Transcendence of Hecke-Mahler Series

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Abstract

We prove transcendence of the Hecke-Mahler series $\sum_{n=0}^{\infty} f(\lfloor n\theta+\alpha\rfloor)\beta^{-n}$, where $f(x)\in\mathbb{Z}[x]$ is a non-constant polynomial, α is a real number, θ is an irrational real number, and β is an algebraic number such that $|\beta|>1$.

1 Introduction

This paper concerns Hecke-Mahler series of the form

$$\sum_{n=0}^{\infty} f(\lfloor n\theta+\alpha\rfloor)\beta^{-n}\,, \tag{1}$$

where $f(x)\in\mathbb{Z}[x]$ is a non-constant polynomial, α and θ are real numbers, and β is an algebraic number such that $|\beta|>1$. It straightforward that (1) is algebraic if θ is rational. Henceforth we assume that θ is irrational and seek to prove transcendence of (1).

Transcendence of (1) in the case $\alpha=0$ was proved by Loxton and Van der Poorten [7, Theorem 7]. Transcendence with α unrestricted and $f(x)=x$ was recently shown by Bugeaud and Laurent [3] and by the present authors [8]. The papers [3, 7] use the Mahler method to prove transcendence, while [8] relies on the p-adic Subspace Theorem. Linear independence over $\overline{\mathbb{Q}}$ of expressions of the form (1), again with $f(x)=x$, is studied in [2, 8]. Meanwhile, Masser [9] proves algebraic independence results for Hecke-Mahler series of the form $\sum_{n=0}^{\infty}\lfloor n\theta\rfloor\beta^{-n}$ in case θ is a quadratic irrational number.

Our main result shows the transcendence of (1) for any non-constant polynomial f and all real α:

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Theorem 1. Let $\theta, \alpha \in \mathbb{R}$, with θ irrational, and let β be an algebraic number with $|\beta| > 1$. Given a non-constant polynomial $f(x) \in \mathbb{Z}[x]$, the Hecke-Mahler series $\sum_{n=0}^{\infty} f(\lfloor n\theta + \alpha \rfloor)\beta^{-n}$ is transcendental.

To prove Theorem 1 we introduce a new combinatorial condition on a sequence of numbers $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ that, via the p-adic Subspace Theorem, entails transcendence of the sum $\sum_{n=0}^{\infty} u_n \beta^{-n}$ for algebraic β with $|\beta| > 1$. This condition is a development of those presented in [5, 8] (which are in turn inspired by [1, 4]), but with significant new elements. satisfies a linear recurrence of a prescribed form.

2 Preliminaries

Let K be a number field of degree d over $\mathbb{Q}$ and let $M(K)$ be the set of *places* of K. We divide $M(K)$ into the collection of *Archimedean places*, which are determined either by an embedding of K in $\mathbb{R}$ or a complex-conjugate pair of embeddings of K in $\mathbb{C}$, and the set of *non-Archimedean places*, which are determined by prime ideals in the ring $\mathcal{O}_K$ of integers of K.

For $a \in K$ and $v \in M(K)$, define the absolute value $|a|_v$ as follows: $|a|_v := |\sigma(a)|^{1/d}$ if v corresponds to a real embedding $\sigma : K \to \mathbb{R}$; $|a|_v := |\sigma(a)|^{2/d}$ if v corresponds to a complex-conjugate pair of embeddings $\sigma, \overline{\sigma} : K \to \mathbb{C}$; and $|a|_v := N(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(a)/d}$ if v corresponds to a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ and $\mathrm{ord}_{\mathfrak{p}}(a)$ is the order to which $\mathfrak{p}$ divides the ideal $a\mathcal{O}$. With the above definitions we have the *product formula*: $\prod_{v \in M(K)} |a|_v = 1$ for all $a \in K^{\times}$. Given a set of places $S \subseteq M(K)$, the ring $\mathcal{O}_S$ of *S-integers* is the subring comprising all $a \in K$ such $|a|_v \le 1$ for all non-Archimedean places $v \notin S$.

