Linear dynamical systems with continuous weight functions

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ABSTRACT

In discrete-time linear dynamical systems (LDSs), a linear map is repeatedly applied to an initial vector yielding a sequence of vectors called the orbit of the system. A weight function assigning weights to the points in the orbit can be used to model quantitative aspects, such as resource consumption, of a system modelled by an LDS. This paper addresses the problems to compute the mean payoff, the total accumulated weight, and the discounted accumulated weight of the orbit under continuous weight functions and polynomial weight functions as a special case. Besides general LDSs, the special cases of stochastic LDSs and of LDSs with bounded orbits are considered. Furthermore, the problem of deciding whether an energy constraint is satisfied by the weighted orbit, i.e., whether the accumulated weight never drops below a given bound, is analysed.

CCS CONCEPTS

• **Theory of computation** → **Timed and hybrid models**; • ;

KEYWORDS

linear dynamical systems, mean payoff, total reward, discounted reward

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1 INTRODUCTION

Dynamical systems describing how the state of a system changes over time constitute a prominent modelling paradigm in a wide variety of fields. A *discrete-time linear dynamical system* (LDS) in ambient space $\mathbb{R}^d$ starts at some initial point $q \in \mathbb{R}^d$. The dynamics of the system are given by a linear update function in form of a matrix $M \in \mathbb{R}^{d \times d}$ that is applied to the current state of the system at each time step. This gives rise to the *orbit* $(q, Mq, M^2q, \dots)$.

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The investigation of LDSs is particularly important as they are arguably the simplest form of dynamical systems, but nevertheless exhibit many challenging problems. Further, the linearisation of more complex systems is ubiquitous in control theory and engineering (see, e.g., [5, 18]) and so many real world problems are solved via the use of linearisations.

Algorithmic problems concerning LDSs form a lively area of research in computer science. Surprisingly, several seemingly simple decidability questions about the orbit of a given LDS have been open for many decades (for an overview, see [12]). For example, two prominent problems about *linear recurrence sequences*, the Positivity Problem and the Skolem Problem, are subsumed by the following problem: given (M, q) and a target set H, decide whether there exists $n \in \mathbb{N}$ such that $M^n q \in H$. Deep results establish the decidability of special cases of the Skolem [23, 25] and Positivity [21, 22] problems in low dimensions. In general, the decidability status of these two problems has, however, been open for many decades. Furthermore, in order to verify that a system modelled as an LDS satisfies desirable properties, typical formal verification problems such as model-checking problems asking whether the orbit of an LDSs satisfies certain temporal properties have been studied [4, 13].

In this paper, we address quantitative verification questions arising when systems are equipped with a weight function. To the best of our knowledge, such quantitative verification tasks on weighted LDSs have not been investigated in the literature. The work [16] on computing the *density* of the visits of an orbit to a semialgebraic set, however, has direct consequences for weighted LDSs with *non-continuous* weight functions that we explain in more detail in the section on related work below.

We consider *continuous* weight functions $w \colon \mathbb{R}^d \to \mathbb{R}$ assigning a weight to each state in the ambient space. Such weight functions can be used to model various quantitative aspects of a system, such as resource or energy consumption, rewards or utilities, or execution time for example. Given a weight function w, we obtain a sequence of weights of the states in the orbit $(w(q), w(Mq), w(M^2q), \dots)$. The goal of this paper is to provide algorithmic answers to the following typical questions arising for weighted systems:

a) What is the *mean payoff*, i.e., the average weight collected per step?
b) What is the *total accumulated weight* of the orbit and what is the so-called *discounted accumulated weight*, where weights obtained after k time steps are discounted with a factor λ^k for a given $\lambda \in (0, 1)$?

Table 1: Overview of the results.

	LDS type	weight function	algorithmic results	
mean payoff	arbitrary	polynomial	computable	(Thm. 3.4)
	bounded orbit	continuous	integral representation computable	(Thm. 3.8)
	stochastic, irreducible	continuous	computable with polynomially many evaluations of the weight function.	(Thm. 3.10)
	stochastic, reducible	continuous	computable with exponentially many evaluations of the weight function.	(Thm. 3.12)
total/discounted weight	arbitrary	polynomial	computable	(Thm. 4.1)
satisfaction of energy constraints	arbitrary	polynomial	decidable in dimension 3	(Thm. 4.7)
	stochastic	linear	Positivity-hard	(Thm. 4.8)
	dimension 4	polynomial	Diophantine-hard	(Thm. 4.9)

c) Is there an $n \in \mathbb{N}$ such that the sum of weights obtained in the first n steps lies below a given bound? This problem is referred to as *satisfaction of an energy constraint* because it corresponds to determining whether a system ever runs out of energy when weights model the energy used or gained per step.

Example 1.1. Assume a scheduler assigns tasks to d different processors $P_1, \ldots, P_d$ and that the load of the processors at different time steps can be modeled as an LDS with matrix $M \in \mathbb{Q}^{d\times d}$ and orbit $(M^k q)_{k\in\mathbb{N}}$ for a $q \in \mathbb{Q}^d$. Further, assume for each processor P_i there is an optimal load μ_i under which it works most efficiently. To evaluate the scheduler, we want to know how closely the real loads in the long-run match the ideal loads. As a measure for how well a vector x matches the vector μ of ideal loads, we use the average squared distance

$$\delta_\mu(x) = \frac{1}{d}\sum_{i=1}^{d}(x_i - \mu_i)^2.$$

To see how well the scheduler manages to get close to optimal loads in the long-run after a possible intialization phase, we consider the mean payoff of the orbit with respect to the weight function δ_μ, i.e.,

$$\lim_{\ell\to\infty} \frac{1}{\ell+1}\sum_{k=0}^{\ell}\delta_\mu(M^k q).$$

If, on the other hand, we know that the orbit will tend to the optimal loads for $k \longrightarrow \infty$, we might instead also want to measure the total deviation $\sum_{k=0}^{\infty}\delta_\mu(M^k q)$. If this value is small, the orbit converges to the optimal loads rather quickly without large deviations initially.

In order to obtain algorithmic results, we consider different combinations of restricted classes of LDSs and restricted classes of continuous weight functions. Namely, besides arbitrary rational LDSs, we consider also LDSs with bounded orbit and *stochastic LDSs*. Stochastic LDSs occur in the context of the verification of probabilistic systems: For a finite-state Markov chain, the sequence of distributions over the state space naturally forms an LDS. The initial distribution can be written as a vector $\iota_{init} \in [0,1]^d$. Afterwards, the transition probability matrix P can be repeatedly applied to obtain the distribution $P^k\iota_{init}$ over states after k steps. In contrast to the path semantics where a probability measure over infinite paths in a Markov chain is defined, the view of a Markov chain as an LDS is also called the *distribution transformer semantics* of Markov chains (see, e.g., [1]).

For the weight functions, we consider general continuous functions. Of course, for algorithmic results, we have to make additional assumptions on the computability or approximability of these functions. Furthermore, we consider the subclass of polynomial weight functions with rational coefficients.

Contribution. We address the problems mentioned above for weighted LDSs with rational entries under *continuous* weight functions. Our contributions are as follows (see also Table 1).

a) Mean payoff: For rational LDSs equipped with a polynomial weight function, we show that it is decidable whether the mean payoff exists, in which case it is rational and computable. We then show how to decide whether the orbit of a rational LDS is bounded. If the orbit of a rational LDS is bounded, we show how to compute the set of accumulation points of the orbit and prove that the mean payoff of the orbit can be expressed as an integral of the weight function over a computable parametrisation of this set. As the parametrisation can be computed explicitly, the integral can be approximated to arbitrary precision for any weight function w that is sufficiently well-behaved.
Next, we consider stochastic LDSs, which constitute a special case of LDSs with bounded orbits. Here, the orbit only has finitely many accumulation points. We show that in case the transition matrix is irreducible, one can compute polynomially many rational points in polynomial time such that the mean payoff is the arithmetic mean of the weight function evaluated at these points. In the reducible case, on the other hand, exponentially many such rational points have to be computed.
b) Total and discounted accumulated weights: For rational LDSs and polynomial weight functions, we prove that the total as well as the discounted accumulated weight of the orbit is computable and rational if finite.
c) Satisfaction of energy constraints: First we prove that it is decidable whether an energy constraint is satisfied by an orbit under a polynomial weight function for LDSs of dimension $d = 3$. We furthermore provide two different hardness results regarding possible extensions of this decidability result: At $d = 4$, the problem is hard with respect to certain open decision problems in Diophantine approximation that

are at the moment wide open. Further, also restricting to stochastic LDSs and linear weight functions, does not lead to decidability in general: we show that the energy satisfaction problem is at least as hard as the Positivity Problem for linear recurrence sequences in this case. The decidability status of the Positivity Problem is open, and it is known from [21] that its resolution would amount to major mathematical breakthroughs.

