}^{k} q_m$ with $q_m \in \mathbb{Q}(m)$ such that $q_m, 1/q_m \ne 0$ is a limit of a solution to the second-order recurrence $u_n = (1 + q_n)u_{n-1} - q_n u_{n-2}$. We connect such limits to minimal solutions of second-order recurrence relations in the next remark.

▶ Remark 2.14. Consider the recurrence relation

$$u_n = (1 + q_n)u_{n-1} - q_n u_{n-2}. \tag{2.4}$$

The constant sequence $\langle 1\rangle_{n=-1}^{\infty}$ is clearly a solution to the recurrence. By Lemma 2.12, we obtain the second solution $\langle v_n\rangle_{n=-1}^{\infty}$ with initial terms $v_{-1} = 0$, $v_0 = 1$, and for $n \in \mathbb{N}$,

$v_n = \sum_{k=0}^{n} \prod_{m=1}^{k} q_m$ where the empty product is equal to unity. These two solutions are linearly independent and it is interesting to ask whether the above recurrence relation has a minimal solution.

Let $\xi := \lim_{n\to\infty} v_n/u_n = \sum_{k=0}^{\infty} \prod_{m=1}^{k} q_m$ if the limit exists. We have the following characterisation for minimal solutions in terms of ξ. If $\xi = \infty$ then $\langle u_n \rangle_n$ is a minimal solution of (2.4). If $\xi \in \mathbb{R}$ then consider the non-trivial sequence $\langle w_n \rangle_n$ with terms $w_n = v_n - \xi u_n$. Clearly $\lim_{n\to\infty} w_n/u_n = 0$ and so we conclude that $\langle w_n \rangle_n$ is a minimal solution. As a side note in the case that $\xi = 0$, $\langle v_n \rangle_n$ is a minimal solution.

▶ **Example 2.15.** A series $\sum c_k x^k$ is called *hypergeometric* if the ratio of consecutive summands c_{k+1}/c_k is equal to a rational function of k for each $k \in \mathbb{N}_0$. It can be shown (see [3]) that a hypergeometric series can be written as follows

$$\sum_{k=0}^{\infty} c_k = c_0 \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_j)_k}{(\beta_1)_k \cdots (\beta_\ell)_k} \frac{x^k}{k!} =: c_0\, {}_jF_\ell(\alpha_1, \ldots, \alpha_j; \beta_1, \ldots, \beta_\ell; x).$$

For $\alpha \in \mathbb{C}$ the *(rising) Pochhammer symbol* $(\alpha)_n$ is defined as $(\alpha)_0 = 1$, and $(\alpha)_n = \prod_{j=0}^{n-1}(\alpha+j)$ for $n \geq 1$. Here the parameters β_m are not negative integers or zero for otherwise the denominator would vanish for some k. It is useful in the sequel (Proposition 4.11) to connect hypergeometric series and the recurrence relation in Remark 2.14. If we choose

$$q_m = \frac{(\alpha_1 + m - 1) \cdots (\alpha_j + m - 1)}{(\beta_1 + m - 1) \cdots (\beta_\ell + m - 1)m} x$$

in order that $q_m, 1/q_m \neq 0$ for each $m \in \mathbb{N}$, then

$$\xi = \sum_{k=0}^{\infty} \prod_{m=1}^{k} q_m = {}_jF_\ell(\alpha_1, \ldots, \alpha_j; \beta_1, \ldots, \beta_\ell; x).$$

3 Relations Between Oracles for Holonomic Sequences

In this section, we examine how the problems of Minimality, Positivity, and Ultimate Positivity[5] for second-order holonomic sequences relate to each other. We shall assume throughout that each of the polynomial coefficients in the associated recurrence (2.1) has degree at most 1. It is convenient to introduce the following notation: we set $g_3(n) = \alpha_1 n + \alpha_0$, $g_2(n) = \beta_1 n + \beta_0$, and, $g_1(n) = \gamma_1 n + \gamma_0$. Thus the recurrence relation under focus is of the form

$$(\alpha_1 n + \alpha_0)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}. \tag{3.1}$$

We have the following results.

▶ **Theorem 3.1.** *The following hold for the family of holonomic sequences satisfying second-order recurrences of degree at most one.*

1. *The Positivity Problem reduces to the Minimality Problem.*
2. *The Positivity Problem and the Ultimate Positivity Problems are interreducible (see Appendix E).*

[5] The *Ultimate Positivity Problem* asks whether a holonomic sequence takes on non-negative values for all but finitely many terms.

The rest of this section is devoted to the proof of the first item of Theorem 3.1. It is useful to separate the problem into subcases according to the signature of the recurrence relation $u_n = b_n u_{n-1} + a_n u_{n-2}$ with $b_n = g_2(n)/g_3(n)$ and $a_n = g_1(n)/g_3(n)$. When the signature of the recurrence is either $(+,+)$ or $(-,-)$ then the problem of deciding whether a solution sequence $\langle u_n \rangle_n$ with initial terms $u_{-1}, u_0 \geq 0$ is trivial. If the recurrence has signature $(+,+)$ then $\langle u_n \rangle_n$ is positive, whilst if the recurrence has signature $(-,-)$ then $u_1 < 0$ and so the solution sequence is not positive. It remains to consider the cases $(-,+)$ and $(+,-)$. Recall the canonical solutions $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ defined in the preliminaries. For a recurrence relation with signature $(-,+)$, the canonical solutions $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ have terms $A_2 < 0$ and $B_1 = b_1 < 0$. Thus in our discussion of the Positivity Problem for non-trivial solutions we can assume that $u_{-1}, u_0 > 0$. For a recurrence relation with signature $(+,-)$ we have that $A_1 = a_1 < 0$ and so one can assume that $u_0 > 0$. We defer our discussion of the positive terms in the solution sequence $\langle B_n \rangle_{n=-1}^{\infty}$ until later in this section.

We shall treat the two signatures separately. We shall first handle recurrence relations with signature $(-,+)$. In this case we have the following result.

▶ **Proposition 3.2.** *Suppose that $\langle u_n \rangle_n$ with initial values $u_{-1}, u_0 > 0$ is a solution sequence for recurrence* (3.1) *with signature $(-,+)$ and the associated continued fraction $\mathrm{K}(a_n/b_n)$ converges to a finite limit μ. The following statements are equivalent:*

1. *the sequence $\langle u_n \rangle_n$ is positive,*
2. *the sequence $\langle u_n \rangle_n$ is minimal, and*
3. *$-u_0/u_{-1} = \mu$.*

For the proof, we need the next lemma, which links positivity of a solution sequence $\langle u_n \rangle_n$ to the sequence of the convergents $\langle f_n \rangle_n$ of the continued fraction $\mathrm{K}(a_n/b_n)$.

▶ **Lemma 3.3.** *Suppose that $\langle u_n \rangle_n$ is a solution sequence for recurrence* (3.1) *with signature $(-,+)$. Assume that $u_{-1} > 0$. For even $n \in \mathbb{N}$, $u_n > 0$ if and only if $f_n > -u_0/u_{-1}$. For odd $n \in \mathbb{N}$, $u_n > 0$ if and only if $f_n < -u_0/u_{-1}$.*

Proof. For the canonical solution sequences $\langle A_n \rangle_{n=-1}^{\infty}$ and $\langle B_n \rangle_{n=-1}^{\infty}$ we have that $u_n = A_n u_{-1} + B_n u_0$ for each $n \in \{-1, 0, \ldots\}$. For recurrences with signature $(-,+)$, it is easy to show by induction that $B_n < 0$ for each odd $n \in \mathbb{N}$, and $B_n > 0$ for each even $n \in \mathbb{N}$. Thus for even $n \in \mathbb{N}$ we have that $u_n > 0$ if and only if $A_n/B_n + u_0/u_{-1} = f_n + u_0/u_{-1} > 0$. The case for odd $n \in \mathbb{N}$ is handled in the same fashion. ◀

Equipped with the above observation, we are in the position to conclude Proposition 3.2.

Proof of Proposition 3.2. We note that $\langle -f_n \rangle_{n=1}^{\infty}$ is the sequence of convergents associated with the positive continued fraction $\mathrm{K}_{n=1}^{\infty} \frac{a_n}{-b_n}$. By Lemma 2.5, the subsequences $\langle -f_{2n} \rangle_{n=1}^{\infty}$ and $\langle -f_{2n-1} \rangle_{n=1}^{\infty}$ converge to finite limits $-\ell_1$ and $-\ell_2$, respectively. By Lemma 3.3, a solution sequence $\langle u_n \rangle_n$ is positive if and only if $\ell_2 \leq -u_0/u_{-1} \leq \ell_1$. The Stern–Stolz series (from Theorem 2.6) associated to the continued fraction $\mathrm{K}_{n=1}^{\infty} \frac{a_n}{-b_n}$ diverges due to our assumption that each of the coefficients in (3.1) is a polynomial with degree in $\{0, 1\}$. We conclude that $\ell_1 = \ell_2$ by Theorem 2.6. Thus $\langle u_n \rangle_n$ is positive if and only if $-u_0/u_{-1}$ is equal to $\mu = \ell_1 = \ell_2$. From Theorem 2.8, a solution sequence $\langle u_n \rangle_n$ is minimal if and only if $-u_0/u_{-1}$ is the value of the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. ◀

We now consider recurrences $u_n = b_n u_{n-1} + a_n u_{n-2}$ with signature $(+,-)$. Given our restriction on the degrees of the polynomial coefficients, we can assume, without loss of generality, that the sequences of coefficients $\langle b_n \rangle_n$ and $\langle a_n \rangle_n$ are monotonic. In the work that follows we split the $(+,-)$ case into two further subcases depending on the sign of the discriminant of the recurrence $\Delta(n) := b_n^2 + 4a_n$. We shall assume that $\mathrm{sign}(\Delta(n))$ is constant as this can be achieved by a suitable computable shift of the recurrence relation. The discussion of the subcase $\Delta(n) < 0$ is given in Appendix C. Let us summarise the results established therein with the following proposition.

▶ **Proposition 3.4.** *A recurrence relation of the form* (3.1) *with signature* $(+,-)$ *and discriminant* $\Delta(n) < 0$ *for each* $n \in \mathbb{N}$ *has no positive non-trivial solutions.*

Let us turn our attention to the subcase $\Delta(n) \geq 0$. We first need some technical lemmas, the first of which shows that such a recurrence relation admits a non-trivial positive solution.

▶ **Lemma 3.5.** *Consider a normalised recurrence* $u_n = b_n u_{n-1} + a_n u_{n-2}$ *with signature* $(+,-)$ *such that* $\Delta(n) \geq 0$ *for each* $n \in \mathbb{N}$. *Let* $\langle B_n \rangle_{n=-1}^{\infty}$ *be the canonical solution sequence with initial conditions* $B_{-1} = 0$ *and* $B_0 = 1$ *associated to this recurrence. Then for each* $n \in \mathbb{N}$, $B_n > 0$.

Proof. We separate the proof into two cases depending on the monotonicity of the coefficients $\langle a_n \rangle_{n=1}^{\infty}$. Let us suppose that $\langle a_n \rangle_n$ is increasing. It is sufficient to show that $B_n/B_{n-1} \geq \sqrt{-a_n}$ as $B_1 = b_1 \geq 2\sqrt{-a_1} > 0$. For the induction step, we have the inequalities below using our assumptions on the discriminant and the monotonicity of $\langle a_n \rangle_n$:

$$B_n/B_{n-1} = b_n + a_n B_{n-2}/B_{n-1} \geq 2\sqrt{-a_n} + a_n/\sqrt{-a_{n-1}} \geq \sqrt{-a_n}.$$

Now suppose that $\langle a_n \rangle_n$ is decreasing. Consider the recurrence sequence $\langle v_n \rangle_{n=-1}^{\infty}$ with terms $v_{-1} = 0$, $v_0 = 1$, and for $n \in \mathbb{N}$, $v_n = (-1)^n u_n / \prod_{k=1}^{n+1} g_1(k)$. The sequence $\langle v_n \rangle_n$ satisfies the recurrence $v_n = b'_n v_{n-1} + a'_n v_{n-2}$ with coefficients $b'_n = -g_2(n)/(g_1(n+1)g_3(n))$ and $a'_n = 1/(g_1(n+1)g_3(n))$, and signature $(+,-)$. Clearly $B_n > 0$ for each $n \in \mathbb{N}$ if and only if $v_n > 0$ for each $n \in \mathbb{N}$. By assumption, $\langle a'_n \rangle_{n=1}^{\infty}$ is an increasing sequence and so we have that $\langle v_n \rangle_n$ is a positive sequence from the previous case. Note the above transformation does not preserve the degrees of coefficients in the recurrence relation. However, the induction proof above does not depend on the degrees of the polynomial coefficients in the recurrence relation. ◀

▶ **Lemma 3.6.** *Suppose that recurrence* (3.1) *has discriminant* $\Delta(n) \geq 0$ *and signature* $(+,-)$. *Then the sequence of convergents* $\langle f_n \rangle_n$ *associated with the continued fraction* $\mathrm{K}(a_n/b_n)$ *is strictly decreasing.*

Proof. By Lemma 3.5, $B_n > 0$ for each $n \in \mathbb{N}$. From Lemma 2.12 we have that $A_n B_{n-1} - A_{n-1} B_n = -\prod_{k=1}^{n}(-a_k)$. Thus

$$f_n - f_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = -\frac{\prod_{k=1}^{n}(-a_k)}{B_{n-1}B_n} < 0,$$

and so $\langle f_n \rangle_n$ is strictly decreasing. ◀

We again link the positivity of a solution to the sequence of convergents of the associated continued fraction using the lemma that follows.

▶ **Lemma 3.7.** *Suppose that* $\langle u_n \rangle_{n=-1}^{\infty}$ *is a solution of the normalised recurrence* $u_n = b_n u_{n-1} + a_n u_{n-2}$ *with signature* $(+,-)$ *such that* $\Delta(n) \geq 0$ *for each* $n \in \mathbb{N}$. *Assume that* $u_{-1} > 0$. *Given* $N \in \mathbb{N}$, *we have that* $-u_0/u_{-1} < f_N$ *if and only if* $u_N > 0$.

Proof. For each $N \in \mathbb{N}$, $u_N = u_{-1}A_N + u_0 B_N$ where $\langle A_n\rangle_{n=-1}^{\infty}$ and $\langle B_n\rangle_{n=-1}^{\infty}$ are the canonical solutions. It follows that $-u_0/u_{-1} < A_N/B_N = f_N$ if and only if $u_N > 0$. Here we have used the assumption that $u_{-1} > 0$ and $B_N > 0$ (from Lemma 3.5). ◀

We are now in the position to characterise positive solutions to the considered recurrence relations via the ratio of the initial terms.