For $m \ge 1$ the *absolute Weil height* of the projective point $\boldsymbol{a} = [a_0 : a_1 : \cdots : a_m] \in \mathbb{P}^m(K)$ is

$$H(\boldsymbol{a}) := \prod_{v \in M(K)} \max(|a_0|_v, \ldots, |a_m|_v) .$$

This definition is independent of the choice of the field K containing $a_0, \ldots, a_m$. We define the height $H(a)$ of $a \in K$ to be the height $H([1 : a])$ of the corresponding point in $\mathbb{P}^1(K)$. For a non-zero Laurent polynomial $f = x^n \sum_{i=0}^{m} a_i x^i \in K[x, x^{-1}]$, where $m \ge 1$ and $n \in \mathbb{Z}$, following [6] we define its height $H(f)$ to be the height $H([a_0 : \cdots : a_m])$ of the vector of coefficients.

The following version of the p-adic Subspace Theorem of Schlickewei [10] is one of the main ingredients of our approach.[1]

Theorem 2. Let $S \subseteq M(K)$ be a finite set of places of K that contains all Archimedean places. Let $v_0 \in S$ be a distinguished place and choose a continuation of $|\cdot|_{v_0}$ to $\overline{\mathbb{Q}}$, also denoted $|\cdot|_{v_0}$. Given $m \ge 2$, let $L(x_1, \ldots, x_m)$ be a linear form with algebraic coefficients and let $i_0 \in \{1, \ldots, m\}$ be a distinguished index such that x_{i_0} has non-zero coefficient in L. Then for any $\varepsilon > 0$ the set of solutions $\boldsymbol{a} = (a_1, \ldots, a_m) \in (\mathcal{O}_S)^m$ of the inequality

$$|L(\boldsymbol{a})|_{v_0} \cdot \left(\prod_{\substack{(i,v) \in \{1,\ldots,m\} \times S \\ (i,v) \ne (i_0, v_0)}} |a_i|_v \right) \le H(\boldsymbol{a})^{-\varepsilon}$$

is contained in a finite union of proper linear subspaces of K^m.

[1] We formulate the special case of the Subspace Theorem in which all but one of the linear forms are coordinate variables.

We will also need the following additional proposition about roots of univariate polynomials.

Proposition 3. [6, Proposition 2.3] Let $f \in K[x, x^{-1}]$ be a Laurent polynomial with at most $k+1$ terms. Assume that f can be written as the sum of two polynomials g and h, where every monomial of g has degree at most d_0 and every monomial of h has degree at least d_1. Let β be a root of f that is not a root of unity. If $d_1 - d_0 > \frac{\log(k\,H(f))}{\log H(\beta)}$ then β is a common root of g and h.

3 A Transcendence Condition

In this section we introduce a condition on a sequence $\boldsymbol{u}$ that implies that the number $\sum_{m=0}^{\infty} u_m \beta^{-m}$ is transcendental. Intuitively the condition says that $\boldsymbol{u}$ almost satisfies a linear recurrence.

Example 4. Let the sequence $\boldsymbol{u} = \langle u_m \rangle_{m=0}^{\infty}$ be given by $u_m := \left\lfloor \frac{m+1}{2+\sqrt{2}} \right\rfloor$. Consider also the sequence $\langle r_n \rangle_{n=0}^{\infty} = \langle 1, 3, 7, 17, 41, 99, 239, \ldots \rangle$ of numerators of the convergents of the continued fraction expansion of $\sqrt{2}$. For all $n \in \mathbb{N}$ define the sequence $\boldsymbol{w}_n = \langle w_{n,m} \rangle_{m=0}^{\infty}$ by $w_{n,m} := u_{m+2r_n} - 2u_{m+r_n} - u_m$. Then $\boldsymbol{w}_n$ becomes increasing sparse for successively larger n. For example, for $n = 2$ it holds that $w_{n,m}$ is zero for $m \in \{0, \ldots, 70\} \setminus \{9, 16, 26, 33, 50, 57, 67\}$, for $n = 3$ we have that $w_{n,m}$ is zero for $m \in \{0, \ldots, 70\} \setminus \{23, 40, 64\}$, while for $n = 4$ we have that $w_{n,m}$ is zero for $m \in \{0, \ldots, 70\} \setminus \{57\}$.

