Alternating Timed Automata over Bounded Time

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***Abstract*—Alternating timed automata are a powerful extension of classical Alur-Dill timed automata that are closed under all Boolean operations. They have played a key role, among others, in providing verification algorithms for prominent specification formalisms such as Metric Temporal Logic. Unfortunately, when interpreted over an infinite dense time domain (such as the reals), alternating timed automata have an undecidable language emptiness problem.**

The main result of this paper is that, over bounded time domains, language emptiness for alternating timed automata is decidable (but nonelementary). The proof involves showing decidability of a class of parametric McNaughton games that are played over timed words and that have winning conditions expressed in the monadic logic of order augmented with the *distance-one* relation.

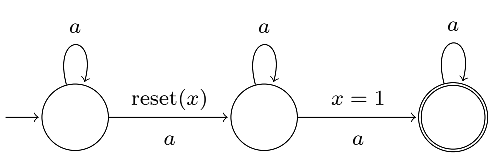
As a corollary, we establish the decidability of the time-bounded model-checking problem for Alur-Dill timed automata against specifications expressed as alternating timed automata.

***Keywords*-Alternation; Timed Automata; Church's Problem**

I. Introduction

Timed automata were introduced by Alur and Dill in [1] as a natural and versatile model for real-time systems. They have been widely studied ever since, both by practitioners and theoreticians. A celebrated result concerning timed automata, which originally appeared in [2], is the PSPACE decidability of the *language emptiness* (or *reachability*) problem.

Unfortunately, the *language inclusion* problem—given two timed automata $\mathcal{A}$ and $\mathcal{B}$, is every timed word accepted by $\mathcal{A}$ also accepted by $\mathcal{B}$?—is known to be undecidable. A related phenomenon is that timed automata are not closed under complementation. For example, the automaton below accepts every timed word in which there are two a-events separated by one time unit:



The complement automaton would have to accept a timed word precisely when no two a-events are separated by one time unit. Intuitively, this is not expressible by a timed automaton, since such an automaton would need a potentially unbounded number of clocks to keep track of the time delay from each a-event. We refer the reader to [3] for a formal analysis of these considerations.

In one sense, the non-closure under complementation is easy to remedy—one simply generalises the transition mode to allow both conjunctive and disjunctive transitions, an idea borrowed from the theory of untimed automata that dates back 30 years [4]. Such untimed *alternating automata* have played key roles in algorithms for complementing Büchi automata (see, e.g., [5]), temporal logic verification [6], [7], and analysis of parity games [8]. In the timed world, the resulting *alternating timed automata* [9], [10], [11], [12] subsume ordinary timed automata and can be shown to be closed under all Boolean operations. They have been used, among others, to provide model-checking algorithms for various fragments of Metric Temporal Logic (MTL); see, e.g., [9], [13], [14]. Unfortunately, the price to pay for the increase in expressiveness is the undecidability of language emptiness!

This undecidability follows immediately from the undecidability of universality for timed automata. The proof of the latter in [1] uses in a crucial way the unboundedness of the time domain. Roughly speaking, this allows one to encode arbitrarily long computations of a Turing machine. On the other hand, many verification questions are naturally phrased over bounded time domains [15], [16], [17]. For example, a run of a communication protocol might normally be expected to have an *a priori* time bound. In fact, most hard real-time problems, which typically involve deadlines, timeouts, and delays, are only pertinent over a finely circumscribed time span.

This leads us to consider the *time-bounded language emptiness problem* for alternating timed automata: given an alternating timed automaton $\mathcal{A}$ and a time bound N, is some finite timed word of duration at most N is accepted by $\mathcal{A}$? (Note that since we are working with a dense model of time, time-bounded words may still contain arbitrarily many events.) The main result of this paper is that this problem is decidable but nonelementary. Since alternating timed automata are closed under all Boolean operations, an immediate corollary is the decidability of the time-bounded model-checking problem for timed automata against specifications expressed as alternating timed automata.

Our proofs exploit the close correspondence between automata and monadic predicate logic. Büchi [18] and Rabin [19] have respectively proven decidability of monadic second-order logic (MSO) over the naturals and the infinite binary tree using translations of MSO to automata. Conversely, automata can easily be transformed into equivalent MSO formulas. The proof of our main result involves a translation from alternating timed automata to monadic predicate logic over the structure $(\mathbb{T}, <, +1)$, where $\mathbb{T}$ is a bounded interval of real numbers and $+1$ is the relation defined by $+1(x, y)$ if and only if $x+1 = y$. The $+1$ relation is used to encode timing constraints in automata.

We translate timed alternating automata into *games* over $(\mathbb{T}, <, +1)$ with winning conditions expressed in monadic predicate logic. The class of games that we obtain is a variation of that introduced by McNaughton [20] in connection with *Church's problem* [21], [22], [23]. Given an $\mathsf{MSO}(<)$ formula $\varphi(X, Y)$, Church's problem asks whether there exists a *causal* operator F such that $\forall X\, \varphi(X, F(X))$. ($F$ is a causal operator if the truth value of $F(X)(n)$ for a monadic predicate X and individual n only depends on the values of $X(m)$ for $m \leq n$.) Thus Church's problem generalises the satisfiability problem for MSO to a *uniformisation* problem. The games that we introduce can be generalised and used to analyse a natural extension of Church's problem for $\mathsf{MSO}(<, +1)$ over bounded intervals of the reals, but we do not pursue this direction here. For our current purpose the key result is a procedure to determine winners of games from this class. This enables us to establish the decidability of the language emptiness problem for alternating timed automata.

It is worth noting that we reduce the language emptiness problem for alternating timed automata to a *uniformisation* problem for *first-order* logic over $(\mathbb{T}, <, +1)$. By contrast, in the classical translations of (untimed) automata to monadic logic, language emptiness is reduced to a *satisfiability* problem for *second-order* logic over $(\mathbb{N}, <)$. In particular the paper [7] shows how to encode run dags of (untimed) alternating automata directly in $\mathsf{MSO}(<)$. It does not seem possible to give a corresponding reduction of the language emptiness problem for alternating timed automata to the satisfiability problem for monadic second-order logic over $(\mathbb{T}, <, +1)$. Intuitively this is because one cannot compute an *a priori* bound on the width of the run dags of a given alternating timed automaton. In Section VI we give a more formal argument that there is no 'reasonable' reduction of language emptiness for alternating timed automata to the satisfiability problem for $\mathsf{MSO}(<, +1)$.

Related Work. The quest for a decidable class of timed automata with good closure properties has led to a considerable body of work, including the introduction of determinisable subclasses of timed automata [24], restrictions to one-clock automata [10], [11], and bounded-variability semantics [25]. See also Henzinger *et al.*'s paper on *fully decidable* formalisms [26].

The present paper substantially generalises some of the main results of [27], in which we established the decidability of both the time-bounded language inclusion problem for timed automata (2EXPSPACE-complete), and the time-bounded MTL model-checking problem for timed automata (EXPSPACE-complete). Both decidability results now easily follow from our new Theorem 13, since MTL formulas can be encoded as one-clock alternating timed automata of a particular type [9]. Theorem 13, in which no restrictions whatsoever are placed on alternating timed automata, does not seem amenable to the proof techniques of [27] and instead requires novel game-theoretic tools. The increased expressiveness, however, comes at the price of a significant blow-up in complexity: from 2EXPSPACE to nonelementary, cf. Theorem 17.

