Extending VASS with Additional Integer Counters

Henry Sinclair-Banks

Based on joint work in LICS'26 with ...

Extending VASS with Additional Integer Counters

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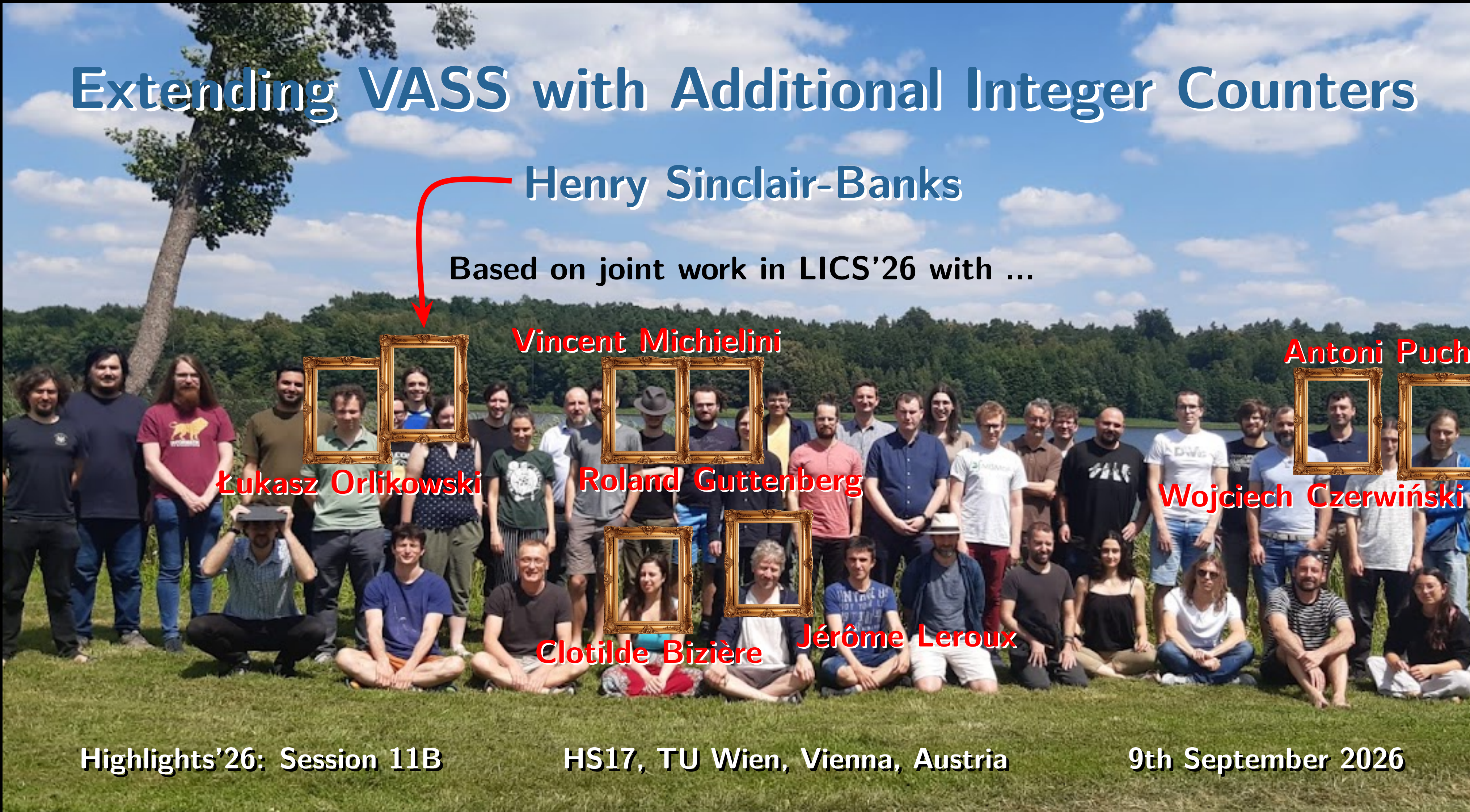
Highlights'26: Session 11B   HS17, TU Wien, Vienna, Austria   9th September 2026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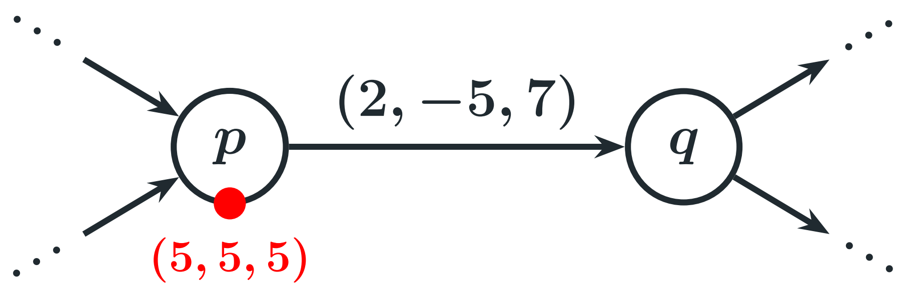
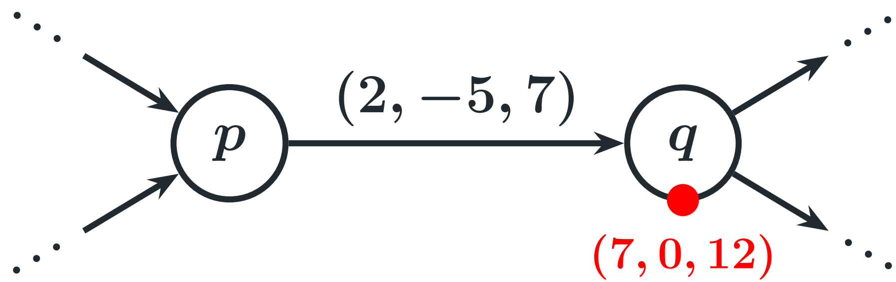
Vector Addition Systems with States



