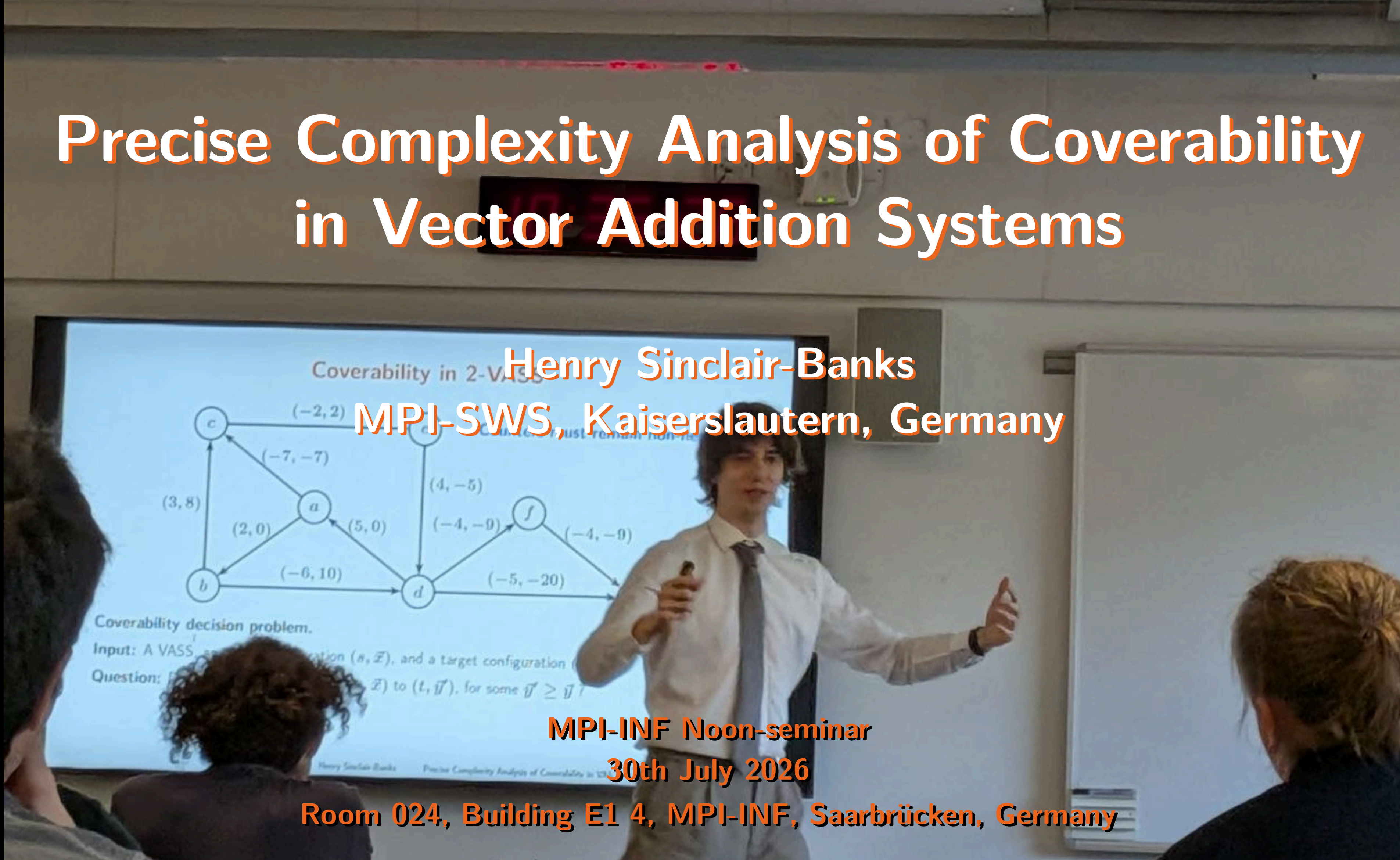
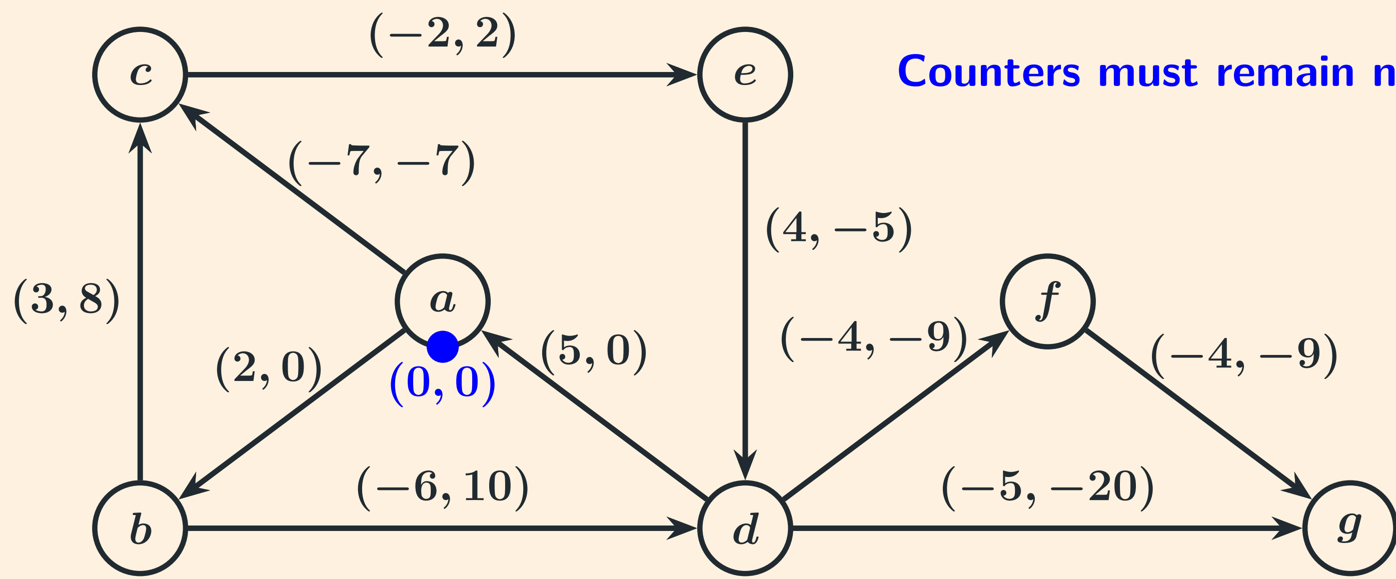


Precise Complexity Analysis of Coverability in Vector Addition Systems

Henry Sinclair-Banks
MPI-SWS, Kaiserslautern, Germany

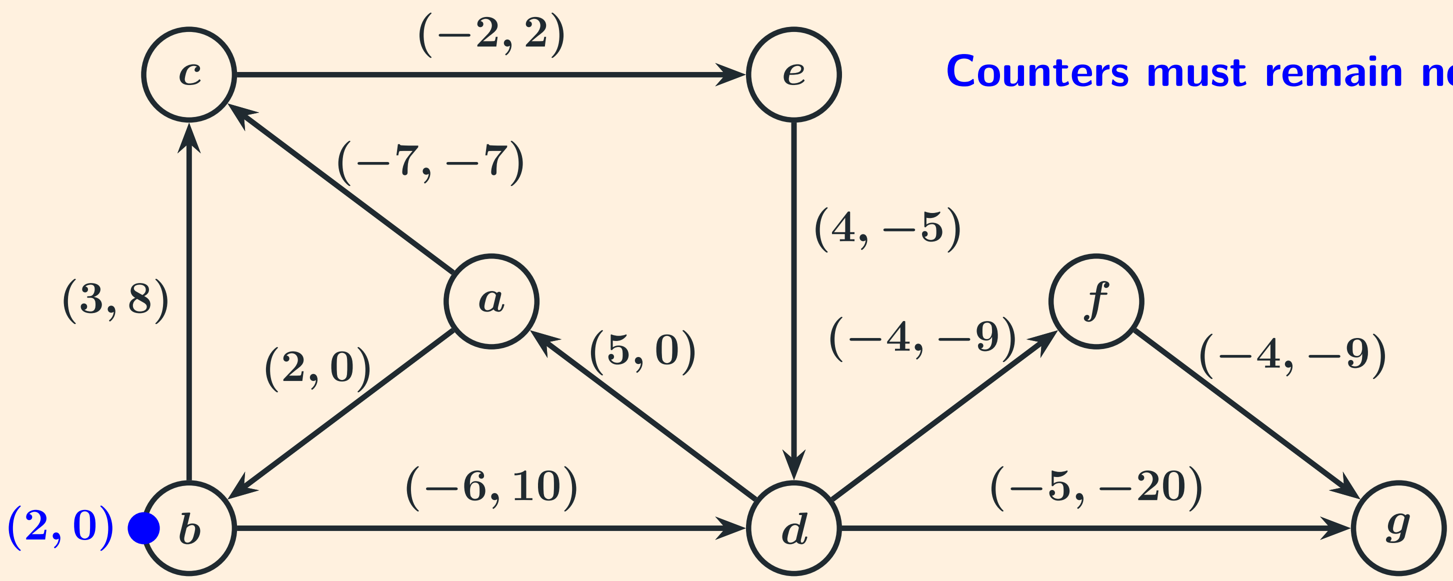
MPI-INF Noon-seminar
30th July 2026
Room 024, Building E1 4, MPI-INF, Saarbrücken, Germany