▶ **Proposition 3.8.** *Suppose that $\langle u_n\rangle_{n=-1}^{\infty}$ is a solution of recurrence (3.1) with signature $(+,-)$ such that $\Delta(n) \geq 0$ for each $n \in \mathbb{N}$. First, the associated continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ converges to a finite limit $\mu < 0$. Second, a solution $\langle u_n\rangle_{n=-1}^{\infty}$ with $u_{-1}, u_0 > 0$ is positive if and only if $-u_0/u_{-1} \leq \mu$.*

Proof. Since the sequence of convergents $\langle f_n\rangle_n$ associated with the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ is strictly decreasing (by Lemma 3.6), it is clear the limit exists. Since $f_1 = a_1/b_1 < 0$, if the value μ is finite then $\mu < 0$.

We claim that μ is finite. Subject to this assumption, let $\langle u_n\rangle_{n=-1}^{\infty}$ be a solution to the recurrence relation. For each $N \in \mathbb{N}_0$, $u_{-1}, u_0, \ldots, u_N > 0$ if and only if $-u_0/u_{-1} < f_N$ by Lemma 3.7. Thus $\langle u_n\rangle_{n=-1}^{\infty}$ is a positive solution if and only if $-u_0/u_{-1} \leq \mu$.

Let us prove that, subject to our assumptions, μ is indeed finite. Suppose, for a contradiction, that μ is infinite. As we have assumed that for each $\ell \in \{1,2,3\}$, $g_\ell(0) \neq 0$, we can define the recurrence corresponding to a one-step backward shift and extend uniquely any given sequence $\langle u_n\rangle_{n=-1}^{\infty}$ to a sequence $\langle u_n\rangle_{n=-2}^{\infty}$. It follows from the recursive definition of the sequence of convergents that if $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ converges to ∞, then $\mathrm{K}_{n=0}^{\infty}(a_n/b_n)$ converges to 0. This conclusion is not possible as the sequence of convergents is strictly decreasing and $f_0 = a_0/b_0 = g_1(0)/g_2(0) < 0$. ◀

We combine the results in Proposition 3.8 and Theorem 2.8 into the following corollary.

▶ **Corollary 3.9.** *Let $\langle u_n\rangle_{n=-1}^{\infty}$ be a solution of recurrence relation (3.1) with signature $(+,-)$ and $\Delta(n) \geq 0$ for each $n \in \mathbb{N}$. Then a solution sequence $\langle u_n\rangle_n$ with $u_{-1}, u_0 > 0$ is positive if and only if $-u_0/u_{-1} \leq \mu$. In addition, if $-u_0/u_{-1} = \mu$ then the sequence $\langle u_n\rangle_n$ is a minimal solution.*

The difficulty one encounters when determining positivity arises when $-u_0/u_{-1}$ is equal to the value μ of the associated continued fraction.

▶ **Proposition 3.10** (Proof in Appendix D)**.** *Let $\langle u_n\rangle_{n=-1}^{\infty}$ be a non-trivial solution sequence for recurrence (3.1) with signature $(+,-)$ and suppose that $\Delta(n) \geq 0$ for each $n \in \mathbb{N}$. Then one can detect if $-u_0/u_{-1} < \mu$.*

We deduce that if one can decide whether a holonomic sequence $\langle u_n\rangle_n$ that solves recurrence (3.1) is minimal, then one can decide whether $\langle u_n\rangle_n$ is a positive solution.

Proof of Theorem 3.1(1). Assume we have an oracle for the Minimality Problem for solutions $\langle u_n\rangle_{n=-1}^{\infty}$ to recurrences of the form (3.1). Given such a recurrence, the existence of a positive solution is decidable by combining Proposition 3.2, Proposition 3.4, and Corollary 3.9. We may thus focus on instances where the associated recurrence relation admits positive solutions. Notice that Proposition 3.2 and Corollary 3.9 imply the existence of minimal solutions. A trivial solution is straightforward to detect. If $\langle u_n\rangle_n$ is minimal, then it is positive by Proposition 3.2 and Corollary 3.9. Assume now that the sequence is dominant. If the signature of the associated recurrence relation is $(-,+)$, then the sequence is not positive by Proposition 3.2. Assume then that the signature is $(+,-)$. By Proposition 3.10, one can detect if $-u_0/u_{-1} < \mu$. The case $-u_0/u_{-1} > \mu$ can also be detected as the sequence contains a negative term. This process is equivalent to deciding whether $\langle u_n\rangle_n$ is positive. ◀

4 Minimality for Degree-1 Holonomic Sequences

Recall from Corollary 2.11 that the problem of whether a recurrence relation of the form (2.1) admits a minimal solution is decidable. In the present section, we focus on the Minimality Problem for such recurrences. For ease of notation, we parametrise the problem as follows.

▶ **Problem 4.1** (Minimality(j,k,ℓ))**.** Given a solution $\langle v_n\rangle_{n=-1}^{\infty}$ to (2.1) with $\deg(g_3)=j$, $\deg(g_2)=k$, and $\deg(g_1)=\ell$, decide whether $\langle v_n\rangle_n$ is minimal.

Problem Minimality$(0,0,0)$ asks one to determine whether a holonomic sequence that solves a second-order C-finite recurrence is a minimal solution. Notice that this is a special case of Minimality$(1,1,1)$ (multiply each of the coefficients by $(n+1)$) and is therefore not treated separately in the sequel. In this section we are interested in the decidability of Minimality(j,k,ℓ) subject to the restriction that $j,k,\ell\leq 1$. The main result of this section is the following.

▶ **Theorem 4.2.** *For $j,k,\ell\leq 1$, Minimality(j,k,ℓ) reduces to determining whether a period, an exponential period, or a period-like integral is equal to zero.*

For definitions and discussion of periods, exponential periods, and period-like integrals see Appendix A.

Observe that the cases for which some of the coefficient polynomials are identically 0 are dealt with in Subsection 2.1. Hence, throughout this section, we assume that $j,k,\ell\in\{0,1\}$, i.e., none of the coefficient polynomials are identically 0. We thus focus on recurrences of the form (3.1), and establish the following conventions. In the case that $\deg(g_3)=0$, we understand that $\alpha_1=0$. A similar convention is applied to the polynomials g_2 and g_1. On the other hand, we shall always assume that the values α_0, β_0, and γ_0 are non-zero in accordance with the assumption that the polynomial coefficients do not vanish on non-negative integers. One further assumption is made: the recurrence relations considered in this section are assumed to admit minimal solutions. This is no loss of generality, as this is a decidable property, as per Corollary 2.11.

The proof of Theorem 4.2 is spread over several subsections with intermediate results. On the face of it, we have eight different problems to consider. We first reduce the number of problems to five by establishing some interreductions between the problems Minimality(j,k,ℓ) for different values of the parameters j, k, ℓ. We further employ minimality-preserving transformations to obtain certain canonical instances of each of the remaining problems (Corollary 4.4). These are then showed to reduce to checking whether a period-like integral vanishes. For four of the cases we analyse an associated *generating function* that connects our sequences to the theory of differential equations. The conclusion of the statement can be can be pieced together from Proposition 4.10, Proposition 4.18, and Subsection 4.2.

4.1 Interreductions of Minimality(j,k,ℓ)

In this subsection we establish some interreductions of the Minimality Problem for degree-1 holonomic sequences. We also identify some canonical instances on which we focus thereafter.

▶ **Proposition 4.3.**

1. Minimality$(0,k,1)$ *reduces to* Minimality$(1,k,0)$ *and vice versa.*
2. Minimality$(1,0,0)$ *reduces to* Minimality$(1,1,1)$.

3. Minimality$(1,1,1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*

$$(n+\alpha)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}, \tag{4.1}$$

where the coefficients are elements of $\mathbb{Q}[n]$, $\beta_1 > 0$ *and* $|\gamma_1| = \beta_1$. *In the case that* $\beta_1^2 + 4\gamma_1 = 0$ *in the above recurrence, then the problem further reduces to the Minimality Problem for solutions to a recurrence of the form*

$$(n+\alpha)u_n = (2n + \beta_0)u_{n-1} - (n + \gamma_0)u_{n-2}, \tag{4.2}$$

where the coefficients are in $\mathbb{Q}[n]$.

4. Minimality$(1,1,0)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*

$$(n+\alpha)u_n = (\beta_1 n + \beta_0)u_{n-1} + \gamma_0 u_{n-2}, \tag{4.3}$$

where the coefficients are in $\mathbb{Q}[n]$, $\beta_1 > 0$ *and* $|\gamma_0| = \beta_1$.

5. Minimality$(1,0,1)$ *reduces to the Minimality Problem for solutions to a recurrence of the form*

$$(n+\alpha)u_n = \beta_0 u_{n-1} + (n + \gamma_0)u_{n-2}, \tag{4.4}$$

where $\alpha_0, \gamma_0 \in \mathbb{Q}$ *and* $\beta_0 \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$.

Notice that sequences satisfying (4.4) are not necessarily holonomic (as defined in this note), as we allow the coefficient β_0 to be irrational. We note that the reduction of Minimality$(1,0,1)$ to (4.4) yields values of the parameter β_0 that are real algebraic numbers of degree at most 2.

Proof. We prove Item 2 below. Item 1 follows immediately from the interreductions of the recurrence relations (2.1) and (2.2) described in the preliminaries. The other items follow from minimality preserving transformations of the form $\langle u_n \rangle_n \mapsto \langle \kappa^n u_n \rangle_n$ for some appropriate constants κ. In the last item we need to know that the recurrence admits a minimal solution if and only if $\alpha_1\gamma_1 > 0$ (compare to Lemma 4.5). See Appendix F for the complete proofs.

Let us normalise the recurrence relation by dividing through by the leading coefficient of g_3. The normalised recurrence is given by

$$(n+\alpha)u_n = \beta u_{n-1} + \gamma u_{n-2} \tag{4.5}$$

with coefficients in $\mathbb{Q}[n]$. We can assume $\alpha := \alpha_0/\alpha_1 > 1$ by taking an appropriate shift.

Suppose that $\langle u_n \rangle_n$ is a solution to (4.5). We observe that $\langle u_n \rangle_n$ satisfies the recurrence

$$(n+\alpha-1)(n+\alpha)u_n = (2\gamma n + \beta^2 + \gamma(2\alpha-3))u_{n-2} - \gamma^2 u_{n-4} \tag{4.6}$$

for $n \geq 3$. This observation follows easily by substituting $\beta u_{n-3} = (n+\alpha-2)u_{n-2} - \gamma u_{n-4}$ performing straightforward algebraic manipulations:

$$\begin{aligned}(n+\alpha-1)(n+\alpha)u_n &= (n+\alpha-1)(\beta u_{n-1} + \gamma u_{n-2}) \\ &= (2\gamma n + \beta^2 + \gamma(2\alpha-3))u_{n-2} - \gamma^2 u_{n-4}.\end{aligned}$$

Let us now define the sequence $\langle z_n \rangle_{n=0}^\infty$ by $z_n = u_{2n}$ for each $n \in \mathbb{N}_0$. Then $z_0 = u_0$ and

$$z_1 = u_2 = \frac{1}{\alpha+2}(\beta u_1 + \gamma u_0) = \left(\frac{\beta^2}{(\alpha+2)(\alpha+1)} + \gamma\right)u_0 + \frac{\beta\gamma}{(\alpha+2)(\alpha+1)}u_{-1}.$$

It is clear that for $n \geq 2$

$$(2n+\alpha-1)(2n+\alpha)z_n = (4\gamma n+\beta^2+\gamma(2\alpha-3))z_{n-1} - \gamma^2 z_{n-2} \tag{4.7}$$

by (4.6) with the mapping $n \mapsto 2n$. This establishes a one-to-one correspondence between the solutions to (4.7) and the solutions to (4.5). We claim that this equivalence of solutions preserves minimality. To conclude the claim, observe that a solution to (4.7) can be transformed into a solution to the recurrence

$$(2n+\alpha-1)v_n = (4\gamma n+\beta^2+\gamma(2\alpha-3))v_{n-1} - \gamma^2(2n+\alpha-2)v_{n-2}$$

such that this transformation preserves minimality (using the same minimality preservation reduction from (2.1) to (2.2)).

Let us prove that minimality is preserved as claimed. We first show that the solutions $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ to (4.5) are linearly independent if and only if the solutions $\langle u_{2n}\rangle_n$ and $\langle v_{2n}\rangle_n$ to (4.7) are linearly independent. One direction is trivial: if $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are linearly dependent, then so are $\langle u_{2n}\rangle_n$ and $\langle v_{2n}\rangle_n$. Assume thus that $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ are linearly independent, but $\langle u_{2n}\rangle_n = \ell\langle v_{2n}\rangle_n$ for some $\ell \in \mathbb{R}$. Consider the solution $\langle U_n\rangle_{n=-1}^\infty := \langle u_n\rangle_n - \ell\langle v_n\rangle_n$ to (4.5). We have $U_{-1} = u_{-1} - \ell v_{-1}$ and $U_0 = 0$. Now $U_{-1} \neq 0$, as otherwise $\langle u_n\rangle_n$ would be proportional to $\langle v_n\rangle_n$. Observe now that $U_1 = \frac{\gamma}{1+\alpha}U_{-1}$ and $U_2 = \frac{\beta}{\alpha+2}U_1 = \frac{\beta\gamma}{(1+\alpha)(2+\alpha)}U_{-1} \neq 0 = u_2 - \ell v_2$. This is a contradiction. We have established that $\langle u_{2n}\rangle_n$ and $\langle v_{2n}\rangle_n$ are necessarily linearly independent.

Assume now that $\langle u_n\rangle_n$ is a minimal solution to (4.5) and let $\langle v_n\rangle_n$ be a dominant solution. Then $0 = \lim_{n\to\infty} u_n/v_n = \lim_{n\to\infty} u_{2n}/v_{2n}$. Since $\langle u_{2n}\rangle_n$ and $\langle v_{2n}\rangle_n$ are linearly independent solutions to (4.7), $\langle u_{2n}\rangle_n$ is necessarily minimal.