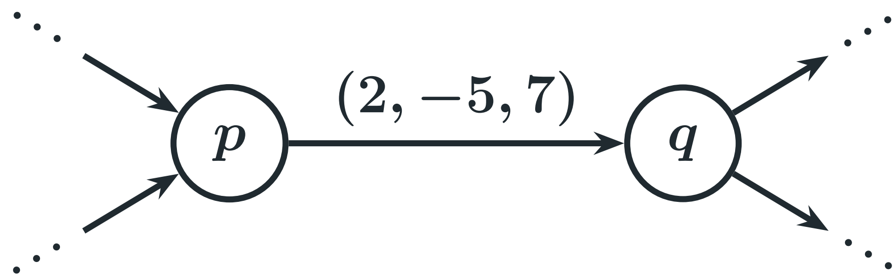
$p(5, 5, 5)$

Vector Addition Systems with States



$p(5, 5, 5) \longrightarrow q(7, 0, 12)$

Vector Addition Systems with States



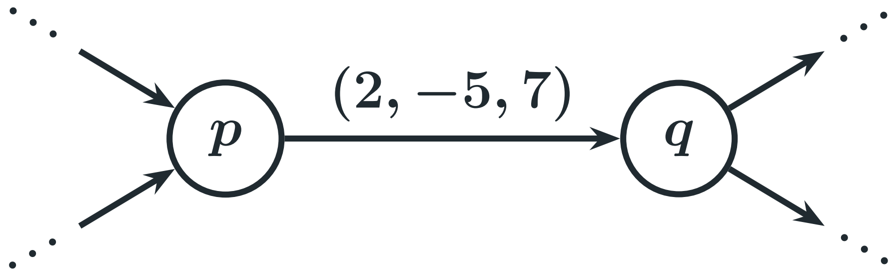
$p(0,0,0) \cancel{\rightarrow} q(2,-5,7)$

VASS

Counters are non-negative

$\mathbb{N}$-counters

Vector Addition Systems with States



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Integer VASS

Counters can go negative

$\mathbb{Z}$-counters

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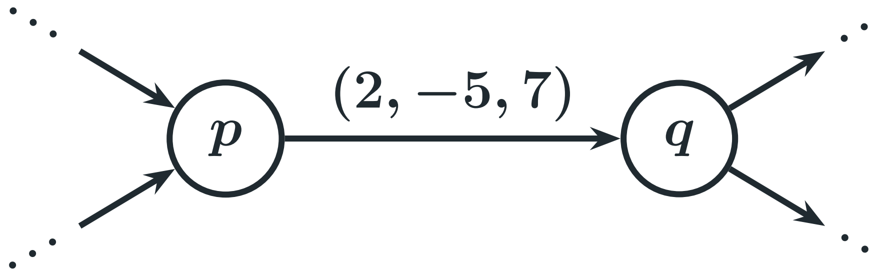

$\boldsymbol{p(0,0,0)} \cancel{\rightarrow} \boldsymbol{q(2,-5,7)}$

d-VASS+$k\mathbb{Z}$

$\boldsymbol{p(0,0,0) \rightarrow q(2,-5,7)}$

VASS

Counters are non-negative

$\mathbb{N}$-counters

VASS extended with additional integer counters

Can have both $\mathbb{N}$- and $\mathbb{Z}$-counters!

Integer VASS

Counters can go negative

$\mathbb{Z}$-counters

Why Study Reachability in VASS+$\mathbb{Z}$?

VASS+$\mathbb{Z}$ have been implicitly studied:

When proving unboundedness is in EXPSPACE.
[Rackoff '78]

In the KLMST algorithm for VASS reachability.
[Kosaraju '82]

When proving that strong promptness detection is in EXPSPACE. [Demri '13]

They also appear implicitly in the analysis of:

- VASS regular separability, [Keskin and Meyer '24]
- mutual reachability in Petri nets with data, [Kamiński and Lasota '24]
- pushdown VASS reachability. [Guttenberg, Keskin, and Meyer '26]

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d-VASS	Unary encoding	Binary encoding
$d = 1$	NL-complete	NP-complete
$d = 2$	NL-complete	PSPACE-complete
$d = 3$	NP-hard	in 2EXPSPACE
$d = 4$	???	in $\mathcal{F}_4$
$d = 5$	PSPACE-hard	in $\mathcal{F}_5$
$d = 6$	???	in $\mathcal{F}_6$
$d = 7$	???	in $\mathcal{F}_7$
$d = 8$	TOWER-hard	in $\mathcal{F}_8$
$\vdots$	$\mathcal{F}_{(d-3)/2}$-hard	in $\mathcal{F}_d$

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Elementary complexity!

???

???

Non-elementary complexity!

The Complexity of 1-VASS+$\mathbb{Z}$

Theorem. Reachability in binary 1-VASS+$\mathbb{Z}$ is NP-complete.