The following definition aims to capture the above behaviour. Here, given $f, g : A \to \mathbb{R}_{\geq 0}$, we write $f \ll g$ if there exists a constant $c > 0$ such that $f(a) \leq c g(a)$ for all $a \in A$.

Definition 5. An integer sequence $\boldsymbol{u} = \langle u_m \rangle_{m=0}^{\infty}$ satisfies Condition (*) if there exist $\sigma \geq 0$, $b_0, \ldots, b_\sigma \in \mathbb{Z}$, and an increasing integer sequence $\langle r_n \rangle_{n=0}^{\infty}$ such that, defining for each $n \in \mathbb{N}$ the sequence $\langle w_{n,m} \rangle_{m=0}^{\infty}$ by $w_{n,m} := \sum_{i=0}^{\sigma} b_i u_{m+ir_n}$, the following two properties are satisfied:

1. *Expanding Gaps*: for all $n \in \mathbb{N}$, the set $\Delta_n := \{m : w_{n,m} \neq 0\}$ is infinite and the minimum distance μ_n between any two elements of Δ_n satisfies $\mu_n \gg r_n$.

2. *Polynomial Variation*: there exists $c_0 \geq 0$ such that for all $n \in \mathbb{N}$ and all $m' > m$ in Δ_n, $|w_{n,m'}| \ll (m' - m)^{c_0} + |w_{n,m}|$.

Theorem 6. Let the integer sequence $\boldsymbol{u} = \langle u_m \rangle_{m=0}^{\infty}$ satisfy Condition (*) and be such that $|u_m| \ll m^{c_1}$ for some $c_1 \geq 0$. Then for any algebraic number β such that $|\beta| > 1$, the sum $\alpha := \sum_{m=0}^{\infty} \frac{u_m}{\beta^m}$ is transcendental.

Proof. Suppose that α is algebraic. We will use the Subspace Theorem to obtain a contradiction. Let S comprise all the Archimedean places of $\mathbb{Q}(\beta)$ and all non-Archimedean places corresponding to prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(\beta)}$ such that $\mathrm{ord}_{\mathfrak{p}}(\beta) \neq 0$. Let $v_0 \in S$ be the place corresponding to the inclusion of $\mathbb{Q}(\beta)$ in $\mathbb{C}$. Recall that $|\cdot|_{v_0} = |\cdot|^{1/\deg(\beta)}$, where $|\cdot|$ denotes the usual absolute value on $\mathbb{C}$. Let $\kappa \geq 2$ be an upper bound of $|\beta|_v$ for all $v \in S$.

By the assumption that $\boldsymbol{u}$ satisfies Condition (*), there is an integer sequence $\langle r_n \rangle_{n=0}^{\infty}$ and $b_0, \ldots, b_\sigma \in \mathbb{Z}$ such that the family of sequences $\langle w_{n,m} \rangle_{m=0}^{\infty}$ defined by $w_{n,m} := \sum_{i=0}^{\sigma} b_i u_{m+ir_n}$ satisfies the Expanding-Gaps and Polynomial-Variation conditions from Definition 5.

Define $\rho := 2\sigma |S| \deg(\beta) \frac{\log \kappa}{\log |\beta|}$. For each $n \in \mathbb{N}$, let $0 \leq m_{n,1} < m_{n,2} < \cdots$ be an increasing enumeration of $\{m : w_{n,m} \neq 0\}$. Since $\mu_n \gg r_n$, we may define $\delta \geq 1$ to be least such that $m_{n,\delta} > \rho r_n$ for infinitely many $n \in \mathbb{N}$. We also define $s_n := m_{n,\delta+1} - 1$ for all $n \in \mathbb{N}$ and note that $s_n - m_{n,\delta} \gg r_n$.

