


Bounding the Escape Time of a Linear Dynamical System over a Compact Semialgebraic Set

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Abstract

We study the Escape Problem for discrete-time linear dynamical systems over compact semialgebraic sets. We establish a uniform upper bound on the number of iterations it takes for every orbit of a rational matrix to escape a compact semialgebraic set defined over rational data. Our bound is doubly exponential in the ambient dimension, singly exponential in the degrees of the polynomials used to define the semialgebraic set, and singly exponential in the bitsize of the coefficients of these polynomials and the bitsize of the matrix entries. We show that our bound is tight by providing a matching lower bound.

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1 Introduction

An *invariant set* of a dynamical system is a set K such that every trajectory that starts in K remains in K . Dually, an *escape set* K is one such that every trajectory that starts in K eventually leaves K (either temporarily or permanently). While it is usually straightforward to establish that a given set K is invariant, it can be challenging to decide whether it is an escape set. Indeed, while the former problem amounts to showing that K is closed under the transition function, the latter potentially involves considering entire orbits. In particular, even in case K has a finite escape time (the maximum number of steps for an orbit to escape the set), it can be highly non-trivial to establish an explicit upper bound on the escape time.

In this paper we focus on escape sets for (discrete-time) linear dynamical systems. Given a rational matrix $A \in \mathbb{Q}^{n \times n}$ we say that $K \subseteq \mathbb{R}^n$ is an escape set for A if for all points $x \in K$, there exists $t \in \mathbb{N}$ such that $A^t x \notin K$. The *compact escape problem (CEP)* asks to decide whether a given compact semialgebraic set K is an escape set for a given matrix A . Decidability of CEP was shown in [17] and its computational complexity was characterised

in [9] as being irreducible with the decision problem for a certain fragment of the theory of real closed fields.

The present paper focusses exclusively on positive instances (A, K) of CEP, that is, we assume that we are given a compact semialgebraic escape set for a linear dynamical system. In this situation it turns out, due to compactness of K , that there exists a finite time T such that for all $x \in K$ there exists $t \leq T$ with $A^t x \notin K$. The least such T is called the *escape time* of (A, K) . Our main result (Theorem 1, shown below) gives an explicit upper bound on the escape time of (A, K) as a function of the length of the description of the matrix A and semialgebraic set K . In general, it is recognised that bounded liveness is a more useful property than mere liveness. Theorem 1 can be used to establish bounded liveness of several kinds of systems. For example, the result gives an upper bound on the termination time of a single-path linear loop with compact guard (cf. [22, 5]); it also gives a bound on the number of steps to remain in a particular control location of a hybrid system before a given (compact) state invariant becomes false, forcing a transition.

We next introduce some terminology to formalise our main contribution. We say that a semialgebraic set S has complexity at most (n, d, τ) if it can be expressed by a boolean combination of polynomial equations and inequalities $P(x_1, \dots, x_n) \bowtie 0$ with $\bowtie \in \{\leq, =\}$, involving polynomials $P \in \mathbb{Z}[x_1, \dots, x_n]$ in at most n variables of total degree at most d with integer coefficients bounded in bitsize by τ . Our main result is as follows:

► **Theorem 1.** *There exists an integer function $\text{CompactEscape}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}$ with the following property. If $K \subseteq \mathbb{R}^n$ is a compact semialgebraic set of complexity at most (n, d, τ) that is an escape set for a matrix $A \in \mathbb{Q}^{n \times n}$ with entries of bitsize at most τ , then the escape time of K is bounded by $\text{CompactEscape}(n, d, \tau)$.*

As explained in the proof sketch below, Theorem 1 relies on the availability of certain quantitative bounds within semialgebraic geometry and number theory, particularly concerning quantifier elimination and Diophantine approximation. The latter results are crucial to handling the case in which the matrix A has complex eigenvalues of absolute value one.

Note that the upper bound on the escape time in Theorem 1 is singly exponential in the degrees and the bitsize of the coefficients of the polynomials used to define K and the bitsize of the coefficients of A . It is doubly exponential in the dimension. In Section 8 we provide two examples, one where A is an isometry and another in which all eigenvalues of A have absolute value strictly greater than one, that yield a corresponding lower bound of this form. It is moreover straightforward to give examples of non-compact escape sets for which the escape time is infinite.

Proof Overview. Let us now give a high-level overview of the proof of Theorem 1. As in the statement of the theorem, let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set of complexity at most (n, d, τ) and let $A \in \mathbb{Q}^{n \times n}$ be a matrix with entries of bitsize bounded by τ , and such that for all $x \in K$ there exists $t \in \mathbb{N}$ such that $A^t x \notin K$.

To facilitate the analysis of the dynamical behaviour of A we first transform our system into real Jordan normal form. A theorem of Cai [6] ensures that this step does not significantly increase the complexity of the system.

The dynamics of A naturally decomposes into a rotational part, corresponding to eigenvalues of modulus one, and an expansive or contractive part, corresponding to eigenvalues of absolute value different from 1 and to generalised eigenvalues of arbitrary moduli. Accordingly, the ambient space \mathbb{R}^n decomposes into two subspaces V_{rec} and $V_{\text{non-rec}}$, such that A exhibits rotational behaviour on V_{rec} and expansive or contractive behaviour on $V_{\text{non-rec}}$. We start by considering the special cases where either $V_{\text{rec}} = 0$ or $V_{\text{non-rec}} = 0$, so that only one of the two types of behaviours occurs.

First, assume that A has no complex eigenvalues of modulus 1. Since every trajectory under A escapes K we have in particular that $0 \notin K$. A theorem due to Jeronimo, Perrucci and Tsigaridas [14] shows that K is bounded away from zero by a function of the form $2^{-(d\tau)^{n^{O(1)}}}$ and a theorem due to Vorobjov [23] establishes an upper bound on the absolute value of every coordinate of every point in K of the form $2^{(d\tau)^{n^{O(1)}}}$. Furthermore, thanks to a result of Mignotte [16], we can bound the eigenvalues of A away from 1 by a function of the form $2^{\tau^{n^{O(1)}}}$. This yields a doubly exponential bound on how long it takes for A to leave the set K (either by converging to 0 or by converging to infinity in some eigenspace).

Now assume that all eigenvalues of A have modulus 1. This case is handled through a combination of two bounds. For the first bound we start by noting that for every $x \in K$ the closure of the orbit $\overline{\mathcal{O}_A(x)}$ is a compact semialgebraic set that is not entirely contained within K . In fact we show that for all $x \in K$ there exists a point $y \in \overline{\mathcal{O}_A(x)}$ whose distance to K is at least $2^{-(d\tau)^{n^{O(1)}}}$. This bound is achieved by applying [14, Theorem 1] to a suitable polynomial on an auxiliary semialgebraic set, which is constructed using quantifier elimination. The singly exponential bounds obtained in [13, 19] are crucial for this step to work. The second step of the argument combines Baker’s theorem on linear forms in logarithms with a quantitative version of Kronecker’s theorem on simultaneous Diophantine approximation to obtain a bound of the form $N_P \in 2^{(\tau P)^{n^{O(1)}}}$ such that for all positive integers P every point $z \in \overline{\mathcal{O}_A(x)}$ is within 2^{-P} of a point of the form $A^t x$ with $0 \leq t \leq N_P$. Combining the two bounds described above, we obtain a doubly exponential bound on the escape time.

In the presence of both types of behaviour, the analysis of each case becomes more involved. We select a parameter $\varepsilon > 0$ and partition K into three sets: $K_{\text{rec}} = K \cap V_{\text{rec}}$, $K_{\geq \varepsilon}$, and $K_{< \varepsilon}$. The matrix A exhibits purely rotational behaviour on K_{rec} . Intuitively, on $K_{\geq \varepsilon}$ the expansive or contractive behaviour of A dominates the overall dynamics, while on $K_{< \varepsilon}$ the rotational behaviour dominates the overall dynamics. We establish in Lemma 14 a bound N_{rec} such that for each initial point $x \in V_{\text{rec}}$, one of its first N_{rec} iterates is bounded away from K . In Lemma 15 we establish a bound $N_{\geq \varepsilon}$ such that every $x \in K_{\geq \varepsilon}$ either escapes or enters $K_{< \varepsilon} \cup K_{\text{rec}}$ within at most $N_{\geq \varepsilon}$ iterations. Finally, in Section 7, we establish a bound on how often the system can switch from a state where rotational behaviour dominates to one where expansive or non-expansive behaviour does and vice versa. We use this to combine the two bounds to an overall bound on the escape time, proving Theorem 1.

Main Contributions. While decidability of CEP was already established in [17], the proof given there was non-effective, combining two unbounded searches. To obtain a uniform quantitative bound on the escape time, the argument given in [17] needs to be refined and extended in two significant ways:

Firstly, one needs to establish non-trivial quantitative refinements of the techniques used in the decidability proof: to bound the escape time for purely expanding or retracting systems, we need to combine the sharp effective bounds on compact semialgebraic sets from real algebraic geometry established in [23, 14] with Mignotte’s root separation bound [16]. The case of purely rotational systems requires an original combination of a quantitative version of Kronecker’s theorem on simultaneous Diophantine approximation [11] and a quantitative version of Baker’s theorem on linear forms in logarithms [1]. All of these techniques were completely absent from the decidability proof.

Secondly, to establish mere decidability of the problem, it was possible to study the possible behaviours of the system – rotating, expanding, or retracting – in isolation. For example, if the set K contains a point which has a non-zero component in an eigenspace

of A for an eigenvalue whose modulus is strictly greater than one, then the system must eventually escape. However, no uniform bound on the escape time may be derived in this situation, for the component is allowed to be arbitrarily close to zero. Therefore, as outlined above, it is necessary in our proof to subdivide K into pieces where rotational, retractive, and expansive behaviour can be present simultaneously. The interaction of the three behaviours significantly increases the difficulty of the analysis and requires completely new ideas.

2 Mathematical Tools

We use the following singly exponential quantifier elimination result given in [2]. For a historical overview on this type of result see [2, Chapter 14, Bibliographical Notes].

► **Theorem 2** ([2, Theorem 14.16]). *Let $S \subseteq \mathbb{R}^{k+n_1+\dots+n_\ell}$ be a semialgebraic set of complexity at most $(k+n_1+n_2+\dots+n_\ell, d, \tau)$. Let $Q_1, \dots, Q_\ell \in \{\exists, \forall\}$ be a sequence of alternating quantifiers. Consider the set $S' \subseteq \mathbb{R}^k$ of all $(x_1, \dots, x_k) \in \mathbb{R}^k$ satisfying the first-order formula*

$$(Q_1(x_{1,1}, \dots, x_{1,n_1})) \dots (Q_\ell(x_{\ell,1}, \dots, x_{\ell,n_\ell})) \cdot \\ ((x_1, \dots, x_k, x_{1,1}, \dots, x_{1,n_1}, \dots, x_{\ell,1}, \dots, x_{\ell,n_\ell}) \in S)$$

Then S' is a semialgebraic set of complexity at most $(k, d^{O(n_1 \dots n_\ell)}, \tau d^{O(n_1 \dots n_\ell \cdot k)})$.

The next theorem is due to Vorobjov [23]. See also [12, Lemma 9] and [3, Theorem 4].

► **Theorem 3.** *There exists an integer function $\text{Bound}(n, d, \tau) \in 2^{\tau d^{O(n)}}$ with the following property:*

Let K be a compact semialgebraic set of complexity at most (n, d, τ) . Then K is contained in a ball centred at the origin of radius at most $\text{Bound}(n, d, \tau)$.

A closely related result, due to [14], yields a lower bound on the minimum of a polynomial over a compact semialgebraic set, provided the minimum is non-zero. The result in [14] mentions explicit constants, which is more than we need.

► **Theorem 4** ([14, Theorem 1]). *There exists an integer function $\text{LowerBound}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ such that the following holds true:*

Let $P \in \mathbb{Q}[x_1, \dots, x_n]$ be a polynomial of degree at most d , whose coefficients have bitsize at most τ . Let K be a compact semialgebraic set of complexity at most (n, d, τ) . If $\min_{x \in K} P(x) > 0$ then $\min_{x \in K} P(x) > 1/\text{LowerBound}$.

With the help of Theorem 2, Theorem 4 can be generalised to yield a lower bound on the distance of two disjoint compact semialgebraic sets. A very similar result is proved in [20] under more general assumptions. Unfortunately, the complexity bound stated there is not sufficiently fine-grained for our purpose, since the author do not distinguish the dimension of a set from the other complexity parameters.

► **Lemma 5.** *There exists an integer function $\text{Sep}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ with the following property:*

Let K and L be compact semialgebraic sets of complexity at most (n, d, τ) . Assume that every $x \in K$ has positive euclidean distance to L . Then $\inf_{x \in K} d(x, L) > 1/\text{Sep}(n, d, \tau)$.

Proof. See Appendix E. ◀

We require a version of Kronecker's theorem on simultaneous Diophantine approximation. See [18, Corollary 3.1] for a proof.

► **Theorem 6.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Consider the free Abelian group*

$$L = \{(n_1, \dots, n_m) \in \mathbb{Z}^m \mid \lambda_1^{n_1} \cdots \lambda_m^{n_m} = 1\}.$$

Let $(\beta_1, \dots, \beta_s)$ be a basis of L . Let $\mathbb{T}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid |z_j| = 1\}$ denote the complex unit m -torus. Then the closure of the set $\{(\lambda_1^k, \dots, \lambda_m^k) \in \mathbb{T}^m \mid k \in \mathbb{N}\}$ is the set $S = \{(z_1, \dots, z_m) \in \mathbb{T}^m \mid \forall j \leq s. (z_1, \dots, z_m)^{\beta_j} = 1\}$.

Moreover, for all $\varepsilon > 0$ and all $(z_1, \dots, z_m) \in S$ there exist infinitely many indexes k such that $|\lambda_j^k - z_j| < \varepsilon$ for $j = 1, \dots, m$.

Moreover, the integer multiplicative relations between given complex algebraic numbers in the unit circle can be elicited in polynomial time. For a proof see [7, 15]. We assume the standard encoding of algebraic numbers, see [8] for details.

► **Theorem 7.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Consider the free Abelian group*

$$L = \{(n_1, \dots, n_m) \in \mathbb{Z}^m \mid \lambda_1^{n_1} \cdots \lambda_m^{n_m}\}.$$

Then one can compute in polynomial time a basis $(\beta_1, \dots, \beta_s) \in (\mathbb{Z}^m)^s$ for L . Moreover, the integer entries of the basis elements β_j are bounded polynomially in the size of the encodings of $\lambda_1, \dots, \lambda_m$.