[Bizière, Czerwiński, Guttenberg, Leroux, Michielini, Orlikowski, Puch, S-B '26]

Theorem. Reachability in binary 1-VASS is NP-hard. [Rosier and Yen '86]

Theorem. Reachability in unary integer VASS is NP-hard. [Folklore SAT reduction]

Theorem. Reachability in binary 1-VASS is in NP. [Haase, Kreutzer, Ouaknine, and Worrell '09]

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Lemma. In a 1-VASS+$\mathbb{Z}$, if there is a run from $p(\vec{u})$ to $q(\vec{v})$, then there is a run:

$$\begin{tikzcd} p(\vec{u}) \arrow[r, "\rho_0"'] & \cdot \arrow[loop above, "x_1 \text{ iterations of } \sigma_1"] \arrow[r, "\rho_1"'] & \cdot \arrow[loop above, "x_2 \text{ iterations of } \sigma_2"] \arrow[r, "\rho_2"'] & \cdots \arrow[r, "\rho_{\ell-1}"'] & \cdot \arrow[loop above, "x_\ell \text{ iterations of } \sigma_\ell"] \arrow[r, "\rho_\ell"'] & q(\vec{v}) \end{tikzcd}$$

such that $\ell \leq$ POLY,

$|\rho_i| \leq$ POLY,

$|\sigma_i| \leq$ POLY,

$x_i \leq$ EXP.

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Lemma. In a 1-VASS+$\mathbb{Z}$, if there is a run from $\boldsymbol{p(\vec{u})}$ to $\boldsymbol{q(\vec{v})}$, then there is a run:

$$\boldsymbol{p(\vec{u})} \xrightarrow[\boldsymbol{\rho_0}]{} \underbrace{\circlearrowleft}_{\boldsymbol{x_1} \text{ iterations of } \boldsymbol{\sigma_1}} \xrightarrow[\boldsymbol{\rho_1}]{} \underbrace{\circlearrowleft}_{\boldsymbol{x_2} \text{ iterations of } \boldsymbol{\sigma_2}} \xrightarrow[\boldsymbol{\rho_2}]{} \cdots \xrightarrow[\boldsymbol{\rho_{\ell-1}}]{} \underbrace{\circlearrowleft}_{\boldsymbol{x_\ell} \text{ iterations of } \boldsymbol{\sigma_\ell}} \xrightarrow[\boldsymbol{\rho_\ell}]{} \boldsymbol{q(\vec{v})}$$

$$\left.\begin{aligned} \text{such that } \ \boldsymbol{\ell} &\leq \text{POLY}, \\ |\boldsymbol{\rho_i}| &\leq \text{POLY}, \\ |\boldsymbol{\sigma_i}| &\leq \text{POLY}, \\ \boldsymbol{x_i} &\leq \text{EXP}. \end{aligned}\right\} \implies \text{polynomial size description.}$$

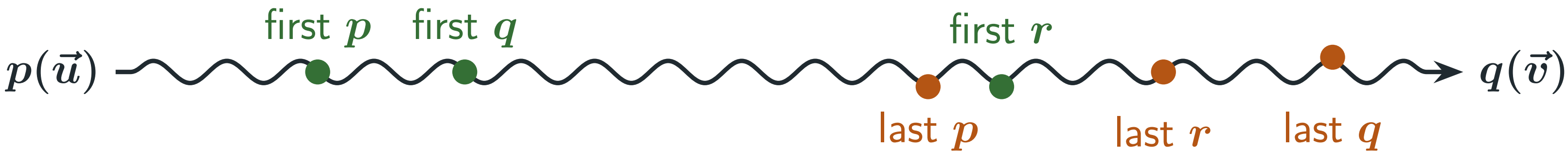
“Proof by Pictures”

$$p(\vec{u}) \rightsquigarrow q(\vec{v})$$

Preliminary step: Obtain an exponential length run from $p(\vec{u})$ to $q(\vec{v})$.

[Atig, Chistikov, Hofman, Kumar, Saivasan, and Zetzsche ’16]

“Proof by Pictures”



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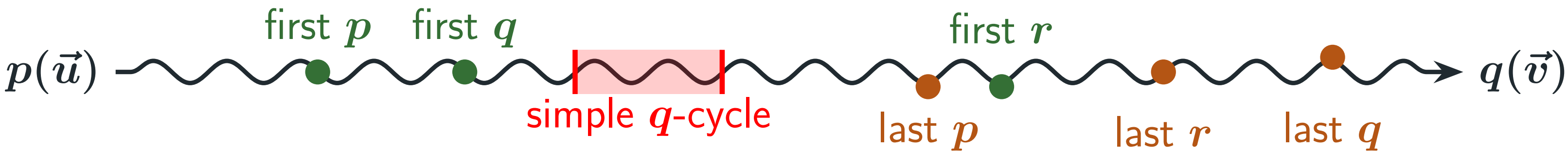

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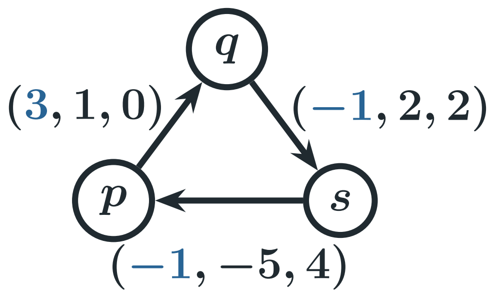
Step 1: Mark the first and last occurrences of each state.

Step 2: Remove simple cycles that do not contain marks.



Long run $\implies$ large gaps between marks.

$\implies$ simple markless cycles.