Conversely, assume that the linearly independent solutions $\langle u_n\rangle_n$ and $\langle v_n\rangle_n$ to (4.5) are such that $\langle u_{2n}\rangle_n$ is a minimal solution to (4.7) (such a solution exists by the previous paragraph, as we assume (4.5) to admit a minimal solution). Then, since $\langle v_{2n}\rangle_n$ is linearly independent to $\langle u_{2n}\rangle_n$, we have $0 = \lim_{n\to\infty} u_{2n}/v_{2n}$. As the recurrence relation 4.5 admits a minimal solution by assumption, it can be shown that the limit $\lim_{n\to\infty} u_n/v_n$ exists ([26, §IV.1.5]), so $\lim_{n\to\infty} u_n/v_n = 0$. We have established that $\langle u_n\rangle_n$ is a minimal solution to (4.5). ◀

▶ **Corollary 4.4.** *The decidability of* MINIMALITY(j,k,ℓ) *with* $j,k,\ell \in \{0,1\}$ *reduces to deciding* MINIMALITY$(0,1,0)$ *and the Minimality Problem for solutions to recurrences of the form* (4.1)*,* (4.2)*,* (4.3)*, and* (4.4)*.*

Before delving into the proof of Theorem 4.2, we establish some notation. Consider the recurrences (4.1)–(4.4). Dividing through by $g_3(n) = n+\alpha$, i.e., putting such recurrences into the form (2.1b), we obtain Poincaré recurrences, since $\lim_{n\to\infty} g_2(n)/g_3(n) = \beta_1$ and $\lim_{n\to\infty} g_1(n)/g_3(n) = \gamma_1$. Let λ and Λ be the roots of the associated characteristic polynomial such that $|\lambda| \leq |\Lambda|$. As β_1 and γ_1 are not zero simultaneously in these recurrences, at least one of the roots is non-zero. Hence $\Lambda \neq 0$.

Now recurrences of the form (4.2) are a subset of recurrences of the form (4.1). To avoid cluttering the text we shall differentiate the two as follows: when referring to recurrences of the form (4.1) we always assume that $\beta_1^2 + 4\gamma_1 \neq 0$. Thus the characteristic roots are always distinct for relation (4.1), while (4.2) has a single repeated characteristic root.

The next lemma gives necessary and sufficient conditions for the existence of minimal solutions.

▶ **Lemma 4.5.**

1. *Recurrence relations associated to* MINIMALITY$(0,1,0)$ *always admits a minimal solution.*
2. *A recurrence of the form* (4.1), (4.3), *or* (4.4) *admits a minimal solution if and only if the associated characteristic roots are real.*
3. *The recurrence* (4.2) *admits a minimal solution if and only if* $\beta_0 - \alpha_0 - \gamma_0 \geq 0$.

Proof. We can determine whether a minimal solution exists using the criteria in Theorem 2.10 that discusses the asymptotic properties of the function $r(n) = 1 + 4\frac{g_1(n)g_3(n-1)}{g_2(n)g_2(n-1)}$.

In Items 2 and 3, notice that the characteristic roots are real if and only if $\beta_1^2 + 4\gamma_1 \geq 0$. Further, the characteristic roots are distinct if and only if $\beta_1^2 + 4\gamma_1 \neq 0$.

1. We have $r(n) = 1 + o(1)$. By Theorem 2.10, the recurrence always admits a minimal solution.
2. First, consider recurrence (4.1). Here we have $r(n) = c_0 + o(1)$, where $c_0 = 1 + 4\gamma_1/\beta_1^2$. Since we assume $\beta_1^2 + 4\gamma \neq 0$, we have that $c_0 \neq 0$. By Theorem 2.10, the recurrence admits a minimal solution if and only if $c_0 > 0$. But $c_0 > 0$ if and only if $\lambda, \Lambda \in \mathbb{R}$. Second, consider recurrence (4.3). In this case $r(n) = 1 + o(1)$. So the recurrence always has a minimal solution and, in addition, we have $\lambda = 0$ and $\Lambda = \beta_1 \in \mathbb{R}$. Finally, consider recurrence (4.4). In this case $r(n) = 1 + 4n^2/\beta_0^2 + o(n^2)$. The recurrence has a minimal solution by Theorem 2.10 and the characteristic roots are ± 1.
3. Consider recurrence (4.2). Here $r(n) = c_1 n^{-1} + c_2 n^{-2} + o(n^{-2})$, where $c_1 = \beta_0 - \alpha - \gamma_0$, and $c_2 = (\beta_0 - \alpha)c_1 + (\alpha - \beta_0/2)(\alpha_0 - \beta_0/2 - 1)$. There are two cases to consider. If $c_1 \neq 0$, then the recurrence admits a minimal solution if and only if $c_1 > 0$. Now assume that $c_1 = 0$, i.e., $\beta_0 = \alpha + \gamma_0$. Then $c_2 = \frac{\alpha - \gamma_0}{2}(\frac{\alpha - \gamma_0}{2} - 1)$. In this case the recurrence admits a minimal solution if and only if $c_2 \geq -1/4$. This inequality is always true since $x(x-1) + 1/4 = (x - 1/2)^2 \geq 0$. ◀

In the following subsection, we show that Theorem 4.2 holds for MINIMALITY$(0,1,0)$. We then consider the other recurrences thereafter.

4.2 Minimality$(0, 1, 0)$

In this subsection we consider solutions to the recurrence

$$u_n = (\beta_1 n + \beta_0)u_{n-1} + \gamma_0 u_{n-2}. \tag{4.8}$$

For a minimal solution $\langle u_n \rangle_n$ to this recurrence, we have

$$-u_0/u_{-1} = \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{\gamma_0}{\beta_1 n + \beta_0}\right) = \beta_0 \frac{{}_0F_1(;\beta_0/\beta_1;\gamma_0/\beta_1^2)}{{}_0F_1(;\beta_0/\beta_1 + 1;\gamma_0/\beta_1^2)} - \beta_0$$

(see [26, §VI.4.1]). Hence the Minimality Problem for the above recurrence reduces to checking the equality

$$(\beta_0 u_{-1} - u_0)\,{}_0F_1(;\beta_0/\beta_1 + 1;\gamma_0/\beta_1^2) = \beta_0\,{}_0F_1(;\beta_0/\beta_1;\gamma_0/\beta_1^2). \tag{4.9}$$

The *Bessel functions of the first kind*, sometimes called *cylinder functions*, $J_s(z)$ are a family of functions that solve Bessel's differential equation [1, 3, 9]. For $z, s \in \mathbb{C}$ the function $J_s(z)$ is defined by the hypergeometric series [1, equation 9.1.69]

$$J_s(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+1)}\left(\frac{z}{2}\right)^{2k+s} = \frac{1}{\Gamma(s+1)}\left(\frac{z}{2}\right)^s {}_0F_1(;s+1;-z^2/4).$$

We obtain the principal branch of $J_s(z)$ by assigning $(z/2)^s$ its principal value. When $\mathrm{Re}(s) > -1/2$ we have the following integral representation [3, equation 4.7.5],

$$J_s(z) = \frac{1}{\sqrt{\pi}\Gamma(s+1/2)}\left(\frac{z}{2}\right)^s \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{s-1/2}\,dt.$$

Hence for $\mathrm{Re}(s) > -1/2$, we have the following integral representation

$${}_0F_1(;s+1;z) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} \int_{-1}^{1} \mathrm{e}^{-2\sqrt{z}t}(1-t^2)^{s-1/2}\,dt.$$

Let us return to minimal solutions of the aforementioned recurrence relation. By substitution into (4.9) and linearity of the integral, we see that MINIMALITY$(0,1,0)$ reduces to checking the equality

$$\int_{-1}^{1} \mathrm{e}^{-2\sqrt{\gamma_0}t/\beta_1}(1-t^2)^{\beta_0/\beta_1-3/2}\left(\frac{2(\beta_0 u_{-1}-u_0)}{2\beta_0-1}(1-t^2)-1\right)dt = 0. \tag{4.10}$$

To ensure that the integral converges absolutely note that we can shift the recurrence so that $\beta_0/\beta_1 > 3/2$. The integral on the left-hand side is an exponential period, and thus we have proved Theorem 4.2 in the case of MINIMALITY$(0,1,0)$.

4.3 Generating function analysis

For the remainder of this section, we only consider recurrences (4.1)–(4.4). By Lemma 4.5, we thus assume that the associated characteristic roots are real. We may now choose $\Lambda > 0$: indeed, $\beta_1 > 0$ in the first three recurrences implies that the dominant root is positive. In the fourth recurrence, the roots are ± 1, and we are free to choose $\Lambda = 1$. (The assumption of $\beta_0 > 0$ in (4.4) is used in the sequel. We will explicitly recall this fact when needed, but, for now, this is not important.)

Let $\langle u_n \rangle_{n\geq -1}$ be a non-trivial solution to one of the recurrences of the form (4.1)–(4.4). We associate to $\langle u_n \rangle_n$ the generating series

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} u_{n-1}x^{n+\alpha}.$$

We consider analytic properties of the generating function defined by the above generating series. We first observe that the series has a positive radius of convergence. Recall that $\Lambda > 0$ in this analysis.

▶ **Lemma 4.6.** *Let $\langle u_n \rangle_n$ be a non-trivial solution to one of aforementioned recurrences. If $\langle u_n \rangle_n$ is a dominant (resp., minimal) solution, then the series $\mathcal{F}(x)$ has radius of convergence $1/\Lambda$ (resp., $1/|\lambda|$, which we understand as ∞ if $\lambda = 0$.)*

Proof. This follows from Theorem 2.2 by the Cauchy–Hadamard theorem. ◀

Now the generating series $\mathcal{F}$ defines a continuous and differentiable function in the interval $[0, 1/|\Lambda|)$ (regardless of whether $\langle u_n \rangle_n$ is dominant or not). Further, $\mathcal{F}$ is analytic in the interval $(0, 1/|\Lambda|)$. It can be shown (see Subsection G.1) that for a given solution $\langle u_n \rangle_n$, the generating function satisfies the differential equation

$$\mathcal{F}'(x) + s(x)\mathcal{F}(x) = t(x),$$

where

$$s(x) = \frac{(\gamma_0 + 2\gamma_1 - \alpha\gamma_1)x + \beta_0 + \beta_1 - \alpha\beta_1}{\gamma_1 x^2 + \beta_1 x - 1} \quad \text{and} \tag{4.11}$$
$$t(x) = \frac{(\beta_0 + \beta_1)u_{-1} - (\alpha + 1)u_0 - \alpha u_{-1}x^{-1}}{\gamma_1 x^2 + \beta_1 x - 1} x^{\alpha}.$$

Note that both s and t are integrable over a neighbourhood of 0 under our assumption that $\alpha > 1$. Standard methods then yield a solution to the differential equation. Namely, by noting that $\mathcal{F}(0) = 0$ as $\alpha > 1$, we have the following solution in the interval $[0, 1/\Lambda)$ as follows:

$$\mathcal{F}(x) = \frac{1}{\mathcal{I}(x)} \int_0^x \mathcal{I}(y)t(y)\, dy, \tag{4.12}$$

where $\mathcal{I}(x)$ is the integrating factor $\exp(\int_0^x s(y)\, dy)$.

Notice here that the denominator $\gamma_1 x^2 + \beta_1 x - 1$ of $t(x)$ and $s(x)$ has roots $1/\Lambda$ and $1/\lambda$ (if $\lambda \neq 0$).

▶ **Lemma 4.7.** *There is at most one non-trivial choice of u_{-1} and u_0, up to scaling, for which $\int_0^{1/\Lambda} \mathcal{I}(y)t(y)\, dy$ vanishes.*

Proof. Observe that the integral is of the form

$$\int_0^{1/\Lambda} \frac{\mathcal{I}(y)}{\gamma_1 y^2 + \beta_1 y - 1} y^{\alpha-1}(Ay + B)\, dy$$

where only A and B depend on u_{-1} and u_0. We note $\mathcal{I}(y)t(y)/(Ay + B)$ has constant sign in the domain $(0, 1/\Lambda)$. This is easily seen by analysing the signs of the numerator and denominator. For the numerator, $\mathcal{I}(y)y^{\alpha-1}$ and for the quadratic in the denominator, we note the domain is either totally contained in $[1/\lambda, 1/\Lambda]$ if $\lambda < 0$ or is disjoint from $[1/\Lambda, 1/\lambda)$ (resp., $[1/\Lambda, \infty)$) if $\lambda > 0$ (resp., $\lambda = 0$).

Assume that the integral vanishes for some choice of u_{-1} and u_0. We first show that $u_{-1} \neq 0$. Indeed, if $u_{-1} = 0$, then the integral takes the form

$$A \int_0^{1/\Lambda} \frac{\mathcal{I}(y)}{\gamma_1 y^2 + \beta_1 y - 1} y^{\alpha}\, dy.$$

The integrand has constant sign and does not vanish on $[0, 1/\Lambda)$ (unless $A = 0$, which occurs precisely when $u_0 = u_{-1} = 0$). We deduce that the integral does not vanish.

Assume now that the integral vanishes for two distinct pairs (u_{-1}, u_0) and (u'_{-1}, u'_0). As $u_{-1}, u'_{-1} \neq 0$ we can, without loss of generality, assume the pairs are of the form $(1, u_0)$ and $(1, u'_0)$ with $u_0 \neq u'_0$. Substitution of these initial values into the integral gives the following

$$\int_0^{1/\Lambda} \frac{\mathcal{I}(y)}{\gamma_1 y^2 + \beta_1 y - 1} y^{\alpha-1}(Ay + B)\, dy = 0 = \int_0^{1/\Lambda} \frac{\mathcal{I}(y)}{\gamma_1 y^2 + \beta_1 y - 1} y^{\alpha-1}(A'y + B)\, dy.$$

By linearity, we conclude that the integral vanishes with the choice $(0, u_1 - u'_1)$. This contradicts our earlier observation and concludes the proof. ◀

The following lemma makes explicit the integrands $\mathcal{I}(y)t(y)$ of the recurrences (4.1)–(4.4).

▶ **Lemma 4.8.** *The integrating factor $\mathcal{I}(x)$ in (4.12), is as follows.*

1. *For recurrences* (4.1) *and* (4.4) *we have*
$$\mathcal{I}(x) = (1/\Lambda - x)^{\nu}|x - 1/\lambda|^{\nu'} = \frac{1}{\Lambda^{\nu}|\lambda|^{\nu'}}(1-\Lambda x)^{\nu}|1-\lambda x|^{\nu'},$$
where $\nu = \mathcal{A}(\lambda)$, $\nu' = -\mathcal{A}(\Lambda)$, *and*
$$\mathcal{A}(x) = \frac{(\alpha - \gamma_0/\gamma_1 - 2)x + (\beta_0 + \beta_1 - \alpha\beta_1)}{\Lambda - \lambda}.$$
2. *For recurrence* (4.2) *we have*
$$\mathcal{I}(x) = \exp\left(\frac{\beta_0 - \alpha - \gamma_0}{x-1}\right)(1-x)^{2+\gamma_0-\alpha}.$$
3. *For recurrence* (4.3) *we have*
$$\mathcal{I}(x) = \mathrm{e}^{\nu' x}(\beta_1^{-1} - x)^{\nu} = \beta_1^{-\nu}\mathrm{e}^{\nu' x}(1-\beta_1 x)^{\nu}$$
where $\nu' = \gamma_0/\beta_1$ *and* $\nu = \beta_0/\beta_1 - \alpha + 1 + \gamma_0/\beta_1^2$.

We have deliberately chosen to share notation between the different items in the above lemma (especially the parameter ν): in the sequel, we shall treat several of the cases simultaneously.