We need to be able to bound away the modulus of eigenvalues that fall outside the unit circle away from 1. This is achieved by combining a classic result due to Mignotte [16] on the separation of algebraic numbers with a bound on the height of the resultant of two polynomials, proved in [4, Theorem 10].

► **Lemma 8.** *Let λ be a complex algebraic number whose minimal polynomial has degree at most d and coefficients bounded in bitsize by τ . Assume that $|\lambda| \neq 1$. Then we have $||\lambda| - 1| > 2^{-(\tau d)^{O(1)}}$.*

Proof. See Appendix C. ◀

3 Preliminaries

3.1 Converting the matrix to real Jordan normal form

To obtain a bound on the escape time it will be important to work with instances of the Escape Problem in real Jordan normal form. In the following, let \mathbb{A} denote the field of algebraic numbers. We establish the following reduction to this case:

► **Lemma 9.** *Let (K, A) be an instance of the Compact Escape Problem. Assume that K is given by a formula involving s polynomial equations and equalities $P \bowtie 0$ where $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial in n variables of degree at most d whose coefficients are bounded in bitsize by τ .*

Let $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ denote the real and imaginary parts of the eigenvalues of A . Let δ be a bound on the degrees of $\gamma_1, \dots, \gamma_m$.

Then there exists an equivalent instance (J, K') of the Compact Escape Problem where $J \in \mathbb{A}^{(n+m) \times (n+m)}$ is in real Jordan normal form and K' is given by a formula involving at most $s + 3m$ polynomial equations and equalities $P \triangleright 0$ where $P \in \mathbb{Z}[x_1, \dots, x_{n+m}]$ is a polynomial in $n + m$ variables of degree at most $\delta \cdot d$ whose coefficients are bounded in bitsize by $\tau + d(\log(2n) + \log(\delta + 1) + \sigma)$, where σ depends polynomially on n and the bitsize of the entries of A .

Proof. See Appendix B. ◀

3.2 Decomposing K

Let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set. Let $A \in \mathbb{R}^{n \times n}$ be a matrix in real Jordan normal form,

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}.$$

Here, each J_i is a real Jordan block of the form

$$J_i = \begin{pmatrix} \Lambda_i & I_i & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_i \\ & & & \Lambda_i \end{pmatrix},$$

where $\Lambda_{i,1}$ is either a real number or a 2×2 real matrix of the form $\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$ and, accordingly, I_i is either the real number 1 or the 2×2 identity matrix. The elements Λ_i correspond to real or complex eigenvalues $\lambda_i \in \mathbb{C}$ of A . By slight abuse of language we call $|\lambda_i|$ the modulus of Λ_i . By further slight abuse of language we define the “eigenspace” of Λ_i as the one- or two-dimensional space spanned by the vectors that correspond to the first entry of the Jordan block J_i . The “generalised eigenspaces” for Λ_i are defined analogously.

Write \mathbb{R}^n as the direct sum of two spaces $\mathbb{R}^n = V_{\text{rec}} \oplus V_{\text{non-rec}}$ where V_{rec} is the direct sum of the eigenspaces for eigenvalues of modulus 1, and $V_{\text{non-rec}}$ is the direct sum of the eigenspaces and generalised eigenspaces for eigenvalues of modulus $\neq 1$ and the generalised eigenspaces for eigenvalues of modulus 1. By convention, if A has no eigenvalues of modulus 1 we let $V_{\text{rec}} = 0$. Similarly, if A has only eigenvalues of modulus 1 and no generalised eigenvalues we let $V_{\text{non-rec}} = 0$. Thus, we decompose the state space \mathbb{R}^n into a part V_{rec} on which A exhibits purely rotational behaviour, and a part $V_{\text{non-rec}}$ where A is additionally expansive or contractive.

We will work with several different norms throughout this paper. In addition to the familiar ℓ^2 and ℓ^∞ norms we introduce a third norm, depending on the matrix A , that combines features of the two. It facilitates block-wise arguments while ensuring that the restriction of A to V_{rec} is an isometry.

Write \mathbb{R}^n as a direct sum $\mathbb{R}^n = V_1 \oplus \dots \oplus V_s \oplus W_1 \oplus \dots \oplus W_t$, where V_1, \dots, V_s correspond to the Jordan blocks of A associated with real eigenvalues and W_1, \dots, W_t correspond to the Jordan blocks of A associated with non-real eigenvalues. Let $\pi_{W_j}: \mathbb{R}^n \rightarrow W_j$ and $\pi_{V_j}: \mathbb{R}^n \rightarrow V_j$ denote the orthogonal projections onto W_j and V_j respectively.

For a vector $x \in V_i$, let $\|x\|_J^{V_i} = \|x\|_\infty$. For a vector $x = (x_1, y_1, \dots, x_k, y_k) \in W_i$, let

$$\|x\|_J^{W_i} = \max_{j=1, \dots, k} \left(\sqrt{x_j^2 + y_j^2} \right).$$

For a vector $x \in \mathbb{R}^n$, let

$$\|x\|_J = \max \left\{ \max_{j=1,\dots,s} \|\pi_{V_j}(x)\|_J^{V_j}, \max_{j=1,\dots,t} \|\pi_{W_j}(x)\|_J^{W_j} \right\}.$$

Call $\|x\|_J$ the Jordan norm of x . Observe that $\|x\|_J$ depends on the choice of the V_i ’s and W_i ’s. The Jordan norm compares to the ℓ^2 - and ℓ^∞ - norms as follows:

$$n^{-1/2} \|x\|_J \leq n^{-1/2} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_J \leq \|x\|_2 \leq n^{1/2} \|x\|_\infty \leq n^{1/2} \|x\|_J.$$

Let $\varepsilon > 0$. Consider the ball $B_J(0, \varepsilon) \subseteq \mathbb{R}^n$ about 0 with respect to the distance induced by the $\|\cdot\|_J$ -norm. We partition K into three sets:

$$\begin{aligned} K_{\text{rec}} &= K \cap V_{\text{rec}} \\ K_{<\varepsilon} &= K \cap (V_{\text{rec}} \oplus ((V_{\text{non-rec}} \cap B_J(0, \varepsilon)) \setminus \{0\})) \\ K_{\geq\varepsilon} &= K \cap (V_{\text{rec}} \oplus (V_{\text{non-rec}} \setminus B_J(0, \varepsilon))) \end{aligned}$$

4 A quantitative version of Kronecker’s theorem for complex algebraic numbers

Our central tool for bounding the escape time in the recurrent case is a quantitative version of Kronecker’s theorem for complex algebraic numbers.

Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Our goal is to find for all $\varepsilon > 0$ a bound N such that for all $(\alpha_1, \dots, \alpha_m) \in \mathbb{T}^m$ contained in the closure of the sequence $(\lambda_1^t, \dots, \lambda_m^t)_{t \in \mathbb{N}}$ there exists $t \leq N$ such that $|\lambda_j^t - \alpha_j| < \varepsilon$ for all $j = 1, \dots, m$.

We first consider the case where the λ_j ’s do not admit any integer multiplicative relations. In this case we can employ the following quantitative version of the continuous formulation of Kronecker’s theorem, proved in [11]:

► **Theorem 10** ([11, Theorem 4.1]). *Let $\varphi_1, \dots, \varphi_N$ and ζ_1, \dots, ζ_N be real numbers. Let $\varepsilon_1, \dots, \varepsilon_N$ be positive real numbers with $\varepsilon_j < 1/2$ for all j . Let $M_j = \left\lceil \frac{1}{\varepsilon_j} \log \frac{N}{\varepsilon_j} \right\rceil$. Let $\varphi = (\varphi_1, \dots, \varphi_N)$. Let $\delta = \min \{ |\varphi \cdot m| \mid m \in \mathbb{Z}^N, |m_j| < M_j, m \neq 0 \}$. Assume that $\delta > 0$. Then in any interval I of length $T \geq 4/\delta$ there is a real number t such that $\|\varphi_j t - \zeta_j\| < \varepsilon_j$, where $\|\cdot\|$ denotes distance to the nearest integer.*

Intuitively, the number δ in Theorem 10 is a quantitative measure of the linear independence of the φ_j ’s, as it bounds away from zero all integer linear combinations of the φ_j ’s with suitably bounded coefficients. In our case we consider the numbers $\varphi_j = \log \lambda_j$. For our purpose we need to obtain a bound on t , and thus a bound on δ , in terms of the algebraic complexity of the numbers $\lambda_1, \dots, \lambda_m$. This is achieved by invoking a quantitative version of Baker’s theorem on linear forms in logarithms due to Baker and Wüstholz [1]. Recall that any algebraic number μ is the root of a unique irreducible polynomial p_μ with pairwise coprime integer coefficients. The *height* of an algebraic number μ is the maximum of the absolute values of the coefficients of p_μ . The *degree* of μ is the degree of p_μ . Recall that a field E is called an *extension* of a field F if E contains F as a subfield. The *degree* of a field extension $E \supseteq F$ is the dimension of E as an F -vector space.

► **Theorem 11.** *Let μ_1, \dots, μ_N be algebraic numbers, none of which is equal to 0 or 1. Let*

$$L(z_1, \dots, z_N) = b_1 z_1 + \dots + b_N z_N$$

be a linear form with rational integer coefficients b_1, \dots, b_N . Let B be an upper bound on the absolute values of the b_j 's. For $j = 1, \dots, N$, let $A_j \geq \exp(1)$ be a bound on the height of μ_j . Let d be the degree of the field extension $\mathbb{Q}(\mu_1, \dots, \mu_N)$ generated by μ_1, \dots, μ_N over \mathbb{Q} . Fix a determination of the complex logarithm \log . Let $\Lambda = L(\log \mu_1, \dots, \log \mu_N)$. If $\Lambda \neq 0$ then

$$\log |\Lambda| > -(16Nd)^{2(N+2)} \log A_1 \cdots \log A_N \log B.$$

Finally, in the case where the λ_j 's admit integer multiplicative relations, we employ Theorem 7 to bound their complexity. We arrive at the following result:

► **Theorem 12.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Assume that the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_s$ are linearly independent over the rationals, where $0 \leq s \leq m$. Let d be the degree of the field extension $\mathbb{Q}(\lambda_1, \dots, \lambda_s)$. Let $A_1, \dots, A_s \geq \exp(1)$ be upper bounds on the heights of $\lambda_1, \dots, \lambda_s$. Let $\ell \in \mathbb{N}$, and $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$ be such that*

$$\lambda_j^\ell = (\lambda_1, \dots, \lambda_s)^{\varepsilon_j}$$

for all $j = s+1, \dots, m$. By convention, if $s = 0$ the right-hand side of the above equation is to be taken equal to 1.

Let

$$L = \max \left\{ \ell, \sum_{k=1}^s |\varepsilon_{s+1,k}|, \dots, \sum_{k=1}^s |\varepsilon_{m,k}| \right\}.$$

Let $\alpha_1, \dots, \alpha_m \in \mathbb{T}^m$ be such that any rational linear relation between the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_m$ is also satisfied by the numbers $2\pi i, \log \alpha_1, \dots, \log \alpha_m$. Let $\varepsilon > 0$. Then there exists a positive integer

$$t \leq 8\pi\ell \left(\frac{2\pi L}{\varepsilon} \right)^s \left(2s \frac{2\pi L}{\varepsilon} \left\lceil \frac{4\pi L}{\varepsilon} \log \frac{4\pi s L}{\varepsilon} \right\rceil \right)^{(16(s+1)d)^{2(s+3)} \log A_1 \cdots \log A_s} + \ell$$

such that $|\lambda_j^t - \alpha_j| < \varepsilon$ for $j = 1, \dots, m$.

Proof. An outline of the proof is sketched above. See Appendix D for a full proof. ◀

For the purpose of bounding the escape time, the following coarse bound suffices:

► **Corollary 13.** *There exists an integer function $\text{Kron}(n, \tau, P) \in 2^{(\tau P)^n O(1)}$, such that the following holds true:*

Let $\lambda_1, \dots, \lambda_n$ be algebraic numbers of modulus 1. Assume that the degree of each λ_j is bounded by n . Let τ be a bound on the bitsize of the coefficients of the minimal polynomials of the λ_j 's. Let P be a positive integer. Let $\alpha_1, \dots, \alpha_n$ be complex numbers which are contained in the closure of the sequence $(\lambda_1^t, \dots, \lambda_n^t)_{t \in \mathbb{N}}$. Then there exists a $t \leq \text{Kron}(n, \tau, P)$ such that $|\alpha_j - \lambda_j^t| < 2^{-P}$ for all $j \in \{1, \dots, n\}$.

Proof. By Kronecker's theorem, any integer multiplicative relation between the λ_j 's is also satisfied by the α_j 's. Theorem 12 hence yields a bound on t such that $|\alpha_j - \lambda_j^t| < 2^{-P}$ holds for all $j \in \{1, \dots, n\}$.

This bound is given in terms of quantities $s, d, \ell, \varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s, A_1, \dots, A_s$, and L . It remains to show that these quantities can be chosen to be suitably bounded in terms of n and τ .

Proposition 26 in Appendix D, which is mainly based on Theorem 7, shows that numbers ℓ and $\varepsilon_1, \dots, \varepsilon_m$ can be computed in polynomial time. In particular, the absolute size of L and ℓ is of the form $2^{(n\tau)^{O(1)}}$. The numbers $\log A_i$ are bounded by τ by assumption. We have $s \leq m \leq n$ by definition. Finally, we have assumed that each λ_j has degree at most n . It follows that the degree d of the field extension $\mathbb{Q}(\lambda_1, \dots, \lambda_s)$ is bounded by n^n . The result follows from Theorem 12. ◀

5 The recurrent eigenspace

The next lemma establishes as a special case an escape bound for all initial values $x \in K_{\text{rec}}$. In order to combine the recurrent and the non-recurrent case we need a stronger result, however. Thus, we establish not only a bound on the escape time for all initial values $x \in K_{\text{rec}}$, but a bound N such that every $x \in V_{\text{rec}}$ – not just in K_{rec} – has distance at least $1/N$ – not just positive distance – from K . Further, note that Lemma 14 is still applicable in the special cases where $K_{\text{rec}} = \emptyset$ or $V_{\text{rec}} = 0$.