Related work. Verification problems for linear dynamical systems have been extensively studied for decades, starting with the question about the decidability of the Skolem [23, 25] and Positivity [21, 22] problems at low orders, which are special cases of the reachability problem for LDSs. Decidable cases of the more general Model-Checking Problem for LDSs have been studied in [4, 13]. In addition, decidability results for parametric LDSs [6] as well as various notions of robust verification [3, 10] have been obtained. See [12] for a survey of what is decidable about discrete-time linear dynamical systems.

There is very little related work on LDSs with weight functions. Closest to our work is the work by Kelmendi [16]. There, it is shown that the *natural density* (which is a notion of frequency) of visits of a rational LDS in a semialgebraic set always exists and is approximable to arbitrary precision. A consequence of this result is that the mean payoff of a rational LDS with respect to a "semialgebraic step function", which takes a partition of the ambient space $\mathbb{R}^d$ into finitely many semialgebraic sets $S_1, \dots, S_k$ and assigns a rational weight w_i to the points in S_i, can be approximated to arbitrary precision. As these step functions are not continuous, this result is orthogonal to our results.

When it comes to Markov chains viewed as LDSs under the distribution transformer semantics, it is known that Skolem and Positivity-hardness results for general LDSs persist [2]. Vahanwala has recently shown [24] that this is the case even for ergodic Markov chains. In [1], Markov chains under the distribution transformer semantics are treated approximately–in contrast to our work–by discretising the probability value space $[0, 1]$ into a finite set of intervals and the problem to decide whether an approximation of the trajectory obtained in this way satisfies a property is studied.

2 PRELIMINARIES

We briefly present our notation and introduce the concepts used in the subsequent sections.

2.1 Linear dynamical systems

A (discrete-time) *linear dynamical system* (LDS) (M, q) of dimension $d > 0$ consists of an update matrix $M \in \mathbb{R}^{d\times d}$ and an initial vector $q \in \mathbb{R}^d$. If the entries of M and q are rational, we say that the LDS is *rational*. The *orbit* $O(M, q)$ of (M, q) is the sequence $(M^k q)_{k\in\mathbb{N}}$. We say that the orbit of (M, q) is bounded if there exists $c \in \mathbb{R}$ such that the Euclidean length $||M^k q|| < c$ for all $k \in \mathbb{N}$. An LDS is called *stochastic* if the matrix M and the initial vector q have only non-negative entries and the entries of each column of M as well as the entries of q sum up to 1. In this case we refer to the matrix M as stochastic too.[1]

2.2 Algebraic numbers

A number $\alpha \in \mathbb{C}$ is *algebraic* if there exists a polynomial $p \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$. Algebraic numbers form a subfield of $\mathbb{C}$ denoted by $\overline{\mathbb{Q}}$. The minimal polynomial of $\alpha \in \overline{\mathbb{Q}}$ is the (unique) monic polynomial $p \in \mathbb{Q}[X]$ of the smallest degree such that $p(\alpha) = 0$. The *degree* of α, denoted by $\deg(\alpha)$, is the degree of the minimal polynomial of α. For each $\alpha \in \overline{\mathbb{Q}}$ there exists a unique polynomial $P_\alpha = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x]$ with $d = \deg(\alpha)$, called the *defining polynomial* of α, such that $P_\alpha(\alpha) = 0$ and $\gcd(a_0, \dots, a_d) = 1$. The polynomial P_α and the minimal polynomial of α have identical roots, and are *square-free*, i.e., all of their roots appear with multiplicity one. The *(naive) height* of α, denoted by $H(\alpha)$, is equal to $\max_{0\le i\le d} |a_i|$. We represent an algebraic number α in computer memory by its defining polynomial P_α and sufficiently precise rational approximations of $\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)$ to distinguish α from other roots of P_α. We denote by $||\alpha||$ the bit length of a representation of $\alpha \in \overline{\mathbb{Q}}$. We can perform arithmetic effectively on algebraic numbers represented in this way.

2.3 Linear recurrence sequences

A sequence $(u_n)_{n\in\mathbb{N}}$ is a *linear recurrence sequence* over a ring $R \subseteq \mathbb{C}$ if there exists a positive integer d and a *recurrence relation* $(a_0, \dots, a_{d-1}) \in R^d$ such that $u_{n+d} = \sum_{i=0}^{d-1} a_i u_{n+i}$ for all n. The *order* of $(u_n)_{n\in\mathbb{N}}$ is the smallest positive integer d such that $(u_n)_{n\in\mathbb{N}}$ satisfies a recurrence relation in R^d. In this work we will mostly encounter LRSs over $\mathbb{Q}$, called *rational LRSs*. Examples of rational LRSs include the Fibonacci sequence, $u_n = p(n)$ for $p \in \mathbb{Q}[x]$, and $u_n = \cos(n\theta)$ where $\theta \in \{\arg(\lambda) : \lambda \in \mathbb{Q}(i)\}$. We refer the reader to the books by Everest et al. [11] and Kauers & Paule [15] for a detailed discussion of linear recurrence sequences.

An LRS $(u_n)_{n\in\mathbb{N}}$ that is not eventually zero satisfies a unique minimal recurrence relation $u_{n+d} = \sum_{i=0}^{d-1} a_i u_{n+i}$ such that $d > 0$ and $a_0 \neq 0$. Writing $A = \begin{bmatrix} a_1 & \cdots & a_{d-1} \end{bmatrix}$ and $q = \begin{bmatrix} u_0 & \cdots & u_{d-1} \end{bmatrix}^\top$, the matrix

$$C := \begin{bmatrix} \mathbf{0} & I_{d-1} \\ a_0 & A \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{d-1} \end{bmatrix} \in R^{d\times d}$$

is called the *companion matrix* of $(u_n)_{n\in\mathbb{N}}$. We have that

$$C^n q = \begin{bmatrix} u_n & \cdots & u_{n-d+1} \end{bmatrix}^\top$$

and $u_n = e_1 C^n q$ for all $n \in \mathbb{N}$, where e_i denotes the ith standard basis vector. As $a_0 \neq 0$, the matrix C is invertible and does not have zero as an eigenvalue.

The *characteristic polynomial* of $(u_n)_{n\in\mathbb{N}}$ is $p(x) = x^d - \sum_{i=0}^{d-1} a_i x^i$. Note that p is identical to the characteristic polynomial $\det(xI - C)$ of the companion matrix C. The *eigenvalues* (also called the *roots*) of $(u_n)_{n\in\mathbb{N}}$ are the d (possibly non-distinct) roots $\lambda_1, \dots, \lambda_d$ of the characteristic polynomial p. An LRS is

[1] In order to keep the notation in line with the notation for general LDSs, we deviate from the standard convention that rows of stochastic matrices sum up to 1 and that stochastic matrices are applied to distributions by multiplication from the right.

- *simple* (or *diagonalisable*) if its characteristic polynomial does not have a repeated root, and
- *non-degenerate* if (i) all real eigenvalues are non-negative, and (ii) for every pair of distinct eigenvalues λ_1, λ_2, the ratio λ_1/λ_2 is not a root of unity.

For each LRS $(u_n)_{n\in\mathbb{N}}$ there exists effectively computable L such that the sequences $u_n^{(k)} = u_{nL+k}$ for $0 \le k < L$ are all non-degenerate [11, Section 1.1.9]. If $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ are LRSs over a field R, and $\circ \in \{+, -, \cdot\}$, then $w_n = u_n \circ v_n$ also defines an LRS over R [15, Theorem 4.2]. Moreover, if $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are both simple, then so is $(w_n)_{n\in\mathbb{N}}$.

The exponential polynomial representation of an LRS. Every LRS $(u_n)_{n\in\mathbb{N}}$ of order $d > 0$ over $\overline{\mathbb{Q}}$ can be written in the form

$$u_n = \sum_{j=1}^{m} p_j(n)\lambda_j^n \tag{1}$$

where $m \ge 1$, $\lambda_1, \ldots, \lambda_m$ are the distinct non-zero eigenvalues of $(u_n)_{n\in\mathbb{N}}$, and each p_i is a non-zero polynomial with algebraic coefficients; see [11, Chapter 1]. Whenever these conditions on m, λ_i and p_i are met, we say that the right-hand side is in the *exponential polynomial form.* Every LRS $(u_n)_{n\in\mathbb{N}}$ that is not eventually zero has a unique representation as in (1) where the right-hand side is in the exponential polynomial form. Moreover, the right-hand side of (1) is never identically zero. This is a folklore result, but we provide a proof in the extended version of the paper for completeness.