For $n \in \mathbb{N}$, define $\boldsymbol{a}_n := (a_{n,0}, \ldots, a_{n,\sigma+\delta+1}) \in (\mathcal{O}_S)^{\sigma+\delta+2}$ by

$$\begin{array}{rcll} a_{n,i} & := & \beta^{ir_n} \quad \text{for } i \in \{0,\ldots,\sigma\} & \qquad a_{n,\sigma+1} := \sum_{i=0}^{\sigma}\sum_{m=0}^{ir_n-1} b_i u_m \beta^{ir_n-m} \\ a_{n,\sigma+1+j} & := & w_{n,m_{n,j}}\,\beta^{-m_{n,j}} \quad \text{for } j \in \{1,\ldots,\delta\}\,. & \end{array} \tag{2}$$

and consider the linear form

$$L(x_0,\ldots,x_{\sigma+\delta+1}) := \alpha\sum_{i=0}^{\sigma} b_i x_i - \sum_{i=\sigma+1}^{\sigma+\delta+1} x_i\,.$$

Then for all $n \in \mathbb{N}$ we have

$$L(\boldsymbol{a}_n) = \sum_{i=0}^{\sigma}\sum_{m=0}^{\infty} b_i u_m \beta^{ir_n-m} - \sum_{i=0}^{\sigma}\sum_{m=0}^{ir_n-1} b_i u_m \beta^{ir_n-m} - \sum_{i=1}^{\delta} w_{n,m_{n,i}}\,\beta^{-m_{n,i}} = \sum_{m=s_n+1}^{\infty} w_{n,m}\beta^{-m} \tag{3}$$

By the assumption that $|u_m| \ll m^{c_1}$, there exists $c_2 \geq 0$ such that $|w_{n,m}| \ll (m+r_n)^{c_2}$. By (3),

$$|L(\boldsymbol{a}_n)| = \left|\sum_{m=s_n+1}^{\infty} w_{n,m}\beta^{-m}\right| \ll \sum_{m=s_n+1}^{\infty}(m+r_n)^{c_2}|\beta|^{-m} \ll s_n^{c_2}\,|\beta|^{-s_n}\,. \tag{4}$$

For $v \in S$, recalling that $|\beta|_v \leq \kappa$, there is a constant c_3 such that

$$|a_{n,\sigma+1}|_v \ll r_n^{c_3}\kappa^{\sigma r_n}\,. \tag{5}$$

By the product formula we have $\prod_{v\in S}|a_{n,i}|_v = 1$ for $i \in \{0,\ldots,\sigma\}$ and

$$\prod_{j=1}^{\delta}\prod_{v\in S}|a_{n,\sigma+1+j}|_v = \prod_{j=1}^{\delta}|w_{n,m_{n,j}}| \ll s_n^{c_2\delta}\,. \tag{6}$$

The bounds (4), (5), and (6), imply that there is a constant c_4 such that for all n,

$$\begin{array}{rcl} |L(\boldsymbol{a}_n)|_{v_0}\cdot\prod_{\substack{(i,v)\in\{0,\ldots,\sigma+\delta+1\}\times S \\ (i,v)\neq(\sigma+1,v_0)}} |a_{i,n}|_v & \leq & \kappa^{\sigma r_n|S|} s_n^{c_4}\,|\beta|^{-s_n/\deg(\beta)} \\ & \leq & s_n^{c_4}|\beta|^{-s_n/2\deg(\beta)}\,, \end{array} \tag{7}$$

where the second inequality follows from the fact that, since $s_n \geq \rho r_n$ and $\rho = 2\sigma|S|\deg(\beta)\frac{\log\kappa}{\log|\beta|}$, we have $\kappa^{\sigma r_n|S|} = |\beta|^{\rho r_n/2\deg(\beta)} \leq |\beta|^{s_n/2\deg(\beta)}$. On the other hand, there exists a constant $c_5 > 0$ such that the height of $\boldsymbol{a}_n$ satisfies $H(\boldsymbol{a}_n) \ll |\beta|^{c_5 s_n}$. Thus there exists $\varepsilon > 0$ such that the right-hand side of (7) is at most $H(\boldsymbol{a}_n)^{-\varepsilon}$ for n sufficiently large. We can therefore apply Theorem 2 to obtain a non-zero linear form $F(x_0,\ldots,x_{\sigma+\delta+1})$, with coefficients in $\overline{\mathbb{Q}}$, such that $F(\boldsymbol{a}_n) = 0$ for infinitely many n.