Proof. The first two claims follow from a straightforward integration of partial fractions. One only notes that the roots of $\gamma_1 x^2 + \beta_1 x - 1$ are $1/\lambda$ and $1/\Lambda$. Further, in the first claim we have $s(x) = \frac{\mathcal{A}(\lambda)}{x-1/\Lambda} - \frac{\mathcal{A}(\Lambda)}{x-1/\lambda}$. In the second claim we have
$$s(x) = \frac{2+\gamma_0-\alpha_0}{x-1} + \frac{\alpha_0+\gamma_0-\beta_0}{(x-1)^2}.$$
We remark that in this case we substitute $\beta_1 = 2$, $\gamma_1 = -1$, and $\gamma_0 \mapsto -\gamma_0$ in (4.11).

In the last case, we substitute $\gamma_1 = 0$, so
$$s(x) = \frac{\gamma_0}{\beta_1} + \frac{\beta_0/\beta_1 + 1 - \alpha + \gamma_0/\beta_1^2}{x - 1/\beta_1}.$$ ◀

Our approach to deciding minimality hinges on identifying when $\mathcal{F}(x)$ and all of its (left) derivatives exist at $1/\Lambda$. Indeed, when $\langle u_n \rangle_n$ is a minimal solution to (4.1), then $\mathcal{F}(x)$ is known to be analytic in a neighbourhood of $1/\Lambda$. As we shall shortly see, this connection holds for all the cases at hand. Let us first discuss the Minimality Problem for instances of (4.2).

4.4 Minimality for recurrence (4.2)

Recall that recurrence (4.2) admits a minimal solution if and only if $\beta_0 \geq \alpha + \gamma_0$ by Lemma 4.5. We shall consider the cases $\beta_0 > \gamma_0 + \alpha$ and $\beta_0 = \gamma_0 + \alpha$ separately.

Recall from Example 2.3 that in the case $\beta_0 > \gamma_0 + \alpha$, recurrence (4.2) admits two linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ such that
$$u_n^{(1)} \sim n^{(1-2\alpha+2\gamma_0)/4} \exp(2\sqrt{(\beta_0 - \alpha - \gamma_0)n}) \quad \text{and}$$
$$u_n^{(2)} \sim n^{(1-2\alpha+2\gamma_0)/4} \exp(-2\sqrt{(\beta_0 - \alpha - \gamma_0)n}).$$

Notice that $\langle u_n^{(2)} \rangle_n$ is a minimal solution and that $\langle u_n^{(1)} \rangle_n$ is dominant. From the asymptotics above, it is evident that $\sum_{n=0}^{\infty} u_{n-1}^{(1)} = \infty$ and $\sum_{n=0}^{\infty} u_{n-1}^{(2)}$ is finite. Hence, by Abel's theorem, we have that $\lim_{x\to 1-} \mathcal{F}(x) = \infty$ (resp., $\lim_{x\to 1-} \mathcal{F}(x)$ is finite) for the generating function $\mathcal{F}$. In particular, we have proved the following lemma.

▶ **Lemma 4.9.** *A non-trivial solution $\langle u_n \rangle_n$ to (4.2) with $\beta_0 > \gamma_0 + \alpha$ is minimal if and only if the left limit of corresponding generating $\lim_{x \to 1-} \mathcal{F}(x)$ exists and is finite.*

▶ **Proposition 4.10.** *Let $\langle u_n \rangle_n$ be a non-trivial solution to (4.2) where $\beta_0 > \alpha + \gamma_0$. Then $\langle u_n \rangle_n$ is minimal if and only if*

$$\int_0^1 \exp\left(\frac{\beta_0 - \alpha - \gamma_0}{y-1}\right)(1-y)^{\gamma_0 - \alpha} y^{\alpha - 1}(Ay - B)\,dy = 0,$$

where $A = (2+\beta_0)u_{-1} - (\alpha+1)u_0$ and $B = \alpha u_{-1}$.

Proof. Recall that a non-trivial solution to (4.2) defines the generating function $\mathcal{F}(x)$ as in Lemma 4.8(2). The integral in this claim is the integral $\int_0^1 \mathcal{I}(y)t(y)\,dy$. The integral is absolutely converging as $\lim_{x\to 1-} \exp(\frac{\beta_0-\alpha-\gamma_0}{x-1}) = 0$ since $\beta_0 > \alpha + \gamma_0$. (In particular, the factor $(1-y)^{\gamma_0-\alpha}$ does not affect the convergence.)

Suppose that $\langle u_n \rangle_n$ is a minimal sequence and let ζ be the value of the associated integral. Assume, for a contradiction, that $\zeta \neq 0$. Then, by definition, $\lim_{x\to 1-} \mathcal{F}(x) = \operatorname{sign}(\zeta)\infty$, as $\mathcal{I}(x)^{-1}$ has a singularity at $x = 1$. This contradicts the criterion in Lemma 4.9. Thus we conclude that the integral $\int_0^1 \mathcal{I}(y)t(y)\,dy$ associated to a minimal solution $\langle u_n \rangle_n$ vanishes.

The converse argument follows from Lemma 4.7 and this concludes the proof. ◀

We then consider the case of $\beta_0 = \alpha + \gamma_0$. The Minimality Problem for this case is decidable as evidenced by following proposition.

▶ **Proposition 4.11.** *Let $\langle u_n \rangle_n$ be a non-trivial solution to (4.2) with $\beta_0 = \alpha + \gamma_0$. If $\alpha \le \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = 1$. If $\alpha > \gamma_0 + 1$ then $\langle u_n \rangle_n$ is minimal if and only if $u_0/u_{-1} = \frac{\gamma_0+1}{\alpha}$.*

Proof. We are in the setting of Remark 2.14 with $q_n = \frac{n+\gamma_0}{n+\alpha}$. Hence $\langle 1 \rangle_n$ and $\langle v_n \rangle_n$ defined by $v_{-1} = 0$, $v_0 = 1$ such that

$$v_n = \sum_{k=0}^{n} \prod_{m=1}^{k} \frac{m+\gamma_0}{m+\alpha} = \sum_{k=0}^{n} \frac{(\gamma_0+1)_k}{(\alpha+1)_k}$$

are linearly independent solution sequences. By a straightforward application of Stirling's approximation, $(\gamma_0+1)_n/(\alpha+1)_n \sim n^{\gamma_0-\alpha}$ as $n \to \infty$. Hence if $\gamma_0 - \alpha \ge -1$ the series diverges (by comparison to the harmonic series) from which we deduce that $\langle 1 \rangle_n$ is the minimal solution. If $\gamma_0 - \alpha < -1$, then $\lim_{n\to\infty} v_n$ converges to the value

$$\sum_{k=0}^{\infty} \frac{(\gamma_0+1)_k (1)_k}{(\alpha+1)_k} \frac{1}{k!} = {}_2F_1(\gamma_0+1, 1; \alpha+1; 1) = \frac{\Gamma(\alpha+1)\Gamma(\alpha-\gamma_0-1)}{\Gamma(\alpha-\gamma_0)\Gamma(\alpha)} = \frac{\alpha}{\alpha-\gamma_0-1}.$$

In the second equality we again use Theorem 2.2.2 from [3]. It follows that $\langle u_n \rangle_n = \frac{\alpha}{\alpha-\gamma_0-1}\langle 1 \rangle_n - \langle v_n \rangle_n$ is a minimal solution, and we may compute $u_0/u_{-1} = \frac{\gamma_0+1}{\alpha}$. ◀

Notice that the integral in Proposition 4.10 above is an exponential period. As the Minimality Problem in the case $\beta_0 = \alpha + \gamma_0$ is a decidable problem, we conclude that Theorem 4.2 holds for recurrences of the form (4.2).

4.5 Minimality for recurrences (4.1), (4.3), and (4.4)

Consider recurrences (4.1), (4.3), and (4.4). Recall that in these instances, the characteristic roots are distinct (when omitting the subcase (4.2), which was handled above). Define $f(x) = \mathcal{I}(x)t(x)$, i.e., $f(x)$ is the integrand of (4.12), and $\mathcal{I}$ is as in Lemma 4.8(1 & 3). Recall that $\Lambda = \beta_1$ in (4.3) and $(1-\Lambda x)$ is a factor of the denominator of $t(x)$. For $x \in [0, 1/\Lambda)$ sufficiently close to $1/\Lambda$,

$$f(x) = (1-\Lambda x)^{\nu-1}\sum_{n=0}^{\infty} c_n(1-\Lambda x)^n = \sum_{n+\nu\le 0} c_n(1-\Lambda x)^{n+\nu-1} + \mathcal{O}\big((1-\Lambda x)^{n_0+\nu-1}\big) \quad (4.13)$$

where ν is given in Lemma 4.8 and $n_0 \in \mathbb{N}_0$ is the least integer such that $n_0+\nu > 0$. In the sequel we use the notation $H(x) = \sum_{n=0}^{\infty} c_n(1-\Lambda x)^n$ for brevity.

▷ Claim 4.12. For $x \in [0, 1/\Lambda)$ sufficiently close to $1/\Lambda$,

$$\int_0^x f(y)\,dy = \sum_{n<-\nu} \frac{-c_n/\Lambda}{n+\nu}(1-\Lambda x)^{n+\nu} + C_0 + C_1\log(1-\Lambda x) + \mathcal{O}((1-\Lambda x)^{n_0+\nu}), \quad (4.14)$$

where $C_1 = -c_{-\nu}/\Lambda$ if $-\nu \in \mathbb{N}_0$ and $C_1 = 0$ otherwise, and

$$C_0 = \sum_{n<-\nu} \frac{c_n/\Lambda}{n+\nu} + \int_0^{1/\Lambda} f(y) - \sum_{n\le -\nu} c_n(1-\Lambda y)^{n+\nu-1}\,dy. \quad (4.15)$$

Proof. This can be seen by integrating the series termwise (cf. [13, Theorem VI.9(ii)].) See Subsection G.2 for a proof. ◀

We are interested in the behaviour of $\mathcal{F}(x) = \mathcal{I}(x)^{-1}\int_0^x f(y)dy$ as $x \to 1/\Lambda-$. By inspecting Lemma 4.8(1 & 3) we notice that in both cases $\mathcal{I}(x)^{-1} = (1-\Lambda x)^{-\nu}$ multiplied by a function that is analytic at $1/\Lambda$. Multiply (4.14) through by $\mathcal{I}^{-1}(x)$. Then, modulo the addition of analytic terms that vanish at $x = 1/\Lambda$, $\mathcal{F}(x)$ is given by the product of an analytic function that does not vanish at $1/\Lambda$ and

$$\sum_{n<-\nu} \frac{-c_n/\Lambda}{n+\nu}(1-\Lambda x)^n + (C_0 + C_1\log(1-\Lambda x))(1-\Lambda x)^{-\nu}. \quad (4.16)$$

The next lemma identifies when $\lim_{x\to 1/\Lambda-} \mathcal{F}^{(\ell)}(x)$ exists for each $\ell \ge 0$.

▶ **Lemma 4.13.** *Consider the parameter ν in (4.13).*

1. *Suppose that $-\nu \notin \mathbb{N}_0$. Then the lower limits of $\mathcal{F}$ and its derivatives are finite at $1/\Lambda$ if and only if $C_0 = 0$; that is,*

$$\int_0^{1/\Lambda} f(y) - \sum_{n\le -\nu} c_n(1-\Lambda y)^{n+\nu-1}\,dy = \sum_{n<-\nu} \frac{-c_n/\Lambda}{n+\nu}. \quad (4.17)$$

2. *Suppose that $-\nu \in \mathbb{N}_0$. Then the lower limits of $\mathcal{F}$ and its derivatives are finite at $1/\Lambda$ if and only if $C_1 = 0$.*

Proof. From (4.16) we have that the lower limits of $\mathcal{F}$ and its derivatives are finite at $1/\Lambda$ if and only if this is so for the function $C_0(1-\Lambda x)^{-\nu} + C_1\log(1-\Lambda x)(1-\Lambda x)^{-\nu}$.

1. Assume that $-\nu \notin \mathbb{N}_0$. Then $C_1 = 0$ by definition. Hence the lower limits of $\mathcal{F}(x)$ and its left derivatives are finite at $1/\Lambda$ if and only if $C_0 = 0$ and so we have the desired result.

2. Assume that $-\nu \in \mathbb{N}_0$. Then the first term in the above function is analytic at $1/\Lambda$. Hence the function $\mathcal{F}(x)$ and all its left derivatives exist at $1/\Lambda$ if and only if $C_1 = 0$. ◀

Recall from Example 2.3 that recurrence (4.4) admits two linearly independent solutions $\langle u_n^{(1)} \rangle_n$ and $\langle u_n^{(2)} \rangle_n$ such that

$$u_n^{(1)} \sim n^{\frac{1}{2}(\beta_0+\gamma_0-\alpha)}, \quad \text{and} \quad u_n^{(2)} \sim (-1)^n n^{\frac{1}{2}(-\beta_0+\gamma_0-\alpha)}.$$

Since $\beta_0 > 0$, $\langle u_n^{(1)} \rangle_n$ is a dominant solution. For ℓ sufficiently large it is immediate that $\sum_{n=0}^{\infty} n(n-1)\cdots(n-\ell)u_{n-1}^{(1)} = \infty$. By Abel's theorem, $\lim_{x\to 1-} \mathcal{F}^{(\ell)}(x) = \infty$ for the corresponding generating function $\mathcal{F}$. We thus have the following corollary.

▶ **Corollary 4.14.** *For any dominant solution to* (4.4)*, the corresponding generating function has a derivative for which* $\lim_{x\to 1-} \mathcal{F}^{(\ell)}(x) = \pm\infty$.

We will now show for each recurrence relation (4.1), (4.3), and (4.4), there exists a choice of u_{-1} and u_0 such that $\mathcal{F}$ and all its left derivatives exist at $1/\Lambda$.

▶ **Lemma 4.15.** *For each of the recurrences* (4.1)*,* (4.3)*, and* (4.4)*, there is a choice of* u_0 *and* u_{-1} *such that the left limits of the corresponding generating function* $\mathcal{F}$ *and all its derivatives exist at* $1/\Lambda$.

Proof. For recurrences (4.1) and (4.3) the result follows from Lemma 4.6 as a minimal solution necessarily defines such u_{-1} and u_0. We may thus focus on recurrence (4.4). Let us put $u_{-1} = 1$. By Lemma 4.8(3) we have

$$f(y) = (1-y)^{\nu-1}(1+y)^{\nu'-1}y^{\alpha-1}(Ay-\alpha)$$

where $\nu = 1 + \frac{\gamma_0-\alpha+\beta_0}{2}$, $\nu' = \frac{\alpha-\gamma_0+\beta_0}{2} - 1 = -\nu + \beta_0$, $A = \beta_0 - (\alpha+1)u_0$. In terms of (4.13), we have $\sum_{n=0}^{\infty} c_n(1-\Lambda x)^n = (1+y)^{\nu'-1}y^{\alpha-1}(Ay-\alpha)$.