There is an extensive body of work concerning games on timed automata and related timed-graph formalisms. This originates in [28], [29] and encompasses concurrent games [30] and tool support [31]. Turn-based games on timed automata can easily be encoded as McNaughton games in the sense of the present paper, thanks to the expressiveness of the logic $\mathsf{MSO}(<, +1)$. However the generality of our class of McNaughton games entails that our decidability results are restricted to bounded time domains, in contrast with [29].

A key aspect of our technical development is the use of *parametric* games. This is related to the works of [32], [33] on Church's problem with parameters.

II. Alternating Timed Automata

Let Σ be a finite alphabet and let $\mathbb{R}_+$ denote the set of non-negative reals. A *timed event* is a pair (a, t), where $t \in \mathbb{R}_+$ is called the *timestamp* of the *event* $a \in \Sigma$. A *timed word* is a finite sequence $w = (a_1, t_1)(a_2, t_2) \ldots (a_n, t_n)$ of timed events whose corresponding sequence of timestamps is strictly increasing. We denote by $\mathrm{untime}(w)$ the underlying untimed word $a_1 a_2 \ldots a_n$. A set of timed words is called a *timed language*. If $\mathbb{T} \subseteq \mathbb{R}_+$ then $\mathbb{T}\Sigma^*$ denotes the set of timed words $w = (a_1, t_1)(a_2, t_2) \ldots (a_n, t_n)$ such that $t_1, t_2, \ldots, t_n \in \mathbb{T}$. In this paper we are particularly interested in timed languages $L \subseteq \mathbb{T}\Sigma^*$ for bounded $\mathbb{T}$. For uniformity we assume that $\mathbb{T} = [0, N)$, where $N \in \mathbb{N}$, although our results can easily be adapted to allow $\mathbb{T}$ to be an arbitrary bounded interval.

A. Automata

Let C be a set of *clock variables*. A *clock valuation* is a function $\nu : C \to \mathbb{R}_+$. If $r \subseteq C$ is a set of clock variables then $\nu[r := 0]$ denotes the valuation that maps each clock variable $x \in r$ to 0 and agrees with ν on all other clocks. The zero clock valuation $\mathbf{0}$ is defined by $\mathbf{0}(x) = 0$ for all

$x \in C$. The set $\Phi(C)$ of *clock constraints* φ is defined by the following grammar:

$$\varphi := \mathbf{true} \mid \mathbf{false} \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid x \sim k\,,$$

where $k \in \mathbb{N}$ and $\sim \in \{<, \leq, =, \neq, \geq, >\}$.

Recall from [1] that the transition relation of a timed automaton can be seen as a partial function

$$\delta : S \times \Sigma \times \Phi(C) \to \mathcal{P}(S \times \mathcal{P}(C))\,,$$

where S is the set of *locations* and Σ is the alphabet of the automaton. The meaning of $(s', r) \in \delta(s, a, \varphi)$ is that from state (s, ν), where ν satisfies φ, the automaton can read letter a and transition to state $(s', \nu[r := 0])$.

In the definition of an *alternating timed automaton* the transition function is generalised to a partial function

$$\delta : S \times \Sigma \times \Phi(C) \to \mathcal{B}_+(S \times \mathcal{P}(C))\,,$$

where $\mathcal{B}_+(S \times \mathcal{P}(C))$ denotes the set of positive Boolean formulas generated by the grammar

$$\theta := (s, r) \mid \theta_1 \vee \theta_2 \mid \theta_1 \wedge \theta_2\,,$$

where $(s, r) \in S \times \mathcal{P}(C)$. Here disjunction corresponds to nondeterministic choice whereas conjunctive transitions are executed in parallel. For example, if $\delta(s, a, \varphi) = (s_1, r_1) \wedge (s_2, r_2)$ then from state (s, ν), where ν satisfies φ, the automaton can read letter a and simultaneously move to $(s_1, \nu[r_1 := 0])$ *and* $(s_2, \nu[r_2 := 0])$.

Definition 1: An ***alternating timed automaton*** is a tuple $\mathcal{A} = (\Sigma, S, C, s_0, F, \delta)$, where

- Σ is a finite alphabet
- S is a finite set of locations
- C is a finite set of clock variables
- $s_0 \in S$ is the initial location
- $F \subseteq S$ is a set of accepting locations
- $\delta : S \times \Sigma \times \Phi(C) \to \mathcal{B}_+(S \times \mathcal{P}(C))$ is a partial function with finite domain.

To simplify our exposition we assume that the image of the transition function δ consists solely of formulas in disjunctive normal form. We also make the following **Partition Assumption:** for each location s and letter $a \in \Sigma$, the set of constraints φ such that $\delta(s, a, \varphi)$ is defined forms a partition of the set $(\mathbb{R}_+)^C$ of clock valuations. Neither of the above two assumptions affects the expressiveness of the model.

Before formally defining the language accepted by an alternating timed automaton, we give some examples.

Example 2: We define an automaton $\mathcal{A}$ over alphabet $\Sigma = \{a\}$ that accepts those words such that for every timed event (a, t) with $t < 1$ there is an event $(a, t+1)$ exactly one time unit later. $\mathcal{A}$ has a single clock x and set of locations $\{s, u, v\}$, with s initial, s and v accepting and u non-accepting. The transition function is defined by

$$\begin{aligned} &\delta(s, a, x < 1) = (s, \emptyset) \wedge (u, \{x\}) &&\delta(s, a, x \geq 1) = (s, \emptyset) \\ &\delta(u, a, x \neq 1) = (u, \emptyset) &&\delta(u, a, x = 1) = (v, \emptyset)\,. \end{aligned}$$

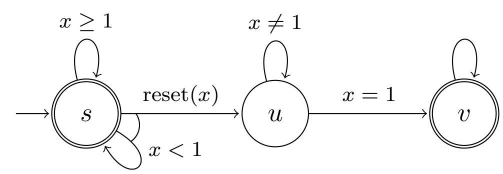


Figure 1. Automaton $\mathcal{A}$

The automaton is illustrated in Figure 1 in which we represent the conjunctive transition out of s by connecting two arrows with an arc.

A run of $\mathcal{A}$ starts in location s. Every time an a-event occurs in the first time unit, the automaton makes a simultaneous transition to both s and u, thus opening up a new thread of computation equipped with a fresh copy of the clock x. The automaton must eventually leave location u, which is non-accepting, and it can only do so exactly one time unit after first entering the location.

Example 3: We define an automaton $\mathcal{B}$, shown in Figure 2, over alphabet $\{a\}$ that accepts those words such that for all consecutive pairs of events (a, t_i) and (a, t_{i+1}) with $t_i, t_{i+1} < 1$ there is no subsequent event (a, t_j) with $t_i + 1 < t_j < t_{i+1} + 1$. Excepting some corner cases[1], this requirement says that for each event (a, t_j) with $1 < t_j < 2$ there is an event $(a, t_j - 1)$ exactly one time unit earlier. Indeed, if there were no such event then letting (a, t_i) be the latest event with $t_i < t_j - 1$ and (a, t_{i+1}) the earliest event with $t_j - 1 < t_{i+1}$ we see that the input word would be rejected by $\mathcal{B}$.