~~2-dimensional Vector Addition Systems with States~~ 2-VASS



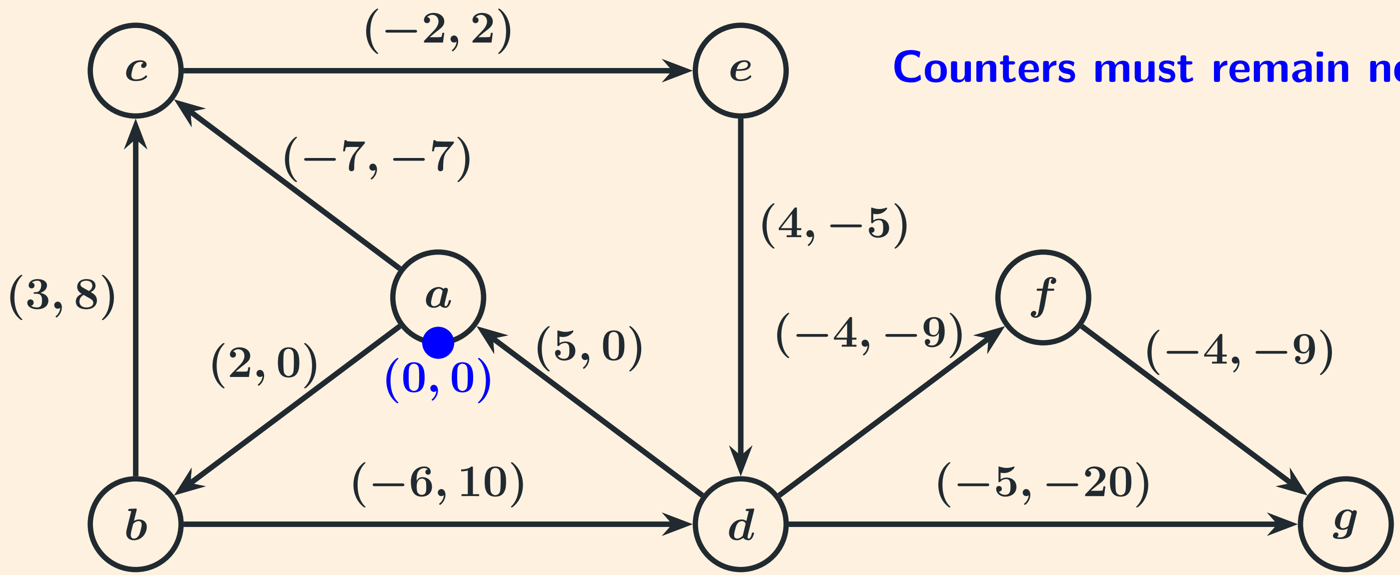
Counters must remain non-negative!

~~2 dimensional Vector Addition Systems with States~~ 2-VASS



Counters must remain non-negative!

Coverability in 2-VASS



Counters must remain non-negative!

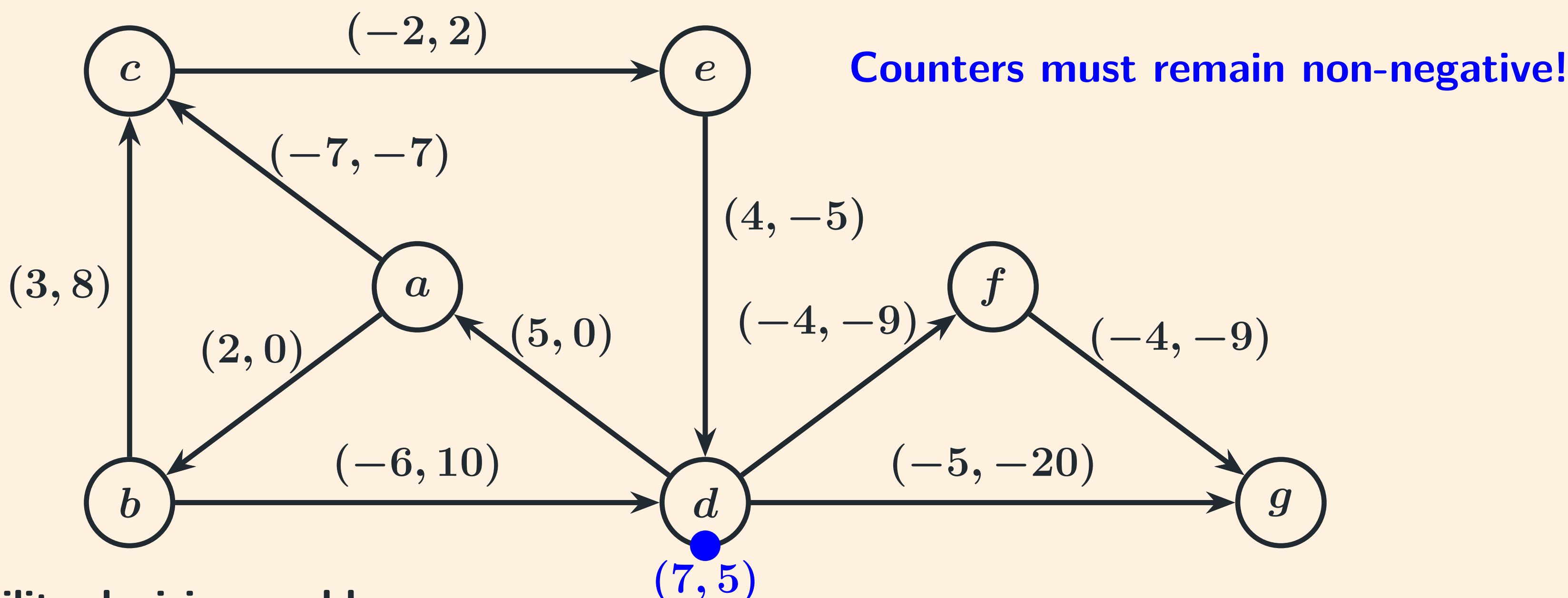
Coverability decision problem.

Input: A VASS, an initial configuration $(s, \vec{x})$, and a target configuration $(t, \vec{y})$.

Question: Does there exist a run from $(s, \vec{x})$ to $(t, \vec{y}')$, for some $\vec{y}' \geq \vec{y}$?

Example: does there exist a run from $(a, (0, 0))$ that covers $(g, (0, 0))$?

Coverability in 2-VASS



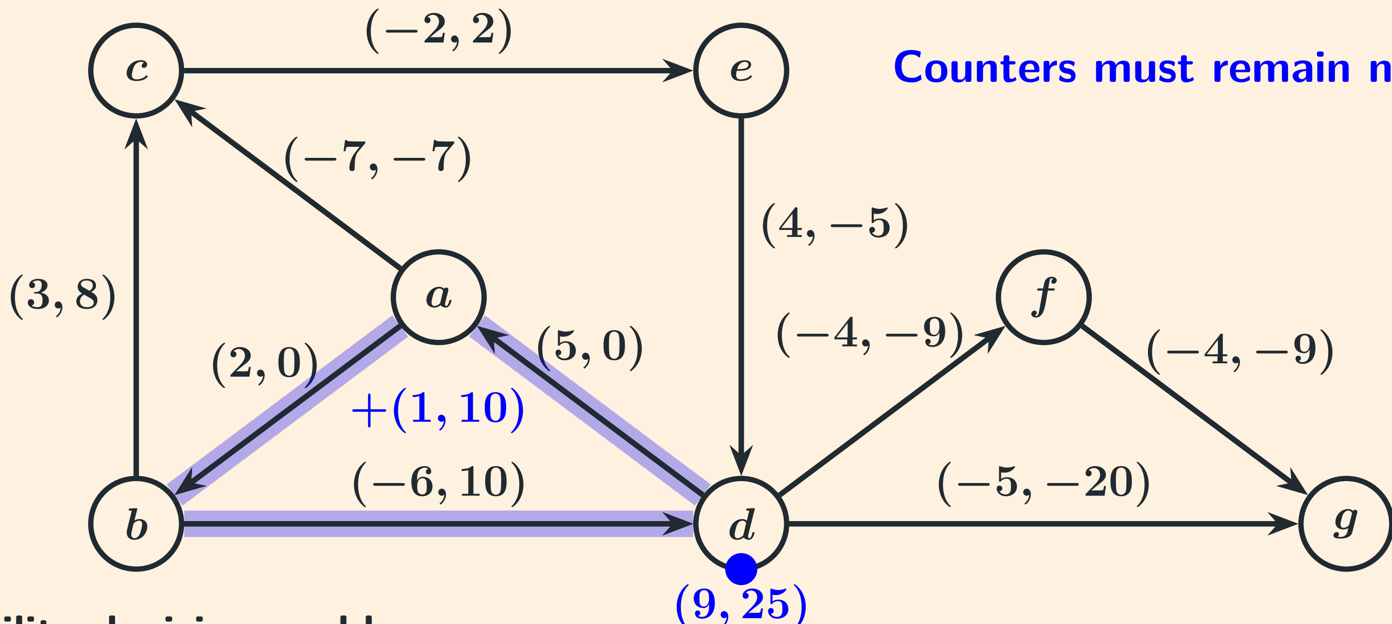
Coverability decision problem.

Input: A VASS, an initial configuration $(s, \vec{x})$, and a target configuration $(t, \vec{y})$.

Question: Does there exist a run from $(s, \vec{x})$ to $(t, \vec{y}')$, for some $\vec{y}' \geq \vec{y}$?

Example: does there exist a run from $(a, (0,0))$ that covers $(g, (0,0))$?

Coverability in 2-VASS



Counters must remain non-negative!