In light of Lemma 4.13, we consider two cases. (Recall here that $\Lambda = 1$ and $\lambda = -1$.)

1. Assume first that $-\nu \notin \mathbb{N}_0$. Then, by Lemma 4.13(1), we need to establish a choice of u_0 for which (4.17) holds. Assume first that $\nu > 0$. Then equation (4.17) is simply $\int_0^1 f(y)\,dy = 0$, which is equivalent to

$$A\int_0^1 (1-y)^{\nu-1}(1+y)^{\nu'-1}y^{\alpha}\,dy = \alpha\int_0^1 (1-y)^{\nu-1}(1+y)^{\nu'-1}y^{\alpha-1}\,dy.$$

Clearly we can choose u_0 for this equation to hold as the integral on the left does not vanish (the integrand is strictly positive at each point in the domain of integration).
Assume that $\nu < 0$ (and recall that $-\nu \notin \mathbb{N}_0$). Consider the parameters in (4.17). We write $(1+y)^{\nu'-1}y^{\alpha-1} = \sum_{n=0}^{\infty} d_n(1-y)^n$ when y is close to 1. Then $\sum_{n=0}^{\infty} c_n(1-y)^n = (Ay-\alpha)\sum_{n=0}^{\infty} d_n(1-y)^n$, so that $c_n = A(d_n - d_{n-1}) - \alpha d_n$ (with the convention $d_{-1} = 0$). Recall that n_0 is the least element of $\mathbb{N}_0$ such that $n_0 + \nu > 0$. We have

$$\sum_{n\le -\nu} c_n(1-y)^n = (Ay-\alpha)\sum_{n\le-\nu} d_n(1-y)^n + Ad_{n_0-1}(1-y)^{n_0}.$$

Now recall that $\Lambda = 1$ and $\lambda = -1$ in (4.4). Then the equality in (4.17) holds if and only if the following expression is equal to zero

$$\int_0^1 (Ay-\alpha)\Big(K(y) - \sum_{n\le-\nu} d_n(1-y)^{n+\nu-1}\Big)\,dy - A\frac{d_{n_0-1}}{n_0+\nu} + \sum_{n<-\nu}\frac{A(d_n-d_{n-1})-\alpha d_n}{n+\nu}.$$

Here $K(y) := (1-y)^{\nu-1}(1+y)^{\nu'-1}y^{\alpha-1}$. We shall henceforth call the above expression $J(A)$. We now show that $J(A)$ is continuous in A (and so is continuous in u_0).

▷ Claim 4.16. The function J is continuous. Moreover, it is differentiable for A in any open, bounded interval (a, b).

Proof. To show that J is differentiable, it suffices to show that one can *pass the differentiation under the integral sign* [41, §20.4] so that

$$\frac{d}{dA}\int_0^1 (Ay-\alpha)\Big(K(y)-\sum_{n\le -\nu} d_n(1-y)^{n+\nu-1}\Big)\,dy = \int_0^1 \frac{\partial}{\partial A} g(y,A)\,dy.$$

This process is permitted because $y \mapsto g(y, A)$ is integrable by construction; the derivative $\partial g(y, A)/\partial A = yK(y) - y\sum_{n\le -\nu} d_n(1-y)^{n+\nu-1}$ exists for each $y \in (0, 1)$; and $|\partial g(y, A)/\partial A|$ is an integrable function independent of A. ◀

Now, by Leibniz's rule we have

$$J'(A) = \int_0^1 yK(y) - y\sum_{n\le -\nu} d_n(1-y)^{n+\nu-1}\,dy - \frac{d_{n_0-1}}{n_0+\nu} + \sum_{n<-\nu}\frac{d_n - d_{n-1}}{n+\nu}.$$

Hence J is either constant or linear in A depending on whether $J'(A) = 0$.

▷ Claim 4.17. We claim that $J'(A) \neq 0$.

Proof. Recall that a dominant solution to (4.4) defines a generating function for which $\lim_{x\to 1-}\mathcal{F}^{(\ell)}(x) = \pm\infty$ for sufficiently large ℓ. From (4.16), we notice that the sign of the above limit is determined by the sign of C_0 (since the analytic term omitted from the expression does not vanish at $1/\Lambda$). It thus suffices to exhibit two choices of u_0 (with $u_{-1} = 1$) for which both the signs in the limit are realised.
Now the solution $\langle v_n\rangle_n$ defined by $v_{-1} = 0$, $v_0 = 1$, to (4.4) is dominant. Indeed, Proposition 3.2 together with the discussion preceding it imply that the sequence $(-1)^n v_n$ is dominant (it satisfies a recurrence with signature $(-,+)$), and dominance is inherited by $\langle v_n\rangle_n$. Let $\langle v'_n\rangle_n$ be a minimal solution to (4.4). We may assume that $v'_{-1} = 1$, as it is linearly independent to $\langle v_n\rangle_n$. Then the solutions given by $\pm\langle v_n\rangle_n + \langle v'_n\rangle_n$ define generating functions that each diverge to $\pm\infty$ (with opposite signs) and $\pm v_0 + v'_0 = 1$ as required. ◀

We deduce that $J(A)$ is a degree one polynomial in A. In particular, there is a choice of A such that $J(A) = 0$. This concludes the proof of the first case.

2. Assume second that $-\nu \in \mathbb{N}_0$. Then, by Lemma 4.13(2), we need to exhibit a choice of u_0 for which $C_1 = 0$. This is equivalent to $c_{-\nu} = 0$. Recall that $\sum_{n=0}^{\infty} c_n(1-y)^n = (Ay-\alpha)(1+y)^{\nu'-1}y^{\alpha-1} =: H(y)$. So, $c_{-\nu} = 0$ if and only if $H^{(-\nu)}(1) = 0$. We further introduce $K(y) = (1+y)^{\nu'-1}y^{\alpha-1}$ (so $H(y) = K(y)(Ay-\alpha)$) and $\ell = -\nu$. Now if $\ell = 0$, then $u_0 = \frac{\beta_0-\alpha}{\alpha+1}$ forces $H(1) = 0$. For $\ell \in \mathbb{N}$ we have

$$H^{(\ell)}(x) = \sum_{k=0}^{\ell}\binom{\ell}{k}K^{(\ell-k)}(x)\frac{d^k}{dx^k}(Ax+B) = K^{(\ell)}(x)(Ax-\alpha) + \ell K^{(\ell-1)}(x)A$$

and so $H^{(\ell)}(1) = A(K^{(\ell)}(1)+\ell K^{(\ell-1)}(1)) - \alpha K^{(\ell)}(1)$. As long as $K^{\ell}(1)+\ell K^{\ell-1}(1) \neq 0$, we can choose u_0 in a suitable way to force $H^{(\ell)}(1) = 0$. Next we show that $K^{(\ell)}(1)+\ell K^{\ell}(1) \neq 0$ to conclude the proof. Recall that $K(y) = (1+y)^{\nu'-1}y^{\alpha-1}$ we have

$$K^{(\ell)}(y) = \sum_{k=0}^{\ell}\binom{\ell}{k}\frac{d^k}{dy^k}(1+y)^{\ell+\beta_0-1}\frac{d^{\ell-k}}{dy^{\ell-k}}y^{\alpha-1}.$$

In our current working $\beta_0, \gamma_0 > 0$ and so we have the following inequalities. First, $\nu' = -\nu + \beta_0 = \ell + \beta_0 > \ell$. Second, $\alpha = 2\nu' + \gamma_0 + 2 - \beta_0 = 2\ell + \gamma_0 + \beta_0 + 2 > \ell$. From the preceding inequalities, analysis of the summands in the above binomial expansion shows that $K^{(\ell)}(1) > 0$ and similarly $K^{(\ell-1)}(1) \geq 0$. We conclude that $K^{(\ell)}(1) + \ell K^{(\ell-1)}(1) \neq 0$ as required. ◀

▶ **Proposition 4.18.** *A solution to recurrence* (4.1), (4.3), *or* (4.4) *is minimal if and only if left limits of the corresponding generating function* $\mathcal{F}$ *and its derivatives exist at* $1/\Lambda$. *Hence the decidability of the minimality problem in these instances reduces to checking the equalities in Lemma 4.13.*

Proof. We have established that, for all the recurrences, a dominant solution defines a generating function for which some derivative tends to $\pm\infty$ as $x \to 1/\Lambda-$. This is evident from (4.16) for the recurrences (4.1) and (4.3), as a minimal solution defines a generating function which is analytic at $1/\Lambda$. For (4.4), this is established in Corollary 4.14.

On the other hand, Lemma 4.13 shows that there exists a solution for which $\mathcal{F}$ and all its derivatives exist at $1/\Lambda$. We conclude that this choice must correspond to a minimal solution. Hence, minimality reduces to checking Lemma 4.13. ◀

The equalities in Lemma 4.13 involve checking whether a period, exponential period, or a period-like integral equals 0 as follows. We conclude that Theorem 4.2 holds for solutions of recurrences of the form (4.1), (4.3), and (4.4). (Notice that it is decidable whether the second equality in Lemma 4.13 holds or not.)

- For (4.1), the equation can be rearranged to obtain

$$\int_0^{1/\Lambda} \frac{1}{\Lambda^\nu |\lambda|^{\nu'}} (1 - \Lambda x)^\nu |1 - \lambda x|^{\nu'} t(x) - \sum_{n \leq -\nu} c_n (1 - \Lambda y)^{n+\nu-1} + \sum_{n < -\nu} \frac{c_n}{n + \nu}\, dy = 0,$$

  where $t(y)$ is an algebraic function, λ and Λ are the characteristic roots of the recurrence. The parameters ν and ν' are algebraic numbers of degree at most 2.[6] If they are rational, then the integral is a period and the parameters c_n are algebraic numbers. If ν and ν' are irrational, then the integral is period-like: the parameters c_n are algebraic multiples of derivatives of $|1 - \lambda x|^{\nu'} t(x)(1 - \Lambda x)$ evaluated at $1/\Lambda$, i.e., are algebraic multiples of algebraic numbers to algebraic powers. Hence the integral on the left is period-like.
- For the recurrence (4.3), the equation can be rearranged to

$$\int_0^{1/\beta_1} \mathrm{e}^{\nu' y} (\beta_1^{-1} - y)^\nu t(y) - \sum_{n \leq -\nu} c_n (1 - \beta_1 y)^{n+\nu-1} + \sum_{n < -\nu} \frac{c_n}{n + \nu}\, dy = 0,$$

  where $t(y)$ is an algebraic function. The integral is an exponential period, as the parameters ν and ν' are rational. In this case the numbers c_n are exponential periods, as they are algebraic multiples of $\mathrm{e}^{\nu'}$: they are derivatives of $\mathrm{e}^{\nu' y} t(y)(1 - y)$ evaluated at 1. The integral on the left is an exponential period.
- For (4.4), the equation can be rearranged to obtain

$$\int_0^1 (1 - y)^{\nu-1} (1 + y)^{\nu'-1} y^{\alpha-1} (Ay - \alpha) - \sum_{n \leq -\nu} c_n (1 - y)^{n+\nu-1} + \sum_{n < -\nu} \frac{c_n}{n + \nu}\, dy = 0.$$

[6] We have $\nu, \nu' \in \mathbb{Q}(\Lambda)$, and they can be rational even if Λ is not.

The parameters ν and ν' here are algebraic numbers, as they are in $\mathbb{Q}(\beta_0)$ and β_0 is a real algebraic number. If they are rational, then the integral is a period, and the numbers c_n are algebraic. If ν and ν' are irrational, then the integral is period-like and the numbers c_n are algebraic multiples of algebraic numbers to algebraic powers. The integral of the left is thus period-like.

5 Conclusions

In the Minimality Problem we are faced with the problem of comparing the ratio of the first two terms of a solution against the value of a polynomial continued fraction. The problem becomes trivial if the value of the polynomial continued fraction is transcendental, as no real algebraic solution is then minimal. For degree-1 holonomic sequences, these values can often be expressed using hypergeometric functions: [26, §6.4] expresses such a value as a quotient of two (contiguous) hypergeometric functions with real algebraic parameters evaluated at real algebraic points. A characterisation of transcendence for such numbers is not known. For holonomic sequences of higher degree, we see values of associated polynomial continued fraction using hypergeometric functions in Example 2.15. One can obtain several transcendental values $1 - 1/\xi$ for the associated continued fraction: using the construction, we see that possible values of ξ include

- $\mathrm{e}^k = {}_0F_0(;;k)$ for $k \in \mathbb{Q}$;
- $\cos(k) = {}_0F_1(;1/2;-k^2/4)$ for $k \in \mathbb{Q}$;
- $\log(1+k)/k = {}_2F_1(1,1;2;-k)$ for $k \in \mathbb{Q}$, $|k| < 1$; and
- $\zeta(s) = {}_{s+1}F_s(1,\dots,1;2,\dots,2;1)$, for $s \in \mathbb{N}$, $s \geq 2$, where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.

(The first three equalities can be seen from the classical Taylor series expansions of the functions. The last equality follows after cancellations.) Many of these values are known to be transcendental. For the Riemann zeta function, Euler proved that if s is a positive integer then $\zeta(2s)$ is a rational multiple of π^{2k} and so it follows that $\zeta(2k)$ is transcendental. The arithmetic study of the values of $\zeta(2k+1)$ is a major undertaking. For example, Apéry's constant $\zeta(3)$ is irrational [4] (not known to be transcendental) and research has shown that infinitely many values of $\zeta(2k+1)$ are irrational [42]. For the time being, there is a lack of understanding of the arithmetic properties of values of hypergeometric functions with rational (or algebraic[7]) parameters evaluated at rational (or algebraic) points, though the study has spanned several decades and several striking results have been established (see, e.g., [47, 6, 5, 17] and references therein).

In general, establishing transcendence of a number is a very challenging task, while establishing irrationality can be easier. This aspect in mind, let us restrict our consideration of holonomic sequences to those with rational elements. The following proposition shows that Minimality$(0,1,0)$ becomes a trivial problem under this restriction.