$\mathcal{B}$ has two clocks x and y, and set of locations $\{s, u, v, w\}$, with s initial and locations s, u and v accepting. The transition function is defined by

$$\begin{aligned} \delta(s, a, x < 1) &= (s, \emptyset) \wedge (u, \{x\}) \\ \delta(s, a, x \geq 1) &= (s, \emptyset) \\ \delta(u, a, \mathbf{true}) &= (v, \{y\}) \\ \delta(v, a, x < 1 \vee y > 1) &= (v, \emptyset) \\ \delta(v, a, x \geq 1 \wedge y \leq 1) &= (w, \emptyset)\,. \end{aligned}$$

From location s every time an a-event occurs the automaton starts a new thread in location u, resetting clock x. On the next a-event this new thread transitions to location v, resetting clock y. Thereafter, if an a-event occurs with $x \geq 1$ and

[1]We should also require that if (a, t_j) is the first or last event with $1 \leq t_j < 2$, then there be an earlier event (a, t_i) with $t_i = t_j - 1$. Based on the same ideas involved in the definition of $\mathcal{B}$, one can easily define an automaton $\mathcal{C}$ that accepts precisely those words satisfying this requirement.

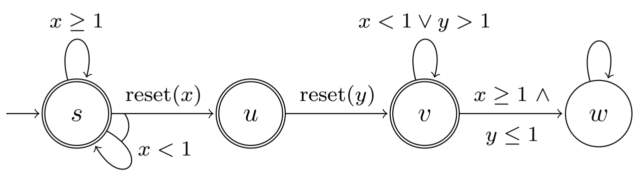


Figure 2. Automaton $\mathcal{B}$

$y \leq 1$, the thread transitions to location w and becomes non-accepting; by this mechanism the automaton blocks timed events (a, t_j), $t > 1$, for which there is a consecutive pairs of events (a, t_i) and (a, t_{i+1}) with $t_i + 1 < t_j < t_{i+1} + 1$.

B. The Acceptance Game.

In general a run of an alternating automaton is defined to be a tree of states. However in this paper we exploit the fact that acceptance of a timed word $w = (a_1, t_1) \ldots (a_n, t_n)$ by an alternating timed automaton $\mathcal{A} = (\Sigma, S, C, s_0, F, \delta)$ has a game-theoretic characterisation [10].

Define the *acceptance game* $\mathbb{G}(\mathcal{A}, w)$ between *Automaton* and *Pathfinder* as follows. A *state* of $\mathbb{G}(\mathcal{A}, w)$ is a triple (i, s, ν), where $0 \leq i \leq n$ is the round number, $s \in S$ is a location and ν is a clock valuation. Play starts in state $(0, s_0, \mathbf{0})$ and consists of n rounds. Suppose that at the beginning of the $(i+1)$-th round, $0 \leq i \leq n-1$, the state is (i, s_i, ν_i) and write $\nu' = \nu_i + t_{i+1} - t_i$. By the Partition Assumption there is a unique clock constraint φ such that ν' satisfies φ and $\delta(s_i, a_i, \varphi)$ is defined. Moreover the latter formula is in disjunctive normal form. Then Automaton moves first by selecting a disjunct θ of $\delta(s_i, a_i, \varphi)$; Pathfinder responds by choosing a conjunct (s, r) of θ, and the next game state is $(i+1, s, \nu'[r := 0])$. Automaton wins if the game ends in an accepting state after the last round.

A *partial play* is a finite sequence of consecutive game states. A *strategy* for Automaton is a mapping that assigns to each such sequence a next move of Automaton. Such a strategy is *winning* if Automaton wins any play in which the strategy is followed. We say that a timed word $w \in \mathbb{T}\Sigma^*$ is *accepted* by $\mathcal{A}$ if and only if Automaton has a winning strategy in $\mathbb{G}(\mathcal{A}, w)$. The *language* $L_{\mathbb{T}}(\mathcal{A})$ is the set of words in $\mathbb{T}\Sigma^*$ that are accepted by $\mathcal{A}$.

One of the motivations for introducing alternating timed automata is that they enjoy better closure properties than ordinary timed automata, cf. [10], [11].

Proposition 4: For any time domain $\mathbb{T} \subseteq \mathbb{R}_+$ the class of languages $L \subseteq \mathbb{T}\Sigma^*$ accepted by alternating timed automata is effectively closed under union, intersection, and complement.

Closure under union and intersection is straightforward since we allow both disjunctions and conjunctions in the transition function. Thanks to the Partition Assumption one can complement an automaton by simply interchanging accepting and non-accepting states and exchanging conjunctions and disjunctions in the transition function. (The convention that transitions be written in disjunctive normal form can easily be re-established.)

Example 5: Taking the intersection of the automaton $\mathcal{A}$ in Example 2, the automaton $\mathcal{B}$ in Example 3, and the automaton $\mathcal{C}$ mentioned in the footnote to Example 3, all defined over alphabet $\{a\}$, one obtains the automaton $\mathcal{A}_{\text{copy}}$. Over time domain $\mathbb{T} = [0, 2)$, $L_{\mathbb{T}}(\mathcal{A}_{\text{copy}})$ consists of those timed words $w = (a, t_1)(a, t_2) \ldots (a, t_{2n})$ such that $t_{i+n} = t_i + 1$ for $1 \leq i \leq n$, i.e., such that the $+1$ function defines a one-to-one correspondence between the set of events in the first time unit and the set of events in the second time unit.

III. MONADIC SECOND-ORDER LOGIC

Throughout this section we assume a fixed time domain $\mathbb{T} = [0, N)$. We consider *monadic second-order logic* (MSO) over the structure $(\mathbb{T}, <, +1)$, where $+1(x, y)$ holds if and only if $x + 1 = y$. The syntax of $\mathsf{MSO}(<, +1)$ has as vocabulary first-order variables $t_1, t_2, \ldots$, monadic predicate variables $X_1, X_2, \ldots$, and the binary relations $+1$ and $<$. Atomic formulas are of the form $X(t)$, $t_1 < t_2$, $+1(t_1, t_2)$, and $t_1 = t_2$. Well-formed formulas are obtained from atomic formulas using Boolean connectives, the first-order quantifiers $\exists t$ and $\forall t$, and the second-order quantifiers $\exists X$ and $\forall X$. If we omit the $+1$ relation then we obtain the sub-logic $\mathsf{MSO}(<)$. We denote sets of monadic predicates in boldface and write $\varphi(\mathbf{X})$ for a formula whose free second-order variables are drawn from the set $\mathbf{X}$. In the sequel we reserve the letters W, X, Y to denote monadic predicate variables, and P, Q, R to denote their interpretations as subsets of $\mathbb{T}$.

Example 6: Fix a finite set $\mathbf{W}$ of monadic predicate variables, and consider the timed word $w = (a_1, t_1) \ldots (a_n, t_n)$ over alphabet $\Sigma = \mathcal{P}_+(\mathbf{W})$ consisting of the nonempty subsets of $\mathbf{W}$. We associate with w a structure M_w that extends $(\mathbb{T}, <, +1)$ with interpretations of the monadic predicate variables $\mathbf{W}$, where $W \in \mathbf{W}$ is interpreted as the set $\{t_i : W \in a_i\}$.

Shelah [34] showed that the satisfiability problem for $\mathsf{MSO}(<)$ over the non-negative reals (and hence also over any nonempty interval of reals) is undecidable. He also proved, however, that decidability can be recovered by restricting second-order quantification to countable sets. Given our interest in modelling finite timed words in the manner of Example 6, the following stronger **Finiteness Assumption** is natural and is assumed henceforth: all free predicate variables are interpreted by finite sets, and second-order quantification ranges over finite sets. Decidability of $\mathsf{MSO}(<)$ under this restriction follows from Shelah's result mentioned above; cf. also [35].