By Claim 7, below, the support of F contains variable $x_{\sigma+1}$ but omits variable $x_{\sigma+\delta+1}$. Thus, by subtracting a suitable multiple of F from L we obtain a linear form L' whose support includes $x_{\sigma+\delta+1}$ but not $x_{\sigma+1}$ and such that $L'(\boldsymbol{a}_n) = L(\boldsymbol{a}_n)$ for infinitely many n.

By Claim 7(i) we have $|L'(\boldsymbol{a}_n)| \gg |a_{n,\sigma+\delta+1}| = \left|w_{n,m_{n,\delta}}\beta^{-m_{n,\delta}}\right|$. Expanding $L'(\boldsymbol{a}_n) = L(\boldsymbol{a}_n)$ as in (3), for infinitely many n we have

$$\left|w_{n,m_{n,\delta}}\beta^{-m_{n,\delta}}\right| \ll \left|L'(\boldsymbol{a}_n)\right| = \left|\sum_{j=\delta+1}^{\infty} w_{n,m_{n,j}}\beta^{-m_{n,j}}\right|. \tag{8}$$

The Polynomial-Variation Condition gives $|w_{n,m_{n,j}}| \ll (m_{n,j}-m_{n,\delta})^{c_0} + |w_{n,m_{n,\delta}}|$ for all $j \geq \delta+1$. Applying this upper bound to the right-hand sum in (8) and dividing through by $|w_{n,m_{n,\delta}}|$, we obtain

$$\left|\beta^{-m_{n,\delta}}\right| \ll \left|(m_{n,\delta+1}-m_{n,\delta})^{c_0}\beta^{-m_{n,\delta+1}}\right| .$$

But this contradicts the fact that $m_{n,\delta+1}-m_{n,\delta} \to \infty$ as $n \to \infty$. □

Claim 7. Let $F(x_0,\ldots,x_{\sigma+\delta+1}) = \sum_{i=0}^{\sigma+\delta+1} \alpha_i x_i$ be a non-zero linear form with coefficients in $\overline{\mathbb{Q}}$. We have (i) if $\alpha_{\sigma+1}=0$ then $|F(\boldsymbol{a}_n)| \gg |a_{n,\sigma+\delta+1}|$; (ii) if $F(\boldsymbol{a}_n)=0$ for infinitely many n, then $\alpha_{\sigma+\delta+1}=0$.

Proof. Let $i,j \in \{1,\ldots,\delta\}$ be such that $i<j$. By the Polynomial-Variation Condition, for all n,

$$\left|\frac{a_{n,\sigma+1+j}}{a_{n,\sigma+1+i}}\right| = \left|\frac{w_{n,m_{n,j}}}{w_{n,m_{n,i}}}\right| |\beta|^{-(m_{n,j}-m_{n,i})} \ll (m_{n,j}-m_{n,i})^{c_0}|\beta|^{-(m_{n,j}-m_{n,i})} .$$

But as $n \to \infty$ we have $m_{n,j}-m_{n,i} \to \infty$ and hence $\left|\frac{a_{n,\sigma+1+j}}{a_{n,\sigma+1+i}}\right| \to 0$. Recalling that $a_{n,i} = \beta^{ir_n}$ for $i \in \{0,\ldots,\sigma\}$, we can say more generally that $\lim_{n\to\infty}\frac{|a_{n,j}|}{|a_{n,i}|}=0$ for $i,j \in \{0,\ldots,\sigma+\delta+1\} \setminus \{\sigma+1\}$ with $i<j$. Item (i) immediately follows.

We next show Item (ii). For all $n \in \mathbb{N}$, by (2) we have $F(\boldsymbol{a}_n) = P_n(\beta)$ for the polynomial $P_n(x) \in \overline{\mathbb{Q}}[x,x^{-1}]$ defined by

$$P_n(x) := \sum_{i=0}^{\sigma} \alpha_i x^{ir_n} + \alpha_{\sigma+1}\sum_{i=0}^{\sigma}\sum_{m=0}^{ir_n-1} b_i u_m x^{ir_n-m} + \sum_{j=1}^{\delta} \alpha_{\sigma+1+j} w_{n,m_{n,j}} x^{-m_{n,j}} .$$