► **Lemma 14.** *There exists an integer function $\text{Rec}(n, d, \tau) \in 2^{(\tau d)^n \text{O}(1)}$ with the following property:*

Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form with algebraic entries. Assume that the minimal polynomial of A has rational coefficients whose bitsize is bounded by τ . Let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set of complexity at most (n, d, τ) . If every point $x \in K_{\text{rec}}$ escapes K under iterations of A then for all $x \in V_{\text{rec}}$ there exists $t \leq \text{Rec}(n, d, \tau)$ such that

$$\text{dist}_{\ell^2}(A^t x, K) > \frac{\sqrt{n}}{\text{Rec}(n, d, \tau)}.$$

Proof. The full proof is given in Appendix F. We only sketch an outline here.

We first prove the result for initial points $x \in K_{\text{rec}}$. For these points, the closure of the orbit $\overline{\mathcal{O}_A(x)}$ of x under A is a compact semialgebraic set. We employ Corollary 13 to obtain for all $\varepsilon > 0$ a doubly exponential bound N such that for all $x \in K_{\text{rec}}$ and all $y \in \overline{\mathcal{O}_A(x)}$ there exists $t \leq N$ such that $\|A^t x - y\|_2 < \varepsilon$. We then use Theorem 4 to obtain a uniform at most doubly exponentially small lower bound on the quantity

$$\inf_{x \in K_{\text{rec}}} \sup_{y \in \overline{\mathcal{O}_A(x)}} \inf_{z \in K} \|y - z\|_2^2.$$

In order to apply this theorem we construct an auxiliary semialgebraic set, whose complexity is controlled by Theorem 2. Combining these two steps, we obtain a function Rec_0 that satisfies the statement of the lemma for all initial points $x \in K_{\text{rec}}$.

Finally, we extend the result to all initial points $x \in V_{\text{rec}}$. The special case where $K_{\text{rec}} = \emptyset$ is treated using Theorem 4.

In the case where K_{rec} is non-empty we obtain from Lemma 5 that every $x \in V_{\text{rec}}$ which is doubly exponentially close to K with a sufficiently large constant in the third exponent is already doubly exponentially close to K_{rec} , with a slightly smaller constant in the third exponent. Now, any point that is sufficiently far away from K trivially satisfies the claim. By the preceding discussion, points $x \in V_{\text{rec}}$ that are sufficiently close to K are already sufficiently close to K_{rec} , so that there exists an escaping orbit $\overline{\mathcal{O}_A(x')}$ with $x' \in K_{\text{rec}}$ which is close to the orbit of x since A is an isometry on V_{rec} . This allows us to reduce the result to the already established result for initial values in K_{rec} . ◀

6 The non-recurrent eigenspace

The next lemma concerns the subset $K_{\geq \varepsilon}$ of K containing the points in K that are bounded away from V_{rec} by some $\varepsilon > 0$.

For any such point, there exist coordinates (or pairs of coordinates if the corresponding eigenvalues are not real) whose contribution to the Jordan norm is greater than ε . Moreover, the contribution to the Jordan norm of these coordinates does not stay constant under

applications of A . If the contribution to the norm of at least one such coordinate is increasing under applications of A , the orbit will eventually leave K , since K is compact. Moreover, Theorem 3 yields an upper bound on the escape time.

Coordinates whose contribution to the norm is decreasing under applications of A will, after sufficiently many iterations, contribute less than ε . We establish a uniform upper bound on the number of iterations required to ensure this for all such coordinates. Combining this with the previous bound, we obtain a number N such that after at most N applications of A , every $x \in K_{\geq \varepsilon}$ has either escaped K , entered $K_{< \varepsilon} \cup K_{\text{rec}}$, or it remains in $K_{\geq \varepsilon}$ because it has a component whose contribution to the norm was initially smaller than ε , but grew beyond ε under iteration of A . In the last case, the point will grow in norm beyond the bound established in Theorem 3 and thus escape K after a further N applications of A . This yields a uniform bound on the number of iterations that are required for any point $x \in K_{\geq \varepsilon}$ to either leave K entirely or move into $K_{< \varepsilon} \cup K_{\text{rec}}$.

The overall structure of this proof closely follows the one given in [10], where the assumptions allow the authors to restrict the discussion to real eigenvalues.

► **Lemma 15.** *There exists an integer function $\text{NonRec}(n, d, \tau, P) \in 2^{(d\tau P)^n}{}^{O(1)}$ with the following property:*

Let K be a compact semialgebraic set of complexity at most (n, d, τ) . Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form. Assume that the characteristic polynomial of A has rational coefficients whose bitsize is bounded by τ . Let P be a positive integer.

Then for all $x \in K_{\geq 2^{-P}}$ there exists $t \leq \text{NonRec}(n, d, \tau, P)$ such that $A^t x \notin K_{\geq 2^{-P}}$.

Proof. See Appendix G for details. ◀

7 Proof of Theorem 1

In the previous two sections, we successively showed how to establish a bound on the escape time for an instance (A, K) when the orbit remains in the recurrent eigenspace and how the orbit behaves when it starts away from the recurrent eigenspace. In this section, we show how to combine both results in order to establish an escape bound for any starting point in K . This will thus prove Theorem 1.

Let (A_0, K_0) be an instance of the compact escape problem, where $K_0 \subseteq \mathbb{R}^n$ is a compact semialgebraic set of complexity at most (n_0, d_0, τ_0) and $A_0 \in \mathbb{Q}^{n \times n}$ is a square matrix with rational entries whose bitsize is bounded by τ_0 . Assume that every point $x \in K_0$ escapes K_0 under iterations of A_0 .

Apply Lemma 9 to convert the instance (A_0, K_0) into an equivalent instance (A, K) such that $A \in \mathbb{A}^{n \times n}$ is in real Jordan normal form. Then the set K has complexity at most (n, d, τ) , where $n = 2n_0$, $d = n_0 d_0$, and $\tau = (n_0 \tau_0 d_0)^{C_\tau}$ for some absolute constant C_τ . By construction, the characteristic polynomial of A has rational coefficients of bitsize at most τ .

Let Rec be the function from Lemma 14. Let $\varepsilon = \frac{1}{\text{Rec}(n, d, \tau)}$ and $N_{\text{rec}} = \text{Rec}(n, d, \tau)$. Let $x \in K$. If $x \in K_{\text{rec}}$ then x escapes within N_{rec} steps. Suppose that $x \in K_{< \varepsilon}$.

Then there are two possibilities:

1. We have $A^t x \notin K_{\geq \varepsilon}$ for all $t \leq N_{\text{rec}}$.
2. We have $A^t x \in K_{\geq \varepsilon}$ for at least one $t \leq N_{\text{rec}}$.

In the first case, the orbit of x remains close to V_{rec} for long enough that we can rely on Lemma 14. Indeed, let x_0 denote the orthogonal projection of x onto V_{rec} . Let $t \leq N_{\text{rec}}$ be such that $\text{dist}_{\ell^2}(A^t x_0, K) > \sqrt{n} \varepsilon$. Since $A^t x \notin K_{\geq \varepsilon}$, we have $\|A^t x - A^t x_0\|_J < \varepsilon$, so that

$\|A^t x - A^t x_0\|_2 < \sqrt{n}\varepsilon$. Let $y \in K$. Then

$$\|A^t x - y\|_2 \geq \|A^t x_0 - y\|_2 - \|A^t x - A^t x_0\|_2 > \sqrt{n}\varepsilon - \sqrt{n}\varepsilon = 0.$$

Thus, x escapes K under iterations of A .

In the second case, let t_1 be such that $A^{t_1} x \in K_{\geq \varepsilon}$. Let NonRec be the function from Lemma 15. Let $N_{\geq \varepsilon} = \text{NonRec}(n, d, \tau, \lceil \log(1/\varepsilon) \rceil)$. By Lemma 15 there exists $t_2 \leq N_{\geq \varepsilon}$ such that $A^{t_2} A^{t_1} x$ is contained either in $K_{< \varepsilon} \cup K_{\text{rec}}$ or in the complement of K . In the latter case we are done. In the former case we apply the initial case distinction: either for all $t \leq N_{\text{rec}}$ we have $A^t A^{t_2} A^{t_1} x \notin K_{\geq \varepsilon}$ or we have $A^{t_3} A^{t_2} A^{t_1} x \in K_{\geq \varepsilon}$ for at least one $t_3 \leq N_{\text{rec}}$. Once again, in the first case, the point has escaped. By repeating this reasoning, we construct a (finite or infinite) sequence t_1, t_2, \dots such that $t_i \leq N_{\text{rec}}$ if i is odd and $t_i \leq N_{\geq \varepsilon}$ if i is even and

$$A^{t_s} \dots A^{t_1} x \in \begin{cases} K_{< \varepsilon} \cup K_{\text{rec}} & \text{if } s \text{ is even,} \\ K_{\geq \varepsilon} & \text{if } s \text{ is odd.} \end{cases}$$

We claim that the sequence t_1, t_2, \dots is finite and contains at most n^3 elements.

Consider a real Jordan block of A of size $m \leq n$ associated to the eigenvalue Λ . Denote by x_J the orthogonal projection of x onto the dimensions associated with this block.

Assume first that Λ is a real eigenvalue (as opposed to a 2×2 block representing a complex eigenvalue). If $\Lambda = 0$, then clearly $\|J^k x_J\|_J$ is monotonically decreasing. Thus, assume in the sequel that $\Lambda \neq 0$.

Let $j \in \{1, \dots, m\}$. The $m - j + 1$ 'th component of the vector $J^k x_J$, viewed as a function of t , is an exponential polynomial $E_j(t) = \Lambda^t P(t)$, where $P \in \mathbb{R}[z]$ is a real polynomial of degree $j - 1$. Consider the real function

$$(E_j(\cdot))^2: \mathbb{R} \rightarrow \mathbb{R}, \quad (E_j(t))^2 = |\Lambda|^{2t} |P(t)|^2.$$

This function is differentiable in t with derivative

$$\frac{d}{dt} (E_j(t))^2 = \Lambda^{2t} (\log(\Lambda^2)(P(t)^2) + 2P(t)P'(t)).$$

This derivative vanishes if and only if the factor $(\log(\Lambda^2)(P(t)^2) + 2P(t)P'(t))$ vanishes. This factor is a polynomial of degree $2j - 2$, so that it has at most $2j - 2$ real zeroes. It follows that there exist numbers $t_{j,1}, \dots, t_{j,m_j}$ with $m_j \leq 2j - 2$ such that the function $(E_j(t))^2 - \varepsilon^2$ does not change its sign in any of the open intervals

$$(0, t_{j,1}), (t_{j,1}, t_{j,2}), \dots, (t_{j,m_j-1}, t_{j,m_j}), (t_{j,m_j}, +\infty).$$

Thus, the norm $\|J^t x_J\|_J$ changes from smaller than ε to bigger than ε at most

$$\sum_{j=1}^m (2j - 2) = 2 \sum_{j=1}^m j - 2m = (m + 1)m - 2m = m^2 - m$$

times.

The case where Λ represents a complex eigenvalue λ is similar. However, we now consider the evolution of the two coordinates corresponding to one Λ -block simultaneously.

For $j \in \{1, \dots, m\}$, write $E_j(t)$ for the $m - j + 1$ 'th component of the vector $J^t x_J$, viewed as a function of t . We have for all $j \in \{1, \dots, m/2\}$ that the function

$$F_j(t) = (E_{2j}(t))^2 + (E_{2j-1}(t))^2$$

is an exponential polynomial $F_j(t) = |\lambda|^t P_j(t)$, where $P_j \in \mathbb{R}[z]$ is a real polynomial of degree $j - 1$. Therefore, exactly as in case where Λ is a real eigenvalue, the derivative of F_j vanishes at most $2j - 2$ times. From which we can deduce that the norm $\|J^t x_j\|_J$ crosses the ε -threshold at most $m^2 - m$ times.

Estimating generously, we have at most n Jordan blocks of size at most n , each of which crosses the ε -threshold at most $n^2 - n$ times. In total, we cross the threshold at most $n^3 - n^2$ times. The total escape bound is hence $n^3 \max\{N_{\text{rec}}, N_{\geq \varepsilon}\}$. By the same argument, the same escape bound holds true when the initial point x lies in $K_{\geq \varepsilon}$.

Substituting the constants N_{rec} , $N_{\geq \varepsilon}$, n , d , and τ with their definitions, we obtain the upper bound

$$\begin{aligned} \text{CompactEscape}(n_0, d_0, \tau_0) = & \\ & (2n_0)^3 \max \left\{ \text{Rec} \left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau} \right), \right. \\ & \left. \text{NonRec} \left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau}, \log \left[\text{Rec} \left(2n_0, n_0 d_0, (n_0 d_0 \tau_0)^{C_\tau} \right) \right] \right) \right\}. \end{aligned}$$

One easily verifies that $\text{CompactEscape}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$ as claimed.

8 A matching lower bound on escape time

In Theorem 1 we established a uniform upper bound on the escape time for all positive instances of the Compact Escape Problem. Our bound is doubly exponential in the ambient dimension and singly exponential in the rest of the data. We will now show that this bound cannot be significantly improved by showing that a doubly exponential bound cannot be avoided even for purely rotational systems. A second example displaying a doubly exponential lower bound is presented in Appendix H.

► **Example 16.** For $(n, d, \tau) \in \mathbb{N}^3$, let $K_{(n,d,\tau)} \subseteq \mathbb{R}^{n+2}$ be the set of all points (x, y, u_1, \dots, u_n) satisfying the (in)equalities: $x^2 + y^2 = 1$, $u_1 = 2^{-\tau}$, $(x-1)^2 + y^2 \geq u_n$ and for $1 \leq i \leq n-1$, $u_{i+1} = (u_i)^d$.

Hence, $K_{(n,d,\tau)} = \left(S^1 \setminus B \left((1, 0), 2^{-\tau d^{n-1}} \right) \right) \times \left\{ \left(2^{-\tau}, 2^{-\tau d}, \dots, 2^{-\tau d^{n-1}} \right) \right\}$, where $S^1 \subseteq \mathbb{R}^2$ is the unit circle. Let $a = \frac{3}{5}$, $b = \frac{4}{5}$. Let

$$A_{(n,d,\tau)} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

where I_n is the $n \times n$ -identity matrix. It is easy to see that the complex number $\frac{3}{5} + i\frac{4}{5}$ has modulus 1 and is not a root of unity. It follows from Dirichlet's theorem on simultaneous Diophantine approximation that the orbit of A is equal to $S^1 \times \left\{ \left(2^{-\tau}, 2^{-\tau d}, \dots, 2^{-\tau d^{n-1}} \right) \right\}$, so that every initial point escapes under A .