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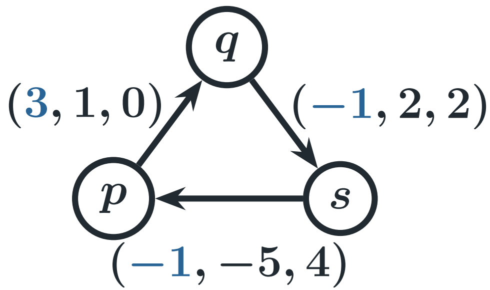

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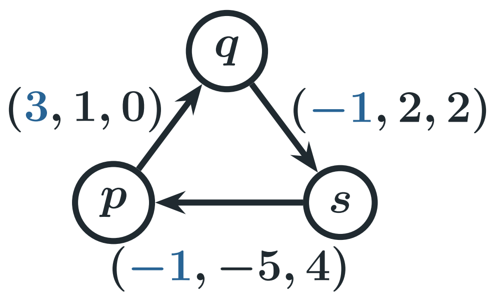

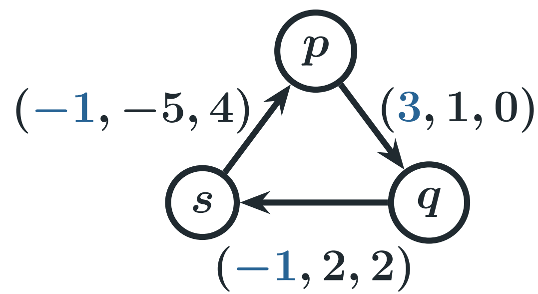

Step 3: “Rotate” the cycle to its minimum $\mathbb{N}$ value.

We now have a p-cycle.

"Proof by Pictures"

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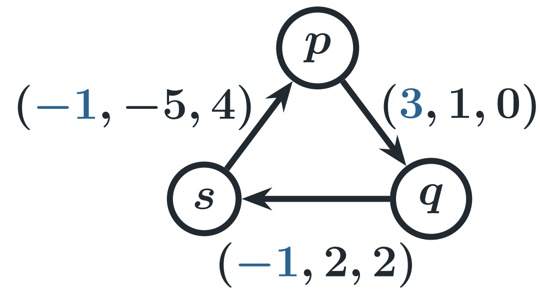

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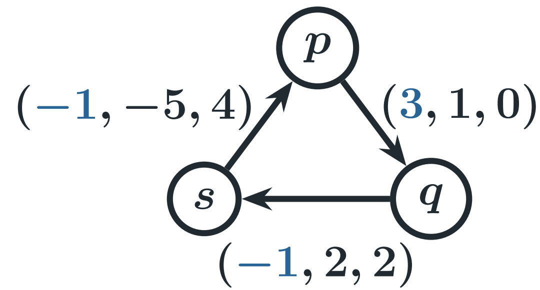

Step 4: Identify the sign of the effect on the $\mathbb{N}$-counter.

"Proof by Pictures"

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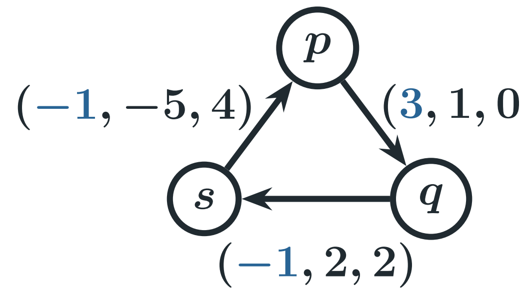

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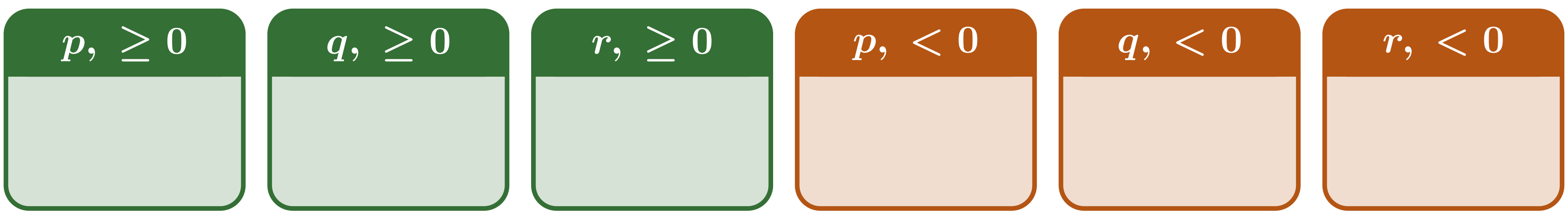
Step 5: Put it in the correct bin.



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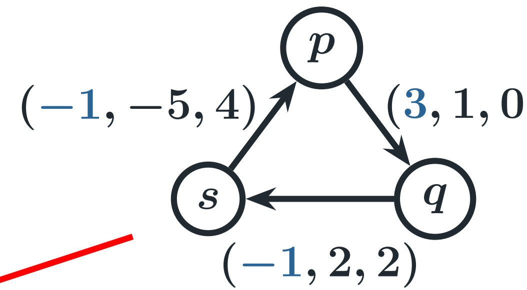