Let $\mathbf{W}$ be a set of predicate variables. Observe that the interpretation M_w of $\mathbf{W}$ arising from a timed word w, as described in Example 6, obeys the Finiteness Assumption. Conversely any interpretation of $\mathbf{W}$ satisfying the Finiteness Assumption arises from a unique timed word. Thus we may identify the set of models of a formula $\varphi(\mathbf{W})$ of $\mathsf{MSO}(<,+1)$ with a timed language. If we furthermore assume, without loss of generality, that $\mathbf{W}$ contains a distinguished predicate W_{init} with a fixed interpretation as the set $\{0\}$, then for timed words w and w', the structures M_w and $M_{w'}$ are order isomorphic if and only if $\text{untime}(w) = \text{untime}(w')$.[2] Building on this observation, we can represent the set of models of an $\mathsf{MSO}(<)$ formula (up to order isomorphism) as a regular untimed language.

Definition 7: A collection $\mathcal{M}$ of interpretations of $\mathbf{W}$ in $\mathbb{T}$ is said to be ***regular*** if there exists a regular (untimed) language L on alphabet $\Sigma = \mathcal{P}_+(\mathbf{W})$ such that

$$\mathcal{M} = \{M_w : w \in \mathbb{T}\Sigma^*, \, \text{untime}(w) \in L\} .$$

The following proposition follows from [35, Theorem 1].

Proposition 8: The set of models of $\varphi(\mathbf{W}) \in \mathsf{MSO}(<)$ is effectively regular.

IV. McNaughton Games

McNaughton [20] gave a formulation of Church's problem in terms of two-player games over the structure $(\mathbb{N}, <)$ with $\mathsf{MSO}(<)$ winning conditions. In this section we introduce a class of McNaughton-type games over bounded time domains $\mathbb{T} = [0, N)$ with $\mathsf{MSO}(<,+1)$ winning conditions. These games are shown to generalise the class of acceptance games for alternating timed automata as defined in Section II. Following [32], [36] we consider games with parameters. The presence of parameters is crucial both in setting up the correspondence with the acceptance game and in our inductive proof of decidability for the McNaughton games.

Let $\varphi(\mathbf{W}, \mathbf{X}, \mathbf{Y})$ be an $\mathsf{MSO}(<,+1)$ formula with free variables among $\mathbf{W}$, $\mathbf{X}$ and $\mathbf{Y}$. We think of $\mathbf{X}$ as a set of variables under the control of *Player I*, $\mathbf{Y}$ as a set of variables under the control of *Player II*, and $\mathbf{W}$ as a set of *parameters*: each instantiation of $\mathbf{W}$ yields a different game. Let $\mathbf{P}$ be an interpretation of $\mathbf{W}$ and, recalling the Finiteness Assumption, let $t_1 < t_2 < \cdots < t_n$ be an enumeration of $\bigcup \mathbf{P}$, i.e., the set of points at which some predicate in $\mathbf{P}$ holds. The *McNaughton game* $\mathbb{G}(\varphi, \mathbf{P})$ is a turn-based game in which Player I and Player II run through the sequence of timestamps $t_1, \ldots, t_n$, successively choosing values for their predicates at each timestamp. More formally:

(i) The game consists of n rounds. In the i-th round Player I chooses a bit vector $b_i \in \{0,1\}^{\mathbf{X}}$ and then Player II chooses a bit vector $b'_i \in \{0,1\}^{\mathbf{Y}}$;

(ii) At the conclusion of the game, Player I has constructed an interpretation $\mathbf{Q}$ of $\mathbf{X}$ that assigns to $X \in \mathbf{X}$ the set $\{t_i : 1 \le i \le n, b_i(X) = 1\}$. Likewise, Player II has constructed an interpretation $\mathbf{R}$ of $\mathbf{Y}$ that assigns to $Y \in \mathbf{Y}$ the set $\{t_i : 1 \le i \le n, b'_i(Y) = 1\}$;

(iii) If $\mathbb{T} \models \varphi(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ then Player I is the winner; otherwise Player II is the winner.

We say that the sequence of moves in the first i rounds of $\mathbb{G}(\varphi, \mathbf{P})$ determines a *partial play* $(b_1, b'_1) \ldots (b_i, b'_i)$, where $b_j \in \{0,1\}^{\mathbf{X}}$ and $b'_j \in \{0,1\}^{\mathbf{Y}}$ for $1 \le j \le i$. A strategy for Player I is a function from the set of partial plays to the set $\{0,1\}^{\mathbf{X}}$. Such a strategy is *winning* if Player I wins all plays in which the strategy is followed.

In the next section we consider the decidability of the following two questions:

Decision Problem. Given a formula $\varphi(\mathbf{W}, \mathbf{X}, \mathbf{Y})$, does there exist an interpretation $\mathbf{P}$ of the parameters $\mathbf{W}$ for which Player I has a winning strategy in the game $\mathbb{G}(\varphi, \mathbf{P})$?

Winning-Parameters Problem. Compute a representation of the set of parameters $\mathbf{P}$ for which Player I has a winning strategy in the game $\mathbb{G}(\varphi, \mathbf{P})$.

In Section V we prove the computability of the winning-parameters problem in case φ is an $\mathsf{MSO}(<)$ formula, and as a consequence establish decidability of the decision problem for $\mathsf{MSO}(<,+1)$ winning conditions.

In the remainder of this section we sketch a reduction of the language emptiness problem for alternating timed automata over time domain $\mathbb{T}$ to the decision problem for McNaughton games also over $\mathbb{T}$. To this end, given an alternating timed automaton $\mathcal{A} = (\Sigma, S, C, s_0, F, \delta)$, we construct a formula $\varphi_{\mathcal{A}}(\mathbf{W}, \mathbf{X}, \mathbf{Y})$ of $\mathsf{MSO}(<,+1)$ so as to establish a correspondence between the acceptance game $\mathbb{G}(\mathcal{A}, w)$ for a given timed word w (cf. Section II) and the McNaughton game $\mathbb{G}(\varphi_{\mathcal{A}}, \mathbf{P})$ for a given interpretation $\mathbf{P}$ of $\mathbf{W}$ that is determined by w.

The set of parameters $\mathbf{W} = \{W_\sigma : \sigma \in \Sigma\}$ contains a predicate variable for each alphabet symbol. A timed word $w \in \mathbb{T}\Sigma^*$ naturally determines an interpretation $\mathbf{P}$ of $\mathbf{W}$ over $\mathbb{T}$, cf. Example 6.

We imagine that Player I and Player II in $\mathbb{G}(\varphi_{\mathcal{A}}, \mathbf{P})$ respectively take the roles of Automaton and Pathfinder in $\mathbb{G}(\mathcal{A}, w)$. For each expression $\theta \in \mathcal{B}_+(S \times \mathcal{P}(C))$ that is a conjunction of atoms we postulate a variable X_θ whose truth value encodes the choice of Automaton to select θ in a given round of $\mathbb{G}(\mathcal{A}, w)$. Similarly, for each atom $\alpha \in S \times \mathcal{P}(C)$ we postulate a variable Y_α whose truth value encodes the choice of Pathfinder to select atom α in a given round of $\mathbb{G}(\mathcal{A}, w)$.

We instrument the formula $\varphi_{\mathcal{A}}$ so that Player I wins the McNaughton game $\mathbb{G}(\varphi_{\mathcal{A}}, \mathbf{P})$ if and only if Automaton wins the acceptance game $\mathbb{G}(\mathcal{A}, w)$. The key components of $\varphi_{\mathcal{A}}$ are subformulas $\varphi_{\text{aut}}(t)$ and $\varphi_{\text{path}}(t)$ respectively ensuring that Player I and Player II correctly simulate Automaton and Pathfinder at each time point t.