Suppose that $P_n(\beta)=0$ for infinitely many n. We will apply Proposition 3 to show that $\alpha_{\sigma+\delta+1}=0$. To this end, write $Q_n := \alpha_{\sigma+\delta+1}x^{-m_{n,\delta}}$ and $R_n := P_n - \alpha_{\sigma+\delta+1}x^{-m_{n,\delta}}$. Then $P_n = Q_n + R_n$, every monomial in Q_n has degree at most $-m_{n,\delta}$, and every monomial in R_n has degree at least $-m_{n,\delta-1}$. By the Expanding-Gaps condition we have $m_{n,\delta}-m_{n,\delta-1} \gg r_n$. Moreover P_n has at most $k_n := (\sigma+1)r_n+\delta \ll m_{n,\delta}-m_{n,\delta-1}$ monomials. Next, from the bounds $|u_m| \ll m^{c_1}$ and $|w_{n,m}| \ll (m+r_n)^{c_1}$, established in the proof of Theorem 6, we see that P_n has height bounded polynomially in $m_{n,\delta}$. But $m_{n,\delta} = m_{n,\delta}-m_{n,\delta-1}+m_{n,\delta-1} \ll m_{n,\delta}-m_{n,\delta-1}$, since $m_{n,\delta-1} \leq \rho r_n \ll m_{n,\delta}-m_{n,\delta-1}$. We thus have $m_{n,\delta}-m_{n,\delta-1} > \frac{\log(k_n H(P_n))}{\log H(\beta)}$ provided that n is sufficiently large. Since $P_n(\beta)=0$, applying Proposition 3 we have $Q_n(\beta)=0$ and hence $\alpha_{\sigma+\delta+1}=0$. □

4 Hecke-Mahler Series

Write $I := [0,1)$ for the unit interval. Every $r \in \mathbb{R}$ can be written uniquely in the form $r = \lfloor r \rfloor + \{r\}$, where $\lfloor r \rfloor \in \mathbb{Z}$ is the *integer part* of r and $\{r\} \in I$ is the *fractional part* of r. Write also $\|r\|$ for the distance of r to the nearest integer. Let $0<\theta<1$ be an irrational number and define the *rotation map* $R = R_\theta : I \to I$ by $R(r) = \{r+\theta\}$.

Write $[a_0,a_1,a_2,a_3,\ldots]$ for the simple continued-fraction expansion of θ. Given $n \in \mathbb{N}$, we write $p_n/q_n := [a_0,a_1,\ldots,a_n]$ for the n-th convergent. It is well known that for all $n \in \mathbb{N}$ we have

$$\frac{1}{(a_{n+1}+2)q_n} < |q_n\theta - p_n| < \frac{1}{a_{n+1}q_n} . \tag{9}$$

We moreover have the *law of best approximation*: $q \in \mathbb{N}$ occurs as one of the q_n just in case $\|q\theta\| < \|q'\theta\|$ for all q' with $0<q'<q$.

Theorem 8. Let $\theta, \alpha \in (0,1)$ with θ irrational. Given a non-constant polynomial $f(x) \in \mathbb{Z}[x]$, the series $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ given by $u_n := f(\lfloor n\theta + \alpha \rfloor)$ satisfies Condition (*).

Proof. Referring to Definition 5, we define the sequence $\langle r_n \rangle_{n=0}^{\infty}$ via the continued-fraction expansion $[a_0, a_1, a_2, a_3, \ldots]$ of θ. If the expansion is unbounded, then choose $\ell_1 < \ell_2 < \cdots$, either all odd or all even, such that $a_{\ell_n+1} \geq a_m$ for all $n \in \mathbb{N}$ and all $m \leq \ell_n$. If the expansion is bounded, then choose $\ell_1 < \ell_2 < \cdots$ to be the even natural numbers. In either case, there exists a constant $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ and $m \leq \ell_n$, we have $\frac{a_{\ell_n+1}}{a_{m+1}+2} \geq \varepsilon$. Now define r_n to be the denominator q_{ℓ_n} of the ℓ_n-th convergent. Since the r_n all have the same parity, we have either that $\|r_n\theta\| = \{r_n\theta\}$ for all n or $\|r_n\theta\| = 1 - \{r_n\theta\}$ for all n. We assume the former case; the reasoning in the latter case requires minor modifications.