We claim that there exists a point $x \in K_{(n,d,\tau)}$ that requires $2^{\tau d^{n-1}}$ steps to escape. Indeed, let $x_0 \in K_{(n,d,\tau)}$ be an arbitrary initial point. Consider the orbit $x_t = A^t x_0$. Let $N < 2^{\tau d^{n-1}}$. By the pigeonhole principle, the finite set of points x_0, \dots, x_N contains at least one consecutive pair of points x_i, x_j on the circle such that the points x_i and x_j are joined by an arc of the circle of length strictly greater than $2/N$. It follows that we can ensure that none of the points x_1, \dots, x_N is outside of $K_{(n,d,\tau)}$ by applying a suitable planar rotation to all points. Since all planar rotations commute, there exists for each angle θ an initial point $x_\theta \in S^1 \times \left\{ \left(2^{-\tau}, 2^{-\tau d}, \dots, 2^{-\tau d^{n-1}} \right) \right\}$, such that the orbit of x_θ under A is equal to the orbit of x_0 under A rotated by θ . This proves the claim.

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A

 Computing real JNF in polynomial time

Given a matrix A with rational entries, we discuss how to compute the real Jordan normal form J of A and the associated change of basis matrix Q in polynomial time. First compute, in polynomial time, the (complex) Jordan normal form J' and change of basis matrix T such that $A = TJ'T^{-1}$ using the algorithm from [6].

Computing J : Suppose, without loss of generality, that

$$J' = \text{diag}(J'_1, J'_2, \dots, J'_{2k-1}, J'_{2k}, J'_{2k+1}, \dots, J'_{2k+z})$$

where for $1 \leq j \leq k$, the Jordan blocks J'_{2j-1} and J'_{2j} have the same dimension and have conjugate eigenvalues $\lambda_j = a_j + b_j i$ and $\bar{\lambda} = a_j - b_j i$, respectively. The blocks $J'_{2k+1}, \dots, J'_{2k+z}$, on the other hand, have real eigenvalues. J is obtained by replacing, for each $1 \leq j \leq k$, $\text{diag}(J'_{2j-1}, J'_{2j})$ with a real Jordan block of the same dimension with $\Lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and keeping the blocks $J'_{2k+1}, \dots, J'_{2k+z}$ unchanged.

Computing Q : Let $\kappa(j)$ denote the multiplicity of the Jordan block J'_i for $1 \leq i \leq 2k+z$, and $v_1^1, \dots, v_{\kappa(1)}^1, \dots, v_1^{2k}, \dots, v_{\kappa(2k)}^{2k}, \dots, v_1^{2k+z}, \dots, v_{\kappa(2k+z)}^{2k+z} \in \mathbb{C}^m$ be the columns of T . It will be the case that for all $1 \leq j \leq k$ and l , $v_l^{2j-1} = \overline{v_l^{2j}}$ in the sense that $v_l^{2j-1} = x_l^j + y_l^j i$ and $v_l^{2j} = x_l^j - y_l^j i$ for vectors $x_l^j, y_l^j \in \mathbb{R}^m$. Moreover, for $j > 2k$, $v_l^{2j} \in \mathbb{R}^m$. Finally, columns of Q are obtained from columns of T as follows. For $1 \leq j \leq k$ and all l , replace v_l^{2j-1} with x_l^j and v_l^{2j} with y_l^j and keep v_l^{2k+z} for all l and $m > 0$ unchanged, in the same way the proof of existence of real Jordan normal form proceeds.

B

 Proof of Lemma 9

► **Lemma 9.** *Let (K, A) be an instance of the Compact Escape Problem. Assume that K is given by a formula involving s polynomial equations and equalities $P \bowtie 0$ where $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial in n variables of degree at most d whose coefficients are bounded in bitsize by τ .*

Let $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ denote the real and imaginary parts of the eigenvalues of A . Let δ be a bound on the degrees of $\gamma_1, \dots, \gamma_m$.

Then there exists an equivalent instance (J, K') of the Compact Escape Problem where $J \in \mathbb{A}^{(n+m) \times (n+m)}$ is in real Jordan normal form and K' is given by a formula involving at most $s + 3m$ polynomial equations and equalities $P \bowtie 0$ where $P \in \mathbb{Z}[x_1, \dots, x_{n+m}]$ is a polynomial in $n + m$ variables of degree at most $\delta \cdot d$ whose coefficients are bounded in bitsize by $\tau + d(\log(2n) + \log(\delta + 1) + \sigma)$, where σ depends polynomially on n and the bitsize of the entries of A .

By Appendix A we can compute in polynomial time real algebraic numbers $\gamma_1, \dots, \gamma_m$ and a matrix $Q \in \mathbb{Q}(\gamma_1, \dots, \gamma_m)^{n \times n}$ such that $A = QJQ^{-1}$, where J is in real Jordan normal form.

More precisely, we can compute in polynomial time:

1. Univariate polynomials with integer coefficients f_1, \dots, f_m such that $f_j(\gamma_j) = 0$ for all $j = 1, \dots, m$.
2. Rational numbers $a_1, b_1, \dots, a_m, b_m$, such that γ_j is the unique root of f_j in the interval $[a_j, b_j]$.

3. For $i = 1, \dots, n$ and $j = 1, \dots, n$ polynomials of degree at most δ $Q_{i,j} \in \mathbb{Q}[x]$ and indexes $\ell_{i,j}$ such that the matrix Q at row i and column j is given by the algebraic number $Q_{i,j}(\gamma_{\ell_{i,j}})$.

Let $\sigma \in \mathbb{N}$ be a common bound on the following quantities:

1. The bitsize of the coefficients of f_1, \dots, f_m .
2. The bitsize of the endpoints of the isolating intervals $[a_j, b_j]$.
3. The bitsize of the coefficients of the polynomials $Q_{j,k}$.

Then σ is computable in polynomial time from A , so that it depends polynomially on n and the bitsize of the entries of A .

We fix $K' = (Q^{-1}K) \times \{\gamma_1, \dots, \gamma_m\}^1$ and note that $A^k x \in K$ if and only if

$$(J \times I_m)^k (Q^{-1}x, (\gamma_1, \dots, \gamma_m)) \in (Q^{-1}K) \times \{(\gamma_1, \dots, \gamma_m)\}.$$

Thus, it remains to show that there exists a description of the set K' with the claimed complexity.

Let $\Phi(x_1, \dots, x_n)$ be the formula that describes K . We introduce fresh variables z_1, \dots, z_m and consider the formula

$$\Psi(z_1, \dots, z_m) \wedge \widehat{\Phi}(x_1, \dots, x_n, z_1, \dots, z_m)$$

where $\Psi(z_1, \dots, z_m)$ is the conjunction of the terms

$$f_j(z_j) = 0 \wedge z_j \geq a_j \wedge z_j \leq b_j$$

which ensures $z_j = \gamma_j$ for $j = 1, \dots, m$, and $\widehat{\Phi}$ is obtained from Φ by replacing each atom $P(x_1, \dots, x_n) \bowtie 0$ in Φ by the atom

$$P\left(\sum_{k=1}^n Q_{1,k}(z_{\ell_{1,k}})x_k, \dots, \sum_{k=1}^n Q_{n,k}(z_{\ell_{n,k}})x_k\right) \bowtie 0.$$

It is not hard to see that this new formula describes the set K' . Evidently, the number of variables in this description is $n + m$. The formula Ψ involves $3m$ polynomials of degree at most δ whose coefficients are bounded in bitsize by σ .

It remains to determine the complexity of the formula $\widehat{\Phi}$. We claim that the degrees of the polynomials in $\widehat{\Phi}$ are bounded by $\delta \cdot d$ and that the bitsize of their coefficients is bounded by $\tau + d(\log(n) + \log(\delta + 1) + \sigma)$. This is established by a straightforward but cumbersome calculation. We recall the multinomial theorem:

► **Lemma 17** (Multinomial theorem). *Let R be a ring. Let N be a positive integer. Let $z_1, \dots, z_N \in R$. Then*

$$\left(\sum_{k=1}^N z_k\right)^e = \sum_{j_1 + \dots + j_N = e} \binom{e}{j_1, \dots, j_N} \prod_{t=1}^N z_t^{j_t},$$

where

$$\binom{e}{j_1, \dots, j_N} = \frac{e!}{j_1! \cdots j_N!}.$$

¹ The last m coordinates are added to allow us to manipulate these constants within the description of K' through a formula.

It will be convenient to make use of the following straightforward application of the distributivity law of multiplication over addition:

► **Lemma 18.** *Let I be a finite set. Let J be a set-valued function that sends each $i \in I$ to a finite set $J(i)$. Let R be a ring. For all $(i, j) \in I \times \prod_{i \in I} J(i)$, let $a_{i,j}$ be an element of R . Then we have*

$$\prod_{i \in I} \sum_{j \in J(i)} a_{i,j} = \sum_f \prod_{i \in I} a_{i,f(i)}$$

where f ranges over all functions

$$f: I \rightarrow \prod_{i \in I} J(i)$$

satisfying $f(i) \in J(i)$ for all $i \in I$.

Now, let $P(x_1, \dots, x_n) \bowtie 0$ be an atom in Φ . This atom is a sum of monomials

$$C \cdot x_1^{e_1} \cdots x_n^{e_n}$$

with $\log |C| \leq \tau$ and $e_1 + \dots + e_n \leq d$. It suffices to bound the degrees and the bitsize of the coefficients of the polynomials that are obtained by applying our substitution of variables to monomials of this form.

Under our substitution such a monomial becomes:

$$C \cdot \left(\sum_{j=1}^n Q_{1,j}(z_{\ell_{1,j}}) x_j \right)^{e_1} \cdots \left(\sum_{j=1}^n Q_{n,j}(z_{\ell_{n,j}}) x_j \right)^{e_n} = C \prod_{k=1}^n \left(\sum_{j=1}^n Q_{k,j}(z_{\ell_{k,j}}) x_j \right)^{e_k}.$$

Apply the multinomial theorem to the expressions $\left(\sum_{j=1}^n Q_{k,j}(z_{\ell_{k,j}}) x_j \right)^{e_k}$ to obtain:

$$C \cdot \prod_{k=1}^n \left(\sum_{j_{k,1} + \dots + j_{k,n} = e_k} \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n (Q_{k,t}(z_{\ell_{k,t}}) x_t)^{j_{k,t}} \right)$$

Write

$$Q_{k,t}(z_{\ell_{k,t}}) = \sum_{p=0}^{\delta} \alpha_{k,t,p} z_{\ell_{k,t}}^p.$$

Applying the multinomial theorem to the terms

$$(Q_{k,t}(z_{\ell_{k,t}}) x_t)^{j_{k,t}} = \left(\sum_{p=0}^{\delta} \alpha_{k,t,p} z_{\ell_{k,t}}^p x_t \right)^{j_{k,t}}$$

we obtain

$$(Q_{k,t}(z_{\ell_{k,t}}) x_t)^{j_{k,t}} = \sum_{r_0 + \dots + r_{\delta} = j_{k,t}} \binom{j_{k,t}}{r_0, \dots, r_{\delta}} \prod_{s=0}^{\delta} \alpha_{k,t,s}^{r_s} z_{\ell_{k,t,s}}^{r_s} x_t^{r_s}.$$

The full expression is hence:

$$C \cdot \prod_{k=1}^n \left(\sum_{j_{k,1} + \dots + j_{k,n} = e_k} \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n \sum_{r_0 + \dots + r_{\delta} = j_{k,t}} \binom{j_{k,t}}{r_0, \dots, r_{\delta}} \prod_{s=0}^{\delta} \alpha_{k,t,s}^{r_s} z_{\ell_{k,t,s}}^{r_s} x_t^{r_s} \right).$$

Write this as:

$$C \cdot \prod_{k=1}^n \sum_{j_{k,1}+\dots+j_{k,n}=e_k} \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n \sum_{r_0+\dots+r_\delta=j_{k,t}} c_{k,j_{k,1}, \dots, j_{k,n}, t, r_0, \dots, r_\delta}.$$

Apply Lemma 18 to move out the innermost sum, thus obtaining an equal expression:

$$C \cdot \prod_{k=1}^n \left(\sum_{j_{k,1}+\dots+j_{k,n}=e_k} \sum_f \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n c_{k,j_{k,1}, \dots, j_{k,n}, t, f(t)} \right).$$

where the sum \sum_f ranges over all functions $f: \{1, \dots, n\} \rightarrow \mathbb{N}^\delta$ with $f(t) = (r_0, \dots, r_\delta)$ satisfying $r_0 + \dots + r_\delta = j_{k,t}$.

Write the result as:

$$C \cdot \prod_{k=1}^n \sum_{j_{k,1}+\dots+j_{k,n}=e_k} \sum_f d_{k,j_{k,1}, \dots, j_{k,n}, f}.$$

Apply Lemma 18 again to obtain that this is equal to:

$$\sum_g C \cdot \prod_{k=1}^n d_{k,g(k)},$$

where g ranges over all functions $g: \{1, \dots, n\} \rightarrow \mathbb{N}^n \times (\mathbb{N}^\delta)^{\{1, \dots, n\}}$ with $g(k) = (j_{k,1}, \dots, j_{k,n}, f)$ satisfying $j_{k,1} + \dots + j_{k,n} = e_k$ and f as above.

Thus, the final result is a sum of monomials of the form

$$\begin{aligned} C \cdot \prod_{k=1}^n d_{k,g(k)} &= C \cdot \prod_{k=1}^n \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n c_{k,j_{k,1}, \dots, j_{k,n}, t, f(t)} \\ &= C \cdot \prod_{k=1}^n \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n \binom{j_{k,t}}{r_0(t), \dots, r_\delta(t)} \prod_{s=0}^{\delta} \alpha_{k,t,s}^{r_s(t)} z_{\ell_{k,t,s}}^{sr_s(t)} x_t^{r_s(t)}, \end{aligned}$$

Where $j_{k,1} + \dots + j_{k,n} = e_k$ and $r_0(t), \dots, r_\delta(t)$ are functions of t satisfying $r_0(t) + \dots + r_\delta(t) = j_{k,t}$.

The degrees of these monomials are bounded by $\delta \cdot d$.