▶ **Proposition 5.1.** *If $\langle u_n \rangle_n$ is a minimal solution to recurrence* (4.8) *then u_0/u_{-1} irrational.*

Proof. Normalise the recurrence relation as follows. First, multiply through by $\alpha \in \mathbb{N}$ to obtain the recurrence $\alpha u_n = (\beta_1' n + \beta_0')u_{n-1} + \gamma_0' u_{n-2}$ with $\alpha, \beta_1', \beta_0', \gamma_0' \in \mathbb{Z}$. Second, use the transformation described in (2.2) to obtain a recurrence of the form $v_n = b_n v_{n-1} + a_n v_{n-2}$ where $b_n = \beta_1' n + \beta_0' \in \mathbb{Z}[n]$ and $a_n = \alpha\gamma_0' \in \mathbb{Z}$ is constant. Without loss of generality we may assume that each b_n is positive by considering the sequence $\langle (-1)^n v_n \rangle_n$ if necessary. The continued fraction $\mathrm{K}_{n=1}^{\infty}(\alpha\gamma_0'/b_n)$ associated to minimal solutions $\langle v_n \rangle_n$ of the transformed

[7] Notice that the parameters of the hypergeometric functions found in Example 2.15 can be algebraic.

recurrence satisfies $\alpha\gamma_0' < b_n + 1$ for sufficiently large $n \in \mathbb{N}$. Continued fractions of this form converge to irrational values (see [8, §XXXIV, pp. 512–514]). This is sufficient to prove the result because the bijection between solution sequences $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ preserves rationality and minimality. ◀

A similar conclusion holds for certain instances of MINIMALITY$(j, 1, \ell)$ with $j + \ell = 1$ (precisely when $|g_3(n-1)g_1(n)| < |g_2(n)|+1$ assuming $g_i \in \mathbb{Z}[n]$). Again, for such recurrences the restricted Minimality Problem becomes trivial.

Let us pursue this line of thought and discuss the restricted Minimality Problem for degree-1 holonomic sequences in general. Consider then recurrences of the form (4.1), where we understand that the roots are of distinct moduli. We may invoke a conjecture of Zudilin [50] (see also Conjecture A.1 for the precise statement) that makes the following prediction: if a second-order Poincaré recurrence relation, in which the coefficients are in $\mathbb{Q}(n)$, has irrational characteristic roots, then all rational solutions to the recurrence are dominant. If the conjecture is true then the restricted Minimality Problem can be trivially answered for such recurrences with irrational characteristic roots. Hence, the only interesting instances are the recurrences which have rational characteristic roots. By the discussion at the end of Subsection 4.5, the Minimality Problem of rational solutions to such recurrences reduces to checking whether a period is equal to 0. This is conjectured to be decidable by Kontsevich and Zagier [20] (see Conjecture A.2 for the precise statement and discussion).

Consider then recurrences of the form (4.2). In this case the (unrestricted) Minimality Problem reduces to checking whether an exponential period is zero (Proposition 4.11), which is also conjectured to be decidable [20]. We conclude that MINIMALITY$(1, 1, 1)$ restricted to rational solution sequences is decidable subject to Zudilin's conjecture and the aforementioned conjectures on periods and exponential periods.

We may say something a bit stronger. From the interreductions between the different parametrized versions of the Minimality Problem established in Subsection 4.1, and subject to the aforementioned conjectures by Kontsevich and Zagier, and Zudilin, restrictions of the problems MINIMALITY(j, k, ℓ) with $j, k, \ell \in \{0, 1\}$ are decidable except for the case $j = \ell = 1$, $k = 0$.

A Conjectures

A.1 A rationality conjecture for holonomic sequences

The following conjecture is due to to Zudilin [50]:

▶ **Conjecture A.1.** *Suppose that $u_n = b_n u_{n-1} + a_n u_{n-2}$ is a second-order Poincaré recurrence with $a_n, b_n \in \mathbb{Q}(n)$. Assume that the characteristic roots λ and Λ associated to the recurrence satisfy $0 < |\lambda| < |\Lambda|$. Suppose that there exist two rational linearly independent solutions $\langle v_n \rangle_n$ and $\langle w_n \rangle_n$ satisfying $v_{n+1}/v_n \sim \lambda$ and $w_{n+1}/w_n \sim \Lambda$ as $n \to \infty$. Then λ and Λ are rational numbers.*

A.2 A decidability conjecture for periods

Kontsevich and Zagier's seminal paper [20] defines a *period* to be a complex number whose real and imaginary parts can be written as absolutely convergent integrals of the form

$$\int_\sigma \frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)} \, dx_1 \cdots dx_n$$

where $P, Q \in \mathbb{Q}[x_1, \dots, x_n]$, Q is not the zero polynomial, and the domain $\sigma \subset \mathbb{R}^n$ is given by polynomial inequalities with rational coefficients. It can be shown that one can replace rational numbers by algebraic numbers, and rational functions by algebraic functions (with algebraic coefficients) in the above definition. The set of periods $\mathcal{P}$ form a countable sub-algebra of $\mathbb{C}$ and it is easily seen that $\overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}$. Two initial examples are:

$$\log(\alpha) = \int_1^\alpha \frac{1}{x} \, dx \quad \text{with } \alpha \in \overline{\mathbb{Q}} \qquad \text{and} \qquad \pi = \int_{x^2+y^2 \le 1} dx dy.$$

Given two algebraic numbers α and β, the problem of determining algorithmically whether $\alpha = \beta$ is known to be decidable. The decidability of the equality of two periods—that is, a decision procedure determining whether two periods (given by two explicit integrals) are equal—is currently open. The next conjecture, see [20, Conjecture 1], by Kontsevich and Zagier, would entail that equality of periods is decidable.

▶ **Conjecture A.2.** *Suppose that a period has two integral representations. One can pass between the representations via a finite sequence of admissible transformations where each transformation preserves the structure that all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$. The admissible transformations are: linearity of the integral, a change of variables, and Stokes's formula.*

It is currently not known whether Euler's number e is a period. The following notion of exponential period was introduced in [20] to extend the definition of period to a larger class containing e. An *exponential period* is a complex number that can be written as an absolutely convergent integral of the form

$$\int_\sigma \mathrm{e}^{-f(x_1,\dots,x_n)} g(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

where f and g are algebraic functions with algebraic coefficients and the domain $\sigma \subset \mathbb{R}^n$ is a semi-algebraic set defined by polynomials with algebraic coefficients. Conjecture A.2 is predicted to generalise to exponential periods in [20]. An overview discussing both periods and exponential periods can be found in [44].

In this paper we encounter integrals that generalise the above concepts of period and exponential period. A *period-like integral* is a number that can be written as an absolutely convergent integral of the form

$$\int_\sigma \mathrm{e}^{-f(x_1,\ldots,x_n)} g(x_1,\ldots,x_n)\, dx_1 \cdots dx_n.$$

Here f is an algebraic function, g is the sum of algebraic functions raised to algebraic powers, and the domain $\sigma \subset \mathbb{R}^n$ is a semi-algebraic set defined by polynomials with algebraic coefficients.

B The PCF Equality Problem and the Minimality Problem

Corollary 2.9 is a straightforward application of Pincherle's Theorem (Theorem 2.8). Given a solution sequence $\langle v_n \rangle_{n=-1}^\infty$ to recurrence relation (3.1), let us consider the corresponding sequence $\langle u_n \rangle_{n=-1}^\infty$ to the normalised recurrence (using the transformation described for (2.2)). This transformation preserves minimality so that $\langle v_n \rangle_n$ is a minimal solution of (3.1) if and only if $\langle u_n \rangle_n$ with initial terms $u_{-1} = v_{-1}$ and $u_0 = g_3(0)v_0$ is a minimal solution of (2.2). The sequence $\langle u_n \rangle_n$ is associated to the polynomial continued fraction $\mathrm{K}(a_n/b_n)$ with partial quotients $b_n = g_2(n)$ and $a_n = g_1(n)g_3(n-1)$ for each $n \in \mathbb{N}$. By Theorem 2.8, $\langle u_n \rangle_n$ is a minimal solution to (2.2) if $\mathrm{K}(a_n/b_n)$ converges to the limit $-u_0/u_{-1}$. Thus if one can determine the value of a polynomial continued fraction then one can determine whether $\langle u_n \rangle_n$ is a minimal solution of (2.2). It follows that one can decide whether $\langle v_n \rangle_n$ is a minimal solution of (3.1), as desired.

Conversely, given a polynomial continued fraction $b_0 + \mathrm{K}(a_n/b_n)$ and a real-algebraic number ξ, let us construct the holonomic sequence $\langle u_n \rangle_{n=-1}^\infty$ such that for each $n \in \mathbb{N}$, $u_n = b_n u_{n-1} + a_n u_{n-2}$ with initial conditions $u_{-1} = 1$ and $u_0 = b_0 - \xi$. By Theorem 2.8, sequence $\langle u_n \rangle_n$ is a minimal solution of the recurrence relation if and only if the continued fraction $\mathrm{K}(a_n/b_n)$ converges to the value $-u_0/u_{-1} = \xi - b_0$. So given a holonomic sequence, if one can determine whether the sequence is a minimal solution of the associated recurrence relation then one can test the value of a polynomial continued fraction.

C Complex characteristic roots

We study the recurrence relation (3.1) under the assumptions that the recurrence relation has signature $(+,-)$ and the discriminant $\Delta(n) = g_2(n)^2 + 4g_3(n)g_1(n) < 0$ for each $n \in \mathbb{N}$. Our aim is to establish Proposition 3.4.

Let $\langle u_n \rangle_{n=-1}^\infty$ be a non-trivial solution to recurrence (3.1) and $\langle x_n \rangle_{n=0}^\infty$ be the associated sequence with terms $x_n = u_n/u_{n-1}$ consider the function $f_n \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f_n(x) = g_3(n)x - g_2(n) - g_1(n)/x$. Observe that f_n is continuous and has no real roots since $\Delta(n) < 0$. Furthermore, we have $\lim_{x\to 0^+} f_n(x) = \lim_{x\to\infty} f_n(x) = \infty$. We thus conclude that for each $n \in \mathbb{N}$, f_n is a strictly positive function on $(0,\infty)$.

▶ **Lemma C.1.** *We have that $x_n = x_{n-1} - f_n(x_{n-1})/g_3(n)$ for each $n \geq 1$. Moreover, we have $x_n = x_0 - \sum_{j=1}^n f_j(x_{j-1})/g_3(j)$. Thus, if $\langle x_n \rangle_{n=0}^\infty$ is a positive sequence, then it is strictly decreasing.*

Proof. Substitution shows that $f_n(x_{n-1}) = g_3(n)(x_{n-1} - x_n)$, and further we have

$$x_n - x_0 = \sum_{j=1}^n x_j - x_{j-1} = -\sum_{j=1}^n f_j(x_{j-1})/g_3(j).$$

It is now evident that the sequence $\langle x_n\rangle_{n=0}^\infty$ is strictly decreasing since both $g_3(j)$ and $f_j(x_{j-1})$ are strictly positive for each $j \in \mathbb{N}$ under the assumption that $\langle x_n\rangle_{n=0}^\infty$ is positive. ◀

We define the functions $h_0(x) = f_0(x)$ and $h_\infty(x) = \alpha_1 x - \beta_1 - \gamma_1/x = \lim_{n\to\infty} f_n(x)/n$. Note that $f_n(x) = h_0(x)+nh_\infty(x)$ and the two functions $h_0(x)$ and $h_\infty(x)/x$ are differentiable and non-negative in the domain $\{x \in \mathbb{R} : x > 0\}$. The function $h_0(x)$ has no real roots, and $h_\infty(x)/x$ has a single real root if $\beta_1^2 + 4\alpha_1\gamma_1 = 0$ and no root otherwise. By continuity, it follows that there exists an $\varepsilon_0 > 0$ such that for all $\{x \in \mathbb{R} : x > 0\}$, $h_0(x)/x > \varepsilon_0$.

Proof of Proposition 3.4. Let $\langle u_n\rangle_{n=-1}^\infty$ be a non-trivial positive solution. If there is an $N \in \mathbb{N}$ such that $u_N = 0$ then it is clear that a subsequent term is negative and so we can assume that $u_n > 0$ for each $n \in \{-1, 0, 1, \ldots\}$. Thus for each $n \in \mathbb{N}_0$, $x_n = u_n/u_{n-1} > 0$. Since $h_0(x)$ and $h_\infty(x)$ are both non-negative on the domain $\{x \in \mathbb{R} : x > 0\}$ and, in addition, there exists an $\varepsilon_0 > 0$ such that $h_0(x)/x > \varepsilon_0$, we have that $f_j(x_{j-1}) = h_0(x_{j-1})+jh_\infty(x_{j-1}) > \varepsilon_0$ too. We combine this uniform bound and Lemma C.1 to obtain

$$x_n = x_0 - \sum_{j=1}^{n} \frac{f_j(x_{j-1})}{g_3(j)} \le x_0 - \sum_{j=1}^{n} \frac{\varepsilon_0}{\alpha_1 j + \alpha_0}.$$

Since the harmonic series diverges, we deduce that there exists an $N \in \mathbb{N}$ such that $x_N < 0$, a contradiction. It follows that $\langle u_n\rangle_n$ is not positive. ◀

D Testing the initial ratio

The goal of this section is to prove Proposition 3.10: given a non-minimal solution $\langle u_n\rangle_{n=-1}^\infty$ to recurrence relation (3.1) with a $(+,-)$ signature and positive discriminants, decide if $\langle u_n\rangle_n$ is positive. By Corollary 3.9, an equivalent problem is to decide if $u_0/u_{-1} \ge \mu$. Together Corollary 3.9 and Lemma D.5 determine a computable threshold for the sequence of ratios $\langle x_n\rangle_{n=0}^\infty$ associated with $\langle u_n\rangle_{n=-1}^\infty$ such that $\langle x_n\rangle_n$ crosses this threshold if and only if $u_0/u_{-1} = x_0 > \mu$. Note that an upper bound on the number of steps taken in computing this threshold depends on the distance $|u_0/u_{-1} - \mu|$.

We define the nth characteristic polynomial χ_n for recurrence relation (3.1) as $\chi_n(x) = g_3(n)x^2 - g_2(n)x - g_1(n)$ for each $x \in \mathbb{R}$. In this section we shall assume that the associated sequences of characteristic roots $\langle \lambda_n\rangle_{n=1}^\infty$ and $\langle \Lambda_n\rangle_{n=1}^\infty$ are both real. Let λ_∞ and Λ_∞ be the corresponding limits, if defined, of these sequences[8]. Note that, from the closed form of $\langle \lambda_n\rangle_{n=1}^\infty$ associated to recurrence relations considered in this section, one can observe that the limit λ_∞ is always finite and thus well-defined (which is not the case of the limit for Λ_∞). If $\langle \Lambda_n\rangle_{n=1}^\infty$ diverges, we choose $\Lambda_\infty = +\infty$.

D.1 Monotonicity and the characteristic roots

The threshold described in the opening of this section depends on the monotonicity of the associated sequences of nth characteristic roots.

▶ **Lemma D.1.** *The sequences $\langle \lambda_n\rangle_{n=1}^\infty$ and $\langle \Lambda_n\rangle_{n=1}^\infty$ are eventually monotonic.*

[8] In the case of Poincaré recurrences, λ_∞ and Λ_∞ coincide with the roots λ and Λ of the associated characteristic polynomial.