Lemma 2.1. *Let $u_n = \sum_{i=1}^{m} p_i(n)\lambda_i^n$ where $m \ge 1$, $\lambda_1, \ldots, \lambda_m \in \overline{\mathbb{Q}}$ are non-zero and pairwise distinct, and each $p_i \in \overline{\mathbb{Q}}[x]$ is non-zero. Then, the sequence $(u_n)_{n\in\mathbb{N}}$ is not identically zero. Specifically, there exists $0 \le n < d$, where $d = \sum_{i=1}^{m}(\deg(p_i) + 1)$, such that $u_n \ne 0$.*

We can also characterise the exponential polynomial representations of real-valued LRS.

Lemma 2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be as in the statement of Lemma 2.1. If $u_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then for every $1 \le i \le m$ there exists j with $1 \le j \le m$ such that $p_j(n) = \overline{p_i}(n)$ and $\lambda_j = \overline{\lambda_i}$.*

Proof. We have $\overline{u_n} = \sum_{j=1}^{m} \overline{p_j}(n)\overline{\lambda_j}^n$. Moreover, $u_n = \overline{u_n}$ since $u_n \in \mathbb{R}$ for all n. The result then follows from the uniqueness of the exponential polynomial representation. □

Throughout this work we will encounter sequences of the form $u_n = p(M^n q)$ where p is a polynomial with rational coefficients and q is a vector with rational entries. Since

$$p(M^n q) = p(e_1 M^n q, \ldots, e_d M^n q),$$

each $u_n^{(k)} = e_k M^n q$ is an LRS over $\mathbb{Q}$ (this can be seen, e.g., by applying the Cayley-Hamilton theorem), and LRS over $\mathbb{Q}$ are closed under addition and multiplication, the sequence $(p(M^n q))_{n\in\mathbb{N}}$ is itself an LRS over $\mathbb{Q}$.

Decision problems about LRS. Sign patterns of LRS have been studied for a long time. Two prominent open problems in this area are the *Skolem Problem* and the *Positivity Problem.* The Skolem Problem is to find an algorithm that, given an LRS u_n, decides if the set $Z = \{n \colon u_n = 0\}$ is non-empty. The most well-known result in this direction is the celebrated Skolem-Mahler-Lech theorem, which (non-constructively) shows that Z is semilinear. In particular, it shows that a non-degenerate $(u_n)_{n\in\mathbb{N}}$ can have only finitely many zeros. The Positivity Problem, on the other hand, asks to find an algorithm that determines if $u_n \ge 0$ for all n.

2.4 Markov Chains

A finite-state *discrete-time Markov chain* (DTMC) M is a tuple (S, P, ι_{init}), where S is a finite set of states, $P : S \times S \to [0, 1]$ is the transition probability function where we require $\sum_{s'\in S} P_{ss'} = 1$ for all $s \in S$ and $\iota_{init} : S \to [0, 1]$ is the initial distribution, such that $\sum_{s\in S} \iota_{init}(s) = 1$. For algorithmic problems, all transition probabilities are assumed to be rational. A finite path ρ in M is a finite sequence $s_0 s_1 \ldots s_n$ of states such that $P(s_i, s_{i+1}) > 0$ for all $0 \le i \le n-1$. We say that a state t is reachable from s if there is a finite path from s to t. If all states are reachable from all other states, we say that M is *irreducible*; otherwise, we say it is *reducible.* A set $B \subseteq S$ of states is called a bottom strongly connected component (BSCC) if it is strongly connected, i.e., all states in B are reachable form all other states in B and if there are no outgoing transitions, i.e., $P(s, t) > 0$ and $s \in B$ implies $t \in B$.

W.l.o.g., we identify S with $\{1, \ldots, d\}$ for $d = |S|$. Then, overloading notation, we consider $P \in \mathbb{R}^{d\times d}$ as a matrix with $P_{ij} = P(j, i)$ for $i, j \le d$.[2] Likewise, we consider ι_{init} to be a (column[3]) vector in $\mathbb{R}^d$ with $(\iota_{init})_i = \iota_{init}(i)$ for $i \le d$. Then, the sequence of distributions over states after k steps is given by $P^k \iota_{init}$, which forms a stochastic LDS. We also write $P_{ij}^{(k)}$ for $(P^k)_{ij}$, which is the probability to move from state j to i in exactly k steps. Further, we say that the matrix P is irreducible if the underlying Markov chain is irreducible. The period d_i of a state i is given by: $d_i = \mathbf{gcd}\{m \ge 1 : P_{ii}^{(m)} > 0\}$. If $d_i = 1$, then we call the state i aperiodic. A Markov chain (and its matrix) are aperiodic if and only if all its states are aperiodic. The period of a Markov chain M and of its transition probability matrix P is the least common multiple of the periods of the states of M.

A vector $\pi \in \mathbb{R}^d$ is called a stationary distribution of the Markov chain if: a) π is a distribution, i.e., $\pi_j \ge 0$ for all j with $1 \le j \le d$, and $\sum_{j=1}^{d} \pi_j = 1$; b) π is stationary, i.e., $\pi = P\pi$, which is to say that $\pi_i = \sum_{i\in S} P_{ij}\pi_j$ for all $j \in S$. For aperiodic Markov chains, it is known that the sequence of distributions over states $(P^k \iota_{init})_{k\in\mathbb{N}}$ converges to a stationary distribution π, which can be computed in polynomial time (see [7, 17]).

3 MEAN PAYOFF

In this section, we address the computation of the *mean payoff* of an orbit. The mean payoff is the average weight collected per step in the long-run. For an LDS given by $M \in \mathbb{Q}^{d\times d}$ and $q \in \mathbb{Q}^d$ and a weight function $w \colon \mathbb{R}^d \to \mathbb{R}$, we define the mean payoff of the orbit as

$$MP_w(M, q) \coloneqq \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} w(M^i q).$$

In the sequel, we address the problem to compute the mean payoff of the orbit of an LDS with respect to continuous weight

[2]This is the transpose of the transition matrix usually defined so that we are in line with our notation for LDSs.

[3]Also here, usually, this is defined as a row vector.

functions. To this end, we restrict either the class of weight functions or the class of LDSs. In Section 3.1, we address the problem for polynomial weight functions. In Sections 3.2 and 3.3 , we consider continuous weight functions on LDSs with bounded orbit and stochastic LDSs, respectively.

3.1 Polynomial weight-functions

In order to compute the mean payoff of the orbit of a rational LDS (M, q) with respect to a polynomial weight function p with rational coefficients, we first recall that the sequence $(p(M^n q))_{n\in\mathbb{N}}$ is an LRS. The following lemma states that the sequence of partial sums of the weights is also an LRS.

Lemma 3.1. *Let (M, q) be an LDS with $M \in \mathbb{Q}^{d\times d}$ and $q \in \mathbb{Q}^d$, and let $p \in \mathbb{Q}[X_1, \dots, X_d]$ be a polynomial weight function with rational coefficients. The sequence*

$$u_n = \sum_{i=0}^{n} p(M^i q)$$

is a rational LRS.

Proof. As discussed in subsection 2.3, $w_n = p(M^i q)$ is a rational LRS. Suppose $(w_n)_{n\in\mathbb{N}}$ satisfies a recurrence relation $w_{n+k} = a_0 w_n + \dots + a_{k-1} w_{n+k01}$, where $a_0, \dots, a_{k-1} \in \mathbb{Q}$. Then $u_{n+k+1} = u_{n+k} + a_{k-1}(w_{n+k} - w_{n+k-1}) + \dots + a_0(w_{n+1} - w_n)$. Hence $(u_n)_{n\in\mathbb{N}}$ itself is an LRS of order at most $k+1$. □

Computing $MP_w(M, q)$ therefore boils down to determining whether the limit $\lim_{n\to\infty} u_n/n$ exists for an LRS $(u_n)_{n\in\mathbb{N}}$ and computing the limit in case it exists (See the extended version of the paper for the proof).