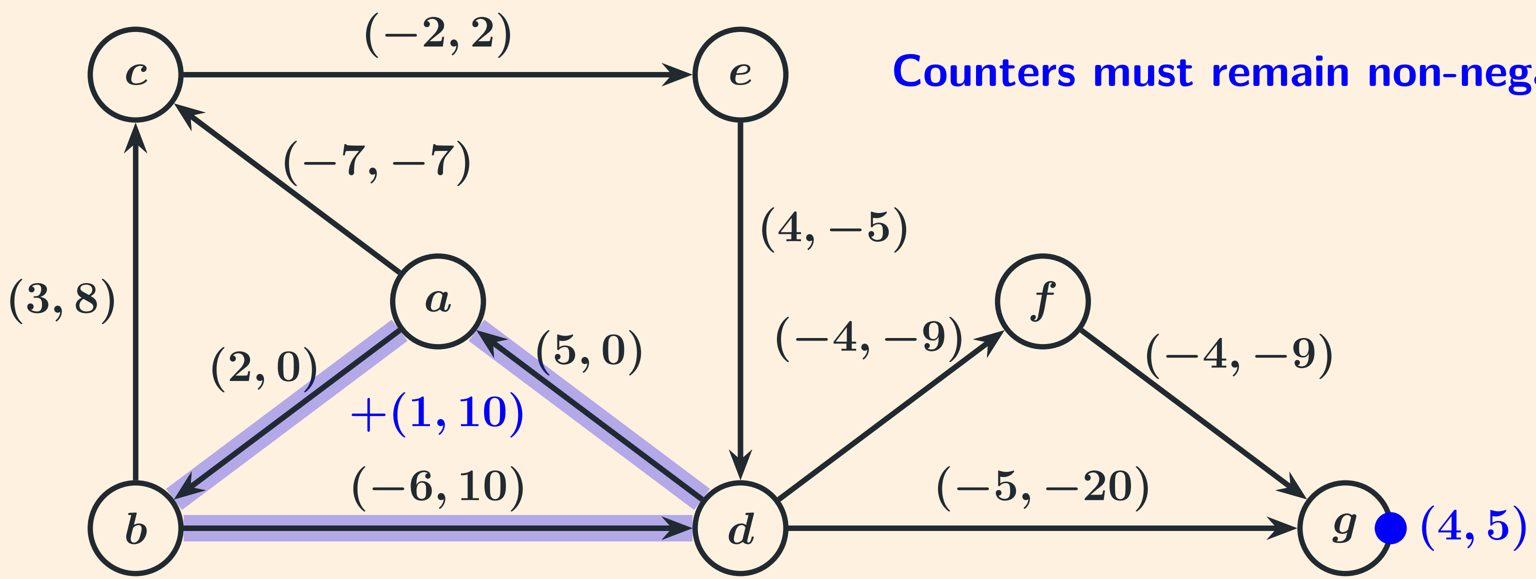
Coverability decision problem.

Input: A VASS, an initial configuration $(s, \vec{x})$, and a target configuration $(t, \vec{y})$.

Question: Does there exist a run from $(s, \vec{x})$ to $(t, \vec{y}')$, for some $\vec{y}' \geq \vec{y}$?

Example: does there exist a run from $(a, (0, 0))$ that covers $(g, (0, 0))$?

Coverability in 2-VASS



Counters must remain non-negative!

Coverability decision problem.

Input: A VASS, an initial configuration $(s, \vec{x})$, and a target configuration $(t, \vec{y})$.

Question: Does there exist a run from $(s, \vec{x})$ to $(t, \vec{y}')$, for some $\vec{y}' \geq \vec{y}$?

Example: does there exist a run from $(a, (0, 0))$ that covers $(g, (0, 0))$?

For the remainer of this talk...

d **is the dimension (number of counters).**

n **is the size of the instance in unary.**

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^{C} time.

Coverability in 2-VASS conditionally requires n^{2} time.

Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

The Complexity of Coverability in d-VASS

Theorem. In a d-VASS, if there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$, then there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$ of length at most $n^{2^{\mathcal{O}(d)}}$. [Künnemann, Mazowiecki, Schütze, S-B, Węgrzycki '23]

[Rackoff '78]

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

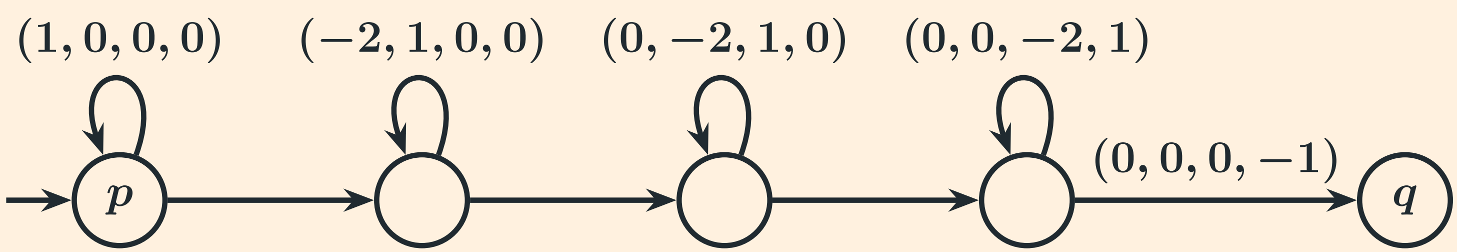
The Complexity of Coverability in d-VASS

Theorem. In a d-VASS, if there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$, then there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$ of length at most $n^{2^{\mathcal{O}(d)}}$. [Künnemann, Mazowiecki, Schütze, S-B, Węgrzycki '23]

[Rackoff '78]

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Theorem. There exists a family of instance of coverability in d-VASS for which the shortest covering runs have length $n^{2^{\Omega(d)}}$. [Lipton '76]



The Complexity of Coverability in d-VASS

Theorem. In a d-VASS, if there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$, then there is a run from $(s, \vec{x})$ that covers $(t, \vec{y})$ of length at most $n^{2^{\mathcal{O}(d)}}$. [Künnemann, Mazowiecki, Schütze, S-B, Węgrzycki '23]

[Rackoff '78]

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Theorem. There exists a family of instance of coverability in d-VASS for which the shortest covering runs have length $n^{2^{\Omega(d)}}$. [Lipton '76]

$\not\Downarrow$ does not directly imply :(

Theorem. Deciding coverability in d-VASS requires $n^{2^{\Omega(d)}}$ time.

Conditionally Optimal Lower Bound for Coverability

Theorem. Assuming the Exponential Time Hypothesis, coverability in d-VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Exponential Time Hypothesis (ETH): 3-SAT with k-variables requires $2^{\Omega(k)}$ time.

$$\Downarrow$$

Detecting whether there is a k-clique in an n-vertex graph requires $n^{\Omega(k)}$ time.

Conditionally Optimal Lower Bound for Coverability

Theorem. Assuming the Exponential Time Hypothesis, coverability in d-VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Exponential Time Hypothesis (ETH): 3-SAT with k-variables requires $2^{\Omega(k)}$ time.

$$\Downarrow$$

Detecting whether there is a k-clique in a k-partite n-vertex graph requires $n^{\Omega(k)}$ time.

[Chen, Chor, Fellows, Huang, Juedes, Kanj, and Xia '05]

[Chen, Huang, Kanj, and Xia '06]

[Cygan, Fomin, Kowalik, Lokshtanov, Marx, Ma. Pilipczuk, and Mi. Pilipczuk '15]

Conditionally Optimal Lower Bound for Coverability

Theorem. Assuming the Exponential Time Hypothesis, coverability in d-VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Idea. Reduce detecting a 2^d-clique in a 2^d-partite n-vertex graph to coverability.

Exponential Time Hypothesis (ETH): 3-SAT with k-variables requires $2^{\Omega(k)}$ time.

$$\Downarrow$$

Detecting whether there is a k-clique in a k-partite n-vertex graph requires $n^{\Omega(k)}$ time.

[Chen, Chor, Fellows, Huang, Juedes, Kanj, and Xia '05]

[Chen, Huang, Kanj, and Xia '06]

[Cygan, Fomin, Kowalik, Lokshtanov, Marx, Ma. Pilipczuk, and Mi. Pilipczuk '15]

Proof Idea: Use Bounded Two-Counter Machines

Idea: Reduce detecting a $\boldsymbol{2^d}$-clique in a $\boldsymbol{2^d}$-partite $\boldsymbol{n}$-vertex directed graph to coverability.

First, reduce to coverability in a $\boldsymbol{n^{2^{\mathcal{O}(d)}}}$-bounded two-counter machine.

Then, simulate a $\boldsymbol{n^{2^{\mathcal{O}(d)}}}$-bounded two-counter machine using an $\mathcal{O}(\boldsymbol{n})$-state $\mathcal{O}(\boldsymbol{d})$-VASS.

Proof Idea: Use Bounded Two-Counter Machines

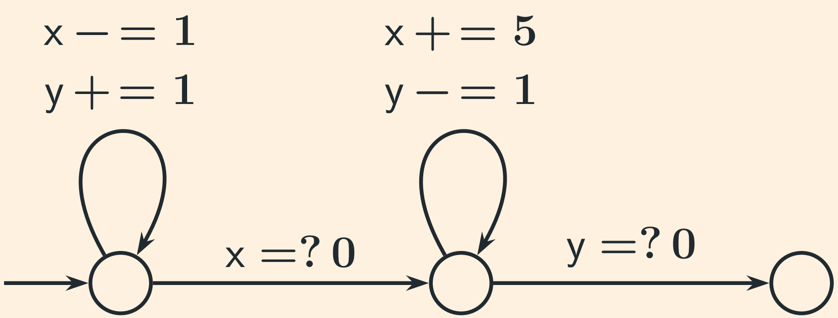
Idea: Reduce detecting a 2^d-clique in a 2^d-partite n-vertex directed graph to coverability.

First, reduce to coverability in a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine.

Then, simulate a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine using an $\mathcal{O}(n)$-state $\mathcal{O}(d)$-VASS.

An $n^{2^{\mathcal{O}(d)}}$**-bounded two-counter machine** has two counters $x, y \in \{0, 1, \ldots, n^{2^{\mathcal{O}(d)}}\}$ that can be added to ($x \mathrel{+}= 2$), subtracted from ($y \mathrel{-}= 3$), and zero-tested ($x =?\, 0$).

1. LOOP($x \mathrel{-}= 1$, $y \mathrel{+}= 1$)
2. $x =?\, 0$
3. LOOP($x \mathrel{+}= 5$, $y \mathrel{-}= 1$)
4. $y =?\, 0$



Proof Idea: Use Bounded Two-Counter Machines

Idea: Reduce detecting a 2^d-clique in a 2^d-partite n-vertex directed graph to coverability.

First, reduce to coverability in a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine.