Let $\sigma := \deg(f) + 1$ and define the sequence $\boldsymbol{w}_n = \langle w_{n,m} \rangle_{m=0}^{\infty}$ by $w_{n,m} := \sum_{k=0}^{\sigma} b_k u_{m+kr_n}$ where $b_k := (-1)^k \binom{\sigma}{k}$ for $k \in \{0, \ldots, \sigma\}$. We rely on the following two claims, whose proofs are given below.

Claim 9. For $0 \leq q < r_n$, if $\|q\theta\| < \sigma\|r_n\theta\|$ then $q \geq \frac{\varepsilon}{\sigma} r_n$.

Claim 10. Let $n \in \mathbb{N}$ be sufficiently large that $\sigma\{r_n\theta\} < 1$. Then there exist $M \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for all $m \in \mathbb{N}$,

1. if $\{m\theta + \alpha\} + \{\sigma r_n \theta\} < 1$, then $w_{n,m} = 0$;
2. if $m > Mr_n$ and $\{m\theta + \alpha\} + \{\ell r_n\theta\} < 1 < \{m\theta + \alpha\} + \{(\ell+1) r_n \theta\}$ for some $\ell \in \{0, \ldots, \sigma - 1\}$, then $\varepsilon_1 m^{\sigma-2} \leq |w_{n,m}| \leq \varepsilon_2 m^{\sigma-2}$.

For $n \in \mathbb{N}$ let $0 \leq m_{n,1} < m_{n,2} <$ be an increasing enumeration of $\Delta_n := \{m : w_{n,m} \neq 0\}$. Note that by Item 2 of Claim 10 and equidistribution of the sequence $\{m\theta + \alpha\}$, for all n the set Δ_n is infinite. Moreover, by Item 1 of Claim 10, for $j \in \{1, 2, \ldots\}$ we have $\|m_{n,j}\theta + \alpha\| < \sigma\|r_n\theta\|$ and $\|m_{n,j+1}\theta + \alpha\| < \sigma\|r_n\theta\|$ and hence $\|(m_{n,j+1} - m_{n,j})\theta\| < \sigma\|r_n\theta\|$. By Claim 9 it follows that $m_{n,j+1} - m_{n,j} \geq \varepsilon r_n \sigma^{-1}$, which establishes the Expanding-Gaps condition in Definition 5.

It remains to show the Polynomial-Variation property: there exists $c_0 \geq 0$ such that for all $m' > m$ in Δ_n we have $|w_{n,m'}| \ll (m' - m)^{c_0} + |w_{n,m}|$. We rely on Item 2 of Claim 10. There are two cases. First suppose that $m > Mr_n$ for M as in the claim. Then

$$|w_{n,m'}| \ll (m')^{\sigma-2} = (m' - m + m)^{\sigma-2} \ll (m' - m)^{\sigma-2} + m^{\sigma-2} \ll (m' - m)^{\sigma-2} + |w_{n,m}| \,.$$

The second case is that $m \leq Mr_n$. Since $m' - m \geq \varepsilon\sigma^{-1} r_n$ we have $m' - m \geq \frac{\varepsilon}{\varepsilon + \sigma M} m'$. From the fact that $|w_{n,m'}| \ll (m' + r_n)^{c_0}$ for some $c_0 \geq 0$, it follows that $|w_{n,m'}| \ll (m' - m)^{c_0}$. ☐

It remains to prove the two claims in the body of the proof of Theorem 8.