Lemma 3.2. *Let $(u_n)_{n\in\mathbb{N}}$ be an LRS over $\mathbb{Q}$. It is decidable whether $\lim_{n\to\infty} u_n/n$ exists, in which case the limit is rational and effectively computable.*

Corollary 3.3. *For a rational LRS $(u_n)_{n\in\mathbb{N}}$, it is decidable whether $\lim_{n\to\infty} u_n$ exists, in which case the limit is rational and effectively computable.*

Proof. The sequence $(n)_{n\in\mathbb{N}}$ is an LRS over $\mathbb{Q}$, and since rational LRS are closed under multiplication, $v_n = nu_n$ also defines a rational LRS. It remains to observe that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n/n$ and apply Lemma 3.2. □

From the lemma and the corollary above, for both $(u_n)_{n\in\mathbb{N}}$ and $(u_n/n)_{n\in\mathbb{N}}$, if the limit exists it is equal to a coefficient of some p_i appearing in the exponential polynomial solution of $(u_n)_{n\in\mathbb{N}}$. Hence the complexity of computing the limit is bounded by the complexity of computing the exponential polynomial; The latter is known to be in polynomial time if we assume the order of the LRS is fixed [21]; in general, if the description length of $(u_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$ and its order is d, the time required to compute the exponential polynomial representation of u_n is polynomial in $\mathcal{I}^d$. Lemma 3.2 puts us into the position to prove the first main result on the computation of the mean payoff:

Theorem 3.4. *Let (M, q) be an LDS with $M \in \mathbb{Q}^{d\times d}$ and $q \in \mathbb{Q}^d$, and $p \in \mathbb{Q}[X_1, \dots, X_d]$ be a polynomial weight function with rational coefficients. It is decidable whether the mean payoff*

$$MP_p(M, q) = \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} p(M^i q)$$

exists, in which case it is rational and computable.

Proof. Immediate by Lemma 3.2 and Lemma 3.1. □

3.2 Bounded LDSs

If the orbit of an LDS is bounded, we can get our hands on the mean payoff with respect to a continuous weight function. We exploit that the orbit of an LDS approaches a limiting shape – which is the set of accumulation points of the orbit – closer and closer in this case. This allows us to express the mean payoff in terms of an integral of the weight function over this limiting shape. This integral computes the "average" value of the weight function on the limiting shape. Of course, we have to carefully ensure that we also know how "frequently" the orbit approaches different parts of the limiting shape. Let us illustrate this idea first:

Example 3.5. Let $w: \mathbb{R}^3 \to \mathbb{R}$ be a continuous weight function and consider the LDS

$$M = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Looking only at the first two coordinates a rotation is repeatedly applied in this LDS. In the complex plane, this rotation is given by multiplication with $3/5 - 4/5i$. As $3/5 - 4/5i$ is not a root of unity, the orbit never reaches a point with $(1, 0)$ in the first two coordinates again. In fact, the first two components of the orbit are dense in the unit circle. Furthermore, these components visit each interval of the same length on the circle with the same frequency. The third component is halved at every step and converges to 0. As the weight function is continuous, we can hence treat the third coordinate as equal to 0 when determining the mean payoff. So, the set of accumulation points of the orbit is $L = \{v \in \mathbb{R}^3 \mid v_3 = 0, |v| = 1\}$, which we can parametrise via $T: [0, 1) \to \mathbb{R}^3$ with $T: \alpha \mapsto \begin{bmatrix} \cos(2\pi\alpha) & \sin(2\pi\alpha) & 0 \end{bmatrix}^\top$. As this parametrisation moves through the circle with constant speed reflecting the fact that the orbit is "equally distributed" over the circle in the first two components, we can now express the mean payoff of the orbit with respect to the weight function w as

$$MP_w(M, q) = \int_0^1 w\left(\begin{bmatrix} \cos(2\pi\alpha) & \sin(2\pi\alpha) & 0 \end{bmatrix}^\top\right) \, \mathrm{d}\alpha.$$

In the sequel, we work out all the necessary steps to check whether the orbit of an LDS is bounded and to obtain such an expression for the mean payoff as an integral for arbitrary rational LDSs with bounded orbit.

Jordan normal form and boundedness of the orbit. Throughout this section, fix a matrix $M \in \mathbb{Q}^{d\times d}$, an initial vector $q \in \mathbb{Q}^d$, and a continuous weight function $w: \mathbb{R}^d \to \mathbb{R}$. We first transform the matrix M into Jordan normal form by computing matrices J and B as well as the inverse B^{-1} with algebraic entries such that

$$M = B \cdot J \cdot B^{-1}$$

where J is in Jordan form with the eigenvalues of M on the diagonal and B is an invertible matrix with generalized eigenvectors of M as columns in polynomial time [8]. Since multiplication with B is a linear bijection, $(M^k \cdot q)_{k\in\mathbb{N}}$ is bounded if and only if the sequence $(J^k \cdot (B^{-1}q))_{k\in\mathbb{N}}$ is bounded. To check whether this is the case, we first simplify the sequence.

We use the notation $J_{\alpha,\ell}$ to denote a Jordan block of size ℓ with α on the diagonal. Observe that multiplying a Jordan block to a vector $q = [q_1, \dots, q_k, 0, \dots, 0,]^\top$ in which the last $\ell - k$ components are 0 results in a vector where this is still the case:

$$J_{\alpha,\ell} \cdot q = \begin{bmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha & 1 & 0 \\ \vdots & & \ddots & \alpha & 1 \\ 0 & \dots & \dots & 0 & \alpha \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ \dots \\ q_k \\ 0 \\ \dots \\ 0 \end{bmatrix} = \begin{bmatrix} J_{\alpha,k} \cdot \begin{bmatrix} q_1 \\ \dots \\ q_k \end{bmatrix} \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Looking at the initial vector $B^{-1}q$, this allows us to simplify the LDS by determining the coordinates at which the orbit $(J^k B^{-1} q)_{k\in\mathbb{N}}$ always stays 0. Suppose the Jordan blocks in J end at coordinates $i_1, \dots, i_m$, respectively, with $1 \le i_1 < i_2 < i_m = d$. Now, let

$$I = \{i \in \{1, \dots, d\} \mid \text{for some index } h, \text{ all } j \text{ with } i \le j \le i_h \text{ satisfy } (B^{-1}q)_j = 0\}.$$

So, I contains only dimensions j such that $(J^k (B^{-1}q))_j = 0$ for all k. We now set all columns and rows of J with an index in I to 0. This does not affect the orbit $(BJ^k B^{-1} q)_{k\in\mathbb{N}}$. After this simplification, the following condition, which we can assume w.l.o.g., is satisfied.

Assumption 1. The LDS given by $M \in \mathbb{Q}^{d\times d}$ and $q \in \mathbb{Q}^d$ has the following property: For the Jordan normal form $M = B \cdot J \cdot B^{-1}$ of M and $v \overset{\text{def}}{=} B^{-1}q$, we have that $v_i \neq 0$ for any coordinate $1 \le i \le d$ at which a non-zero Jordan block of J ends.

Proposition 3.6. *Under Assumption 1, the orbit $(J^k q)_{k\in\mathbb{N}}$ is bounded if and only if all eigenvalues on the diagonal of J have modulus at most 1 and the Jordan blocks in J with an eigenvalue α with $|\alpha| = 1$ have size 1.*

We delegate the proof to the extended version of the paper. Proposition 3.6 allows us to decide whether the orbit of the LDS given by M and v is bounded. From now on, we assume that it is bounded. We now further simplify the LDS by removing all eigenvalues with modulus less than 1: For a Jordan block $J_{\alpha,\ell}$ with $|\alpha| < 1$, we know $J_{\alpha,\ell} \to 0$ for $k \to \infty$. As we apply the function B viewed as a linear map and the *continuous* function w to the points in the orbit and as the mean payoff does not depend on a prefix of the orbit, we can set all such Jordan blocks to 0 without affecting the mean payoff. So, w.l.o.g. we can work under the following assumption after this simplification because the Jordan blocks with eigenvalues with modulus 1 have size 1 in the light of Proposition 3.6:

Assumption 2. The matrix M of the rational LDS (M, q) is diagonalisable and all non-zero eigenvalues have modulus 1. So, there is a computable algebraic matrix B with computable inverse B^{-1} and a computable algebraic diagonal matrix D whose entries all have modulus 1 or 0 with $M = B \cdot D \cdot B^{-1}$.

Multiplicative relations between the eigenvalues. Before we can parametrise the set of accumulation points of the orbit, we have to detect *multiplicative relations* between the elements on the diagonal of D. Before defining (the group of) multiplicative relations, let us illustrate this concept in an example:

Example 3.7. Consider the matrix $D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$ for an algebraic number λ with $|\lambda| = 1$ that is not a root of unity. Then, $\lambda \cdot \bar{\lambda} = 1$ is a multiplicative relation between λ and $\bar{\lambda}$. Further, $(\lambda^k)_{k\in\mathbb{N}}$ is dense in the torus $\mathbb{T} \coloneqq \{x \in \mathbb{C} \mid |x| = 1\}$. Now, the sequence $\left(\lambda^k, \bar{\lambda}^k\right)_{k\in\mathbb{N}}$ is dense in $L \coloneqq \{(x, y) \in \mathbb{T}^2 \mid x \cdot y = 1\}$, but not in $\mathbb{T}^2$. So, for an initial vector v, the set of accumulation points of $(D^k v)_{k\in\mathbb{N}}$ is $L \cdot v$ and not $\mathbb{T}^2 \cdot v$.