Proof. Let us define a function $\lambda\colon \mathbb{R} \to \mathbb{R}$ given by

$$\lambda(x) = \frac{g_2(x) - \sqrt{g_2(x)^2 + 4g_1(x)g_3(x)}}{2g_3(x)}.$$

Note that for each $n \in \mathbb{N}$, $\lambda(n) = \lambda_n$. Then we can write the derivative of the function in terms of constants A, B and C (see [25]) as follows:

$$\lambda'(x) = -\frac{g_2(x)C + 2g_3(x)B - C\sqrt{g_2(x)^2 + 4g_3(x)g_1(x)}}{2g_3(x)^2\sqrt{g_2(x)^2 - 4g_3(x)g_1(x)}}$$
$$= -\frac{2(B^2 - AC)}{\sqrt{g_2(x)^2 + g_3(x)g_1(x)}(g_2(x)C + 2g_3(x)B + C\sqrt{g_2(x)^2 + g_3(x)g_1(x)})}.$$

From the above equations, we have the following cases:

- If $C \geq 0$ and $g_2(x)C + 2g_3(x)B \geq 0$, then $\operatorname{sign}(\lambda'(x)) = -\operatorname{sign}(B^2 - AC)$.
- If $C \leq 0$ and $g_2(x)C + 2g_3(x)B \geq 0$, then $\operatorname{sign}(\lambda'(x)) = -1$.
- If $C \geq 0$ and $g_2(x)C + 2g_3(x)B \leq 0$, then $\operatorname{sign}(\lambda'(x)) = 1$.
- If $C \leq 0$ and $g_2(x)C + 2g_3(x)B \leq 0$, then $\operatorname{sign}(\lambda'(x)) = \operatorname{sign}(B^2 - AC)$.

The sign of $g_2(x)C + 2g_3(x)B$ changes at most once. Thus, the sign of λ' is eventually constant and therefore $\lambda\colon \mathbb{R} \to \mathbb{R}$ is eventually monotonic. It follows that $\langle \lambda_n \rangle_{n=1}^{\infty}$ is eventually monotonic. A similar argument proves that $\langle \Lambda_n \rangle_{n=1}^{\infty}$ is eventually monotonic. ◀

D.2 A threshold for positivity

Let $\langle u_n \rangle_{n=-1}^{\infty}$ be a solution sequence of the recurrence with a $(+,-)$ signature (3.1) and let $\langle x_n \rangle_{n=0}^{\infty}$ be its associated sequence of ratios. Without loss of generality, we shall assume that the sequence $\langle \lambda_n \rangle_{n=1}^{\infty}$ is monotonic. We first consider the case where $\langle \lambda_n \rangle_{n=1}^{\infty}$ is decreasing.

▶ **Proposition D.2.** *Suppose that $\langle \lambda_n \rangle_{n=1}^{\infty}$ is decreasing, that there exists a $k \in \mathbb{N}$ such that $x_k \geq \lambda_k$ and that $u_{-1}, u_0, \ldots, u_k > 0$, then $\langle u_n \rangle_{n=-1}^{\infty}$ is positive.*

Proof. From the assumptions and the recurrence relation for $\langle x_n \rangle_{n=0}^{\infty}$, we obtain the following inequalities: $g_3(k+1)x_{k+1} \geq g_2(k+1) + g_1(k+1)/\lambda_k \geq g_3(k+1)\lambda_{k+1}$ and so $x_{k+1} \geq \lambda_{k+1}$. It follows by induction that $x_n \geq \lambda_n > \lambda_\infty$ for all $n > k$. Thus $\langle u_n \rangle_{n=-1}^{\infty}$ is positive. ◀

We obtain a similar threshold for positivity when $\langle \lambda_n \rangle_{n=1}^{\infty}$ is increasing.

▶ **Proposition D.3.** *Suppose that $\langle \lambda_n \rangle_{n=1}^{\infty}$ is increasing, $\lambda_\infty < \Lambda_\infty$, there exists $k \in \mathbb{N}$ such that $x_k \geq \lambda_\infty$ and that $u_{-1}, u_0, \ldots, u_k > 0$, then $\langle u_n \rangle_n$ is positive.*

Proof. We can assume without loss of generality that $\lambda_\infty < \Lambda_n$. As a consequence, $\lambda_\infty \in [\lambda_n, \Lambda_n)$, and so we have $g_3(n)\lambda_\infty \leq g_2(n) + g_1(n)/\lambda_\infty$. From this result and the existence of $k \in \mathbb{N}$ such that $x_k \geq \lambda_\infty$, we have $g_3(k+1)x_{k+1} \geq g_2(k+1) + g_1(k+1)/\lambda_\infty \geq g_3(k+1)\lambda_\infty$ and so $x_{k+1} \geq \lambda_\infty$. It follows by induction that $x_n \geq \lambda_\infty$ for all $n > k$. Thus $\langle u_n \rangle_{n=-1}^{\infty}$ is positive. ◀

The case when we have a single repeated characteristic root $\lambda_\infty = \Lambda_\infty$ is more involved.

▶ **Proposition D.4.** *Suppose that recurrence (3.1) has a single repeated characteristic root. Let us assume that $\langle \lambda_n \rangle_{n=1}^{\infty}$ is increasing, there exists an $k \in \mathbb{N}$ such that $x_k \geq \sqrt{-g_1(k+1)/g_3(k+1)}$ or $x_k \geq g_2(k)/(2g_3(k))$, and $u_{-1}, u_0, \ldots, u_k > 0$. Then $\langle u_n \rangle_{n=-1}^{\infty}$ is positive.*

Proof. Consider the constant $C = \alpha_0\beta_1 - \alpha_1\beta_0$. We start with the case $C < 0$. In this case the sequence with terms given by $g_2(n)/g_3(n)$ is decreasing as

$$\frac{g_2(n+1)}{g_3(n+1)} - \frac{g_2(n)}{g_3(n)} = \frac{C}{g_3(n)g_3(n+1)},$$

and additionally $g_2(n)/(2g_3(n)) \geq \lambda_\infty$. We obtain the following inequalities using our assumption on $k \in \mathbb{N}$:

$$x_{k+1} \geq \frac{g_2(k+1)}{g_3(k+1)} - \frac{\sqrt{-g_3(k+1)g_1(k+1)}}{g_3(k+1)} \geq \frac{g_2(k+1)}{2g_3(k+1)} \geq \lambda_\infty.$$

The result in this case follows similarly to the method outlined in Proposition D.3.

Consider the case $C \geq 0$. Let us show that for all $n \geq k+1$ we have $x_n \geq g_2(n)/(2g_3(n))$. We outline the inductive step of the proof. Suppose that $n \geq k+1$ and assume the inductive hypothesis holds for n. Then

$$x_{n+1} \geq \frac{g_2(n+1)}{2g_3(n+1)} + \frac{g_2(n+1)g_2(n) + 4g_1(n+1)g_3(n)}{2g_2(n)g_3(n+1)} \geq \frac{g_2(n+1)}{2g_3(n+1)} + \frac{\beta_1\beta_0 + 4\alpha_0\gamma_1}{2g_2(n)g_3(n+1)}.$$

As $C \geq 0$, we have that $\beta_0 \geq \beta_1\alpha_0/\alpha_1$. Thus we obtain

$$x_{n+1} \geq \frac{g_2(n+1)}{2g_3(n+1)} + \frac{\beta_1\beta_0 + 4\alpha_0\gamma_1}{2g_2(n)g_3(n+1)} \geq \frac{g_2(n+1)}{2g_3(n+1)},$$

which concludes the induction step. It follows that the sequence $\langle x_n \rangle_n$, and so $\langle u_n \rangle_n$ remains positive. ◀

Let $\langle \mu_n \rangle_{n=0}^\infty$ denote the sequence of ratios associated to a solution to recurrence (3.1) with initial ratio $\mu_0 = \mu$. We have the following:

▶ **Lemma D.5.** *Let $\langle u_n \rangle_{n=-1}^\infty$ be a solution to recurrence (3.1) and $\langle x_n \rangle_{n=0}^\infty$ the sequence of consecutive ratios. Suppose that there exists $\varepsilon > 0$ such that $x_0 > \mu + \varepsilon$. Then for each $n \in \mathbb{N}$, we have the following results.*

1. *If $\langle \lambda_n \rangle_{n=1}^\infty$ is decreasing, then for all $n \in \mathbb{N}$, either $x_n > \mu_n + \varepsilon$ or $x_n \geq \lambda_n$.*
2. *If $\langle \lambda_n \rangle_{n=-1}^\infty$ is increasing, then for all $n \in \mathbb{N}$ one of the following occurs: $x_n > \mu_n + \varepsilon$, $x_n \geq \lambda_\infty$, $x_n \geq g_2(n)/(2g_3(n))$ or $x_n \geq \sqrt{-g_1(n+1)/g_3(n+1)}$.*

Proof.

1. Suppose that $\langle \lambda_n \rangle_{n=1}^\infty$ is decreasing. We proceed by induction. The base case is given by hypothesis. Assume the induction hypothesis holds for $n \in \mathbb{N}$. If $x_n \geq \lambda_n$, then, as in the proof of Proposition D.2, $x_{n+1} \geq \lambda_{n+1}$. Similarly, if $x_n \geq \lambda_{n+1}$, we have

$$g_3(n+1)x_{n+1} = g_2(n+1) + \frac{g_1(n+1)}{x_n} \geq g_2(n+1) + \frac{g_1(n+1)}{\lambda_{n+1}} \geq g_3(n+1)\lambda_{n+1}$$

and so $x_{n+1} \geq \lambda_{n+1}$. Otherwise, we have the following inequalities:

$$x_{n+1} - \mu_{n+1} = \frac{g_1(n+1)}{x_n g_3(n+1)} - \frac{g_1(n+1)}{\mu_n g_3(n+1)} > \frac{-g_1(n+1)\varepsilon}{g_3(n+1)x_n\mu_n} > \frac{-g_1(n+1)\varepsilon}{g_3(n+1)\lambda_{n+1}^2} > \varepsilon.$$

The last inequality holds since $\chi_{n+1}(\sqrt{-g_1(n+1)/g_3(n+1)}) < 0$ when $\Delta(n+1) \geq 0$.

2. Suppose that $\langle \lambda_n \rangle_{n=1}^\infty$ is increasing. The respective inductive proofs for the $x_n \geq \lambda_\infty$, $x_n \geq g_2(n)/2g_3(n)$ and $x_n \geq \sqrt{-g_1(n+1)/g_3(n+1)}$ follow by Propositions D.3 and D.4. Otherwise, as before, we have:

$$x_{n+1} - \mu_{n+1} > \frac{-g_1(n+1)\varepsilon}{g_3(n+1)x_n\mu_n} \geq \varepsilon.$$

The proof is complete. ◀

Proof of Proposition 3.10. Let $\langle y_n \rangle_{n=0}^{\infty}$ denote a sequence of ratios of consecutive terms of a solution to recurrence (3.1). If there exists $n_0 \geq 0$ such that $y_{n_0} < \min(\lambda_\infty, \lambda_{n_0+1})$, then it can be shown that for all $n \geq n_0$, $y_n < \min(\lambda_\infty, \lambda_{n+1})$ and $\langle y_n \rangle_{n=n_0}^{\infty}$ is a decreasing sequence. If $\langle y_n \rangle_{n=0}^{\infty}$ is positive, then it is converging, which is impossible as the only possible limits of such a sequence of ratios are λ_∞ and Λ_∞.

The sequence $\langle \mu_n \rangle_n$ being positive, it thus satisfies for all $n \geq 0$ that $\mu_n \geq \min(\lambda_\infty, \lambda_{n+1})$. It follows from Lemma D.5 that the positivity of a solution sequence $\langle u_n \rangle_n$ and its sequence of consecutive ratios $\langle x_n \rangle_{n=0}^{\infty}$ is determined by one the threshold crossings given in Propositions D.2, D.3 and D.4. One can thus detect whether $x_0 > \mu$ by computing an initial number of terms in the sequence $\langle x_n \rangle_n$. On the one hand, this algorithm is guaranteed to terminate. On the other hand, the number of steps does not have an upper bound independent on the distance between x_0 and μ. ◀

E The Positivity and the Ultimate Positivity Problems

In this section we establish the second statement in Theorem 3.1: that the Positivity and Ultimate Positivity Problems in this setting are interreducible problems. As before we separate our discussion according to the signature of recurrence (3.1). The two degenerate cases that are solved using first-order recurrence relations are discussed in Subsection 2.1. In the following discussion we assume that a given sequence $\langle u_n \rangle_{n=-1}^{\infty}$ satisfying recurrence (3.1) has $u_{-1}, u_0 \neq 0$. Otherwise, if $u_{-1}, u_0 = 0$ then the sequence is trivially positive, and if only one of the initial terms is zero then a suitable shift gives initial terms that are non-zero.

Let us first consider that recurrence (3.1) has signature $(+,-)$ and, without loss of generality, assume that $\operatorname{sign}(\Delta(n))$ is constant. Suppose that $\langle u_n \rangle_{n=-1}^{\infty}$ is a solution of (3.1). We can assume that the initial terms u_{-1} and u_0 have the same sign. For otherwise u_0 and u_1 have the same sign, so one can shift the sequence by one step to obtain this property. As the first two terms have the same sign, the ratio is positive. We can thus rely on the results of Section 3. If the sequence of discriminants is negative, we can deduce from Proposition 3.4 that the sign of $\langle u_n \rangle_{n=-1}^{\infty}$ changes infinitely often and thus there are no positive nor ultimately positive sequences. If the sequence of discriminants is positive, with an initial shift, we can assume that for all $n \in \mathbb{N}, g_2(n)/g_3(n) \geq \max(\lambda_n, \lambda)$ (as λ is finite). We then have that the sequence $\langle u_n \rangle_{n=-1}^{\infty}$ changes sign at most once. Indeed, if there exists $n_0 \geq 0$ such that u_{n_0} does not have the same sign as u_{n_0-1}, then $u_{n_0+1}/u_{n_0} \geq g_2(n_0+1)/g_3(n_0+1)$ and from Proposition D.2, Proposition D.3 and Proposition D.4, this implies that the sequence of ratios will remain positive. As the sign of the sequence changes at most once, a sequence is positive if and only if it is ultimately positive.

Let us now consider the case that (3.1) has signature $(-,+)$. Assume first that $u_{-1}, u_0 > 0$. Then, as shown in Proposition 3.2, the only positive solution sequences are those that are minimal. Moreover, as this holds also for any shift of the sequence, the only ultimately positive sequences are those that are minimal. Now if $u_{-1}, u_0 < 0$, then through similar reasoning, $\langle u_n \rangle_{n=-1}^{\infty}$ is neither positive nor ultimately positive. Now assume that the two initial terms have opposite signs. Without loss of generality, one can assume that $u_0 > 0$. Consider the sequence $\langle v_n \rangle_{n=-1}^{\infty}$ such that for all $n \in \mathbb{N}, v_n = (-1)^n u_n$. This sequence starts with two positive terms and satisfies the recurrence relation

$$g_3(n)v_n = -g_2(n)v_{n-1} + g_1(n)v_{n-2}, \tag{E.1}$$

which has signature $(+,+)$. We conclude that the transformed sequence $\langle v_n\rangle_{n=-1}^{\infty}$ has constant sign, which implies that the signs in sequence $\langle u_n\rangle_{n=-1}^{\infty}$ alternate. Thus $\langle u_n\rangle_{n=-1}^{\infty}$ is neither positive nor ultimately positive.