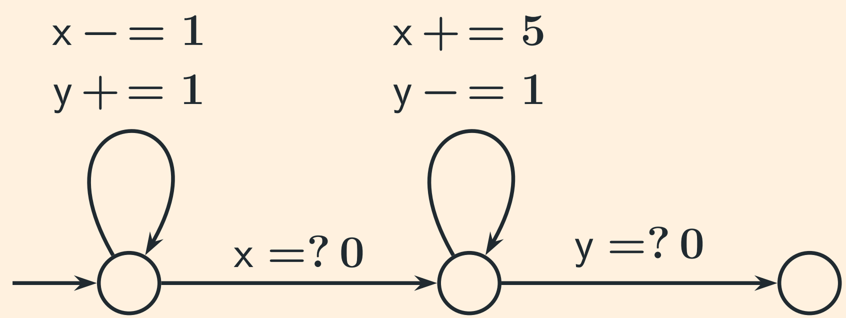
Then, simulate a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine using an $\mathcal{O}(n)$-state $\mathcal{O}(d)$-VASS.

An **$n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine** has two counters $\mathsf{x}, \mathsf{y} \in \{0, 1, \ldots, n^{2^{\mathcal{O}(d)}}\}$ that can be added to ($\mathsf{x} += 2$), subtracted from ($\mathsf{y} -= 3$), and zero-tested ($\mathsf{x} =? 0$).

Pre: $\mathsf{x} = x,\ \mathsf{y} = 0$

1. $\mathtt{LOOP}(\mathsf{x} -= 1, \mathsf{y} += 1)$
2. $\mathsf{x} =? 0$
3. $\mathtt{LOOP}(\mathsf{x} += 5, \mathsf{y} -= 1)$
4. $\mathsf{y} =? 0$

Post: $\mathsf{x} = x \cdot 5,\ \mathsf{y} = 0$



$\mathtt{MULTIPLY}(\mathsf{x}, 5)$

Proof Idea: Use Bounded Two-Counter Machines

Idea: Reduce detecting a 2^d-clique in a 2^d-partite n-vertex directed graph to coverability.

First, reduce to coverability in a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine.

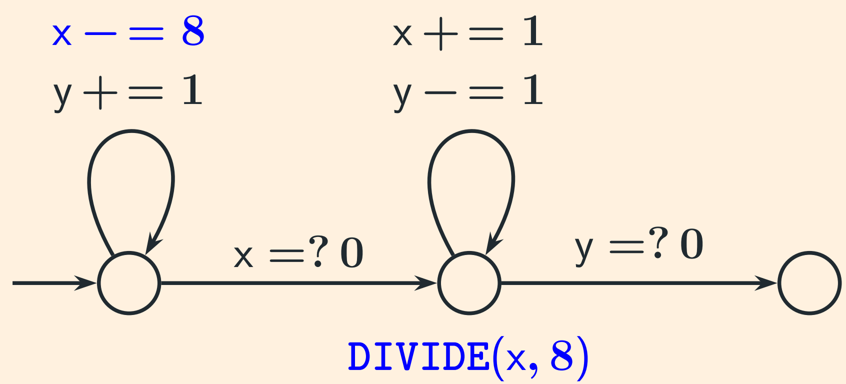
Then, simulate a $n^{2^{\mathcal{O}(d)}}$-bounded two-counter machine using an $\mathcal{O}(n)$-state $\mathcal{O}(d)$-VASS.

An $n^{2^{\mathcal{O}(d)}}$-**bounded two-counter machine** has two counters $\mathsf{x}, \mathsf{y} \in \{0, 1, \ldots, n^{2^{\mathcal{O}(d)}}\}$ that can be added to ($\mathsf{x} \mathrel{+}= 2$), subtracted from ($\mathsf{y} \mathrel{-}= 3$), and zero-tested ($\mathsf{x} =? 0$).

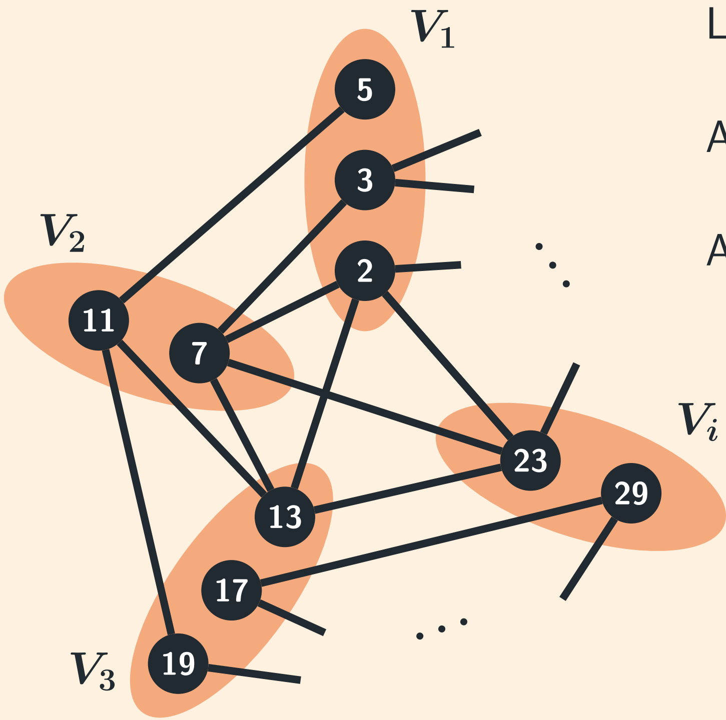
Pre: $\mathsf{x} = x,\ \mathsf{y} = 0$

1. $\texttt{LOOP}(\mathsf{x} \mathrel{-}= 8, \mathsf{y} \mathrel{+}= 1)$
2. $\mathsf{x} =? 0$
3. $\texttt{LOOP}(\mathsf{x} \mathrel{+}= 1, \mathsf{y} \mathrel{-}= 1)$
4. $\mathsf{y} =? 0$

Post: $\mathsf{x} = x \div 8, \mathsf{y} = 0$



Detecting Cliques using Divisibility Tests

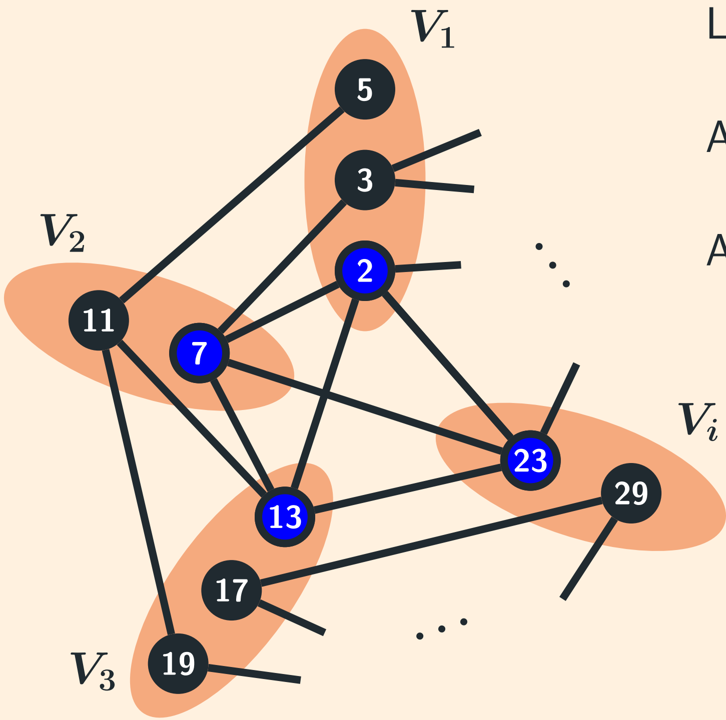


Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

Associate the first n primes with the verticies.

A candidate k-clique is represented by a product of k primes.

Detecting Cliques using Divisibility Tests



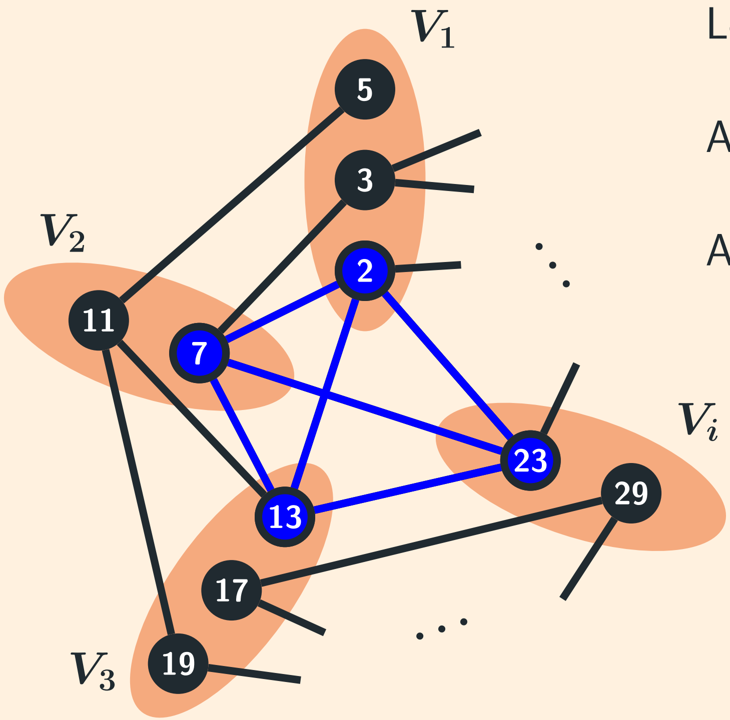
Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

Associate the first n primes with the verticies.

A candidate k-clique is represented by a product of k primes.

Example: $c = 2 \cdot 7 \cdot 13 \cdot \ldots \cdot 23$.

Detecting Cliques using Divisibility Tests



Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

Associate the first n primes with the verticies.

A candidate k-clique is represented by a product of k primes.