We follow an approach also taken in [16] to detect multiplicative relations between the algebraic numbers $\lambda_1, \dots, \lambda_d \in \overline{\mathbb{Q}}$. We work under Assumption 2 and we first reorder the coordinates such that the entries on the diagonal of D are $\lambda_1, \dots, \lambda_\ell, \lambda_{\ell+1}, \dots, \lambda_d$ where λ_i is not 0 or 1 for $i \le \ell$ and the entries λ_j with $j > \ell$ are all equal to 0 or 1. The group

$$G \coloneqq G(\lambda_1, \dots, \lambda_\ell) = \{(m_1, \dots, m_\ell) \in \mathbb{Z}^\ell \mid \lambda_1^{m_1} \cdots \lambda_\ell^{m_\ell} = 1\}$$

is called the *group of multiplicative relations* between $\lambda_1, \dots, \lambda_\ell$. If this group is consists only of the neutral element, we say that $\lambda_1, \dots, \lambda_\ell$ are *multiplicatively independent.*

Note that G is a free abelian group, and has a basis of at most ℓ elements from $\mathbb{Z}^\ell$. By a deep result of Masser [19], G has a basis B such that for each $v \in B$, $\|v\|_\infty < p(\|\lambda_1\| + \ldots + \|\lambda_\ell\|)^\ell$, where p is an absolute polynomial. Hence a basis of G can be computed in polynomial space (given $\lambda_1, \dots, \lambda_\ell$) by simply enumerating all possible bases satisfying Masser's bound. As described in detail in [16], each element $(b_1, \dots, b_\ell) \in B$ of the basis allows us to express one of the eigenvalues in terms of the others: Suppose $b_j \neq 0$. Then, the equation $\lambda_1^{b_1} \cdots \lambda_\ell^{b_\ell} = 1$, allows us to conclude

$$\lambda_j^{b_j} = \prod_{i \neq j} \lambda_i^{-b_i} \qquad \text{and hence} \qquad \lambda_j = \rho_j \prod_{i \neq j} \lambda_i^{-b_i/b_j}$$

where ρ_j is a b_jth root of unity. Applying this procedure consecutively to all elements of the basis B, we can divide and reorder the eigenvalues $\lambda_1, \dots, \lambda_\ell$ as $\lambda_1, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_\ell$ such that $\lambda_1, \dots, \lambda_m$ are multiplicatively independent and such that each λ_j with $m + 1 \le j \le \ell$ is not 1 and can be written as

$$\lambda_j = \rho_j \cdot \prod_{i=1}^{m} \lambda_i^{q_{j,i}}$$

where ρ_j is a root of unity and $q_{j,i} \in \mathbb{Q}$ for $1 \le i \le m$.

Subsequences without periodicity. The fact that expression for the eigenvalues λ_j with $m + 1 \le j \le \ell$ contains the b_jth root of unity ρ_j introduces a periodic behavior to the sequence $(\lambda_j^k)_{k\in\mathbb{N}}$. In order to eliminate this periodic behavior, we divide the orbit into subsequences as follows: We let P be the least common multiple of the values b_j for $m+1 \le j \le \ell$. As ρ_j is a b_jth root of unity, $\rho_j^P = 1$ for all j with $m+1 \le j \le \ell$. We now split the sequence $(D^k)_{k\in\mathbb{N}}$ into the P subsequences of the form $(D^{Pk+r})_{k\in\mathbb{N}}$ for $r \in \{0,\ldots,P-1\}$. The diagonal entries of D^{kP} are

$$\lambda_1^{Pk},\ldots,\lambda_m^{Pk},\prod_{i=1}^{m}(\lambda_i^{Pk})^{q_{m+1,i}},\ldots,\prod_{i=1}^{m}(\lambda_i^{Pk})^{q_{\ell,i}},\lambda_{\ell+1},\ldots,\lambda_d.$$

Recall here that $\lambda_{\ell+1},\ldots,\lambda_d$ are all 0 or 1.

We can now express any point in the orbit $BD^{Pk+r}B^{-1}q$ in terms of $\lambda_1^k,\ldots,\lambda_m^k$ and D^r. To this end, we define the map $T_r\colon \mathbb{T}^m \to \mathbb{R}^d$ by setting

$$\Delta = \operatorname{diag}\left(\mu_1^P,\ldots,\mu_m^P,\prod_{i=1}^{m}(\mu_i^P)^{q_{m+1,i}},\ldots,\prod_{i=1}^{m}(\mu_i^P)^{q_{\ell,i}},\lambda_{\ell+1},\ldots,\lambda_d\right),$$

where $\operatorname{diag}(x_1,\ldots,x_d)$ denotes a diagonal matrix with $x_1,\ldots,x_d$ on the diagonal, and

$$T_r(\mu_1,\ldots,\mu_m) = BD^r\Delta B^{-1}q.$$

The map T is chosen such that

$$T_r(\lambda_1^k,\ldots,\lambda_m^k) = BD^{Pk+r}B^{-1}q.$$

This is also the reason why T_r maps into $\mathbb{R}^d$.

Parametrising the set of accumulation points. For a real x, we define $x \mod 1 := x - \lfloor x\rfloor$. For $1 \le j \le m$, we define the number $\alpha_j \in [0,1)$ as the unique number with $\lambda_j = e^{2\pi i\alpha_j}$. Let $S\colon [0,1)^m \to \mathbb{T}^m$ (recall that $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\}$) be the map

$$(\beta_1,\ldots,\beta_m) \mapsto (e^{2\pi i\beta_1},\ldots,e^{2\pi i\beta_m}).$$

So, we get $(\lambda_1^k,\ldots,\lambda_m^k) = S(k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1)$ and hence

$$BD^{Pk+r}B^{-1}q = T_r(S(k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1)).$$

Following the exposition in [16], we can now apply an equidistribution theorem by Weyl [26]. First, observe that the fact that $\lambda_1,\ldots,\lambda_m$ are multiplicatively independent means that the values $1,\alpha_1,\ldots,\alpha_m$ are linearly independent over $\mathbb{Q}$: If there were a non-zero vector $c_0,c_1,\ldots,c_m$ with $c_0+\sum_{j=1}^m c_j\alpha_j = 0$, this vector would witness a multiplicative relation between $\lambda_1,\ldots,\lambda_m$. In [26], it is now shown that for any measurable set $U \subseteq [0,1)^m$, we have

$$\lim_{n\to\infty}\frac{|\{0\le k\le n \mid (k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1)\in U\}|}{n+1} = \mathcal{L}(U) \tag{$*$}$$

where $\mathcal{L}$ is the Lebesgue measure. For more details, we also refer to the exposition of this argument in [16].

This means that the sequence of arguments $((k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1))_{k\in\mathbb{N}}$ is dense and "equally distributed" in the cube $[0,1)^m$, and hence the sequence $((\lambda_1^k,\ldots,\lambda_m^k))_{k\in\mathbb{N}}$ is dense and "equally distributed" in the m-dimensional torus $\mathbb{T}^m$ where "equally distributed" means that every subset of the same size is hit equally often in the sense of Equation $(*)$.

Mean payoff as integral. Now, we are in the position to prove the main result of this subsection: The mean payoff of a bounded orbit wrt a continuous weight function can be expressed as an integral.

Theorem 3.8. *Let $M \in \mathbb{Q}^{d\times d}$ be a matrix and $q \in \mathbb{Q}^d$ an initial vector satisfying Assumption 2. Let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous weight function. Let $P \in \mathbb{N}$ and $T_r\colon \mathbb{T}^m \to \mathbb{R}^d$ for $r < P$, and $S\colon [0,1)^m \to \mathbb{T}^m$ be as above. Then, for each r with $0 \le r < P$, the mean payoff of the sub-orbit $(M^{kP+r}q)_{k\in\mathbb{N}}$ wrt w exists and can be expressed as*

$$MP_w(M^P,M^rq) = \lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k} w(M^{kP+r}q) = \int_{[0,1)^m} w\circ T_r\circ S \,\mathrm{d}\mathcal{L}$$

where $\mathcal{L}$ is the Lebesgue measure on $[0,1)^m$. The mean payoff of the original orbit is then the arithmetic mean

$$MP_w(M,q) = \frac{\sum_{r=0}^{P-1} MP_w(M^P,M^rq)}{P}.$$

Proof. Let $\alpha_1,\ldots,\alpha_m \in [0,1)$ be such that $\lambda_j = e^{2\pi i\alpha_j}$ as above. For $r < P$, we have constructed S and T_r such that

$$M^{kP+r}q = T_r(S(k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1))$$

for all k. As w is continuous, it can be written as sum of Lebesgue measurable step functions $w = \sum_{j=0}^{\infty} f_j\cdot \mathbb{1}_{A_j}$ where, for all j, the coefficient f_j is in $\mathbb{R}$, the set $A_j \subseteq \mathbb{R}^d$ is measurable, and $\mathbb{1}_{A_j}$ is 1 on points in A_j and 0 otherwise. For $\mathbb{1}_{A_j}$, we now observe

$$\begin{aligned}
&\lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k}\mathbb{1}_{A_j}(M^{kP+r}q)\\
&= \lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k}\mathbb{1}_{A_j}(T_r(S(k\alpha_1 \mod 1,\ldots,k\alpha_m \mod 1)))\\
&= \lim_{k\to\infty}\frac{|\{i\le k \mid T_r(S(i\alpha_1 \mod 1,\ldots,i\alpha_m \mod 1))\in A_j\}|}{k+1}\\
&= \mathcal{L}((T_r\circ S)^{-1}(A_j))
\end{aligned}$$

where the last equality follows from equation $(*)$ that is stated above and shown in [26]. But, we also have

$$\int_{[0,1)^m}\mathbb{1}_{A_j}\circ T_r\circ S \,\mathrm{d}\mathcal{L} = \mathcal{L}((T_r\circ S)^{-1}(A_j)).$$