The formula $\varphi_{\text{path}}(t)$ ensures that at time t Pathfinder only chooses one atom α, and moreover that α is a conjunct

[2] We include the predicate W_{init} since an order isomorphism from M_w to $M_{w'}$ must map 0 to itself.

of the expression θ chosen by Automaton at time t (written $\theta \models \alpha$). This is expressed as:

$$\bigwedge_{\alpha} \left(Y_\alpha(t) \rightarrow \bigvee_{\theta \models \alpha} X_\theta(t) \right) \wedge \bigwedge_{\alpha \neq \beta} \neg (Y_\alpha(t) \wedge Y_\beta(t)) \, .$$

The formula $\varphi_{\mathrm{aut}}(t)$ ensures that at time t Automaton chooses a disjunct θ of the transition function δ. It is the conjunction over all locations $s \in S$, inputs $\sigma \in \Sigma$ and guards $\psi \in \Phi(C)$, such that $\delta(s, \sigma, \psi)$ is defined, of the formulas

$$\forall v \, ((\mathit{state}_s(v) \wedge \mathit{next}(v,t) \wedge W_\sigma(t) \wedge \mathit{const}_\psi(t)) \rightarrow \bigvee_{\theta \models \delta(s,\sigma,\psi)} X_\theta(t)) \, .$$

Here $\mathit{state}_s(v)$ and $\mathit{next}(v,t)$ are easily defined auxiliary formulas, respectively expressing that the automaton is in state s at time v, and that v and t are consecutive timestamps in the input word. Similarly, $\mathit{const}_\psi(t)$ expresses that the clock constraint $\psi \in \Phi(C)$ holds at time t. For example, in case $\psi \equiv x \sim k$ we define $\mathit{const}_\psi(t)$ to be the formula

$$\exists u \left(\begin{array}{c} u < t \wedge \mathit{reset}_x(u) \wedge \forall w (u < w < t \rightarrow \neg \mathit{reset}_x(w)) \\ \wedge \, t - u \sim k \end{array} \right)$$

where $\mathit{reset}_x(u)$ is an auxiliary formula expressing that clock x was reset at time u (which information is available from $Y_\alpha(u)$).

φ_{aut} and φ_{path} are components of a formula $\varphi_{\mathcal{A}}$ which encodes the winning condition of Automaton in the acceptance game. Specifically, $\varphi_{\mathcal{A}}$ expresses that play must start in an initial location, it must end in an accepting location, and for all time points t at which some W_σ holds (i.e., all time points of the input word) $\varphi_{\mathrm{aut}}(t)$ holds unless φ_{path} had previously failed. This is all straightforward to formalise, and in fact this can be accomplished such that $\varphi_{\mathcal{A}}$ has no second-order quantifiers, that is, it is a formula in the first-order fragment of $\mathsf{MSO}(<, +1)$.

The games $\mathbb{G}(\mathcal{A}, w)$ and $\mathbb{G}(\varphi_{\mathcal{A}}, \mathbf{P})$ are essentially isomorphic, and it is now straightforward to prove the following:

Proposition 9: Automaton wins the acceptance game $\mathbb{G}(\mathcal{A}, w)$ if and only if Player I wins the McNaughton game $\mathbb{G}(\varphi_{\mathcal{A}}, \mathbf{P})$, where $\mathbf{P}$ is the set of parameters associated with the timed word w.

V. Main Result

In this section we show decidability of the decision problem for McNaughton games over bounded time domains. From this we derive decidability of the time-bounded language emptiness problem for alternating timed automata. The proof has three parts: first we recall from [27] a satisfiability preserving and reflecting translation from $\mathsf{MSO}(<, +1)$ to $\mathsf{MSO}(<)$ over bounded time domains; next we show how to solve the winning-parameters problem for McNaughton games with $\mathsf{MSO}(<)$ winning conditions; finally we combine the first two contributions to solve the decision problem for games with $\mathsf{MSO}(<, +1)$ winning conditions.

Throughout this section, let $\mathbb{T} = [0, N)$ be a fixed time domain.

A. Eliminating the Metric

Given an $\mathsf{MSO}(<, +1)$ formula φ, we define a straightforward syntactic transformation into an $\mathsf{MSO}(<)$ formula $\overline{\varphi}$ such that there is a natural bijection between models of φ over $[0, N)$ and models of $\overline{\varphi}$ over $[0, 1)$.

Let $\mathbf{X}$ be the set of monadic predicates appearing in φ. With each predicate $X \in \mathbf{X}$, we associate a collection $X_0, \ldots, X_{N-1}$ of N fresh monadic predicates. We then write $\overline{\mathbf{X}} = \{X_i \mid X \in \mathbf{X}, 0 \leq i \leq N-1\}$. Intuitively, each X_i is a predicate on $[0, 1)$ that represents X over the subinterval $[i, i+1)$. Formally, an interpretation of $X \in \mathbf{X}$ over $[0, N)$ yields an interpretation of X_i over $[0, 1)$ by defining $X_i(t)$ if and only if $X(i + t)$. Note that this correspondence yields a bijection between interpretations of $\mathbf{X}$ on $[0, N)$ and interpretations of $\overline{\mathbf{X}}$ on $[0, 1)$.

We can assume that φ does not contain any (first- or second-order) existential quantifiers, by replacing them with combinations of universal quantifiers and negations if need be. It is also convenient to rewrite φ into a formula that makes use of a unary function $+1$ instead of the $+1$ relation. To this end, replace every occurrence of $+1(x, y)$ in φ by $(x < N-1 \wedge x + 1 = y)$.

Next, replace every instance of $\forall x \, \psi$ in φ by the formula

$$\forall x \, (\psi[x/x] \wedge \psi[x+1/x] \wedge \ldots \wedge \psi[x + (N-1)/x]) \, ,$$

where $\psi[t/x]$ denotes the formula resulting from substituting every free occurrence of the variable x in ψ by the term t. Intuitively, this transformation is legitimate since first-order variables in our target formula will range over $[0, 1)$ rather than $[0, N)$.

Having carried out these substitutions, use simple arithmetic to rewrite every term in φ as $x + k$, where x is a variable and $k \in \mathbb{N}$ is a non-negative integer constant.

Every inequality occurring in φ is now of the form $x + k < N - 1$ or $x + k_1 < y + k_2$. Replace every inequality of the first kind by $\mathbf{true}$ if $k + 2 \leq N$ and by $\mathbf{false}$ otherwise, and replace every inequality of the second kind by (i) $x < y$, if $k_1 = k_2$; (ii) $\mathbf{true}$, if $k_1 < k_2$; and (iii) $\mathbf{false}$ otherwise.

Every equality occurring in φ is now of the form $x + k_1 = y + k_2$. Replace every such equality by $x = y$ if $k_1 = k_2$, and by $\mathbf{false}$ otherwise.

Every use of monadic predicates in φ now has the form $X(x + k)$, for $k \leq N - 1$. Replace every such predicate by $X_k(x)$.

Finally, replace every occurrence of $\forall X \, \psi$ in φ by $\forall X_0 \, \forall X_1 \, \ldots \forall X_{N-1} \, \psi$. The resulting formula is the desired $\overline{\varphi}$. Note that $\overline{\varphi}$ does not mention the $+1$ function, and is therefore indeed a sentence in $\mathsf{MSO}(<)$.