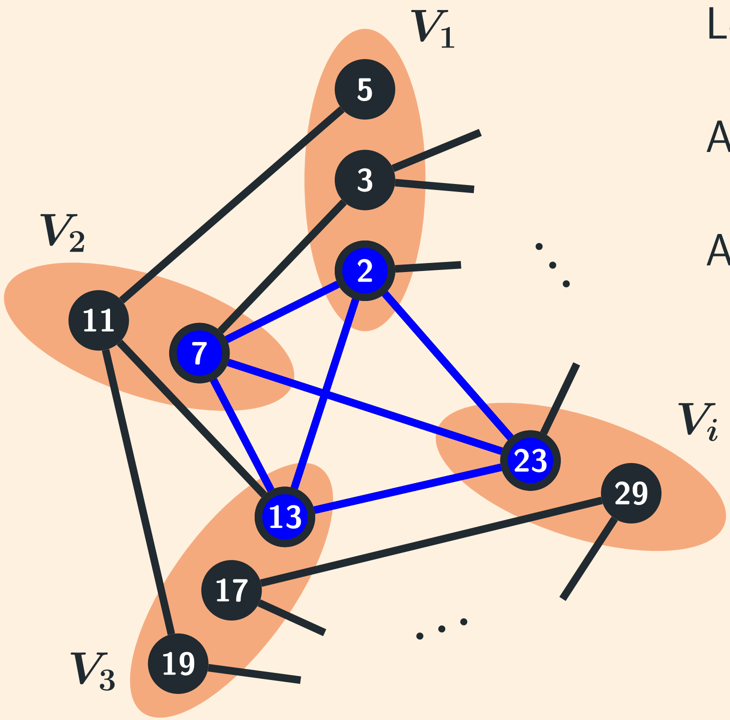
Example: $c = 2 \cdot 7 \cdot 13 \cdot \ldots \cdot 23$.

To check if v represents a clique, use divisibility tests to verify all nodes are adjacent.

Example: $(2 \cdot 7) | c\,?$ $\quad (2 \cdot 13) | c\,?$ $\quad (7 \cdot 13) | c\,?$ $\quad \ldots$

$(2 \cdot 23) | c\,?$ $\quad (7 \cdot 23) | c\,?$ $\quad (13 \cdot 23) | c\,?$

Detecting Cliques using Divisibility Tests



Let $(V_1 \cup V_2 \cup \cdots \cup V_k, E)$ be a k-partite n-vertex graph.

Associate the first n primes with the verticies.

A candidate k-clique is represented by a product of k primes.

Example: $c = 2 \cdot 7 \cdot 13 \cdot \ldots \cdot 23$.

To check if v represents a clique, use divisibility tests to verify all nodes are adjacent.

Example: $(2 \cdot 7) | c\,?$ $(2 \cdot 13) | c\,?$ $(7 \cdot 13) | c\,?$ $\ldots$
$(2 \cdot 23) | c\,?$ $(7 \cdot 23) | c\,?$ $(13 \cdot 23) | c\,?$

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that, for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) | \, p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Bounded Two-Counter Machine Implementation

There e… $\boldsymbol{k}$, there

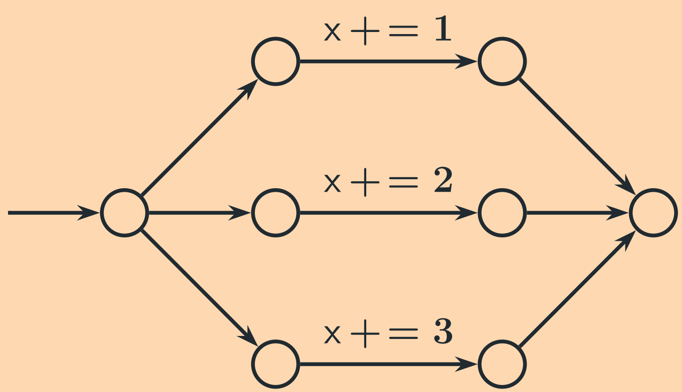
is an e… $\boldsymbol{k}$-clique.

Guessing with Nondeterministic Branching

Pre: $\mathsf{x} = \boldsymbol{x}$

1. $\texttt{GUESS}\ \boldsymbol{c} \in \{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}\}$
2. $\mathsf{x} \mathrel{+}= \boldsymbol{c}$

Post: $\mathsf{x} = \boldsymbol{x} + \boldsymbol{1}$, or
$\mathsf{x} = \boldsymbol{x} + \boldsymbol{2}$, or
$\mathsf{x} = \boldsymbol{x} + \boldsymbol{3}$.



Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Part one: Guess a candidate clique.

Pre: $\mathsf{x} = 1,\ \mathsf{y} = 0$.

$\mathtt{GUESS}\ p_1 \in \text{Primes}(V_1)$

$\mathtt{MULTIPLY}(\mathsf{x}, p_1)$

$\vdots$

$\mathtt{GUESS}\ p_k \in \text{Primes}(V_k)$

$\mathtt{MULTIPLY}(\mathsf{x}, p_k)$

Post: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k,\ \mathsf{y} = 0$.

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Part one: Guess a candidate clique.

Pre: $\mathsf{x} = 1$, $\mathsf{y} = 0$.

$\texttt{GUESS}\ p_1 \in \text{Primes}(V_1)$

$\texttt{MULTIPLY}(\mathsf{x}, p_1)$

$\vdots$

$\texttt{GUESS}\ p_k \in \text{Primes}(V_k)$

$\texttt{MULTIPLY}(\mathsf{x}, p_k)$

Post: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k$, $\mathsf{y} = 0$.

Part two: Check the candidate is a clique.

Pre: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k$, $\mathsf{y} = 0$.

$\texttt{GUESS}\ \{p_1, p_2\} \in (V_1 \times V_2) \cap E$

$\texttt{DIVIDE}(\mathsf{x}, p_1 \cdot p_2)$

$\texttt{MULTIPLY}(\mathsf{x}, p_1 \cdot p_2)$

$\vdots$

$\texttt{GUESS}\ \{p_{k-1}, p_k\} \in (V_{k-1} \times V_k) \cap E$

$\texttt{DIVIDE}(\mathsf{x}, p_{k-1} \cdot p_k)$

$\texttt{MULTIPLY}(\mathsf{x}, p_{k-1} \cdot p_k)$

Post: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k$, $\mathsf{y} = 0$.

Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \ldots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \ldots \cdot p_k \iff$ there exists a k-clique.

Part one: Guess a candidate clique.

Pre: $\mathsf{x} = 1,\ \mathsf{y} = 0$.

$\texttt{GUESS}\ p_1 \in \text{Primes}(V_1)$

$\texttt{MULTIPLY}($

$\texttt{GUESS}\ p_k \in \text{Primes}(V_k)$

$\texttt{MULTIPLY}(\mathsf{x}, p_k)$

Post: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k,\ \mathsf{y} = 0$.

Part two: Check the candidate is a clique.

Pre: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k,\ \mathsf{y} = 0$

$) \cap E$

$\texttt{DIVIDE}(\mathsf{x}, p_1 \cdot p_2)$

$\texttt{MULTIPLY}(\mathsf{x}, p_1 \cdot p_2)$

$\vdots$

$\texttt{GUESS}\ \{p_{k-1}, p_k\} \in (V_{k-1} \times V_k) \cap E$

$\texttt{DIVIDE}(\mathsf{x}, p_{k-1} \cdot p_k)$

$\texttt{MULTIPLY}(\mathsf{x}, p_{k-1} \cdot p_k)$

Post: $\mathsf{x} = p_1 \cdot \ldots \cdot p_k,\ \mathsf{y} = 0$.

This two-counter program terminates $\iff$ there exists a k-clique.

VASS can Simulate Bounded Two-Counter Machines

Counter bound of k-clique detecting two-counter machine: $p^k_{\mathsf{max}} \leq \mathcal{O}(n^k \log(n)^k) \leq \mathcal{O}(n^{2k})$.

Size of k-clique detecting two-counter machine: $\mathcal{O}(n^{11}) \leq \mathit{poly}(n)$.

VASS can Simulate Bounded Two-Counter Machines

Counter bound of k-clique detecting two-counter machine: $p^k_{\mathsf{max}} \leq \mathcal{O}(n^k \log(n)^k) \leq \mathcal{O}(n^{2k})$.

Size of k-clique detecting two-counter machine: $\mathcal{O}(n^{11}) \leq$ *poly*(n).

Lemma. In *poly*(n) time, one can construct a $\mathcal{O}(\log(k))$-VASS that can *simulate* an $\mathcal{O}(n^k)$-bounded two-counter machine of *poly*(n) size. [Rosier and Yen '85]

VASS can Simulate Bounded Two-Counter Machines

Counter bound of k-clique detecting two-counter machine: $p^k_{\textsf{max}} \leq \mathcal{O}(n^k \log(n)^k) \leq \mathcal{O}(n^{2k})$.

Size of k-clique detecting two-counter machine: $\mathcal{O}(n^{11}) \leq \textit{poly}(n)$.

Lemma. In *poly*(n) time, one can construct a $\mathcal{O}(\log(k))$-VASS that can *simulate* an $\mathcal{O}(n^k)$-bounded two-counter machine of *poly*(n) size. [Rosier and Yen '85]

If we set $k = 2^d$, the *poly*(n)-size two-counter machine for detecting 2^d-cliques is $\mathcal{O}(n^{2^d})$-bounded.

$\implies$ In *poly*(n) time, one can construct an $\mathcal{O}(d)$-VASS for detecting 2^d-cliques.

VASS can Simulate Bounded Two-Counter Machines

Counter bound of k-clique detecting two-counter machine: $p^k_{\textbf{max}} \leq \mathcal{O}(n^k \log(n)^k) \leq \mathcal{O}(n^{2k})$.

Size of k-clique detecting two-counter machine: $\mathcal{O}(n^{11}) \leq$ *poly*(n).

Lemma. In *poly*(n) time, one can construct a $\mathcal{O}(\log(k))$-VASS that can *simulate* an $\mathcal{O}(n^k)$-bounded two-counter machine of *poly*(n) size. [Rosier and Yen '85]

If we set $k = 2^d$, the *poly*(n)-size two-counter machine for detecting 2^d-cliques is $\mathcal{O}(n^{2^d})$-bounded.

$\implies$ In *poly*(n) time, one can construct an $\mathcal{O}(d)$-VASS for detecting 2^d-cliques.

Remark. Termination is coverability.

This two-counter program terminates $\iff$ there exists a k-clique.

"Can I get to the end of the program with any (at least zero) value on each of the counters?"