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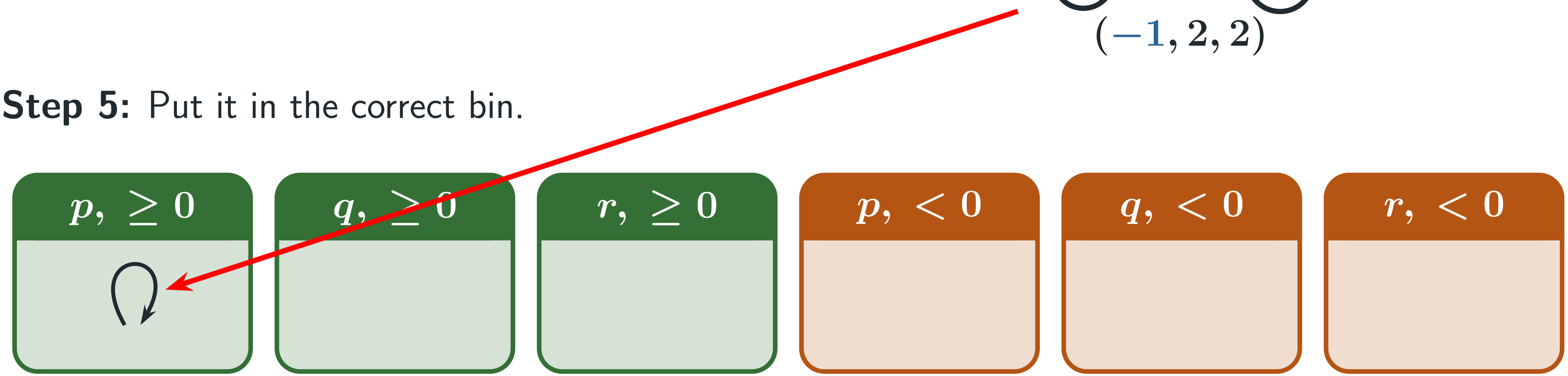

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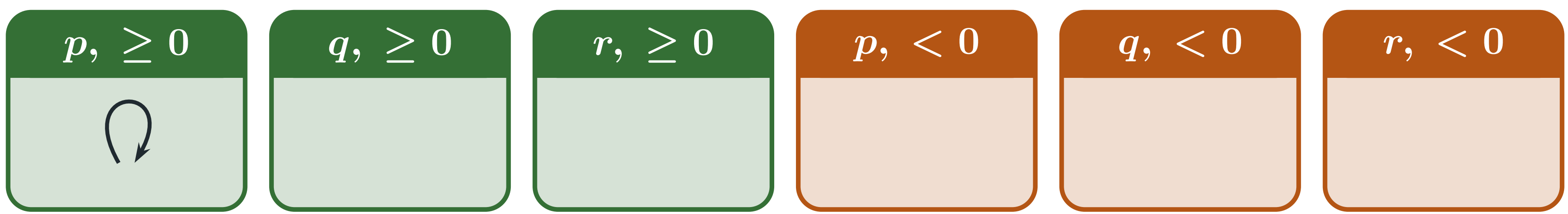
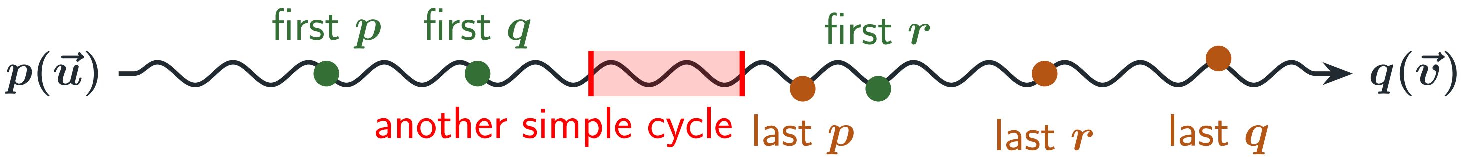


Step 6: Repeat steps 2 – 6.

“Proof by Pictures”



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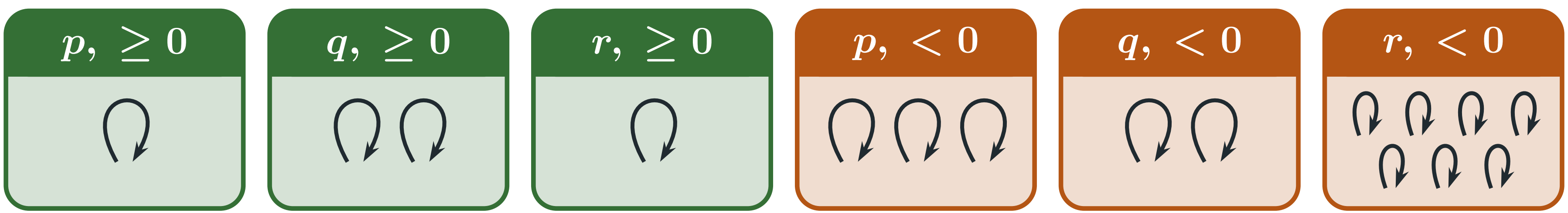

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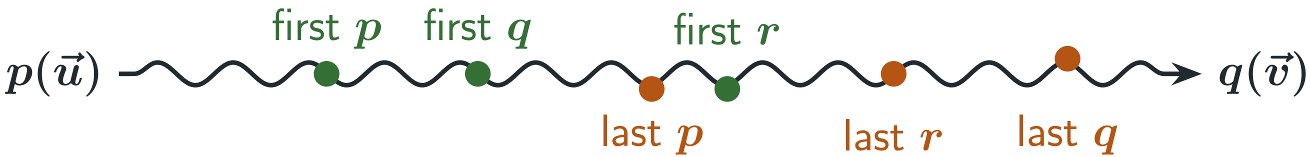