Let us compute a bound on the bitsize of the coefficients. We have:

$$\begin{aligned} &\log \left(|C| \cdot \prod_{k=1}^n \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n \binom{j_{k,t}}{r_0(t), \dots, r_\delta(t)} \prod_{s=0}^{\delta} |\alpha_{k,t,s}|^{r_s(t)} \right) \\ &\leq \tau + \sum_{k=1}^n \log \binom{e_k}{j_{k,1}, \dots, j_{k,n}} + \sum_{k=1}^n \sum_{t=1}^n \log \binom{j_{k,t}}{r_0(t), \dots, r_\delta(t)} + \sum_{k=1}^n \sum_{t=1}^n \sum_{s=0}^{\delta} r_s(t) \sigma. \end{aligned}$$

Use the estimate $\binom{f}{k_1, \dots, k_m} \leq m^f$ to obtain:

$$\begin{aligned} &\log \left(|C| \cdot \prod_{k=1}^n \binom{e_k}{j_{k,1}, \dots, j_{k,n}} \prod_{t=1}^n \binom{j_{k,t}}{r_0(t), \dots, r_\delta(t)} \prod_{s=0}^{\delta} |\alpha_{k,t,s}|^{r_s(t)} \right) \\ &\leq \tau + \sum_{k=1}^n e_k \log(n) + \sum_{k=1}^n \sum_{t=1}^n j_{k,t} \log(\delta + 1) + \sum_{k=1}^n \sum_{t=1}^n \sum_{s=0}^{\delta} r_s(t) \sigma. \\ &\leq \tau + d \log(n) + d \log(\delta + 1) + d \sigma. \\ &= \tau + d(\log(n) + \log(\delta + 1) + \sigma). \end{aligned}$$

Thus, everything is shown.

C Proof of Lemma 8

Recall the following classic theorem due to Mignotte [16].

► **Theorem 19** (Mignotte). *Let $P \in \mathbb{Z}[x]$ be a square-free univariate polynomial of degree at most d , whose coefficients have absolute value bounded by H . Let $\alpha \neq \beta$ be distinct roots of P . Then*

$$|\alpha - \beta| > \frac{\sqrt{3}}{(d+1)^{d+2} H^{d-1}}.$$

We obtain the following more explicit version of Lemma 8:

► **Lemma 20.** *Let λ be a complex algebraic number of degree d and height H . Assume that $|\lambda| \neq 1$. Then*

$$||\lambda| - 1| > \frac{1}{\sqrt{3}(2d+2)^{2d+3} 2^{2d} (d!)^{2d} (d+1)^{2d(d-1)} H^{2d^2}}.$$

Proof. The numbers λ and $\bar{\lambda}$ are roots of the same minimal polynomial P of degree d and height H . It follows that the number $|\lambda|^2 = \lambda\bar{\lambda}$ is a root of the polynomial $Q(x) = \text{res}_z(P(x), x^d P(z/x))$, where $\text{res}_z(A, B)$ denotes the resultant of the polynomials $A, B \in \mathbb{Q}[x][z]$ with coefficients in the integral domain $\mathbb{Q}[x]$, cf. e.g. [8, p. 159].

The degree of $Q(x)$ is at most $2d$. By [4, Theorem 10] the height of $Q(x)$ is bounded by

$$H' = d!(d+1)^{d-1} H^d.$$

The polynomial $Q(x)(x-1)$ has degree at most $2d+1$ and height at most $2H'$.

It follows from Theorem 19 that

$$||\lambda|^2 - 1| > \frac{\sqrt{3}}{(2d+2)^{2d+3} (2H')^{2d}}.$$

Note that $||\lambda|^2 - 1| = ||\lambda| - 1| \cdot ||\lambda| + 1|$. If $|\lambda| > 2$ then the claim is trivial, so we may assume that $||\lambda| + 1| \leq 3$, yielding

$$||\lambda| - 1| > \frac{\sqrt{3}}{3(2d+2)^{2d+3} (2H')^{2d}} = \frac{1}{\sqrt{3}(2d+2)^{2d+3} 2^{2d} (d!)^{2d} (d+1)^{2d(d-1)} H^{2d^2}}.$$

◀

D Proof of Theorem 12

Recall Dirichlet's theorem on simultaneous Diophantine approximation:

► **Theorem 21** (Dirichlet). *Let $\varphi_1, \dots, \varphi_N \in \mathbb{R}$ be arbitrary real numbers. Let $M \in \mathbb{R}$ with $M \geq 1$. Then there exist integers q, p_1, \dots, p_N with $1 \leq q \leq M$ such that*

$$|q\varphi_j - p_j| < \frac{1}{qM^{1/N}}.$$

Throughout this section, let $\|\cdot\|$ denote the distance to the closest integer. We recall that Kronecker's theorem has two equivalent formulations: a discrete one and a continuous one.

► **Theorem 22** (Kronecker's Theorem - Discrete Formulation). *Let $\varphi_1, \dots, \varphi_N$ be real numbers, linearly independent over \mathbb{Q} . Let ζ_1, \dots, ζ_N be arbitrary real numbers. Let $\varepsilon > 0$. Then there exists a real number t such that for all j :*

$$\|\varphi_j t - \zeta_j\| < \varepsilon.$$

► **Theorem 23** (Kronecker's Theorem - Continuous Formulation). *Let $1, \varphi_1, \dots, \varphi_N$ be real numbers, linearly independent over \mathbb{Q} . Let ζ_1, \dots, ζ_N be arbitrary real numbers. Let $\varepsilon > 0$. Then there exists an integer t such that for all j :*

$$\|\varphi_j t - \zeta_j\| < \varepsilon.$$

The standard proof of equivalence of the two formulations in particular allows us to translate a quantitative version of the continuous formulation into a Quantitative version of the discrete formulation:

► **Corollary 24.** *Let $\varphi_1, \dots, \varphi_N$ and ζ_1, \dots, ζ_N be arbitrary real numbers. Let $\varepsilon > 0$. Let q, p_1, \dots, p_N be integers such that $|q\varphi_j - p_j| < \varepsilon$. If there exists a real number $0 \leq t \leq T$ such that for all j we have*

$$\|(q\varphi_j - p_j)t - \zeta_j\| < \varepsilon/2,$$

then there exists an integer $k \leq |q|T$ such that for all j we have

$$\|\varphi_j k - \zeta_j\| < \varepsilon.$$

Proof. By assumption there exist integers r_1, \dots, r_N such that we have

$$|(q\varphi_j - p_j)t - \zeta_j - r_j| < \varepsilon/2.$$

Write $t = \ell + \delta$ with $\ell \in \mathbb{Z}$ and $|\delta| \leq \frac{1}{2}$. We obtain:

$$|q\ell\varphi_j + q\delta\varphi_j - \ell p_j - \delta p_j - \zeta_j - r_j| < \varepsilon/2.$$

It follows that

$$\|q\ell\varphi_j - \zeta_j\| \leq |q\ell\varphi_j - \ell p_j - \zeta_j - r_j| \leq |q\ell\varphi_j + q\delta\varphi_j - \ell p_j - \delta p_j - \zeta_j - r_j| + |q\delta\varphi_j - \delta p_j| < \varepsilon.$$

Thus, we may let $k = q\ell$. ◀

► **Theorem 25.** *Let $\lambda_1, \dots, \lambda_N$ and $\alpha_1, \dots, \alpha_N$ be complex numbers of modulus 1. Let $1/2 > \varepsilon > 0$ be a positive real number. Assume that $\lambda_1, \dots, \lambda_N$ are algebraic numbers such that the numbers $\log \lambda_1, \dots, \log \lambda_N, 2\pi i$ are linearly independent over \mathbb{Q} . Let d be the degree of the field extension $\mathbb{Q}(\lambda_1, \dots, \lambda_N)$ over \mathbb{Q} . Let $A_1, \dots, A_N \geq \exp(1)$ be upper bounds on the heights of $\lambda_1, \dots, \lambda_N$. Then there exists a positive integer*

$$t \leq 8\pi \left(\frac{2\pi}{\varepsilon}\right)^N \left(2N \left(\frac{2\pi}{\varepsilon}\right)^N \left\lceil \frac{4\pi}{\varepsilon} \log \frac{4\pi N}{\varepsilon} \right\rceil\right)^{(16(N+1)d)^{2(N+3)} \log A_1 \cdots \log A_N}$$

such that $|\lambda_j^t - \alpha_j| < \varepsilon$ for all $j \in \{1, \dots, N\}$.

Proof. Let \log denote the determination of the logarithm where the imaginary part of $\log z$ is in the interval $[0, 2\pi)$. Write $\log \lambda_j = 2\pi i \vartheta_j$ and $\log \alpha_j = 2\pi i \beta_j$. Let $B = \left(\frac{2\pi}{\varepsilon}\right)^N$. Using Theorem 21, choose integers q, p_1, \dots, p_N with $1 \leq q \leq B$ such that

$$|q\vartheta_j - p_j| < \frac{1}{B^{1/N}} = \frac{\varepsilon}{2\pi}.$$

Let

$$M = \lceil \frac{4\pi}{\varepsilon} \log \frac{4\pi N}{\varepsilon} \rceil.$$

Let $m \in \mathbb{Z}^N \setminus \{0\}$ with $|m| \leq M$. Then

$$\begin{aligned} & |m_1(q\vartheta_1 - p_1) + \cdots + m_N(q\vartheta_N - p_N)| \\ &= \frac{1}{2\pi} |m_1(q2\pi i\vartheta_1 - p_1 2\pi i) + \cdots + m_N(q2\pi i\vartheta_N - p_N 2\pi i)| \\ &= \frac{1}{2\pi} |qm_1 2\pi i\vartheta_1 + \cdots + qm_N 2\pi i\vartheta_N - (m_1 p_1 + \cdots + m_N p_N) 2\pi i| \\ &= \frac{1}{2\pi} |qm_1 \log \lambda_1 + \cdots + qm_N \log \lambda_N - 2(m_1 p_1 + \cdots + m_N p_N) \log(-1)|. \end{aligned}$$

By assumption the above quantity is non-zero, so Theorem 11 yields a uniform lower bound

$$\delta = \frac{1}{2\pi} \mathcal{B}^{-(16(N+1)d)^{2(N+3)} \log A_1 \cdots \log A_N},$$

where \mathcal{B} is a bound on the size of the coefficients qm_j and $2(m_1 p_1 + \cdots + m_N p_N)$. We have by construction $|q| \leq B$ and $|m_j| \leq M$. Since $\theta_j \leq 1$ we may choose $p_j \leq q \leq B$. It follows that we may choose $\mathcal{B} = 2NMB$.

We have hence established an estimate

$$|m_1(q\vartheta_1 - p_1) + \cdots + m_N(q\vartheta_N - p_N)| > \delta > 0$$

for all $m \in \mathbb{Z}^N \setminus \{0\}$ with $|m| \leq M$. Now, Theorem 10 asserts the existence of a real number $t_0 \in [0, 4/\delta]$ and integers $s_1, \dots, s_N \in \mathbb{Z}$ such that

$$|(q\vartheta_j - p_j)t_0 - \beta_j - s_j| < \frac{\varepsilon}{4\pi}.$$

Corollary 24 yields the existence of a positive integer $t \leq \frac{4}{\delta} \left(\frac{2\pi}{\varepsilon}\right)^N$ and $r_1, \dots, r_N \in \mathbb{Z}$ such that

$$|\vartheta_j t - \beta_j - r_j| < \frac{\varepsilon}{2\pi}.$$

By the mean value inequality it follows that

$$|\lambda^t - \alpha_j| < \varepsilon.$$

◀

D.1 Admitting integer multiplicative relations

► **Proposition 26.** *Given complex algebraic numbers $\lambda_1, \dots, \lambda_m$ of modulus 1 we can compute in polynomial time positive integers $1 \leq s \leq m$, $1 \leq j_1 \leq \cdots \leq j_s \leq m$, $\ell \in \mathbb{N}$, and multi-indexes $\varepsilon_j \in \mathbb{Z}^s$ for $j = 1, \dots, m$ such that $\lambda_{j_1}, \dots, \lambda_{j_s}$ do not admit any integer multiplicative relations and $\lambda_j^\ell = (\lambda_{j_1}, \dots, \lambda_{j_s})^{\varepsilon_j}$ for $j = 1, \dots, m$.*

Proof. By Theorem 7 we can compute in polynomial time a finite sequence of multi-indexes $\beta_1, \dots, \beta_{m-s}$ such that the free Abelian group

$$L = \{\alpha \in \mathbb{Z}^m \mid (\lambda_1, \dots, \lambda_m)^\alpha = 1\}$$

is generated by β_1, \dots, β_m . Further, the size of the β_j 's is bounded polynomially in the sum of the heights and degrees of $\lambda_1, \dots, \lambda_m$.

Bring the matrix with rows $\beta_1, \dots, \beta_{m-s}$ into upper triangular form. This can be done in polynomial time. This yields indexes j_1, \dots, j_s , positive numbers ℓ_1, \dots, ℓ_m , and multi-indexes η_1, \dots, η_m such that

$$\lambda_j^{\ell_j} = (\lambda_{j_1}, \dots, \lambda_{j_s})^{\eta_j}.$$

Let $\ell = \text{lcm}(\ell_1, \dots, \ell_m)$ and $\varepsilon_j = \ell/\ell_j \eta_j$. ◀

Note that the bitsize of ℓ and $\varepsilon_1, \dots, \varepsilon_m$ are bounded polynomially in the input data, but their total size may be exponential.

► **Proposition 27.** *Let $1 \leq s \leq m$ be positive integers. Let $\ell \in \mathbb{Z}$. Let $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$ be multi-indexes. Let*

$$f: \mathbb{T}^s \rightarrow \mathbb{T}^m, f(z_1, \dots, z_s) = (z_1^\ell, \dots, z_s^\ell, (z_1, \dots, z_s)^{\varepsilon_{s+1}}, \dots, (z_1, \dots, z_s)^{\varepsilon_m}).$$

Then, with respect to the ℓ^∞ -norm, f is Lipschitz-continuous with Lipschitz constant

$$\max \left\{ \ell, \sum_{k=1}^s |\varepsilon_{s+1,k}|, \dots, \sum_{k=1}^s |\varepsilon_{m,k}| \right\}.$$

Proof. Observe that f extends to a differentiable function of type $\mathbb{C}^s \rightarrow \mathbb{C}^m$. Let $(z_1, \dots, z_s) \in \mathbb{D}^s$ be a point in the unit polydisk. Let $Df(z_1, \dots, z_s)$ denote the Jacobian of f at (z_1, \dots, z_s) . By the mean value inequality it suffices to compute a bound on the operator norm of $Df(z_1, \dots, z_s)$. An elementary calculation shows

$$\|Df(z_1, \dots, z_s)\|_\infty = \max \left\{ \ell, \sum_{k=1}^s |\varepsilon_{s+1,k}|, \dots, \sum_{k=1}^s |\varepsilon_{m,k}| \right\}.$$

► **Proposition 28.** *Let $1 \leq s \leq m$ be positive integers. Let $\ell \in \mathbb{Z}$. Let $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$ be multi-indexes. Let $\lambda_1, \dots, \lambda_s$ be complex algebraic numbers of modulus 1 which do not admit any integer multiplicative relations. Let*

$$f: \mathbb{T}^s \rightarrow \mathbb{T}^m, f(z_1, \dots, z_s) = (z_1^\ell, \dots, z_s^\ell, (z_1, \dots, z_s)^{\varepsilon_{s+1}}, \dots, (z_1, \dots, z_s)^{\varepsilon_m}).$$

Then the closure of the sequence $(f(\lambda_1^t, \dots, \lambda_s^t))_{t \in \mathbb{N}}$ is equal to the range of f over \mathbb{T}^s .