Reducing Clique Detection to Coverability in VASS

Detecting 2^d-cliques in an n-vertex graph requires $n^{\Omega(2^d)}$ time under the Exponential Time Hypothesis.

$\big\downarrow$ Via divisibilty tests of the product-of-primes encoding.

Construct an instance of termination in a *poly*(n)-size $\mathcal{O}(n^{2^d})$-bounded two-counter machine.

$\big\downarrow$ Using Rosier and Yen's simulation lemma.

Then, in *poly*(n) time, construct an instance of coverability in an $\mathcal{O}(d)$-VASS.

Theorem. Assuming the exponential time hypothesis, coverability in VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^C time.

Coverability in 2-VASS conditionally requires n^2 time.

Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^C time.

Coverability in 2-VASS conditionally requires n^2 time.

Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

The Complexity of Coverability in 2-VASS

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Corollary. Coverability in 2-VASS can be decided in $n^{\overset{C}{\cancel{320}}}$ time.

The Complexity of Coverability in 2-VASS

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Corollary. Coverability in 2-VASS can be decided in $n^{\overset{C}{\cancel{320}}}$ time.

k-cycle detection problem:

Input: A directed graph G.

Question: Does there exist a cycle in G of length k?

k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

[Alon, Yuster, and Zwick '97]

[Yuster and Zwick '04]

[Darlirrooyford, Vuong, and Williams '21]

The Complexity of Coverability in 2-VASS

Corollary. Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Corollary. Coverability in 2-VASS can be decided in $n^{\overset{C}{\cancel{320}}}$ time.

k-cycle detection problem:

Input: A directed graph G.

Question: Does there exist a cycle in G of length k?

k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

[Alon, Yuster, and Zwick '97]

[Yuster and Zwick '04]

[Darlirrooyford, Vuong, and Williams '21]

Theorem. Assuming the k-cycle hypothesis, there does not exist an $\varepsilon > 0$ and an $\mathcal{O}(n^{2-\varepsilon})$-time algorithm that decides coverability in 2-VASS.

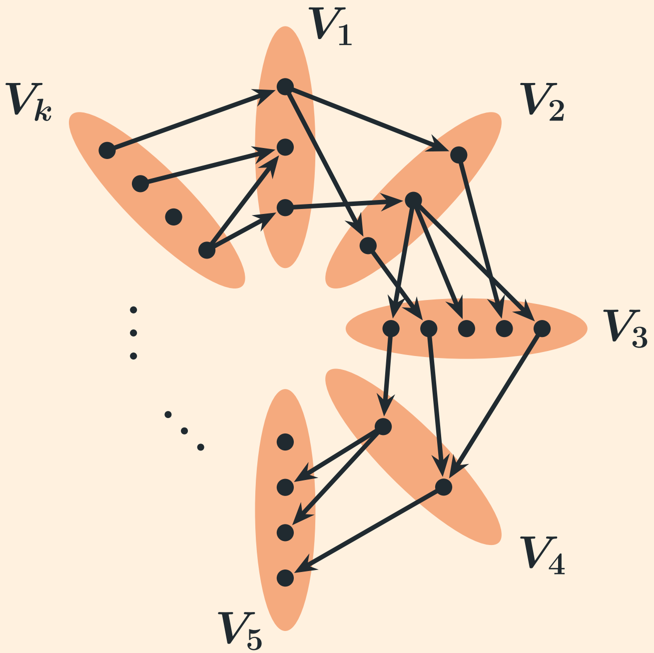
[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

k-Cycle Hypothesis Simplifying Assumption

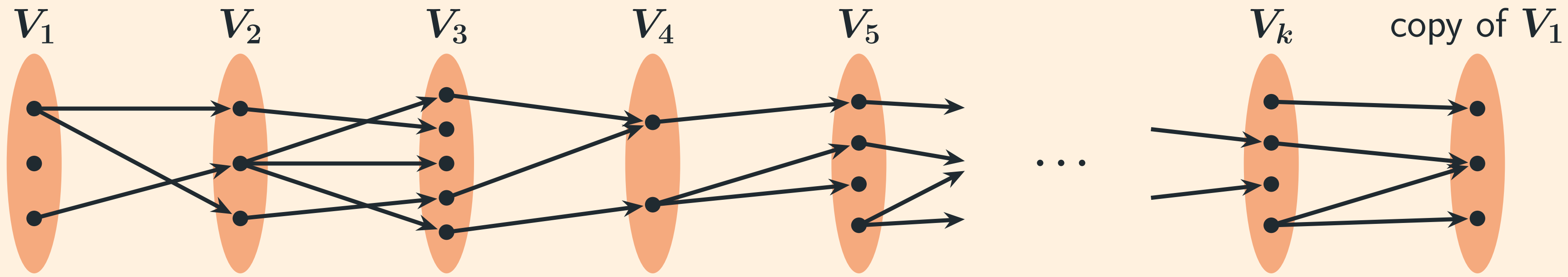
k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

[Alon, Yuster, and Zwick '97]
[Yuster and Zwick '04]
[Darlirrooyford, Vuong, and Williams '21]

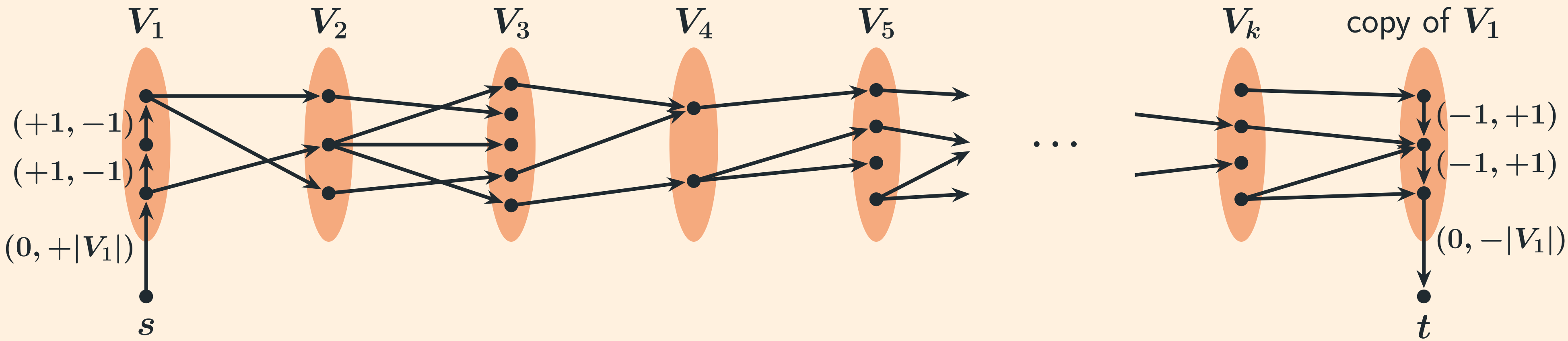
It suffices to only consider **k-circle layered graphs**:
[Lincoln, Williams, and Williams '18]



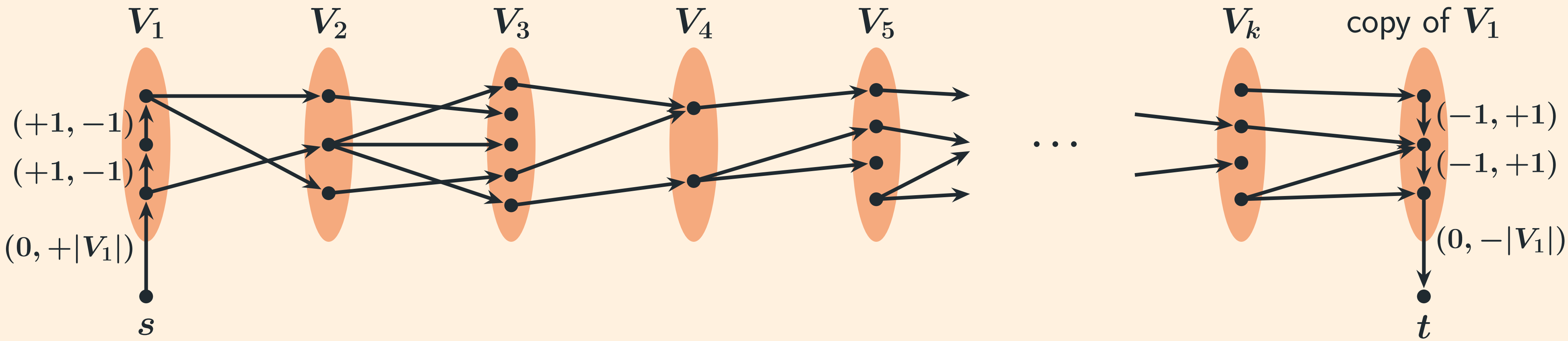
Reduction From k-Cycle Detection



Reduction From k-Cycle Detection

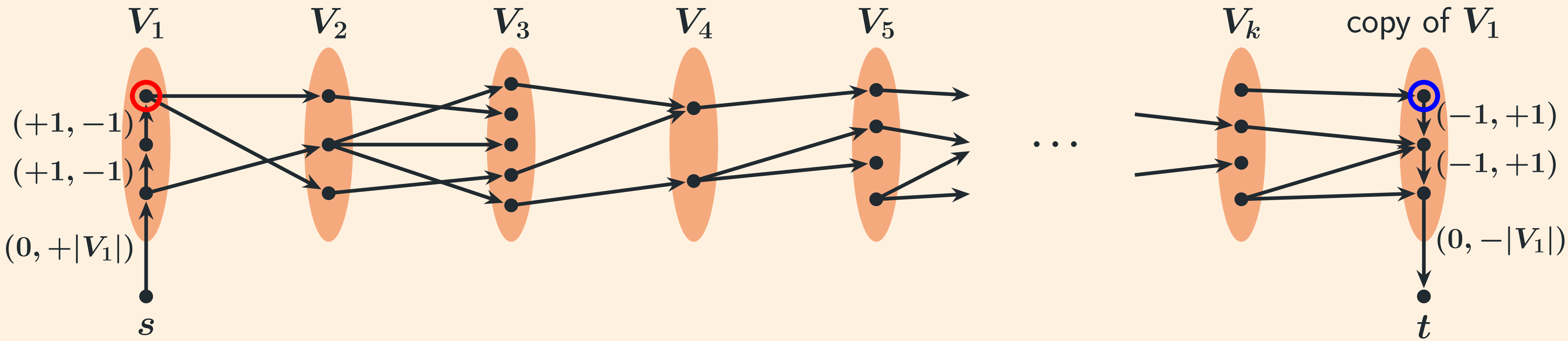


Reduction From k-Cycle Detection



Claim. From $(s,(0,0))$ you can cover $(t,(0,0))$ $\iff$ there is a k-cycle the original graph.