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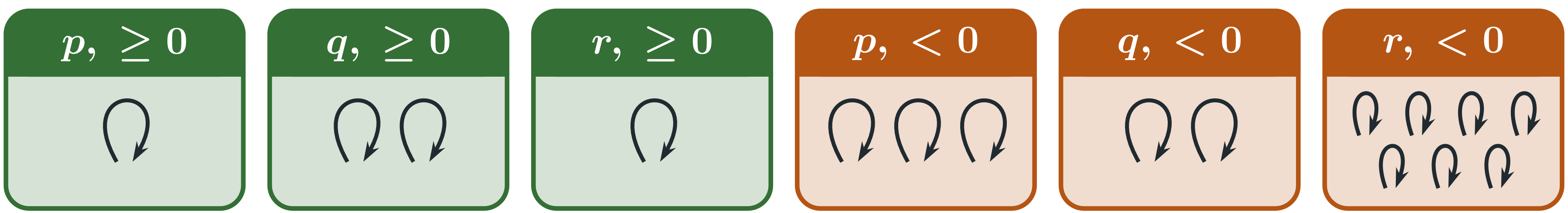

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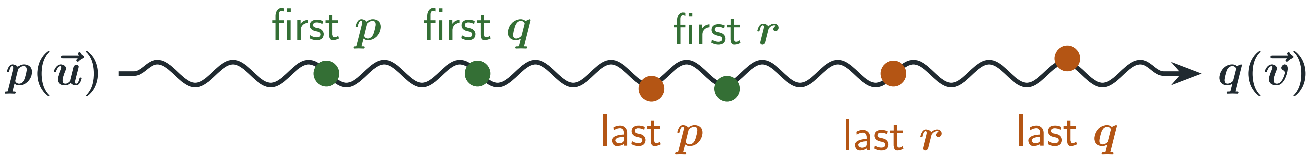
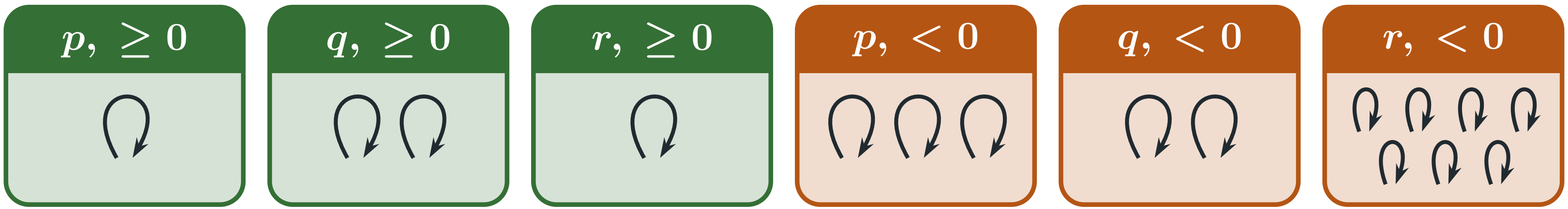


What are we left with?



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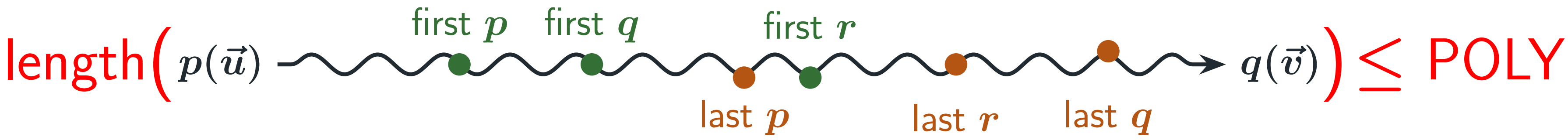

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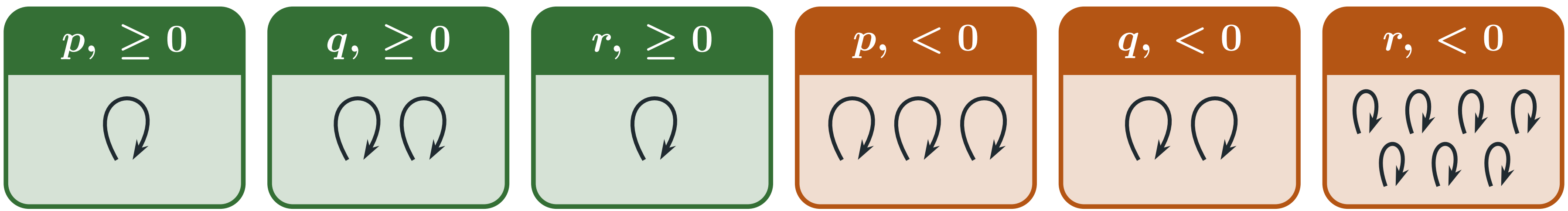
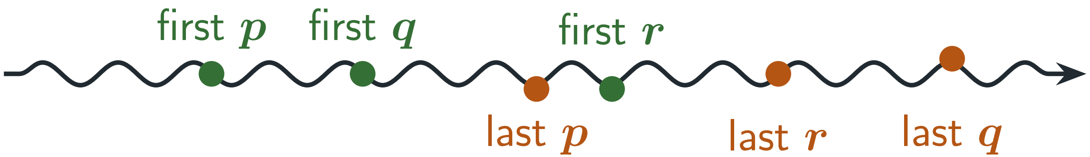