Putting this together, we obtain

$$\begin{aligned}
MP_w(M^P,M^rq) &= \lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k} w(M^{kP+r}q)\\
&= \sum_{j=0}^{\infty} f_j\cdot\lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k}\mathbb{1}_{A_j}(M^{kP+r}q)\\
&= \sum_{j=0}^{\infty} f_j\cdot\int_{[0,1)^m}\mathbb{1}_{A_j}\circ T_r\circ S \,\mathrm{d}\mathcal{L}\\
&= \int_{[0,1)^m} w\circ T_r\circ S \,\mathrm{d}\mathcal{L}.
\end{aligned}$$

This finishes the proof of the first claim. The claim that the mean payoff $MP_w(M,q)$ can now be expressed as the arithmetic mean is obvious. □

Approximation of the mean payoff. Although we can compute explicit representations of the functions T_r and S, it remains unclear – even for simple functions w – how to compute the integrals $\int_{[0,1)^m} w\circ T_r\circ S \,\mathrm{d}\mathcal{L}$ whose arithmetic mean is the mean payoff of the original LDS. Nevertheless, the numerical approximation of integrals is an extensively studied area (see, e.g., [9]). The function S is a simple parametrisation of the d-dimensional torus and the function T_r is a polynomial with algebraic coefficients that we can explicitly compute. In particular, both functions are differentiable and we can bound the modulus of the gradient of $T_r \circ S$ on $[0,1]^m$. So, if the function w can be approximated and is well-behaved, e.g., if w is Lipschitz continuous with known upper bound for its Lipschitz constant, also the integrals that we obtained can be approximated to arbitrary precision. In particular, for a polynomial weight function w, the mean payoff can be approximated to arbitrary precision as polynomials are Lipschitz continuous on the compact set $T_r \circ S([0,1]^m)$ and a bound for the Lipschitz constant can be computed from the gradient of w. For more details on conditions under which the integral can be approximated to arbitrary precision, we refer the reader to [9].

3.3 Stochastic LDSs

Stochastic LDSs are a special case of LDSs with a bounded orbit. In this section, we will show that in the case of stochastic LDSs, we can compute the mean payoff of the orbit under a continuous weight function by evaluating the weight function on finitely many points. In the aperiodic case, the orbit even converges to a single point so that it suffices to evaluate the weight function once:

Lemma 3.9. *Let $P \in \mathbb{Q}^{d\times d}$ be a stochastic, aperiodic matrix and $\iota_{init} \in \mathbb{Q}^d$ an initial distribution. Furthermore, let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous weight function. Then, $MP_w(P, \iota_{init}) = w(\pi)$ where π is the stationary distribution $\lim_{k\to\infty} P^k \iota_{init}$ of P, which is computable in polynomial time.*

Proof. As described in Section 2.4, we know that the orbit $(P^k \iota_{init})_{k\in\mathbb{N}}$ converges to a stationary distribution π in this case, which can be computed in polynomial time [7, 17]. So, $\lim_{k\to\infty} P^k \iota_{init}$ exists and, as w is continuous, we know $\lim_{k\to\infty} w(P^k \iota_{init}) = w(\pi)$. It is straightforward to observe that

$$MP_w(P, \iota_{init}) \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} w(P^i \iota_{init}) = w(\lim_{k\to\infty} P^k \iota_{init}) = w(\pi).$$

□

Hence the computation of the mean payoff boils down to evaluating the function w once on a rational point computable in polynomial time in this case. We next address the periodic case by splitting up the orbit into subsequences.

For an irreducible and periodic Markov chain with period L, we have that P^L is aperiodic and $L \le d$ by [20, Theorem 1.8.4]. Together with Lemma 3.9, this allows us to characterize $MP_w(P^L, P^r \iota_{init})$, which is the mean payoff of $(P^{Lk+r} \iota_{init})_{k\in\mathbb{N}}$. We conclude

$$MP_w(P, \iota_{init}) = \frac{1}{L} \sum_{r=0}^{L-1} MP_w(P^L, P^r \iota_{init}).$$

So, for irreducible stochastic LDSs, we can divide $(P^{Lk+r} \iota_{init})_{k\in\mathbb{N}}$ into L equally spaced subsequences and compute the mean payoff $MP_w(P, \iota_{init})$ as the arithmetic mean of the mean payoffs of these subsequences.

Theorem 3.10. *Let $P \in \mathbb{Q}^{d\times d}$ be a stochastic, irreducible matrix and $\iota_{init} \in \mathbb{Q}^d$ an initial distribution. Let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous weight function. Then, we can compute points $\pi_0, \dots, \pi_{L-1} \in \mathbb{Q}^d$ in polynomial time for some $L \le d$ such that $MP_w(P, \iota_{init}) = \frac{1}{L} \sum_{i=0}^{L-1} w(\pi_i)$.*

As the points $\pi_0, \dots, \pi_{L-1}$ can be computed in polynomial time and are hence of length at most polynomial in the length of the original input, we can conclude the following statement about the approximation of the mean payoff:

Corollary 3.11. *Assume that the value $w(a)$ can be approximated in time $f_w(\|a\|, \epsilon)$ up to some precision $\epsilon \ge 0$ (where $\epsilon = 0$ corresponds to exact computation) for all rational inputs $a \in \mathbb{Q}^d$, where $\|a\|$ is the bitlength of a. There is a fixed polynomial p such that the mean payoff $MP_w(P, \iota_{init})$ can be approximated up to precision ϵ in time at most $d \cdot f_w(p(\|(P, \iota_{init})\|, \epsilon) + p(\|(P, \iota_{init})\|)$ where $\|(P, \iota_{init})\|$ is the bit length of the original input.*

When a Markov chain is reducible, the states can be renamed in a way such that, the matrix representation of the Markov chain will contain distinct blocks corresponding to the bottom strongly connected components (BSCCs) on the diagonal along with additional columns at the right representing states that do not belong to any BSCC:

$$\begin{bmatrix} \square & 0...0 & 0...0 & * & * \\ 0...0 & \square & 0...0 & * & * \\ 0...0 & 0...0 & \square & * & * \\ 0...0 & 0...0 & 0....0 & * & * \end{bmatrix}$$

Each block representing a BSCC constitutes an irreducible Markov chain. Assume we have k blocks with periods $L_1, L_2, ..., L_k$ correspondingly. Let l be the least common multiple of the periods. Now we will have l subsequences of the orbit each of which will converge. The convergence of the rows in the bottom is a result of the fact that Markov chain will enter a BSCC with probability 1. So, in general, we have l subsequences of the orbit, all of which converge. We observe that $l \le d^d$, from which the following result follows:

Theorem 3.12. *Let $P \in \mathbb{Q}^{d\times d}$ be a stochastic matrix and $\iota_{init} \in \mathbb{Q}^d$ an initial distribution. Let $w\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous weight function. Then, we can compute points $\pi_0, \dots, \pi_{l-1} \in \mathbb{Q}^d$ in exponential time for some $l \le d^d$ such that $MP_w(P, \iota_{init}) = \frac{1}{l} \sum_{i=0}^{l-1} w(\pi_i)$.*

As the transition matrix P^l of the l subsequences as well as the initial values $P^r \iota_{init}$ with $0 \le r < l$ can be computed in polynomial time by repeated squaring, each of the points π_i with $0 \le i < l$ can be computed in polynomial time. Assuming that the value $w(a)$ can be approximated in time $f_w(\|a\|, \epsilon)$ for all rational inputs $a \in \mathbb{Q}^d$, we can hence conclude that there is again a fixed polynomial q such that the mean payoff of reducible stochastic LDSs can be approximated to precision ϵ in time bounded by $d^d \cdot f_w(q(\|(P, \iota_{init})\|, \epsilon) + d^d \cdot q(\|(P, \iota_{init})\|)$ analogously to Theorem 3.11.