Reduction From k-Cycle Detection



Claim. From $(s,(0,0))$ you can cover $(t,(0,0))$ $\iff$ there is a k-cycle the original graph.

Idea. Suppose you leave V_1 via the i-th node and arrive at the copy of V_1 at the j-th node.

First component $\implies i \geq j$.
Second component $\implies j \geq i$.
$\implies$ Any run from s that reaches t must have $i = j$.

Reduction From k-Cycle Detection

k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

[Alon, Yuster, and Zwick '97]

[Yuster and Zwick '04]

[Darlirrooyford, Vuong, and Williams '21]

$$+$$

Claim. From $(s, (0,0))$ you can cover $(t, (0,0))$ $\iff$ there is a k-cycle the original graph.

Linear-time reduction!

Idea. Suppose you leave V_1 via the i-th node and arrive at the copy of V_1 at the j-th node.

First component $\implies i \geq j$.

Second component $\implies j \geq i$.

$\implies$ Any run from s that reaches t must have $i = j$.

Reduction From k-Cycle Detection

k-cycle hypothesis: for every $\varepsilon > 0$ and m, there exists k such that there is no $\mathcal{O}(m^{2-\varepsilon})$-time algorithm for detecting k-cycles in m-edge graphs.

[Alon, Yuster, and Zwick '97]
[Yuster and Zwick '04]
[Darlirrooyford, Vuong, and Williams '21]

$+$

Claim. From $(s, (0,0))$ you can cover $(t, (0,0))$ $\iff$ there is a k-cycle the original graph.

Linear-time reduction!

$\Downarrow$

Theorem. Assuming the k-cycle hypothesis, there does not exist an $\varepsilon > 0$ and an $\mathcal{O}(n^{2-\varepsilon})$-time algorithm that decides coverability in 2-VASS. [Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^C time.

Coverability in 2-VASS conditionally requires n^2 time.

Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

Preview of this talk

Part 1: d-VASS

Coverability in d-VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Coverability in d-VASS conditionally requires $n^{2^{\Omega(d)}}$ time.

Parameterised reduction from k-clique detection.

Part 2: 2-VASS

Coverability in 2-VASS can be decided in n^{C} time.

Coverability in 2-VASS conditionally requires n^{2} time.

Linear-time reduction from k-cycle detection.

Part 3: 1-VASS

Coverability in 1-VASS can be decided in n^2 time

Coverability in (binary) 1-VASS is in AC^1.

Open Problem

What is the *exact* complexity of of coverability in 1-VASS ?

Complexity of Coverability in 1-VASS

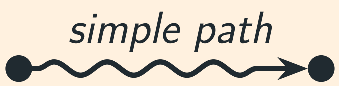
Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Complexity of Coverability in 1-VASS

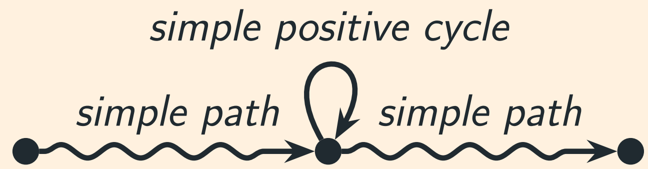
Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Observation. One need not take non-positive weight cycles.

$\implies$



or

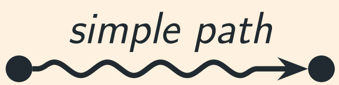


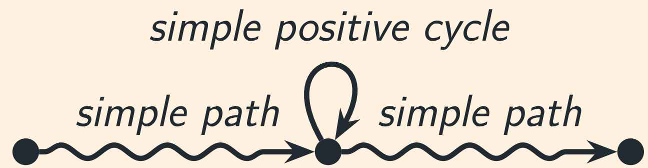
witnesses coverability.

Complexity of Coverability in 1-VASS

Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Observation. One need not take non-positive weight cycles.

$\implies$ 

or 

witnesses coverability.

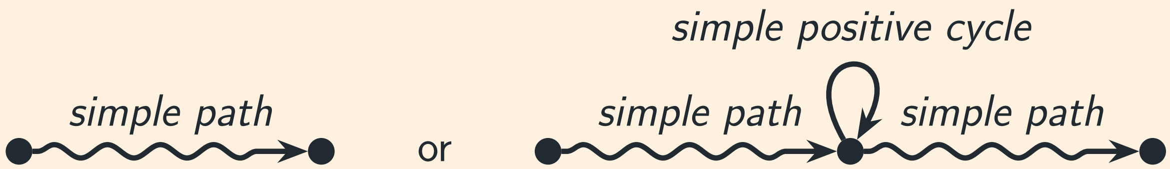
Idea. Modify the Bellman-Ford $\mathcal{O}(|V||E|)$-time algorithm for shortest paths in integer-weighted graphs.

Complexity of Coverability in 1-VASS

Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Observation. One need not take non-positive weight cycles.

$\implies$



witnesses coverability.

Idea. Modify the Bellman-Ford $\mathcal{O}(|V||E|)$-time algorithm for shortest paths in integer-weighted graphs.

Theorem. Coverability in (binary-encoded) 1-VASS is in AC^1. [Shakiba, S-B, and Zetzsche '25]

This matches the state-of-the-art for shortest paths in $\mathbb{N}$-weighted directed graphs. [Cook '85]

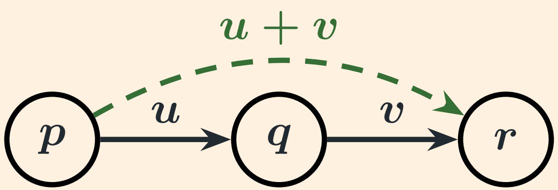
An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

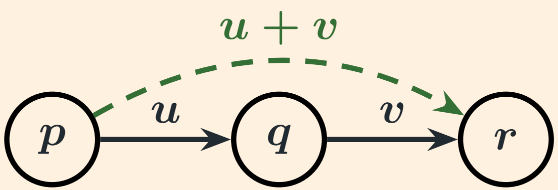
An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



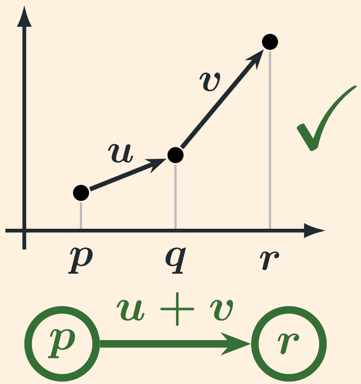
An Efficient Algorithm for Coverability in 1-VASS



`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

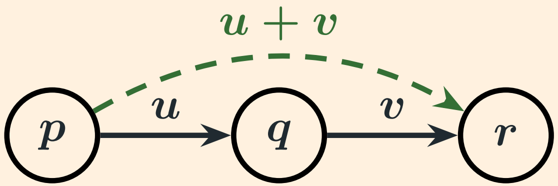
`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":

Scenario 1

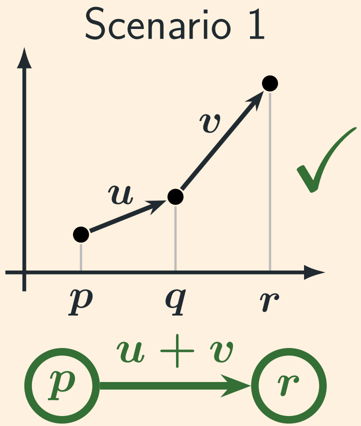


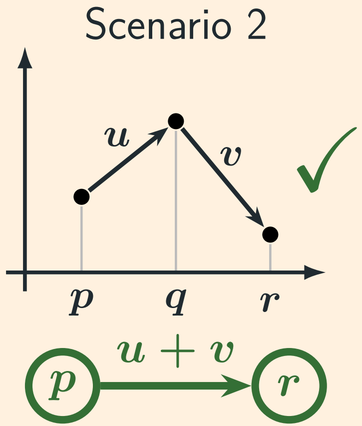
An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.



`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":

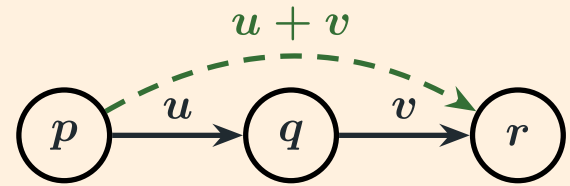




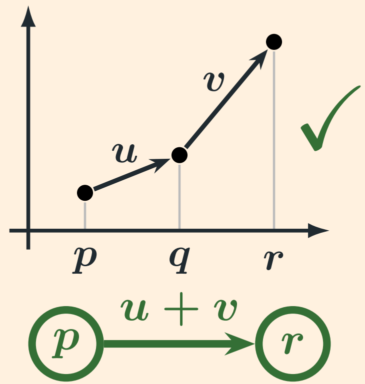
An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

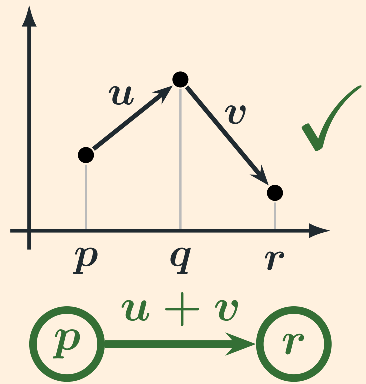
`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



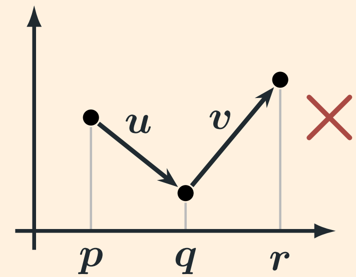
Scenario 1



Scenario 2



Scenario 3

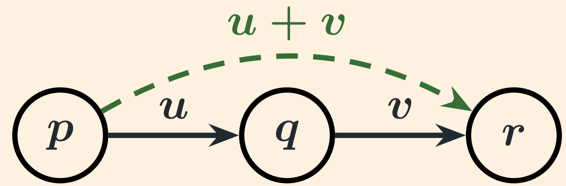


Do not add shortcut transition

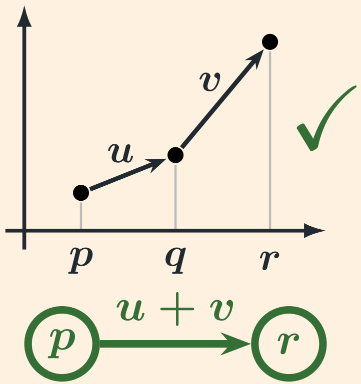
An Efficient Algorithm for Coverability in 1-VASS

STEP 0: Let $\mathcal{V}_0$ be the given 1-VASS.