"Proof by Pictures"

length$\Big(p(\vec{u})$



$q(\vec{v})\Big) \leq$ POLY

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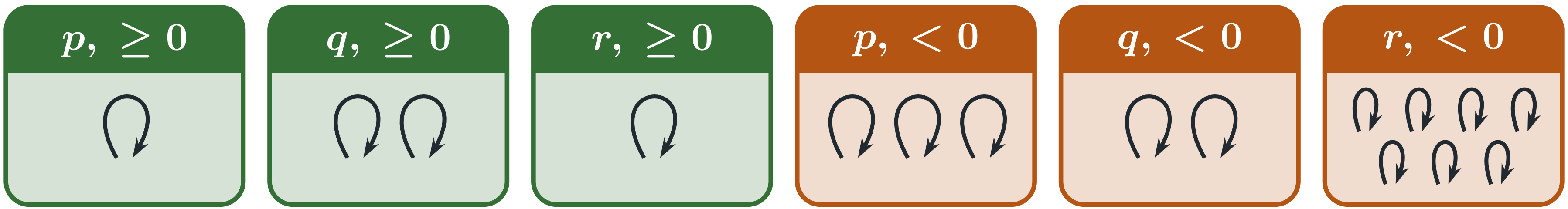
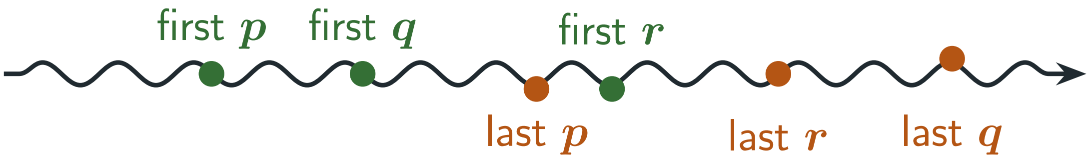


Claim. If bin$(q, \geq 0)$ is re-inserted at first q and bin$(q, < 0)$ is re-inserted at last q, we get a valid run!

"Proof by Pictures"

length$\Big($ $p(\vec{u})$



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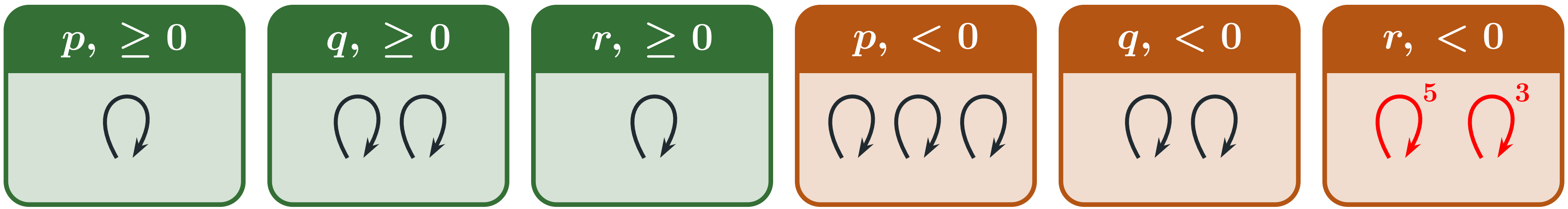


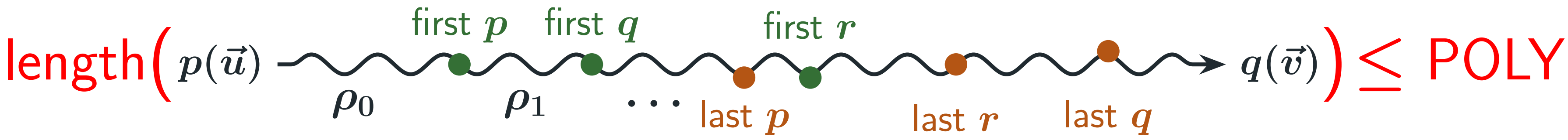
Claim. If bin$(q, \geq 0)$ is re-inserted at first q and bin$(q, < 0)$ is re-inserted at last q, we get a valid run!

Step 7: Use Carathéodory bounds for integer cones (small supports) to replace each bin.

[Eisenbrand and Shmonin '05]

If $\vec{s} = n_1\,\vec{x}_1 + \ldots + n_k\,\vec{x}_k$, then there is a small subset $Y \subseteq \{x_1, \ldots, x_k\}$ such that $\vec{s} \in \text{span}(Y)$.

"Proof by Pictures"



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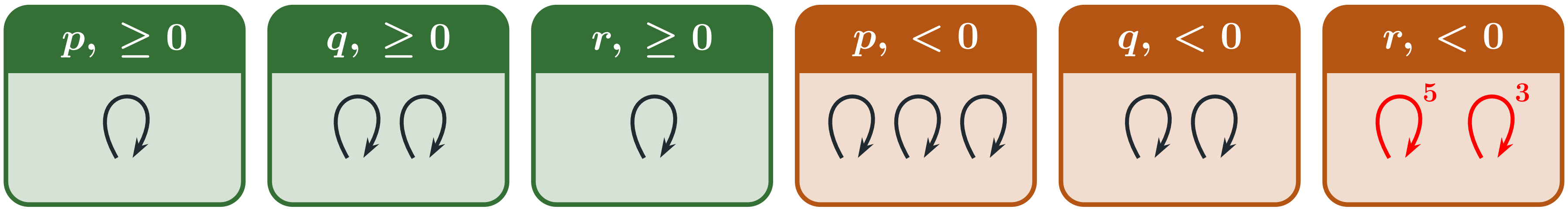