Proof. By Kronecker's theorem 6 the sequence $((\lambda_1^t, \dots, \lambda_s^t))_{t \in \mathbb{N}}$ is dense in the torus \mathbb{T}^s . Let $A = \{(\lambda_1^t, \dots, \lambda_s^t) \mid t \in \mathbb{N}\}$. Since f is continuous we have $f(\overline{A}) \supseteq \overline{f(A)}$. Since \mathbb{T}^s is compact, the range $f(\mathbb{T}^s)$ is closed, so that we have $f(\mathbb{T}^s) \supseteq \overline{f(A)}$. It follows that $f(\overline{A}) = f(\mathbb{T}^s) = \overline{f(A)}$. ◀

► **Theorem 29.** *Let $(\lambda_1, \dots, \lambda_m)$ be complex algebraic numbers of modulus 1. Assume that the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_s$ are linearly independent over the rationals, where $0 \leq s \leq m$. Let d be the degree of the field extension $\mathbb{Q}(\lambda_1, \dots, \lambda_s)$. Let $A_1, \dots, A_s \geq \exp(1)$ be upper bounds on the heights of $\lambda_1, \dots, \lambda_s$. Let $\ell \in \mathbb{N}$, and $\varepsilon_{s+1}, \dots, \varepsilon_m \in \mathbb{Z}^s$ be such that*

$$\lambda_j^\ell = (\lambda_1, \dots, \lambda_s)^{\varepsilon_j}$$

for all $j = s+1, \dots, m$. By convention, if $s = 0$ the right-hand side of the above equation is to be taken equal to 1.

Let

$$L = \max \left\{ \ell, \sum_{k=1}^s |\varepsilon_{s+1,k}|, \dots, \sum_{k=1}^s |\varepsilon_{m,k}| \right\}.$$

Let $\alpha_1, \dots, \alpha_m \in \mathbb{T}^m$ be such that any rational linear relation between the numbers $2\pi i, \log \lambda_1, \dots, \log \lambda_m$ is also satisfied by the numbers $2\pi i, \log \alpha_1, \dots, \log \alpha_m$. Let $\varepsilon > 0$. Then there exists a positive integer

$$t \leq 8\pi\ell \left(\frac{2\pi L}{\varepsilon} \right)^s \left(2s \frac{2\pi L}{\varepsilon} \left\lceil \frac{4\pi L}{\varepsilon} \log \frac{4\pi s L}{\varepsilon} \right\rceil \right)^{(16(s+1)d)^{2(s+3)} \log A_1 \cdots \log A_s} + \ell$$

such that $|\lambda_j^t - \alpha_j| < \varepsilon$ for $j = 1, \dots, m$.

Proof. Divide the sequence $(\lambda_1^t, \dots, \lambda_m^t)_{t \in \mathbb{N}}$ into ℓ disjoint subsequences

$$(\lambda_1^{\ell t+j}, \dots, \lambda_m^{\ell t+j})_{t \in \mathbb{N}} = (\lambda_1^j \lambda_1^{\ell t}, \dots, \lambda_s^j \lambda_s^{\ell t}, \lambda_{s+1}^j (\lambda_1, \dots, \lambda_s)^{t\varepsilon_{s+1}}, \dots, \lambda_m^j (\lambda_1, \dots, \lambda_s)^{t\varepsilon_m})_{t \in \mathbb{N}}.$$

for $j = 0, \dots, \ell - 1$.

By Proposition 27, the closure of the sequence $(\lambda_1^{\ell t+j}, \dots, \lambda_m^{\ell t+j})_t$ is the set

$$C_j = \left\{ (\lambda_1^j z_1^e, \dots, \lambda_s^j z_s^e, \lambda_{s+1}^j (z_1, \dots, z_s)^{\varepsilon_{s+1}}, \dots, \lambda_m^j (z_1, \dots, z_s)^{\varepsilon_m}) \mid (z_1, \dots, z_s) \in \mathbb{T}^s \right\}.$$

Hence, the closure of the sequence $(\lambda_1^t, \dots, \lambda_m^t)_t$ is the union of the sets C_j .

Let (z_1, \dots, z_m) be contained in the closure of the sequence $(\lambda_1^t, \dots, \lambda_m^t)$. Let j be such that $(z_1, \dots, z_m) \in C_j$.

Since the numbers λ_j have modulus 1, the function

$$f_j(z_1, \dots, z_s) = \left(\lambda_1^j z_1^\ell, \dots, \lambda_s^j z_s^\ell, \lambda_{s+1}^j (z_1, \dots, z_s)^{\varepsilon_{s+1}}, \dots, \lambda_m^j (z_1, \dots, z_s)^{\varepsilon_m} \right)$$

is Lipschitz-continuous with Lipschitz constant L . Let $t(\varepsilon)$ denote the bound from Theorem 25, as a function of ε . By definition there exists $t \leq t(\varepsilon/L)$ such that

$$|f_j(\lambda_1^t, \dots, \lambda_s^t) - (z_1, \dots, z_m)| < \varepsilon.$$

We have $f_j(\lambda_1^t, \dots, \lambda_s^t) = (\lambda_1^{\ell t+j}, \dots, \lambda_m^{\ell t+j})$. The sequence index $\ell t + j$ is smaller than $\ell(t(\varepsilon/L) + 1)$. \blacktriangleleft

E Proof of Lemma 5

► **Lemma 30.** *There exists an integer function $\text{Sep}(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ with the following property:*

Let K and L be compact semialgebraic sets of complexity at most (n, d, τ) . Assume that every $x \in K$ has positive euclidean distance to L . Then $\inf_{x \in K} \text{dist}_{\ell^2}(x, L) > 1/\text{Sep}(n, d, \tau)$.

Proof. If either K or L are empty then the result is trivial. Thus, let us assume that both sets are non-empty.

Consider the semialgebraic set

$$S = \left\{ (x, y) \in K \times L \mid \forall z \in L. \left(\|x - z\|_2^2 \geq \|x - y\| \right) \right\}.$$

By Theorem 2 the set S has complexity $(2n, (d\tau)^{n^{O(1)}}, (d, \tau)^{n^{O(1)}})$. By compactness, the distance $\text{dist}_{\ell^2}(x, L)$ is attained in a point $y \in L$ for all $x \in K$, so that for all $x \in K$ there exists $y \in L$ such that $(x, y) \in S$.

We clearly have

$$\inf_{x \in K} \text{dist}_{\ell^2}(x, L) = \inf_{(x,y) \in S} \|x - y\|_2^2. \quad (1)$$

The right-hand side of (1) is a polynomial, so that the result follows from Theorem 4. ◀

F Proof of Lemma 14

We will first prove the following weaker version of Lemma 14, where we only establish an escape bound and a lower bound on the distance to K for initial points in K_{rec} .

► **Lemma 31.** *There exists an integer function $\text{Rec}_0(n, d, \tau) \in 2^{(\tau d)^{n^{O(1)}}}$ with the following property:*

Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form. Assume that the minimal polynomial of A has rational coefficients whose bitsize is bounded by τ . Let $K \subseteq \mathbb{R}^n$ be a semialgebraic set of complexity at most (n, d, τ) . If every point $x \in K_{\text{rec}}$ escapes K under iterations of A then for all $x \in K_{\text{rec}}$ there exists $t \leq \text{Rec}_0(n, d, \tau)$ such that

$$\text{dist}_{\ell^2}(A^t x, K) > \frac{\sqrt{n}}{\text{Rec}_0(n, d, \tau)}.$$

For $x \in V_{\text{rec}}$, let

$$\mathcal{O}_A(x) = \{A^t x \mid t \in \mathbb{N}\}$$

denote the orbit of x under A . Let $\overline{\mathcal{O}_A(x)}$ denote its closure. By Kronecker's theorem 6, the set $\overline{\mathcal{O}_A(x)}$ is semialgebraic.

The sequence $(A^t x)_{t \in \mathbb{N}}$ is dense in $\overline{\mathcal{O}_A(x)}$ by definition. A combination of Theorem 12 and Theorem 3 yields a quantitative refinement of this qualitative statement:

► **Lemma 32.** *There exists an integer function $D(n, d, \tau, P) \in 2^{(\tau P d)^{n^{O(1)}}}$ with the following property:*

Let A be a matrix in real Jordan normal form. Assume that the characteristic polynomial of A has rational coefficients whose bitsize is bounded by τ . Let K be a compact semialgebraic set of complexity at most (n, d, τ) . Let P be a positive integer. Then for all $x \in K_{\text{rec}}$ and all $y \in \overline{\mathcal{O}_A(x)}$ there exists $t \leq D(n, d, \tau, P)$ such that

$$\|A^t x - y\|_2 < 2^{-P}.$$

Proof. Let $\text{Kron}(n, \tau, P) \in 2^{(\tau P)^{n^{O(1)}}}$ be the function from Corollary 13. Let $\text{Bound}(n, d, \tau) = 2^{\tau d^{\beta(n+1)}}$, where β is the constant from Theorem 3.

Put

$$D(n, d, \tau, P) = \text{Kron}(n, \tau, P + \lceil \log(n) + \log(\text{Bound}(n, d, \tau)) \rceil).$$

It is easy to see that $D(n, d, \tau, P) \in 2^{(\tau P d)^{n^{O(1)}}}$ as claimed.

To prove that D has the desired properties, let A and K be a matrix and a compact semialgebraic set as above. Let P be a positive integer. For a matrix $Q = (q_{i,j})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ Let

$$\|Q\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n q_{i,j}^2 \right)^{1/2}$$

denote the Frobenius norm of Q . The Frobenius norm is sub-multiplicative and hence satisfies

$$\|Q \cdot x\|_2 \leq \|Q\|_F \cdot \|(x, \dots, x)\|_F = \|Q\|_F \cdot \sqrt{n} \cdot \|x\|_2$$

for all $x \in \mathbb{R}^n$.

Let $C = \text{Bound}(n, d, \tau)$. Let $x \in K_{\text{rec}}$. Let $y \in \overline{\mathcal{O}_A(x)}$. Then, by Corollary 13, there exist $t \leq \text{Kron}(n, \tau, P + \lceil \log(n) + \log(\text{Bound}(n, d, \tau)) \rceil)$ and $Q \in \mathbb{R}^{n \times n}$ such that $y = Qx$ and

$$\|A^t - Q\|_F < 2^{-P - \lceil \log(n) + \log(C) \rceil} \leq \frac{2^{-P}}{\sqrt{n}C}$$

Hence:

$$\|A^t x - y\|_2 = \|(A^t - Q)x\|_2 \leq \|A^t - Q\|_F \cdot \sqrt{n} \cdot |x| < 2^{-P}.$$

The last inequality uses that by 3 we have $|x| \leq C$. ◀

Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form. Assume that the characteristic polynomial of A has rational coefficients of bitsize at most τ . Let $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set of complexity at most (n, d, τ) . Assume that every point $x \in K_{\text{rec}}$ escapes K under iterations of A .

By definition, a point $x \in K_{\text{rec}}$ escapes K under iterations of A if and only if there exists $y \in \overline{\mathcal{O}_A(x)}$ such that $\text{dist}_{\ell^2}(y, K) > 0$. Our next goal is to sharpen this to a uniform lower bound on $\inf_{x \in K} \sup_{y \in \overline{\mathcal{O}_A(x)}} \text{dist}_{\ell^2}(y, K)$. The main idea is to employ Theorem 4. Since the function $g(x) = \sup_{y \in \overline{\mathcal{O}_A(x)}} \text{dist}_{\ell^2}(y, K)$ is not a polynomial, we construct an auxiliary compact semialgebraic set in higher dimension that allows us to reduce the problem of finding a uniform lower bound on $g(x)$ to the problem of finding a uniform lower bound on a polynomial. The idea is essentially the same as that of the proof of Lemma 5.

Let

$$\overline{\mathcal{O}_A} = \left\{ (x, y) \in \mathbb{R}^{2n} \mid x \in K_{\text{rec}}, y \in \overline{\mathcal{O}_A(x)} \right\}.$$

Let

$$S = \left\{ (x, y, z) \in \mathbb{R}^{3n} \mid x \in K_{\text{rec}}, y \in \overline{\mathcal{O}_A(x)}, z \in K \right\} = \overline{\mathcal{O}_A} \times K.$$

By compactness of K , the number

$$\eta = \min_{x \in K_{\text{rec}}} \max_{y \in \overline{\mathcal{O}_A(x)}} \min_{z \in K} \|y - z\|_2^2 \tag{2}$$

is strictly positive. Letting $D(n, d, \tau, P)$ denote the function from Lemma 32, observe that every point $x \in K_{\text{rec}}$ escapes K under at most

$$D(n, d, \tau, \lceil \log(1/\eta) \rceil)$$

iterations of A . To obtain a bound on the escape time it hence suffices to obtain a bound of (2) away from zero. This is achieved by expressing (2) as the minimum of a polynomial over a compact semialgebraic set. To this end, we consider the set

$$S' = \left\{ (x, y, z) \in \mathbb{R}^{3n} \mid x \in K_{\text{rec}}, y \in \overline{\mathcal{O}_A(x)}, z \in K, \forall w \in \overline{\mathcal{O}_A(x)}. \text{dist}_{\ell^2}(y, K) \geq \text{dist}_{\ell^2}(w, K) \right\}.$$

Observe that

$$\min_{x \in K_{\text{rec}}} \max_{y \in \overline{\mathcal{O}_A(x)}} \min_{z \in K} \|y - z\|_2^2 = \min_{(x, y, z) \in S'} \|y - z\|_2^2. \tag{3}$$

This leads to two problems:

► **Lemma 33.** *Let M_1, \dots, M_N be 2×2 matrices with entries in a polynomial ring $\mathbb{Z}[x_1, \dots, x_k]$. Let d be a bound on the total degree of all matrix entries. Let τ be a bound on the bitsize of the coefficients of all matrix entries. Then the entries of the product $M_1 \cdots M_N$ have total degree at most Nd and coefficients bounded in bitsize by $N\tau + k(N-1)\log(d+1)$.*

Thus, the closure \mathcal{M} of $(A_0^t)_{t \in \mathbb{N}}$ is an algebraic set that can be described using at most n^2 polynomials in n^2 variables, each of which has its degree bounded by a polynomial in n and τ , and coefficients bounded in bitsize by a polynomial in n and τ .