Let us now consider the case the (3.1) has signature $(+,+)$. Suppose that $\langle u_n\rangle_{n=-1}^{\infty}$ is a solution to recurrence (3.1). If u_{-1} and u_0 have the same sign, then trivially the sequence has constant sign. Assume that u_{-1} and u_0 have opposite signs, let us assume without loss of generality that $u_0 > 0$. Consider the sequence $\langle v_n\rangle_{n=-1}^{\infty}$ such that for each $n \in \mathbb{N}, v_n = (-1)^n u_n$. This sequence starts with two positive terms and satisfies recurrence relation (E.1). As seen earlier, in this case we can detect with either a positivity or an ultimate positivity oracle whether the sequence $\langle v_n\rangle_{n=-1}^{\infty}$ remains positive or if its sign alternates. In the later case, need only determine whether the sign alternates on the even or odd terms to decide whether $\langle u_n\rangle_{n=-1}^{\infty}$ is positive, which can be achieved by computing a finite number of terms.

It is trivial to see that there are no positive nor ultimately positive non-trivial solutions when (3.1) has signature $(-,-)$.

F Interreductions Between Degree-1 Holonomic Sequences

In this short appendix, we prove the remaining cases of Proposition 4.3.

Proof of Proposition 4.3.

1. Follows immediately from the interreductions of the recurrence relations (2.1) and (2.2) described in the preliminaries.
3. We proceed as follows. As $\alpha_1 \neq 0$, we may divide through by α_1 if necessary. We set $\alpha := \alpha_0/\alpha_1$. Next, if $\beta_1 \neq |\gamma_1|$, we consider the sequence $\langle(\operatorname{sign}(\gamma_1)\beta_1/\gamma_1)^n u_n\rangle_n$ instead, as this sequence satisfies the recurrence

$$(n+\alpha)v_n = \operatorname{sign}(\gamma_1)\frac{\beta_1}{\gamma_1}(\beta_1 n + \beta_0)v_{n-1} + \frac{\beta_1^2}{\gamma_1^2}(\gamma_1 n + \gamma_0)v_{n-2}.$$

Clearly minimality is preserved in this translation and $\operatorname{sign}(\operatorname{sign}(\gamma_1)\beta_1^2/\gamma_1) = 1$. Hence the desired result follows.

Assume now that $\beta_1^2 + 4\gamma_1 = 0$ in (4.1). It follows that $\gamma_1 = -\beta_1$ (as $|\gamma_1| = |\beta_1|$), and since $\beta_1(\beta_1 - 4) = 0$ with $\beta_1 \neq 0$, it follows that $\beta_1 = 4$. Now the sequence $\langle(1/2)^n u_n\rangle_n$ satisfies recurrence (4.2) and minimality is clearly preserved in this transformation.
4. Analogous to the first part of the above case.
5. In this case the recurrence admits a minimal solution if and only if $\gamma_1\alpha_1 > 0$. This follows by an application of Theorem 2.10 with $r(n) = 1 + 4\frac{g_1(n)g_3(n-1)}{g_2(n)g_2(n-1)} = 4\frac{\alpha_1\gamma_1}{\beta_0^2}n^2 + o(n^2)$ The reduction to (4.4) then follows by considering the sequence $\langle(\operatorname{sign}(\beta_0)\sqrt{\alpha_1/\gamma_1})^n u_n\rangle_n$. ◀

G Analytic properties of the generating function

G.1 Associated differential equation

We consider a differential equation associated to the recurrence relation (3.1). We assume here that $\deg(a) = 1$. In particular, we have $\alpha_1, \alpha_0 > 0$. By dividing through by α_1, we may take $\alpha_1 = 1$. By shifting, we may further assume that $\alpha := \alpha_0 > 1$. The recurrence relation we consider is thus of the form

$$(n+\alpha)u_n = (\beta_1 n + \beta_0)u_{n-1} + (\gamma_1 n + \gamma_0)u_{n-2}. \qquad \text{(G.1)}$$

We allow here $\beta_1 = 0$ or $\gamma_1 = 0$.

Consider the generating series $\mathcal{F}(x) = \sum_{n=-1}^{\infty} u_n x^{n+\alpha}$. By relation (G.1), we have that

$$\sum_{n=2}^{\infty} g_3(n)u_{n-1}x^{n+\alpha} = \sum_{n=2}^{\infty} g_2(n)u_{n-2}x^{n+\alpha} + \sum_{n=2}^{\infty} g_1(n)u_{n-3}x^{n+\alpha}.$$

Observe now that

$$\sum_{n=2}^{\infty} g_3(n)u_{n-1}x^{n+\alpha} = x\sum_{n=2}^{\infty}(n+\alpha)u_{n-1}x^{n+\alpha-1} = x\mathcal{F}'(x) - (\alpha+1)u_0x^{\alpha+1} - \alpha u_{-1}x^{\alpha}.$$

In a similar fashion one can write

$$\sum_{n=2}^{\infty} g_2(n)u_{n-1}x^{n+\alpha} = \beta_1 x^2\mathcal{F}'(x) + (\beta_0 + (1-\alpha)\beta_1)x\mathcal{F}(x) - (\beta_0+\beta_1)u_{-1}x^{\alpha+1},$$

and

$$\sum_{n=2}^{\infty} g_1(n)u_{n-2}x^{n+\alpha} = \gamma_1 x^3\mathcal{F}'(x) + (\gamma_0 + (2-\alpha)\gamma_1)x^2\mathcal{F}(x).$$

When we combine the above three results we obtain the first-order differential equation $\mathcal{F}'(x) + s(x)\mathcal{F}(x) = t(x)$ with the functions s and t defined in (4.11).

G.2 Proof of Claim 4.12

We will show that

$$\int_0^x f(y)\,dy = \sum_{n<-\nu} \frac{-c_n/\Lambda}{n+\nu}(1-\Lambda x)^{n+\nu} + C_0 + C_1\log(1-\Lambda x) + \mathcal{O}((1-\Lambda x)^{A+1}),$$

where $C_1 = -c_{-\nu}/\Lambda$ if ν is a non-positive integer and $C_1 = 0$ otherwise, and

$$C_0 = \sum_{n<-\nu}\frac{c_n/\Lambda}{n+\nu} + \int_0^{1/\Lambda} f(y) - \sum_{n\le -\nu} c_n(1-\Lambda y)^{n+\nu-1}\,dy.$$

Proof. Let us write $r(y) = \sum_{n\le-\nu} c_n(1-\Lambda y)^{n+\nu-1}$. Then

$$\int_0^x f(y)\,dy = \int_0^{1/\Lambda} f(y) - r(y)\,dy - \int_x^{1/\Lambda} f(y) - r(y)\,dy + \int_0^x r(y)\,dy.$$

We study the three integrals on the right-hand side. The first integral can be written in terms of C_0 as follows:

$$\int_0^{1/\Lambda} f(y) - r(y)\,dy = \int_0^{1/\Lambda} f(y) - \sum_{n\le-\nu} c_n(1-\Lambda y)^{n+\nu-1}\,dy = C_0 - \sum_{n<-\nu}\frac{c_n/\Lambda}{n+\nu}.$$

The integrand in the second integral is analytic in its domain and so, by integrating the power series expansion, we obtain the estimate

$$\int_x^{1/\Lambda} f(y) - r(y)\,dy = \mathcal{O}((1-\Lambda x)^{n_0+\nu}).$$

For the third integral we have

$$\int_0^x r(y)\,dy = \sum_{n\le -\nu}\int_0^x c_n(1-\Lambda y)^{n+\nu-1}\,dy$$
$$= \sum_{n<-\nu}\frac{-c_n/\Lambda}{n+\nu}(1-\Lambda x)^{n+\nu} + C_1\log(1-\Lambda x) + \sum_{n<-\nu}\frac{c_n/\Lambda}{n+\nu}.$$

In the above $C_1 = 0$ if $-\nu \notin \mathbb{N}_0$, otherwise $C_1 = -c_{-\nu}/\Lambda$. Combining these three results gives the desired form. ◀

H Justification for Example 2.3

The aim of this appendix is to establish the claimed asymptotic behaviours of solutions to the recurrence relations in Example 2.3. The proof of this is a straightforward application of the framework given by Kooman in [23], but we give a proof for the sake of completeness.

Recall that the recurrence relations in hand are

$$(n+\alpha)v_n = \beta v_{n-1} + (n+\gamma)v_{n-2}, \quad \text{and} \tag{H.1a}$$
$$(n+\alpha)v'_n = (2n+\beta)v'_{n-1} - (n+\gamma)v'_{n-2}. \tag{H.1b}$$

In the former recurrence, we assume $\beta > 0$, and in the latter we assume $\beta > \alpha+\gamma$.

For the duration of this appendix, the sequence $\langle v_n\rangle_n$ (resp., $\langle v'_n\rangle_n$) always refers to a solution to (H.1a) (resp., (H.1b)). We first describe a minimality preserving transformation to obtain recurrences of a suitable form.

Given a solution $\langle u_n\rangle_{n=-1}^\infty$ to (2.1), we define the sequence $\langle w_n\rangle_{n=-1}^\infty$ so that $w_{-1} = v_{-1}$ and $v_n = w_n\prod_{j=-1}^n \frac{g_2(j)}{2g_3(j)}$ for each $n\in\mathbb{N}_0$. It is easily shown that $\langle v_n\rangle_n$ satisfies recurrence (2.1) if and only if $\langle w_n\rangle_n$ satisfies the following recurrence

$$w_n = 2w_{n-1} + \frac{4g_1(n)g_3(n-1)}{g_2(n-1)g_2(n)}w_{n-2}$$

Let $\langle w_n\rangle_n$ (resp., $\langle w'_n\rangle_n$) be the sequence obtained by applying the above transformation to $\langle v_n\rangle_n$ (resp., $\langle v'_n\rangle_n$). The recurrence relations satisfied by $\langle w_n\rangle_n$ and $\langle w'_n\rangle_n$ take the respective forms

$$w_n = 2w_{n-1} + \frac{4(n+\gamma)(n+\alpha-1)}{\beta^2}w_{n-2} \tag{H.2a}$$
$$w'_n = 2w'_{n-1} - \frac{4(n+\gamma)(n+\alpha-1)}{(2n+\beta)(2n+\beta-2)}w'_{n-2}. \tag{H.2b}$$

Now Kooman's characterisation deals with recurrences of the above form. In order to establish the asymptotic behaviour of solutions to recurrences (H.1a) and (H.1b), it suffices to combine the asymptotic equalities of solutions to (H.2a) and (H.2b) with the asymptotic behaviour of the product $\prod_{j=-1}^n \frac{g_2(j)}{2g_3(j)}$ as $n\to\infty$. Let us first take care of the asymptotics of the latter term.

▶ **Lemma H.1.**
1. *We have* $\prod_{j=-1}^n \frac{\beta}{2(n+\alpha)} \sim C\frac{\beta^n}{2^n n!}n^{-\alpha}$ *for some constant* $C\neq 0$.
2. *We have* $\prod_{j=-1}^n \frac{2n+\beta}{2(n+\alpha)} \sim Cn^{\beta/2-\alpha}$ *for some constant* $C\neq 0$.

Proof. The claims follow quite straightforwardly from the following observations. First, for $a \neq 0$, $an + b \neq 0$ for all $n \in \{-1, 0, \dots\}$ and

$$\prod_{j=-1}^{n}(aj+b) = a^{n+2}\prod_{j=0}^{n+1}(j-1+b/a) = a^{n+2}(b/a-1)_{n+1} = a^{n+2}\frac{\Gamma(b/a+n)}{\Gamma(b/a-1)}.$$

Second, by Stirling's formula, we have $\Gamma(x) \sim \sqrt{2\pi}\mathrm{e}^{-x}x^{x-1/2}$ as $\mathrm{Re}(x) \to \infty$. ◀

We then establish the asymptotic behaviour of solutions to recurrences (H.2a) and (H.2b).

▶ **Lemma H.2.**

1. *Recurrence relation* (H.2a) *admits two linearly independent solutions that have the following asymptotic equalities*

$$w_n \sim (n-1)!(\pm 2/\beta)^n n^{\frac{1}{2}(\pm\beta+\gamma+\alpha)+1}.$$

2. *Recurrence relation* (H.2b) *admits two linearly independent solutions that have the following asymptotic equalities*

$$w'_n \sim n^{1/4+(\alpha-\beta+\gamma)/2}\exp(\pm 2\sqrt{(\beta-\alpha-\gamma)n}).$$

Proof. To apply Kooman's characterisation, we require knowledge of the asymptotic behaviour of the coefficient of w_{n-2} (resp., w'_{n-2}) in the corresponding recurrence relation. In fact, Kooman studies recurrences of the form $x_{n+2} = 2x_{n+1} - \mathfrak{C}_n x_n$. (Notice the signature of this recurrence relation.) So, writing $\mathfrak{C}_{n-2} = -\frac{4g_3(n-1)g_1(n)}{g_2(n)g_2(n-1)}$, we need the knowledge of the terms in the asymptotic expansion of $\mathfrak{C}_n$.

1. We may express $\mathfrak{C}_n = -\frac{4(n+\alpha+1)(n+\gamma+2)}{\beta^2} = -(\frac{2}{\beta})^2 n^2 - (\frac{2}{\beta})^2(\gamma+3+\alpha)n + \mathcal{O}(1)$. Now [23, Ex. 1], case $a = 2$, establishes asymptotic equalities for solutions to this recurrence. (The parameters C and A there are assigned the values $(\frac{2}{\beta})^2$ and $(\frac{2}{\beta})^2(\gamma+3+\alpha)$, respectively.) Recalling that $\beta > 0$, the claimed asymptotic equalities are seen to hold after cancellations.
2. For n large enough, we may express $\mathfrak{C}_n$ as a Laurent series: $\mathfrak{C}_n = \frac{4(n+\alpha+1)(n+\gamma+2)}{(2n+\beta+4)(2n+\beta+2)} = 1 + (\alpha-\beta+\gamma)\frac{1}{n} + \mathcal{O}(n^{-2})$. Now [23, Ex. 1], case $a = -1$, establishes asymptotics for this recurrence, as we assume $\beta - \alpha - \gamma \neq 0$. (Again, the parameter C there is assigned the value $\beta - \alpha - \gamma$). The claimed asymptotics equalities follow. ◀

The asymptotic equalities in Example 2.3 follow from the above two lemmas.

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