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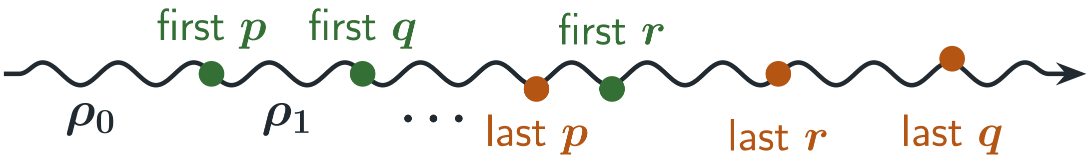
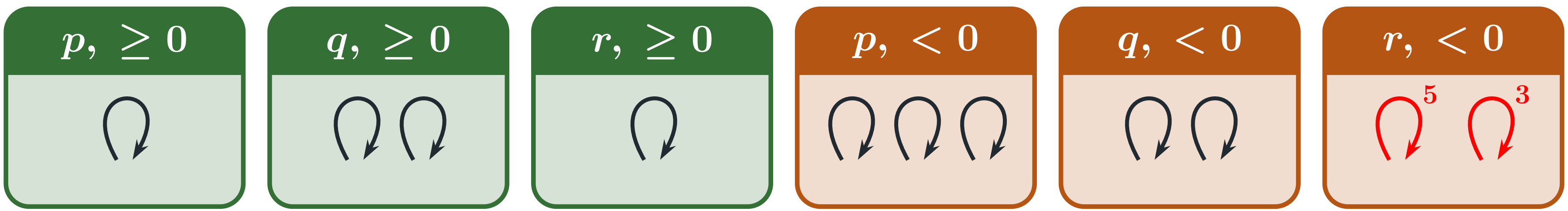
"Proof by Pictures"

length$\Big($ $p(\vec{u})$  $q(\vec{v})$ $\Big) \leq$ POLY

What are we left with?



Claim. If bin$(q, \geq 0)$ is re-inserted at first q and bin$(q, < 0)$ is re-inserted at last q, we get a valid run!

Small number of paths and cycles ✓ Short paths ρ_i ✓

Short cycles σ_i ✓ Exponential exponents x_i ✓

Extending VASS with Additional Integer Counters

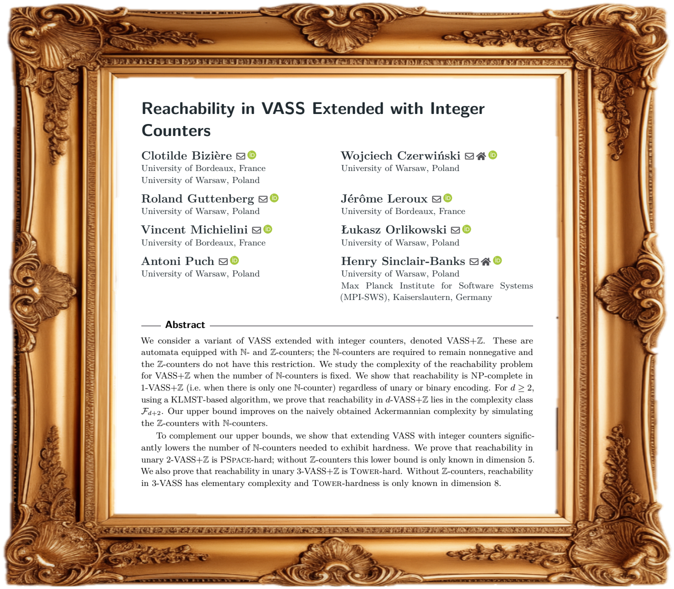
Theorem. Reachability in binary $\mathbf{1}$-VASS+$\mathbb{Z}$ is NP-complete.

Theorem. Reachability in unary $\mathbf{2}$-VASS+$\mathbb{Z}$ is PSPACE-hard.

⚠ *New Turing machine simulation technique* ⚠

Theorem. Reachability in $\mathbf{3}$-VASS+$\mathbb{Z}$ is Tower-hard.

Theorem. Reachability in $\boldsymbol{d}$-VASS+$\mathbb{Z}$ is in $\boldsymbol{\mathcal{F}_{d+2}}$.



Reachability in VASS Extended with Integer Counters

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Max Planck Institute for Software Systems (MPI-SWS), Kaiserslautern, Germany

Abstract

We consider a variant of VASS extended with integer counters, denoted VASS+$\mathbb{Z}$. These are automata equipped with $\mathbb{N}$- and $\mathbb{Z}$-counters; the $\mathbb{N}$-counters are required to remain nonnegative and the $\mathbb{Z}$-counters do not have this restriction. We study the complexity of the reachability problem for VASS+$\mathbb{Z}$ when the number of $\mathbb{N}$-counters is fixed. We show that reachability is NP-complete in 1-VASS+$\mathbb{Z}$ (i.e. when there is only one $\mathbb{N}$-counter) regardless of unary or binary encoding. For $d \geq 2$, using a KLMST-based algorithm, we prove that reachability in d-VASS+$\mathbb{Z}$ lies in the complexity class $\mathcal{F}_{d+2}$. Our upper bound improves on the naively obtained Ackermannian complexity by simulating the $\mathbb{Z}$-counters with $\mathbb{N}$-counters.

To complement our upper bounds, we show that extending VASS with integer counters significantly lowers the number of $\mathbb{N}$-counters needed to exhibit hardness. We prove that reachability in unary 2-VASS+$\mathbb{Z}$ is PSPACE-hard; without $\mathbb{Z}$-counters this lower bound is only known in dimension 5. We also prove that reachability in unary 3-VASS+$\mathbb{Z}$ is TOWER-hard. Without $\mathbb{Z}$-counters, reachability in 3-VASS has elementary complexity and TOWER-hardness is only known in dimension 8.

Thank You!

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