We can hence describe $\overline{\mathcal{O}_A}$ by the following formula:

$$\exists a_{1,1}, \dots, a_{1,n}, \dots, a_{n,1}, \dots, a_{n,n} \cdot \left((x_1, \dots, x_n) \in K \wedge \left(\bigwedge_{i=m_1+m_2+m_3+1}^n (x_i = 0) \right) \wedge (a_{i,j})_{i,j=1,\dots,n} \in \mathcal{M} \wedge \left(\bigwedge_{i=1}^n \left(y_i = \sum_{j=1}^n a_{i,j} x_j \right) \right) \right).$$

This formula involves polynomials in $n^2 + n$ variables. Their degrees are bounded by an expression in $(dn\tau)^{O(1)}$ and their coefficients are bounded in bitsize by an expression in $(n\tau)^{O(1)}$. By applying singly-exponential quantifier elimination (Theorem 2) we obtain that $\overline{\mathcal{O}_A}$ can be defined by a quantifier free formula involving polynomials of degree at most $(d\tau)^{n^{O(1)}}$ whose coefficients have bitsize at most $(d\tau)^{n^{O(1)}}$. The same bounds hold true for the complexity of the set $S = \overline{\mathcal{O}_A} \times K$.

Let us now bound the complexity of the set S' . For $w, y \in \overline{\mathcal{O}_A(x)}$ the predicate $\text{dist}_{\ell^2}(y, K) \geq \text{dist}_{\ell^2}(w, K)$ is expressed by the following first-order formula:

$$\exists m, n, v. (v \in K \wedge n \in K \wedge v \in K \wedge (d_{\ell^2}(y, v) < d_{\ell^2}(y, m) \vee d_{\ell^2}(w, v) < d_{\ell^2}(w, n) \vee d_{\ell^2}(y, m) \geq d_{\ell^2}(w, n))).$$

Let us write this as $\exists m, n, v. \Phi(y, w, m, n, v)$. Hence, the predicate $\forall w \in \overline{\mathcal{O}_A(x)}. \text{dist}_{\ell^2}(y, K) \geq \text{dist}_{\ell^2}(w, K)$ can be written as

$$\Psi(x, y) = \forall w. \exists m, n, v. (w \notin \overline{\mathcal{O}_A(x)} \vee \Phi(y, w, m, n, v)).$$

Thus,

$$S' = \{(x, y, z) \in \mathbb{R}^{3n} \mid (x, y, z) \in S \wedge \Psi(x, y)\}.$$

The formula $\Psi(x, y)$ involves polynomials of degree at most $(d\tau)^{n^{O(1)}}$ and coefficients bounded in bitsize by $(d\tau)^{n^{O(1)}}$. Now, Theorem 2 yields a complexity bound for S' of at most $(3n, (d\tau)^{n^{O(1)}}, (d\tau)^{n^{O(1)}})$.

By Theorem 4 we obtain the existence of a function

$$\text{LowerBound}'(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$$

such that the minimum of the polynomial (3) over the set S' is bounded from below by $\frac{2\sqrt{n}}{\text{LowerBound}'(n, d, \tau)}$. This means that for all $x \in K_{\text{rec}}$ there exists $y \in \overline{\mathcal{O}_A(x)}$ such that

$$\text{dist}_{\ell^2}(y, K) > \frac{2\sqrt{n}}{\text{LowerBound}'(n, d, \tau)}.$$

Now, consider the function

$$\text{Rec}_0(n, d, \tau) = \max\{\text{LowerBound}'(n, d, \tau), D(n, d, \tau, \lceil \log(\text{LowerBound}'(n, d, \tau)) \rceil)\} \in 2^{(d\tau)^{n^{O(1)}}}.$$

We claim that the function Rec_0 has the property stated in Lemma 31. Let $x \in K_{\text{rec}}$. Let $y \in \overline{\mathcal{O}_A}(x)$ with $\text{dist}_{\ell^2}(y, K) > \frac{2\sqrt{n}}{\text{LowerBound}'(n, d, \tau)}$. Then there exists $t \leq \text{Rec}_0(n, d, \tau)$ such that

$$\|A^t x - y\|_2 < \frac{1}{\text{LowerBound}'(n, d, \tau)} \leq \frac{\sqrt{n}}{\text{LowerBound}'(n, d, \tau)}.$$

For all $z \in K$ we then have

$$\|A^t x - z\|_2 > \|y - z\|_2 - \|A^t x - y\|_2 \geq \frac{\sqrt{n}}{\text{LowerBound}'(n, d, \tau)} > \frac{\sqrt{n}}{\text{Rec}(n, d, \tau)}.$$

This concludes the proof of Lemma 31. It remains to extend the result to all initial points $x \in V_{\text{rec}}$. We first treat the special case where K_{rec} is empty.

► **Lemma 34.** *There exists an integer function $\text{EmptyBound}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$ with the following property:*

Assume that K_{rec} is empty. Let $x \in V_{\text{rec}}$. Then $\text{dist}_{\ell^2}(x, K) > 1/\text{EmptyBound}(n, d, \tau)$.

Proof. The function $\text{dist}_{\ell^2}(\cdot, V_{\text{rec}})$ is linear, and thus in particular a polynomial. By assumption,

$$\inf_{x \in K} \text{dist}_{\ell^2}(x, V_{\text{rec}}) > 0.$$

Now, Theorem 4 yields a lower bound of the desired shape. ◀

We begin with a technical lemma, which combines quantifier elimination with Lemma 5.

► **Lemma 35.** *Assume that K_{rec} is non-empty. Let C be a positive integer. Further assume that K_{rec} is contained in a ball of radius $2^{(d\tau)^{n^C}}$. Let $x \in V_{\text{rec}}$. If $\text{dist}_{\ell^2}(x, K_{\text{rec}}) > 2^{-(d\tau)^{n^C}}$ then $\text{dist}_{\ell^2}(x, K) > 2^{-(d\tau)^{n^{C+O(1)}}}$.*

Proof. Consider the function $\text{dist}_{\ell^2}(x, K)$ on the set

$$L = \left\{ x \in V_{\text{rec}} \mid \|x\|_2 \leq 2^{(d\tau)^{n^C}} \wedge \text{dist}_{\ell^2}(x, K_{\text{rec}}) \geq 2^{-(d\tau)^{n^C}} \right\}.$$

This set can be defined by the following first-order formula:

$$\forall (y_1, \dots, y_n) \in \mathbb{R}^n. \forall (b_0, \dots, b_m) \in \mathbb{R}^{n^C}. \\ \left((y_1, \dots, y_n) \in K_{\text{rec}} \wedge b_0 = 2 \wedge b_{i+1} = b_i^{(d\tau)} \right) \rightarrow \|x\|_2^2 \leq b_m^2 \wedge \|x - y\|_2^2 \geq (1/b_m)^2$$

Applying quantifier elimination (Theorem 2), we find that the set L can be defined by a quantifier-free formula involving polynomials of degree $(d\tau)^{O(n^C)}$ whose coefficients are bounded in bitsize by $(d\tau)^{O(n^C)}$.

It follows from Lemma 5 that every point in L has distance from K at least

$$1/\text{Sep}(n, (d\tau)^{O(n^C)}, (d\tau)^{O(n^C)}) = 2^{\left(-(d\tau)^{O(n^C)} \right)^{n^{O(1)}}} = 2^{-(d\tau)^{n^{C+O(1)}}}$$

◀

Let us now turn to the proof of Lemma 5.

The function $\text{Rec}(n, d, \tau)$ is majorised by $2^{(d\tau)^{nR}}$ for some constant R . Let $C \geq R + 1$ be such that K is contained in a ball of radius $2^{(d\tau)^{nC}}$. Such a constant exists by Theorem 3. Let D be a constant such that all $x \in V_{\text{rec}}$ with $\text{dist}_{\ell^2}(x, K_{\text{rec}}) > 2^{-(d\tau)^{nC}}$ satisfy $\text{dist}_{\ell^2}(x, K) > 2^{-(d\tau)^{n^{C+D}}}$. The constant D exists thanks to Lemma 35.

Let

$$\text{Rec}(n, d, \tau) = \max \left\{ \text{EmptyBound}(n, d, \tau), \left\lceil \sqrt{n} 2^{(d\tau)^{n^{C+D}}} \right\rceil, \left\lceil \frac{\sqrt{n} \text{Rec}_0(n, d, \tau)}{\sqrt{n} - 2^{-(d\tau)^{nC}} \text{Rec}_0(n, d, \tau)} \right\rceil \right\}$$

Then, since we have chosen $C \geq R + 1$, we have $\text{Rec}(n, d, \tau) \in 2^{(d\tau)^{n^{O(1)}}}$.

Let us now show that Rec has the desired property. If K_{rec} is empty, then Rec has the desired property by construction. Let us hence assume for the rest of the proof that K_{rec} is non-empty.

Let $x \in V_{\text{rec}}$. If $\text{dist}_{\ell^2}(x, K) > 2^{-(d\tau)^{C+D}}$ then $\text{dist}_{\ell^2}(x, K) > \sqrt{n}/\text{Rec}(n, d, \tau)$

Assume that $\text{dist}_{\ell^2}(x, K) \leq 2^{-(d\tau)^{C+D}}$. Then by 35 we have $\text{dist}_{\ell^2}(x, K_{\text{rec}}) \leq 2^{-(d\tau)^{nC}}$. Choose $y \in K_{\text{rec}}$ such that $\|x - y\|_2 < 2^{-(d\tau)^{nC}}$. Then, by Lemma 31 there exists $t \leq \text{Rec}_0(n, d, \tau)$ such that

$$\text{dist}_{\ell^2}(A^t y, K) > \sqrt{n}/\text{Rec}_0(n, d, \tau).$$

Since A is an isometry on V_{rec} , we have

$$\|A^t x - A^t y\|_2 = \|x - y\|_2 \leq 2^{-(d\tau)^{nC}}.$$

It follows that

$$\text{dist}_{\ell^2}(A^t x, K) > \sqrt{n}/\text{Rec}_0(n, d, \tau) - 2^{-(d\tau)^{nC}} > \sqrt{n}/\text{Rec}(n, d, \tau).$$

G Proof of Lemma 15

Let J_k be a real Jordan block of multiplicity k corresponding to either a real eigenvalue Λ or a complex pair $\Lambda = a \pm ib$. We use t to denote positive integer time-steps.

$$J_k^t = \begin{bmatrix} \Lambda^t & t\Lambda^{t-1} & \binom{t}{2}\Lambda^{t-1} & \dots & \binom{t}{k-1}\Lambda^{t-k+1} \\ 0 & \Lambda^t & t\Lambda^{t-1} & \dots & \binom{t}{k-2}\Lambda^{t-k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t\Lambda^{t-1} \\ 0 & 0 & 0 & \dots & \Lambda^t \end{bmatrix}.$$

where Λ can be considered $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ or a scalar quantity depending on the type of eigenvalue.

Thus we see that $x(t) := J_k^t x$ can be written component-wise as

$$x_j(t) = \sum_{i=j}^k \binom{t}{i-j} \Lambda^{t-(i-j)} x_i,$$

where the ‘components’ $x_1 \dots x_k$ refer to scalars (if the eigenvalue is real) or vectors $(x_j^{(r)}, x_j^{(i)})$ if the eigenvalue is complex.

We will use absolute value signs to denote the Jordan norm $\|\cdot\|_J$ of a component x_j . Note this is the same as the ℓ_2 norm since we consider only one component.

Then we have the following three lemmas, where we case split on the modulus of the eigenvalue Λ , which we notate as γ , to obtain block-wise bounds on escape times for non-recurrent eigenspaces:

These lemmas have the same structure as those in [10], however for convenience we present the discrete case in full here, as [10] formulates them in the continuous case.

► **Lemma 36** (Polynomially expanding Jordan block - $\gamma = 1$). *Let C be a positive number such that K is contained in the ℓ^2 -ball centred at 0 with radius C . Let $x \in K$. Assume there exists $j \geq 2$ with $|x_j| > \varepsilon$ in the Jordan block. Let $N = \frac{1}{k} \left(\frac{k^2 C}{\varepsilon} \right)^{2^{k-1}}$. Then there exists j and $t \leq N$ with $|x_j(t)| > C$.*

Proof. Given the set of equations

$$x_j(t) = \sum_{i=j}^k \binom{t}{i-j} \Lambda^{t-(i-j)} x_i,$$

we want an N such that there exists j such that $|x_j(N)| > C$.

Let $j_1 \geq 2$ be the smallest j such that $|x_{j_1}| > \varepsilon$ (we are given that such a component exists). Consider the component

$$x_{j_1-1}(t) = \sum_{i=j_1-1}^k \binom{t}{i-j_1+1} \Lambda^{t-(i-j_1+1)} x_i.$$

We set $N_{j_1} = kC/\varepsilon$. Observe that (note $\gamma = 1$)

$$|x_{j_1-1}(N_{j_1})| \geq |(kC/\varepsilon)x_{j_1}| - |x_{j_1-1}| - \sum_{i=j_1+1}^k \left| \binom{N_{j_1}}{i-j_1+1} \Lambda^{t-(i-j_1+1)} x_i \right|$$

Since the first term is larger than kC and the second term is smaller than C , the only way $|x_{j_1-1}(N_{j_1})|$ could be less than C (and thus not escape K) is if one of the later terms is larger than C . Let j_2 be the highest index such that $\left| x_{j_2} \binom{N_{j_1}}{j_2-j_1+1} \right| \geq C$. Note that $j_2 > j_1$. We now have a lower bound on a higher index coefficient, namely $|x_{j_2}| \binom{N_{j_1}}{j_2-j_1+1} \geq C$. We now repeat the process with the component

$$x_{j_2-1}(t) = \Lambda^t x_{j_2-1} + \binom{t}{1} \Lambda^{t-1} x_{j_2} + \sum_{i=j_2+1}^k \binom{t}{i-j_2+1} \Lambda^{t-(i-j_2+1)} x_i$$

We have $|x_{j_2}| \binom{N_{j_1}}{j_2-j_1+1} \geq C$ thus setting $N_{j_2} > k \binom{N_{j_1}}{j_2-j_1+1}$ ensures that $|N_{j_2} x_{j_2}| > kC$.