4 TOTAL (DISCOUNTED) REWARD AND SATISFACTION OF ENERGY CONSTRAINTS

In this section, again let $M \in \mathbb{Q}^{d\times d}$ be a matrix, $q \in \mathbb{Q}^d$ be an initial vector, and $w\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial weight function with rational coefficients. We define the *total reward* as

$$\mathrm{tr}(M, q, w) := \sum_{k=0}^{\infty} w(M^k q).$$

Likewise, for a rational discount factor $\delta \in (0,1)$ we define the *total discounted reward* as

$$\mathrm{dr}(M, q, w, \delta) := \sum_{k=0}^{\infty} \delta^k w(M^k q).$$

Both of these quantities, when they exist, can be determined effectively.

Theorem 4.1. *It is decidable whether the series $\sum_{k=0}^{\infty} w(M^k q)$ and $\sum_{k=0}^{\infty} \delta^k w(M^k q)$ converge, in which case their values are rational and can be computed.*

Proof. Let $u_n = \sum_{k=0}^{n} w(M^k q)$. As discussed in subsection 3.1, $(u_n)_{n\in\mathbb{N}}$ is a rational LRS, and we can apply Corollary 3.3. Similarly, let $v_n = \sum_{k=0}^{\infty} \delta^k w(M^k q)$. As $(\delta^n)_{n\in\mathbb{N}}$ is itself a (rational) LRS and such LRS are closed under pointwise multiplication, v_n is also a rational LRS. We again apply Corollary 3.3. □

We next discuss *energy constraints*. We say that a series of real weights $(w_i)_{i\in\mathbb{N}}$ satisfies the energy constraint with budget B if

$$\sum_{i=0}^{k} w_i \geq -B$$

for all $k \in \mathbb{N}$. We will prove that for LDS (M, q) of dimension at most 3, satisfaction of energy constraints is decidable. The proof is based on the fact that three-dimensional systems are tractable thanks to Baker's theorem [14]. For higher-dimensional systems, no such tractability result is known. We will show that deciding satisfaction of energy constraints is, in general, at least as hard as the Positivity Problem, already with linear weight functions.

4.1 Baker's theorem and its applications

A *linear form in logarithms* is an expression of the form $\Lambda = b_1 \operatorname{Log} \alpha_1 + \ldots + b_m \operatorname{Log} \alpha_m$ where $b_i \in \mathbb{Z}$ and $\alpha_i \in \overline{\mathbb{Q}}$ for all $1 \leq i \leq m$. Here Log denotes the principal branch of the complex logarithm. The celebrated theorem of Baker places a lower bound on $|\Lambda|$ in case $\Lambda \neq 0$. Baker's theorem, as well as its p-adic analogue, play a critical role in the proof of [23] that the Skolem Problem is decidable for LRS of order at most 4, as well as decidability of the Positivity Problem for low-order LRS.

Theorem 4.2 (Special case of the main theorem in [27]). *Let $\Lambda = b_1 \operatorname{Log} \alpha_1 + \ldots + b_m \operatorname{Log} \alpha_m$ be as above, $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_m) : \mathbb{Q}]$, and suppose $A, B \geq e$ are such that $A > H(\alpha_i)$ and $B > |b_i|$ for all $1 \leq i \leq m$. If $\Lambda \neq 0$, then*

$$\log|\Lambda| > -(16mD)^{2(m+2)} (\log A)^m \log B.$$

A direct consequence of Baker's theorem is the following [22, Corollary 8]. Recall that $\mathbb{T}$ denotes $\{z \in \mathbb{C} : |z| = 1\}$.

Lemma 4.3. *Let $\alpha \in \mathbb{T} \cap \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}$. For all $n \geq 2$, if $\alpha^n \neq \beta$ then $|\alpha^n - \beta| > n^{-C}$ where C is an effective constant that depends on α and β.*

If α is not a root of unity, $\alpha^n = \beta$ holds for at most one n and n can be effectively bounded.

Lemma 4.4. *Let $\alpha, \beta \in \overline{\mathbb{Q}}$ be non-zero, and suppose α is not a root of unity. There exists effectively computable $N \in \mathbb{N}$ such that $\alpha^n \neq \beta$ for all $n \in \mathbb{N}$ with $n > N$.*

Combining the two lemmas above, we obtain the following.

Theorem 4.5. *Let $\alpha \in \mathbb{T}$, $\beta \in \mathbb{Q}$, and suppose α is not a root of unity. There exists effectively computable $N, C \in \mathbb{N}$ such that for $n > N$, $|\alpha^n - \beta| > n^{-C}$.*

The next lemma summarises the family of linear recurrence sequences to which we can apply Baker's theorem. For reasons of space we delegate the proof to the extended version of the paper.

Lemma 4.6. *Let $\gamma \in \mathbb{T}$ be not a root of unity, $r_1, \ldots, r_\ell \in \mathbb{R}$ be non-zero, and $u_n = \sum_{i=1}^{m} c_i \Lambda_i^n$ be an LRS over $\mathbb{R}$ where $m \geq 1$, $c_i, \Lambda_i \in \overline{\mathbb{Q}}$ are non-zero for all i, and $\Lambda_1, \ldots, \Lambda_m$ are pairwise distinct. Suppose each Λ_i is in the multiplicative group generated by $\{\gamma, r_1, \ldots, r_\ell\}$.*

(a) *There exists effectively computable N_1 such that $u_n \neq 0$ for all $n > N_1$.*
(b) *For $n > N_1$, $|u_n| > L^n n^{-C}$, where $L = \max_i |\Lambda_i|$ and C is an effectively computable constant.*
(c) *It is decidable whether $u_n \geq 0$ for all n.*

4.2 Satisfaction of energy constraints

Before giving our decidability result, we need one final ingredient about partial sums of LRS. Let $w_n = n^l \lambda^n$ for some $l \geq 0$ and $\lambda \in \overline{\mathbb{Q}}$, and $u_n = \sum_{k=0}^{n} w_k$. If $\lambda = 1$, then $u_n = p(n)$, where p is a polynomial of degree $l + 1$. If $\lambda \neq 1$, then $u_n = q(n)\lambda^n$, where q is a polynomial of degree at most l with algebraic coefficients. In particular, $q(n)$ is a solution of the functional equation $\lambda f(n) - f(n-1) = n^l$. It follows that if the LRS $(w_n)_{n\in\mathbb{N}}$ has only real eigenvalues, then so does the sequence given by $u_n = \sum_{k=0}^{n} w_k$. Similarly, if $(w_n)_{n\in\mathbb{N}}$ is diagonalisable and does not have 1 as an eigenvalue, then the same applies to $(u_n)_{n\in\mathbb{N}}$. In fact, the eigenvalues of $(u_n)_{n\in\mathbb{N}}$ form a subset of the eigenvalues of $(w_n)_{n\in\mathbb{N}}$.

Theorem 4.7. *Let $M \in \mathbb{Q}^{3\times3}$, $q \in \mathbb{Q}^3$, $\delta \leq 1$ be a discount factor, and $w\colon \mathbb{R}^3 \to \mathbb{R}$ be a polynomial weight function with rational coefficients. For $B \in \mathbb{Q}_{\geq 0}$, it is decidable whether the weights $(\delta^n w(M^n q))_{n\in\mathbb{N}}$ satisfy the energy constraint with budget B.*

Proof. Let $w_n = \delta^n w(M^n q)$ and $u_n = B + \sum_{i=0}^{n} w(M^i q)$. We have to decide if $u_n \geq 0$ for all n. First suppose M has only real eigenvalues. Then w_n and u_n are both LRSs with only real eigenvalues. By taking subsequences if necessary, we can assume $(u_n)_{n\in\mathbb{N}}$ is non-degenerate. Write $u_n = \sum_{i=1}^{m} p_i(n)\rho_i^n$ where the right-hand side is in the exponential-polynomial form. In particular, for all i, p_i is not the zero polynomial. Since $(u_n)_{n\in\mathbb{N}}$ is non-degenerate, wlog we can assume $\rho_1 > \ldots > \rho_m > 0$. If $p_1(n)$ is negative for sufficiently large n, then the energy constraint is not satisfied. Otherwise, we can compute N such that for all $n > N$, $u_n > 0$. It remains to check whether $u_n \geq 0$ for $0 \leq n \leq N$. Next, suppose M has non-real eigenvalues $\lambda, \overline{\lambda}$, and a real eigenvalue ρ. Write $\gamma = \lambda/|\lambda|$ and $r = |\lambda|$. Then u_n is of the form

$$u_n = cn + \sum_{i=1}^{m} c_i \Lambda_i^n := cn + v_n$$

where $\Lambda_1, \ldots, \Lambda_m$ are pairwise distinct and in the multiplicative group generated by r, ρ, δ, γ. Wlog we can assume $c_i \neq 0$ for all i, but c may be zero. If γ is a root of unity of order $k > 0$ (i.e. $\gamma^k = 1$), then we can take subsequences $(u_n^{(0)})_{n\in\mathbb{N}}, \ldots, (u_n^{(k-1)})_{n\in\mathbb{N}}$, where

$u_n^{(j)} = u_{nk+j}$ for $n \in \mathbb{N}$ and $0 \leq j < k$, and each $(u_n^{(j)})_{n\in\mathbb{N}}$ has only real eigenvalues. We can then apply the analysis above. Hereafter we assume γ is not a root of unity.