The correspondence between φ and $\overline{\varphi}$ is formalised in the following proposition.

Proposition 10: The formula $\varphi(\mathbf{X})$ holds in the structure $([0,N),<,+1)$ under interpretation $\mathbf{P}$ if and only if the transformed formula $\overline{\varphi}(\overline{\mathbf{X}})$ holds in the structure $([0,1),<)$ under interpretation $\overline{\mathbf{P}}$.

B. The Regularity Lemma

Recall from Definition 7 that, due to the Finiteness Assumption, we can represent the set of interpretations satisfying an $\mathsf{MSO}(<)$ formula $\varphi(\mathbf{X})$ as a regular untimed language.

Lemma 11: Let $\varphi(\mathbf{W},\mathbf{X},\mathbf{Y})$ be an $\mathsf{MSO}(<)$ formula. Then the set $\{\mathbf{P} : \text{Player I wins } \mathbb{G}(\varphi,\mathbf{P})\}$ is effectively regular.

Proof: According to Proposition 8 we can compute a deterministic finite automaton $\mathcal{A}=(Q,\Sigma,q_0,\delta,F)$ over alphabet $\Sigma=\mathcal{P}_+(\mathbf{W}\cup\mathbf{X}\cup\mathbf{Y})$ whose language represents the set of models of φ. We seek an automaton over alphabet $\Sigma_1=\mathcal{P}_+(\mathbf{W})$ representing the set of interpretations $\mathbf{P}$ of $\mathbf{W}$ such that Player I wins $\mathbb{G}(\varphi,\mathbf{P})$.

Let $w=w_1w_2\ldots w_n\in\Sigma_1^*$ represent an interpretation of $\mathbf{W}$. By distinguishing the contribution of input bits between two protagonists, respectively called Player I and Player II, we define a *graph game*[3] $\Gamma(\mathcal{A},w)$, which can be seen as a discrete analogue of $\mathbb{G}(\varphi,\mathbf{P})$.

There are two kinds of vertices in $\Gamma(\mathcal{A},w)$, Player-I vertices and Player-II vertices respectively. For each automaton state $q\in Q$ and position $0\le i\le n$, we include a Player-I vertex (q,i); if moreover $0\le i\le n-1$ then we also include a Player-II vertex (q,i,b) for each bit vector $b\in\{0,1\}^{\mathbf{X}}$. If $0\le i\le n-1$ then we include an edge from (q,i) to (q,i,b), corresponding to a choice of bit vector b by Player I; for $b'\in\{0,1\}^{\mathbf{Y}}$ we also include an edge from (q,i,b) to $(q',i+1)$, where $q'=\delta(q,(w_{i+1},b,b'))$, corresponding to a choice of bit vector b' by Player II.

The rules of the game $\Gamma(\mathcal{A},w)$ are as follows. Player I chooses moves at Player-I vertices and Player II chooses moves at Player-II vertices. Play starts in the vertex $(q_0,0)$ and Player I wins if play reaches a vertex (q,n) with $q\in F$.

The game $\Gamma(\mathcal{A},w)$ essentially represents an *untiming* of $\mathbb{G}(\varphi,\mathbf{P})$: the moves in each game are the same once one elides the timestamp associated with each round in the latter game. In particular, Player I wins $\Gamma(\mathcal{A},w)$ if and only if Player I wins $\mathbb{G}(\varphi,\mathbf{P})$.

For $E\subseteq Q$ a set of automaton states and $w\in\Sigma_1^*$, we define the set $\mathrm{Force}_w(E)$ of states from which Player I can force play into E on input w by

$$\begin{aligned}\mathrm{Force}_\varepsilon(E)&=E\\ \mathrm{Force}_u(E)&=\{q:\exists b_1\,\forall b_2\,\delta(q,(u,b_1,b_2))\in E\}\quad u\in\Sigma_1\\ \mathrm{Force}_{uw}(E)&=\mathrm{Force}_u(\mathrm{Force}_w(E))\qquad u\in\Sigma_1,w\in\Sigma_1^*\end{aligned}$$

Given $w\in\Sigma_1^*$, it is straightforward that Player I wins $\Gamma(\mathcal{A},w)$ if and only if $q_0\in\mathrm{Force}_w(F)$. From this observation one can build an automaton $\mathcal{B}$ on alphabet Σ_1 that accepts those words w such that Player I wins $\Gamma(\mathcal{A},w)$. The set of states of $\mathcal{B}$ is $\mathcal{P}(Q)$, with F the unique final state, and $\{S\subseteq Q: q_0\in S\}$ the set of initial states. We include a transition $S\xrightarrow{b}T$ on input $b\in\Sigma_1$ if and only if $S=\mathrm{Force}_b(T)$. ∎

A result similar to Lemma 11 has been proven in [32], [33] for parametric games over $(\mathbb{N},<)$.

C. The Decision Procedure

Theorem 12: Let $\mathbb{T}=[0,N)$ be a fixed time domain. Given an $\mathsf{MSO}(<,+1)$ formula $\varphi(\mathbf{W},\mathbf{X},\mathbf{Y})$, it is decidable whether there exists an interpretation $\mathbf{P}$ of $\mathbf{W}$ over $\mathbb{T}$ such that Player I wins $\mathbb{G}(\varphi,\mathbf{P})$.

Proof: Applying the transformation described in subsection V-A to $\varphi(\mathbf{W},\mathbf{X},\mathbf{Y})$ yields an $\mathsf{MSO}(<)$ formula $\overline{\varphi}(\overline{\mathbf{W}},\overline{\mathbf{X}},\overline{\mathbf{Y}})$, where $\overline{\mathbf{W}}=\{W_i : W\in\mathbf{W}, 0\le i<N\}$, $\overline{\mathbf{X}}=\{X_i : X\in\mathbf{X}, 0\le i<N\}$, and $\overline{\mathbf{Y}}=\{Y_i : Y\in\mathbf{Y}, 0\le i<N\}$. Then interpretations $\mathbf{P},\mathbf{Q},\mathbf{R}$ of $\mathbf{W},\mathbf{X},\mathbf{Y}$ as predicates on $[0,N)$ naturally yield interpretations $\overline{\mathbf{P}},\overline{\mathbf{Q}},\overline{\mathbf{R}}$ of $\overline{\mathbf{W}},\overline{\mathbf{X}},\overline{\mathbf{Y}}$ as predicates on $[0,1)$ where, e.g., $P_i(t)$ holds if and only if $P(i+t)$ holds, $0\le t<1$ and $0\le i<N$. By Proposition 10 we have that $\varphi(\mathbf{P},\mathbf{Q},\mathbf{R})$ holds if and only if $\overline{\varphi}(\overline{\mathbf{P}},\overline{\mathbf{Q}},\overline{\mathbf{R}})$ holds.

Observe, however, that there is a significant difference between the game $\mathbb{G}(\varphi,\mathbf{P})$, which is played over the interval $[0,N)$, and the game $\mathbb{G}(\overline{\varphi},\overline{\mathbf{P}})$, played over $[0,1)$. For example, in the former, for $X\in\mathbf{X}$ and $0\le t<1$, Player I chooses the value of $X(t)$ before $X(t+1)$. However, in the latter, these values, respectively represented as $X_0(t)$ and $X_1(t)$, are chosen at the same time.