Proof of Claim 9. Assume that $\|q\theta\| < \sigma\|r_n\theta\|$. Choose the largest m such that $q_m \leq q$. By the law of best approximation we have $\|q_m\theta\| \leq \|q\theta\| < \sigma\|r_n\theta\| = \sigma\|q_{\ell_n}\theta\|$. Then (9) gives

$$\frac{1}{(a_{m+1} + 2)q_m} \leq \|q_m\theta\| \leq \sigma\|q_{\ell_n}\theta\| \leq \frac{\sigma}{a_{\ell_n+1} q_{\ell_n}}$$

and hence $q_m \geq \frac{a_{\ell_n+1} q_{\ell_n}}{\sigma(a_{m+1}+2)}$. We also have $m < \ell_n$, since $q_m \leq q < r_n = q_{\ell_n}$; thus, by the defining property of ℓ_n, we have $\frac{a_{\ell_n+1}}{\sigma(a_{m+1}+2)} \geq \frac{\varepsilon}{\sigma}$. Combining the two previous bounds gives

$$q \geq q_m \geq \frac{a_{\ell_n+1} q_{\ell_n}}{\sigma(a_{m+1} + 2)} \geq \frac{\varepsilon}{\sigma} q_{\ell_n} = \frac{\varepsilon}{\sigma} r_n \,.$$

This concluses the proof. ☐

Proof of Claim 10. Given $y \in \mathbb{Z}$, define the difference operator $\Delta_y : \mathbb{Z}[x] \to \mathbb{Z}[x]$ by $\Delta_y(g)(x) = g(x) - g(x+y)$. Since $\Delta_y(g)$ has degree strictly less than that of g, we have that $(\Delta_y)^\sigma(f)(x) = \sum_{k=0}^{\sigma} b_k f(x+yk)$ is identically zero. We now prove the two items of the claim.

1. If $\{m\theta+\alpha\} + \{\sigma r_n \theta\} < 1$, then

$$\begin{aligned} w_{n,m} &= \sum_{k=0}^{\sigma} b_k f(\lfloor (m+kr_n)\theta+\alpha \rfloor) \\ &= \sum_{k=0}^{\sigma} b_k f(\lfloor m\theta+\alpha \rfloor + \lfloor kr_n\theta \rfloor) \\ &= \sum_{k=0}^{\sigma} b_k f(\lfloor m\theta+\alpha \rfloor + k\lfloor r_n\theta \rfloor) \\ &= 0\,. \end{aligned}$$

2. If $\{m\theta+\alpha\} + \{\ell r_n\theta\} < 1 < \{m\theta+\alpha\} + \{(\ell+1) r_n\theta\}$ for some $\ell \in \{0,\dots,\sigma-1\}$, then

$$\begin{aligned} w_{n,m} &= \sum_{k=0}^{\sigma} b_k f(\lfloor (m+kr_n)\theta+\alpha \rfloor) \\ &= \sum_{k=0}^{\ell} b_k f(\lfloor m\theta+\alpha \rfloor + k\lfloor r_n\theta \rfloor) + \sum_{k=\ell+1}^{\sigma} b_k f(\lfloor m\theta+\alpha \rfloor + 1 + k\lfloor r_n\theta \rfloor) \\ &= \sum_{k=0}^{\ell} b_k (f(\lfloor m\theta+\alpha \rfloor + k\lfloor r_n\theta \rfloor) - f(\lfloor m\theta+\alpha \rfloor + 1 + k\lfloor r_n\theta \rfloor))\,. \end{aligned}$$

Define $p \in \mathbb{Z}[x,y]$ by $p(x,y) := \sum_{k=0}^{\ell} b_k(f(x+ky) - f(x+1+ky))$. The equation above can be written $w_{n,m} = p(\lfloor m\theta+\alpha \rfloor, \lfloor r_n\theta \rfloor)$. Since f has degree $\sigma-1$, p has total degree at most $\sigma-2$. (Note that $\sigma \geq 2$ since f is not constant.) By direct calculation, the coefficient of $x^{\sigma-2}$ in p is the product of the leading coefficient of f and

$$(1-\sigma)\sum_{k=0}^{\ell} b_k = (1-\sigma)\sum_{k=0}^{\ell}(-1)^k\binom{\sigma}{k} = (-1)^\ell(1-\sigma)\binom{\sigma-1}{\ell} \neq 0\,.$$

Thus for M suitably large, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $x \geq My$ implies $\varepsilon_0 x^{\sigma-2} \leq |p(x,y)| \leq \varepsilon_2 x^{\sigma-2}$. The claim follows. ☐

Combining Theorems 6 and 8 we obtain our main result, Theorem 1.

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