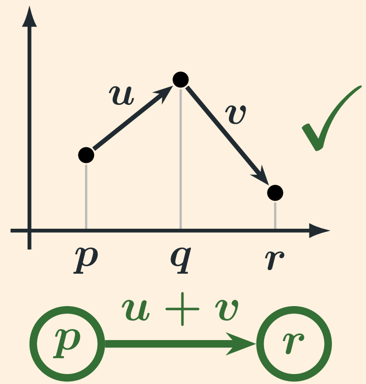
STEP 1: Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



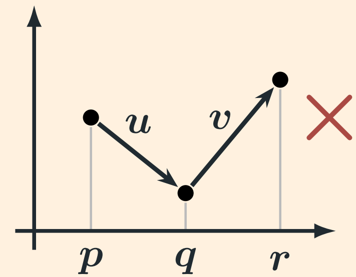
Scenario 1



Scenario 2

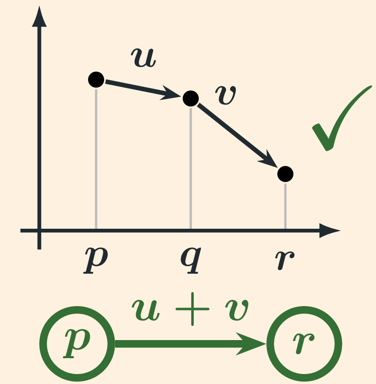


Scenario 3



Do not add shortcut transition

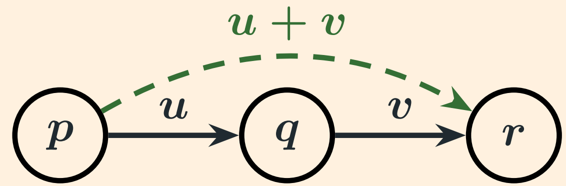
Scenario 4

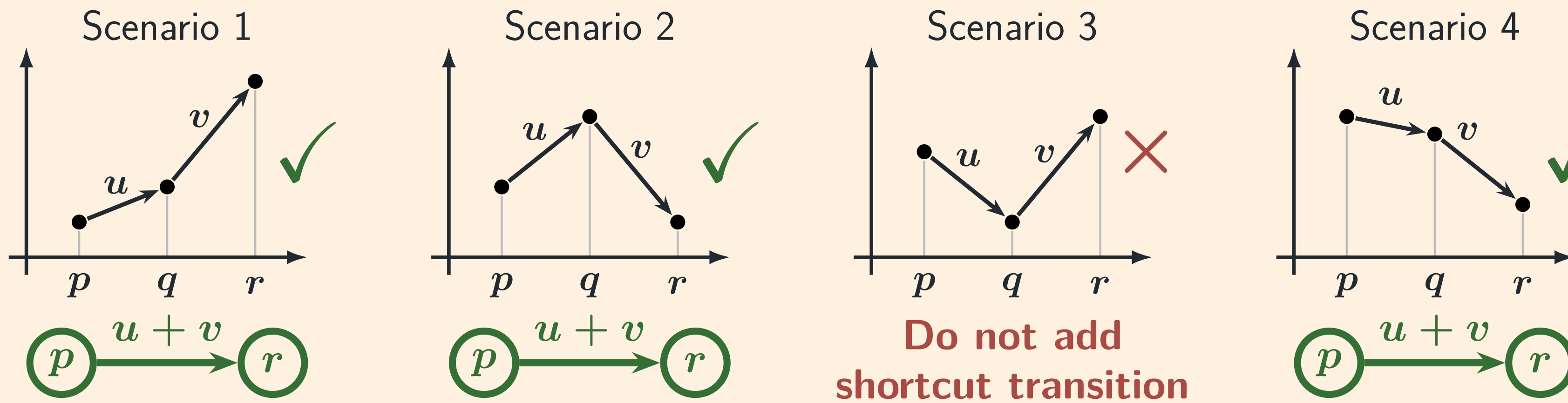


An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":

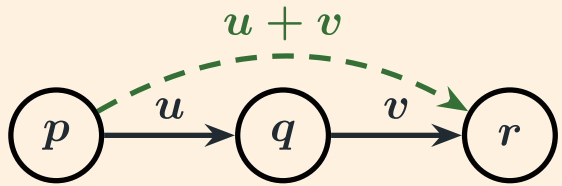




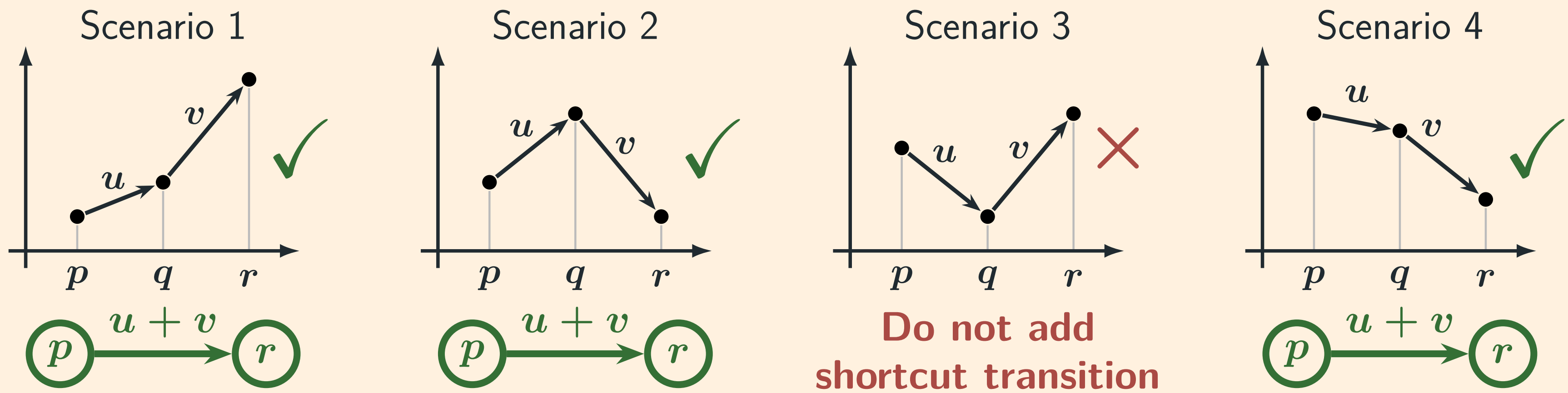
`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.



`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

Claim. There is a covering run in $\mathcal{V}_0$ if and only if there is a covering run of **length** ≤ 2 in $\mathcal{V}_k$.

What is the Fine-Grained Complexity of Coverability in 1-VASS?

Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Observation. One need not take non-positive weight cycles.

Open Problem. Does there exist a $o(n^2)$-time algorithm for coverability in 1-VASS?

witnesses coverability.

Idea. Modify the $\mathcal{O}(|V||E|)$Bellman-Ford algorithm for shortest paths in integer-weighted graphs.

Theorem. Coverability in (binary-encoded) 1-VASS is in AC^1. [Shakiba, S-B, and Zetzsche '25]

This matches the state-of-the-art for shortest paths in $\mathbb{N}$-weighted directed graphs. [Cook '85]

What is the Fine-Grained Complexity of Coverability in 1-VASS?

Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time.

Open Problem. Does there exist a $o(n^2)$-time algorithm for coverability in 1-VASS?

new ideas?

lift ideas?

Theorem. There is an $n\,\mathrm{polylog}(n)$-time algorithm for SSSP (or detecting a negative cycle).

[Bernstein, Nanongkai, and Wulff-Nilsen '22] [Bringmann, Cassis, and Fischer '23]
[Haeupler, Jiang, Saranurak '26] [J. Li '26] [Hair, G. Li, J. Li, and Zhang '26+]

Theorem. Coverability in (binary-encoded) 1-VASS is in AC^1. [Shakiba, S-B, and Zetzsche '25]

This matches the state-of-the-art for shortest paths in $\mathbb{N}$-weighted directed graphs. [Cook '85]

Precise Complexity Analysis of Coverability in Vector Addition Systems

Theorem. Assuming the exponential time hypothesis, coverability in d-VASS requires $n^{2^{\Omega(d)}}$ time.

Idea. Reduce detecting a 2^d-clique in a 2^d-partite n-vertex graph to coverability in d-VASS.

Theorem. Assuming the k-cycle hypothesis, coverability in 2-VASS "requires quadratic time".

Idea. Reduce, in linear time, k-cycle detection in m-edge directed graphs to coverability in 2-VASS.

Theorem. Coverability in 1-VASS can be decided in $\mathcal{O}(n^2)$ time and is in AC^1.

Open problem. Does there exist an $o(n^2)$-time algorithm for coverability in 1-VASS ?

Thank You!

Presented by Henry Sinclair-Banks, MPI-SWS, Kaiserslautern, Germany

Noon-seminar, MPI-INF, Saarbrücken, Germany

Presentation made with BeamerikZ