Instead we associate with $\mathbb{G}(\varphi,\mathbf{P})$ a *sequence* of McNaughton games $G_0,\ldots,G_{N-1}$, each over the interval $[0,1)$, and each with an $\mathsf{MSO}(<)$ winning condition. Intuitively the i-th game G_i corresponds to the restriction of $\mathbb{G}(\varphi,\mathbf{P})$ to the time interval $[i,i+1)$. Accordingly we have Player I choose the value of X_i, $X\in\mathbf{X}$ and Player II choose the value of Y_i, $Y\in\mathbf{Y}$ in G_i. The key insight in defining G_i is to treat the variables X_j, $X\in\mathbf{X}$ and Y_j, $Y\in\mathbf{Y}$ as additional parameters for each $j<i$. That is, the respective choices of Player I and Player II in the preceding games G_j, $j<i$ become parameters in G_i. (Strictly speaking each G_i is a *family* of games, one game for each instantiation of the extra parameters in φ_i.)

To be precise, the winning condition of G_i is an $\mathsf{MSO}(<)$ formula φ_i with free variables $\overline{\mathbf{W}}$, $\{X_j : X\in\mathbf{X}, 0\le j\le$

[3]Graph games are a simple and classical notion [23]. We treat them informally to avoid overburdening the reader with yet more game-theoretic formalism.

$i\}$ and $\{Y_j : Y \in \mathbf{Y}, 0 \le j \le i\}$. Of these, Player I controls X_i, $X \in \mathbf{X}$, Player II controls Y_i, $Y \in \mathbf{Y}$, and the remaining variables are parameters. The definition of the φ_i proceeds backwards, from φ_{N-1} down to φ_0, and is such that Player I wins $\mathbb{G}(\varphi, \mathbf{P})$ if and only if he wins $G_0 = \mathbb{G}(\varphi_0, \overline{\mathbf{P}})$. This equivalence allows us to decide the winner of $\mathbb{G}(\varphi, \mathbf{P})$.

To start with we define $\varphi_{N-1} := \overline{\varphi} \wedge \chi_{N-1}$, where χ_i, which constrains Player I and Player II to move only when one of the predicates W_i, $W \in \mathbf{W}$ is true, is defined by

$$\left(\bigwedge_{X \in \mathbf{X}} X_i \subseteq \bigcup_{W \in \mathbf{W}} W_i \right) \vee \left(\bigvee_{Y \in \mathbf{Y}} Y_i \not\subseteq \bigcup_{W \in \mathbf{W}} W_i \right) .$$

Suppose we have defined the game G_{i+1} involving parameters $\overline{\mathbf{W}}$, $\{X_j : 0 \le j \le i, X \in \mathbf{X}\}$ and $\{Y_j : 0 \le j \le i, Y \in \mathbf{Y}\}$. By the Regularity Lemma the set of interpretations of these parameters such that Player I wins G_{i+1} is effectively regular and is expressible by an $\mathsf{MSO}(<)$ formula ψ_i with the above parameters as its set of free variables. The winning condition for the game G_i is then defined to be $\varphi_i := \psi_i \wedge \chi_i$, where χ_i is as defined above.

This completes the definition of the games $G_0, \ldots, G_{N-1}$. There is a natural bijective correspondence between the set of positions of $\mathbb{G}(\varphi, \mathbf{P})$ and the set of positions of (the various instantiations of) the G_i, where a position of $\mathbb{G}(\varphi, \mathbf{P})$ with timestamp t corresponds to a position of $G_{\lfloor t \rfloor}$ with timestamp $t - \lfloor t \rfloor$. Moreover this association preserves the identity of the winning player. In particular, Player I wins $\mathbb{G}(\varphi, \mathbf{P})$ if and only if he wins $\mathbb{G}(\varphi_0, \overline{\mathbf{P}})$. We omit the details. ■

Recalling from Section IV the reduction of the language emptiness problem for alternating timed automata to the game decision problem for $\mathsf{MSO}(<, +1)$, we obtain the following theorem, which is the central result of our paper:

Theorem 13: The time-bounded language emptiness problem for alternating timed automata is decidable.

Note, as an immediate corollary, that Theorem 13 entails the decidability of the time-bounded model-checking problem of timed automata against alternating timed automata specifications.

VI. Complexity

Define a family of functions $\exp_k : \mathbb{N} \to \mathbb{N}$ by $\exp_0(n) = n$ and $\exp_{k+1}(n) = 2^{\exp_k(n)}$. A function $f : \mathbb{N} \to \mathbb{N}$ is *nonelementary* if it grows faster than any $\exp_k$.

Our procedure for determining language emptiness for alternating timed automata has nonelementary complexity. This blow-up does not arise from the translation of $\mathsf{MSO}(<)$ formulas to automata in the proof of Proposition 8, since the quantifier-alternation depth of the relevant formulas is bounded independently of the automata. Rather, the culprit is the exponential blow-up that occurs with each application of the Regularity Lemma in the proof of Theorem 12.

In this section we give a nonelementary lower bound for the language emptiness problem for alternating timed automata. We prove this by reduction from the emptiness problem for star-free regular expressions.

A *star-free regular expression* over alphabet Σ is built from the symbols $\emptyset$ and σ, for any $\sigma \in \Sigma$, using the operations of union ($+$), concatenation ($\cdot$), and complementation ($\neg$). Such an expression E denotes a language $L(E) \subseteq \Sigma^*$ which is defined as follows:

$L(\emptyset) = \emptyset$ and $L(\sigma) = \{\sigma\}$;
$L(E + E') = L(E) \cup L(E')$;
$L(E \cdot E') = L(E) \cdot L(E')$;
$L(\neg E) = \Sigma^* \setminus L(E)$.

The following result was shown in [37].

Theorem 14: The language emptiness problem for star-free regular expressions is nonelementary.

We give a polynomial-time reduction of the language emptiness problem for star-free regular expressions to the time-bounded language emptiness problem for alternating timed automata. Note that since language emptiness for (untimed) alternating automata is PSPACE-complete [4], such a reduction would not be possible in the untimed setting. Before describing the reduction we need some auxiliary notions concerning regular expressions.

The *operator depth* $\mathrm{odp}(E)$ of a regular expression E is defined as follows:

$\mathrm{odp}(\emptyset) = \mathrm{odp}(\sigma) = 1$;
$\mathrm{odp}(E + E') = \max\{\mathrm{odp}(E), \mathrm{odp}(E')\} + 1$;
$\mathrm{odp}(E \cdot E') = \max\{\mathrm{odp}(E), \mathrm{odp}(E')\} + 1$;
$\mathrm{odp}(\neg E) = \mathrm{odp}(E)$.

Note that negation does not count toward the operator depth.

Given a star-free regular expression E over alphabet Σ and a word $u \in \Sigma^*$ we define the *membership game* $\mathbb{G}(u, E)$. This is a two-player game with N rounds, where N is the operator depth of E. The two players are *Prover*, who is trying to show $u \in E$, and *Refuter*, who is trying to show $u \notin E$. The positions of the game are pairs (v, F) where v is a sub-word of u and F has the form G or $\neg G$ for G a sub-expression of E. The initial position is (u, E). Suppose the position at the start of a given round is (v, F), where $v = v_1 \ldots v_n$; then the round proceeds as follows:

- If $F \equiv F_1 \cdot F_2$ then Prover moves first by choosing an index i in v. Refuter responds by selecting either $(v_1 \ldots v_{i-1}, F_1)$ or $(v_i \ldots v_n, F_2)$ as the position in the next round;
- If $F \equiv \neg(F_1 \cdot F_2)$ then Refuter moves first by choosing an index i in v. Prover responds by selecting either $(v_1 \ldots v_{i-1}, \neg F_1)$ or $(v_i \ldots v_n, \neg F_2)$ as the position in the next round;
- If $F \equiv F_1 + F_2$ then Prover selects either (v, F_1) or (v, F_2) as the position in the next round;
- If $F \equiv \neg(F_1 + F_2)$ then Refuter selects either $(v, \neg F_1)$ or $(v, \neg F_2)$ as the position in the next round.