Suppose $c = 0$. Then Lemma 4.6 (c) applies and we can decide if u_n is positive. Next, suppose $c \neq 0$ and $L := \max_i |\Lambda_i| \leq 1$. We can compute N_2 such that $|cn| > |v_n|$ for all $n > N_2$. Hence in this case $u_n \geq 0$ for all n if and only if $c > 0$ and $u_n > 0$ for $0 \leq n \leq N_2$. Finally, suppose $c \neq 0$ and $L > 1$. Applying Lemma 4.6 (b), there exists effectively computable N_3 such that $|u_n| > |cn|$ for $n > N_3$. Hence $u_n \geq 0$ for all n if and only if $u_n \geq 0$ for $0 \leq n \leq N_3$ and $\sum_{i=1}^m c_i\Lambda_i^n \geq 0$ for $n > N_3$. The latter can be decided by applying Lemma 4.6 (c) to the sequence $v_n = \sum_{i=1}^m c_i\Lambda_i^{N_3+n} = \sum_{i=1}^m (c_i\Lambda_i^{N_3})\Lambda_i^n$. □

4.3 Positivity and Diophantine hardness

Recall that the energy satisfaction problem is to decide, given a matrix $M \in \mathbb{Q}^{d\times d}$, $q \in \mathbb{Q}^d$, $B \in \mathbb{Q}$, and a polynomial p with rational coefficients, whether there exists n such that $\sum_{k=0}^n p(M^k q) < B$. This problem is at least as hard as the Positivity Problem already for stochastic LDSs and linear weight functions.

Theorem 4.8. *The Positivity Problem can be reduced to the energy satisfaction problem above restricted to a Markov chain (M, q) and a linear weight function w.*

Proof. It is known from [2, 24] that the Positivity Problem for arbitrary LRS over $\mathbb{Q}$ can be reduced to the following problem: given a Markov chain (M, q), decide whether there exists n such that $e_1 M^n q \geq 1/2$. We reduce the latter to the energy satisfaction problem. Given a Markov chain $(M, q) \in \mathbb{Q}^{d\times d} \times d$, let

$$P = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}$$

and $t = (1/2q, 1/2Mq) \in \mathbb{Q}^{2d}$. Observe that (P, t) is also a Markov chain. Moreover, $P^n t = (1/2M^n q, 1/2M^{n+1} q)$. We choose the weight function $w(x_1, \ldots, x_{2d}) = 2(x_{d+1} - x_1)$ and $B = 1/2 - e_1 \cdot q$. Then $w(P^n q) = e_1 M^{n+1} q - e_1 M^n q$, and $u_n := \sum_{k=0}^n w(P^n q) \geq B$ if and only if $e_1 M^{n+1} q \geq 1/2$. Hence there does not exist n such that $e_1 M^n q \geq 1/2$ if and only if $e_1 \cdot q < 1/2$ and there does not exist n such that $u_n < B$. □

The Positivity Problem for LRS of order 6 was shown in [21] to be *Diophantine-hard*. Specifically, for $r \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}(i)$ (that is, $\lambda = a + bi$ where $a, b \in \mathbb{Q}$) let

$$u_n^{\lambda,r} = -n + \frac{n}{2}(\lambda^n + \overline{\lambda^n}) + \frac{ri}{2}(\overline{\lambda^n} - \lambda^n) = r\,\mathrm{Im}(\lambda^n) - n\,\mathrm{Re}(\lambda^n) + n.$$

If for all r and λ we can decide whether $u_n^{\lambda,r} \geq 0$ for all $n \geq 0$, then we could compute the *Lagrange constants* of a large class of numbers, which would amount to a major mathematical breakthrough in number theory. The following theorem states that a solution to the energy satisfaction problem for rational LDS in dimension 4 with polynomial weight functions would also yield this breakthrough.

Theorem 4.9. *The energy satisfaction problem for rational LDSs in dimension 4 and polynomial weight functions with rational coefficients is Diophantine-hard.*

Proof. We prove the following: If we can decide the energy satisfaction problem with $d = 4$, then we can decide for every r, λ whether $u_n^{\lambda,r} \geq 0$ for all n. Fix $r \in \mathbb{Q}$ and $\lambda = a + bi \in \mathbb{Q}(i)$. Define

$$M = \begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}$$

and the initial point $q = (0, 0, 0, 1)$. We have

$$M^n q = [-n\,\mathrm{Im}(\lambda^{n-1}) \quad n\,\mathrm{Re}(\lambda^{n-1}) \quad -\mathrm{Im}(\lambda^n) \quad \mathrm{Re}(\lambda^n)]^\top.$$

Recall that for all $n \in \mathbb{N}$, (i) $\mathrm{Re}(\lambda^{n+1}) = a\,\mathrm{Re}(\lambda^n) - b\,\mathrm{Im}(\lambda^n)$ and (ii) $\mathrm{Im}(\lambda^{n+1}) = a\,\mathrm{Im}(\lambda^n) + b\,\mathrm{Re}(\lambda^n)$. Hence there exist polynomials $p_1, \ldots, p_4$ with rational coefficients such that $p_1(M^n q) = n\,\mathrm{Im}(\lambda^n)$, $p_2(M^n q) = n\,\mathrm{Re}(\lambda^n)$, $p_3(M^n q) = \mathrm{Re}(\lambda^n)$ and $p_4(M^n q) = \mathrm{Im}(\lambda^n)$ for all n. Next, consider

$$w_n = u_{n+1}^{\lambda,r} - u_n^{\lambda,r} = r\,\mathrm{Im}(\lambda^{n+1}) - n\,\mathrm{Re}(\lambda^{n+1}) - r\,\mathrm{Im}(\lambda^n) + n\,\mathrm{Re}(\lambda^n) + 1.$$

Since $u_0^{\lambda,r} = 0$, we have $u_{n+1}^{\lambda,r} = \sum_{k=0}^n w_n$. Moreover, using facts (i), (ii) and the polynomials $p_1, \ldots, p_4$ we can construct a polynomial p with rational coefficients such that $w_n = p(M^n q)$ for all n. Hence $u_n^{\lambda,r} \geq 0$ for all n if and only if the weights $(p(M^n q))_{n\in\mathbb{N}}$ satisfy the energy constraint with budget $B = 0$. □

5 CONCLUSION

We have shown how to compute (or approximate) the mean-payoff and the total or discounted weight of the orbit of rational LDSs for several combinations of restricted classes of LDSs and classes of continuous weight functions (see Table 1). Remarkably, these results concerning *infinite horizon* questions do not rely on restrictions of the dimension – in contrast to decidability results for the Skolem [23, 25] and the Positivity [21, 22] problems, which can be seen as special cases of reachability questions about the orbit of an LDS.

For the question whether an orbit of a rational LDS with a polynomial weight function satisfies an energy constraints, on the other hand, we have shown decidability for dimension 3 by utilising results about LRSs based on Baker's theorem and Diophantine-hardness for dimension 4. Further, the restriction to stochastic LDSs and linear weight functions turned out to be Positivity-hard.

Instead of continuous weight functions, also functions w assigning a weight to each semialgebraic set in a collection of semialgebraic sets $S_1, \ldots, S_m$ constitute an interesting class of weight functions. Here, several interesting questions can be asked. E.g., given an LDS $(M, q) \in \mathbb{Q}^{d\times d} \times \mathbb{Q}^d$ and w, compare the (discounted) total reward/mean-payoff to a given threshold. Here at time n the reward received is $\sum_{i=1}^m \mathbb{1}(M^n q \in S_i) w(S_i)$. This problem appears to be difficult with deep connections to Diophantine approximation.

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