Continuing this process, we will either find a component that escapes the set or move on to a component with higher index, which can happen at most $k-1$ times, because we have the constraints $j_1 \geq 2, \forall m, j_m \leq k$, and $j_m > j_{m-1}$. This gives us a recursive definition for the bound, which is

$$N_{j_m} > k \binom{N_{j_{m-1}}}{j_m - j_{m-1} + 1}$$

We wish to find an upper bound on $N = N_M$, the time by which we are guaranteed that at least one component escapes, subject to the constraints $j_1 \geq 2, j_M \leq k$, and $j_m > j_{m-1}$.

We can solve the recursive inequality by weakening it (since we only need an upper bound on N) to

$$N_{j_m} > (kN_{j_{m-1}})^{j_m - j_{m-1} + 1}.$$

Note that pulling the constant k into the exponentiated part is valid because $j_m - j_{m-1} + 1 > 2$ always. Setting $S_{j_m} = kN_{j_m}$, we get $S_{j_m} > S_{j_{m-1}}^{j_m - j_{m-1} + 1}$, $S_{j_1} = k^2C/\varepsilon$, which reduces to

$$S_{j_M} > \left(\frac{k^2C}{\varepsilon}\right)^{\prod_{m=2}^M (j_m - j_{m-1} + 1)}$$

The term $\prod_{m=2}^M (j_m - j_{m-1} + 1)$ is maximised when for all m , $j_m = j_{m-1} + 1$, thus in the worst case we have

$$S_{j_M} > \left(\frac{k^2C}{\varepsilon}\right)^{2^{k-1}}$$

Thus we have a bound for a modulus-1-eigenvalue component to escape, which is

$$N = \frac{1}{k} \left(\frac{k^2C}{\varepsilon}\right)^{2^{k-1}}.$$

◀

► **Lemma 37** (Exponentially expanding Jordan block - $\gamma > 1$). *Let C be a positive number such that K is contained in the ℓ^2 -ball centred at 0 with radius C . Let $x \in K$. Assume there exists $j \geq 2$ with $|x_j| > \varepsilon$ in the Jordan block. Let $N = 2^{k-1} \frac{\log(kC/\varepsilon)}{\log \gamma}$. Then there exists j and $t \leq N$ with $|x_j(t)| > C$.*

Proof. The proof is very similar in structure to the modulus-1-eigenvalue case, though the presence of an exponential factor gives us a much better bound.

By construction of ε , there exists $j_1 \geq 2$ such that $|x_{j_1}| > \varepsilon$. Consider the component

$$x_{j_1}(t) = \sum_{i=j_1}^k \binom{t}{i-j_1} \Lambda^{t-(i-j_1)} x_i.$$

Set $N_{j_1} = \frac{\log(kC/\varepsilon)}{\log \gamma}$ and observe that

$$|x_{j_1}(N_{j_1})| \geq \left| \Lambda^{\frac{\log(kC/\varepsilon)}{\log \gamma}} x_{j_1} \right| - \sum_{i=j_1+1}^k \left| \binom{N_{j_1}}{i-j_1} \Lambda^{N_{j_1}-(i-j_1)} x_i \right|.$$

Since the first term is larger than kC , the only way $|x_{j_1}(N_{j_1})|$ can be less than C (and thus not escape the set) is if one of the later terms is larger than C . Let j_2 be the highest index such that $\left| \binom{N_{j_1}}{j_2-j_1} \Lambda^{N_{j_1}-(j_2-j_1)} x_{j_2} \right| \geq C$. Note that $j_2 > j_1$. We now have a lower bound on a higher index coefficient, namely $\binom{N_{j_1}}{j_2-j_1} \gamma^{N_{j_1}-(j_2-j_1)} |x_{j_2}| \geq C$. Now we repeat the process with the component

$$x_{j_2}(t) = \Lambda^t x_{j_2} + \sum_{i=j_2+1}^k \binom{t}{i-j_2} \Lambda^{t-(i-j_2)} x_i$$

We want $|\Lambda^t x_{j_2}| > kC$, so it is enough to set

$$\gamma^{N_{j_2}} > k \binom{N_{j_1}}{j_2 - j_1} \gamma^{N_{j_1} - (j_2 - j_1)}.$$

Similarly to the previous lemma, this gives us a recursive definition for the bound, which is

$$N_{j_m} - N_{j_{m-1}} > \frac{\log \binom{N_{j_{m-1}}}{j_m - j_{m-1}}}{\log \gamma} + \frac{\log k}{\log \gamma} - (j_2 - j_1).$$

For N sufficiently larger than k (which we may easily assume), we can weaken this inequality to

$$N_{j_m} > 2N_{j_{m-1}} + \frac{\log k}{\log \gamma},$$

which is easily solved to get

$$N_M > 2^{M-1} N_{j_1} - \frac{\log k}{\log \gamma}.$$

As $M \leq k$ and $N_{j_1} = \frac{\log(kC/\varepsilon)}{\log \gamma}$ we have a bound for a positive-eigenvalue component to escape, which is

$$N \leq 2^{k-1} \frac{\log(kC/\varepsilon)}{\log \gamma}.$$

◀

► **Lemma 38** (Exponentially shrinking Jordan block: $\gamma < 1$). *Let $C \geq n$ be a positive number such that K is contained in the ℓ^2 -ball centred at 0 with radius C . Let $x \in K$. Let ε be a positive real number. Let $N = \frac{4k}{\log(1/\gamma)} \log \left(\frac{2k}{\log(1/\gamma)} \right) + \frac{2 \log(kC/\varepsilon)}{\log(1/\gamma)}$. Then there exists $t \leq N$ with $|x_j(t)| < \varepsilon$ for all j in the block.*

Proof. If $\gamma = 0$ then we have $x_j(n) = 0$ for all j in the block, so that we may assume $\gamma > 0$ for the rest of the proof.

We have the equations

$$x_j(t) = \sum_{i=j}^k \binom{t}{i-j} \Lambda^{t-(i-j)} x_i,$$

For any j , we have the result

$$|x_j(t)| \leq \sum_{i=j}^k \binom{t}{i-j} \cdot |\Lambda^{t-(i-j)} x_i| < kC(et/k)^k \gamma^{t-k},$$

where the second inequality is obtained via standard bounds on binomial coefficients

In order to have $|x_j(t)| < \varepsilon$, it is enough to have $kC(et/k)^k \gamma^{t-k} < \varepsilon$, which is equivalent (after weakening slightly by dropping irrelevant terms) to $t > \frac{k}{\log(1/\gamma)} \log t + \frac{\log(kC/\varepsilon)}{\log(1/\gamma)}$.

Here we need a small technical lemma.

► **Lemma 39** ([21, Lemma A.1, Lemma A.2]). *Suppose $a \geq 1$ and $b > 0$, then $t \geq a \log t + b$ if $t \geq 4a \log(2a) + 2b$.*

Applying this lemma we get a bound on N such that for all $j \leq k$, $x_j(N) < \varepsilon$, namely

$$N \leq \frac{4k}{\log(1/\gamma)} \log\left(\frac{2k}{\log(1/\gamma)}\right) + \frac{2\log(kC/\varepsilon)}{\log(1/\gamma)}.$$

◀

We now show that the time to leave $K_{\geq \varepsilon}$ is doubly exponential in the ambient dimension, singly exponential in the rest of the input data, and inverse polynomial in ε .

► **Lemma 40** (Non-recurrent overall bound). *There exists a positive integer constant L with the following property:*

Let K be a semialgebraic set of complexity at most (n, d, τ) . Let $A \in \mathbb{A}^{n \times n}$ be a matrix in real Jordan normal form. Assume that the characteristic polynomial of A has rational coefficients, bounded in bitsize by τ . Let ε be a positive real number. Define a partition of K into K_{rec} , $K_{< \varepsilon}$ and $K_{\geq \varepsilon}$ as described in Section 3. Let

$$N_{\geq \varepsilon} = \left(\frac{1}{\varepsilon}\right)^{2^n} \cdot 2^{(\tau \cdot d)^{Ln}}.$$

Then for all $x \in K_{\geq \varepsilon}$ there exists $t \leq N_{\geq \varepsilon}$ such that $A^t x \notin K_{\geq \varepsilon}$, which is to say, we have either escaped the set K completely or moved into $K_{< \varepsilon} \cup K_{\text{rec}}$.

Proof. Since $x \in K_{\geq \varepsilon}$, we start with at least one component greater than ε in Jordan norm. This allows us to leverage the block-wise bounds.

Let $N_{\geq \varepsilon} = 2 \max_{\gamma} \{N_{\gamma}\}$, where N_{γ} ranges over the possible bounds depending on the size of the eigenvalues of A . Within a time $N_{\geq \varepsilon}/2 = \max_{\gamma} \{N_{\gamma}\}$, thanks to the analysis of the block-wise bounds, there are three possibilities:

- the orbit has increased in size beyond C and has thus left K . This occurs if there was a component associated to an expanding eigenvalue that was larger than ε in Jordan norm;
- all components are now smaller than ε , thus leaving $K_{\geq \varepsilon}$ (by entering $K_{< \varepsilon}$);
- some component corresponding to an expanding eigenvalue which was originally less than ε has become greater than ε . In this case, waiting another $N_{\geq \varepsilon}/2$ amount of time puts the trajectory in the first case, ensuring it escapes.

Thus in all cases the trajectory has escaped by time $N_{\geq \varepsilon}$.

We now explicitly compute $N_{\geq \varepsilon}$ by using the complexity of K and A to bound the three possibilities, namely

$$N = \frac{4k}{\log(1/\gamma)} \log\left(\frac{2k}{\log(1/\gamma)}\right) + \frac{2\log(kC/\varepsilon)}{\log(1/\gamma)}.$$

(shrinking eigenvalue γ),

$$N = 2^{k-1} \frac{\log(kC/\varepsilon)}{\log \gamma}.$$

(exponentially expanding eigenvalue γ) and

$$N = \frac{1}{k} \left(\frac{k^2 C}{\varepsilon}\right)^{2^{k-1}}.$$

(modulus 1 eigenvalue), where k is the multiplicity of the Jordan block and thus $k \leq n$, the dimension.

We now compute bounds on $\log \gamma$ and C in terms of K and A .

Bounding $\log \gamma$: Let τ be a bound on the bitsize of the coefficients of the minimal polynomial of the eigenvalues of A as well as a bound on the total bitsize of K .

Using the fact that $x/2 < \log(1+x) < x$ for $|x| \ll 1$, Lemma 8 yields a constant R satisfying $\frac{1}{\log \gamma} < 2^{(\tau n)^R}$.

Bounding C : Let d bound the degree of the polynomials defining K . Then from Theorem 3 we have the existence of a constant S satisfying $C < 2^{(\tau \cdot d)^{S(n+1)}}$.

Plugging these bounds into the iteration bounds from the previous lemmas, and overapproximating for simplicity, we finally get the following bound: With a new constant Q based on R and S , we have

$$N_{\geq \varepsilon} \leq \left(\frac{1}{\varepsilon}\right)^{2^n} \cdot 2^{(2^n \cdot (\tau \cdot d)^{S(n+1)})} + (2 \cdot \tau \cdot d)^{(\tau n)^Q} + \log(1/\varepsilon) \cdot 2^{(\tau n)^Q}.$$

We can simplify this still further by amalgamating terms. Letting L be a new constant, we set

$$N_{\geq \varepsilon} = \left(\frac{1}{\varepsilon}\right)^{2^n} \cdot 2^{(\tau \cdot d)^{L n}}.$$

Thus the time to leave $K_{\geq \varepsilon}$ is doubly exponential in the dimension, singly exponential in the rest of the input data, and inverse polynomial in ε . ◀

H Example of matching lower bound

In Section 8, we matched the bound using a rotational system which needed a doubly exponential time to escape by the small hole in the circle. Here, we present another example where the doubly exponential bound comes from the size of the set we define.

► **Example 41.** The construction of our first family of instances $(K_{(n,d,\tau)}, A_{(n,d,\tau)})_{(n,d,\tau) \in \mathbb{N}^3}$ relies on the fact that one can define a compact semialgebraic set whose size is doubly-exponential in the ambient dimension.

For $(n, d, \tau) \in \mathbb{N}^3$, define $K_{(n,d,\tau)} \subseteq \mathbb{R}^{n+1}$ as the set of all points (x_1, \dots, x_n, x_u) satisfying the (in)equalities:

$$x_u = 1,$$

$$x_1 = 2^\tau,$$

$$\text{For } 1 \leq i \leq n-2, \ x_{i+1} = x_i^d,$$

$$0 \leq x_n \leq x_{n-1}^d.$$

Thus, a point $x \in \mathbb{R}^{n+1}$ belongs to $K_{(n,d,\tau)}$ if and only if it is of the form $(2^\tau, 2^{\tau d}, \dots, 2^{2^{\tau d^{n-2}}}, y, 1)$ where $y \in [0, 2^{\tau d^{n-1}}]$.

We now define $A_{(n,d,\tau)}$ to be the matrix which only adds 1 (through the coefficient x_u) to the penultimate coordinate:

$$A_{(n,d,\tau)} = \begin{pmatrix} 1 & 0 & \dots & & & \\ 0 & 1 & \dots & & & \\ \vdots & \vdots & \ddots & & & \\ & & & 1 & 1 & \\ & & & 0 & 1 & \end{pmatrix}$$

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Therefore, given an initial point $x_0 \in K_j$, we have that $x_t = A_{(n,d,\tau)}^t x_0 = x_0 + (0, \dots, 0, t, 0)$. This sequence obviously escapes. The point $(2^\tau, 2^{\tau d}, \dots, 2^{\tau d^{n-2}}, 0, 1) \in K_{(n,d,\tau)}$ requires $2^{\tau d^{n-1}+1}$ iterations to escape. This is doubly exponential in the ambient dimension and singly exponential in the rest of the data.