The positions (v, σ), $(v, \neg\sigma)$, $(v, \emptyset)$, and $(v, \neg\emptyset)$ are terminal. In these cases Prover wins if v is a member of the corresponding expression and Refuter wins otherwise.

It is clear that Prover has a winning strategy in $\mathbb{G}(u, E)$ if and only if $u \in L(E)$.

Definition 15: Suppose that E is a star-free regular expression over alphabet Σ. Given a time domain $\mathbb{T} = [0, N)$, we associate with E a timed language $L_{\mathbb{T}}(E) \subseteq \mathbb{T}\Sigma^*$ containing those timed words $w = (a_1, t_1) \ldots (a_n, t_n)$ satisfying the following two properties:

(i) There exists a word $u \in L(E)$ such that $\mathrm{untime}(w) = u^N$ consists of N copies of u;

(ii) Each successive copy of u in w is separated by one time unit, that is, $t_{i+|u|} = t_i + 1$ for $1 \le i \le Nn - |u|$, where $|u|$ denotes the length of u.

Example 16: In case $\mathbb{T} = [0, 2)$ and $\Sigma = \{a\}$ then $L_{\mathbb{T}}(\Sigma^*) = L_{\mathbb{T}}(\mathcal{A}_{\mathrm{copy}})$, where $\mathcal{A}_{\mathrm{copy}}$ is the automaton defined in Example 5.

Let E be a star-free regular expression of operator depth N and write $\mathbb{T} = [0, N)$. It holds by construction that $L_{\mathbb{T}}(E)$ is nonempty if and only if $L(E)$ is nonempty. Next we define an alternating timed automaton $\mathcal{A}_E$ such that $L_{\mathbb{T}}(\mathcal{A}_E) = L_{\mathbb{T}}(E)$.

There are two ideas behind the definition of $\mathcal{A}_E$. The first, following Example 16, is that $L_{\mathbb{T}}(\Sigma^*)$ is the language of a simple variant of $\mathcal{A}_{\mathrm{copy}}$—call it $\mathcal{B}$—that accepts its input if and only if the sub-word occurring in the first time unit is repeated in all subsequent time units. The second idea is that for an arbitrary expression E we can define $\mathcal{A}_E$ as the intersection of $\mathcal{B}$ with another automaton. Note that if w is accepted by $\mathcal{B}$ then $\mathrm{untime}(w) = u^N$ for some $u \in \Sigma^*$. The definition of $\mathcal{A}_E$ is such that $\mathcal{A}_E$ simulates the membership game $\mathbb{G}(u, E)$ by playing one round in each time unit. Intuitively, $\mathcal{A}_E$ simulates moves of Prover by disjunctive transitions and moves of Refuter by conjunctive transitions.

Recall that a position of the membership game $\mathbb{G}(u, E)$ is a pair (v, F), where v is a sub-word of u and F has the form G or $\neg G$ for a sub-expression G of E. In order to remember the game position (v, F) between successive time units, $\mathcal{A}_E$ stores F in its finite control, while it records v by resetting a clock x as it reads the first letter of v and resetting a clock y as it reads the last letter of v. Automaton $\mathcal{A}_E$ consists of N gadgets; one time unit passes between control entering and exiting each gadget.

There is an initialisation gadget that resets clock x on the first event of the timed word and resets clock y on the last event in the first time unit.

The gadget for $F \equiv F_1 \cdot F_2$ is illustrated in Figure 3 (we omit the labels on transitions). Referring to the appropriate clause in the membership game, the choice of when to take the transition exiting location s simulates the move of Prover to select an index of the input word; the two conjunctive branches of this transition simulate the choice of Refuter either to choose the left sub-word or the right sub-word.[4]

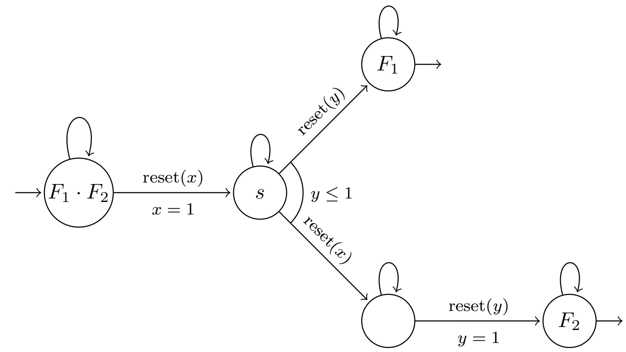


Figure 3. Gadget for $F_1 \cdot F_2$

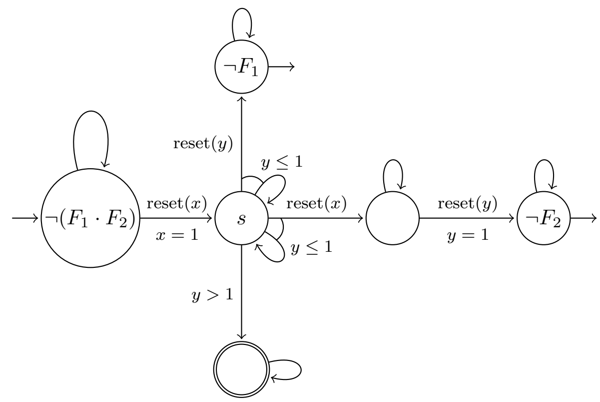


Figure 4. Gadget for $\neg(F_1 \cdot F_2)$

The gadget for $F \equiv \neg(F_1 \cdot F_2)$ is illustrated in Figure 4. It operates along similar lines as the gadget for $F_1 \cdot F_2$ and we omit detailed explanation.

Theorem 17: The time-bounded language emptiness problem for alternating timed automata is nonelementary.

It can be seen from the proof of Theorem 17 that the nonelementary lower bound applies for automata with only two clocks. On the other hand, for a fixed time bound the decision procedure for language emptiness presented in Theorem 13 is elementary. The translation of the language emptiness problem to the decision problem for McNaughton games detailed in Proposition 9 involves a winning condition $\varphi_{\mathcal{A}}$ whose quantifier depth is absolutely bounded. Furthermore the decision procedure for McNaughton games presented in Theorem 12 is elementary if the quantifier depth of the winning condition and the time bound are both fixed.

We conclude by observing that there cannot be an elementary procedure that computes an $\mathsf{MSO}(<, +1)$ for-

[4]Strictly speaking, to handle the case in which v is the empty word, we should include transitions in Figures 3 and 4 that allow clocks x and y to be reset simultaneously. But these are omitted for readability.

mula φ from a given alternating timed automaton $\mathcal{A}$ such that (i) φ and $\mathcal{A}$ define identical timed languages over all time domains $\mathbb{T}$; (ii) the quantifier alternation depth of φ is bounded independently of $\mathcal{A}$. The existence of such a procedure would contradict Theorem 17 given that the satisfiability problem for $\mathsf{MSO}(<, +1)$ over bounded time intervals is elementary for formulas of a fixed quantifier-alternation depth [27]. By contrast there is a straightforward translation from (untimed) automata over words to the existential fragment of $\mathsf{MSO}(<)$ [18], [19].

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