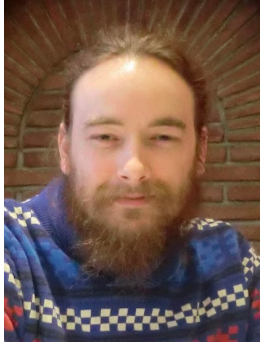


History-Constrained Systems



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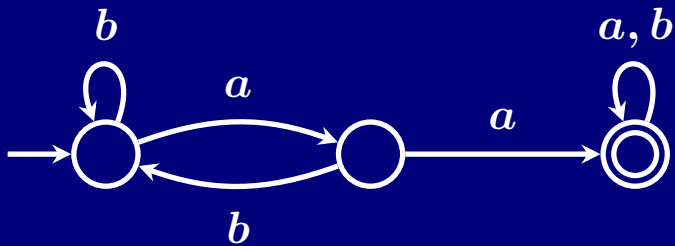
Formal Methods 2026: Information Flow Verification

20th May 2026

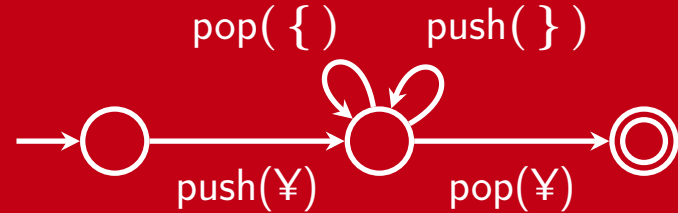
Auditorium (2F), Hitotsubashi Hall, Tokyo, Japan

Basic Models of Computation

Finite Automata (DFAs, NFAs)

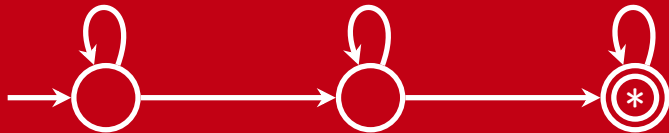


Pushdown Automata (PDAs)



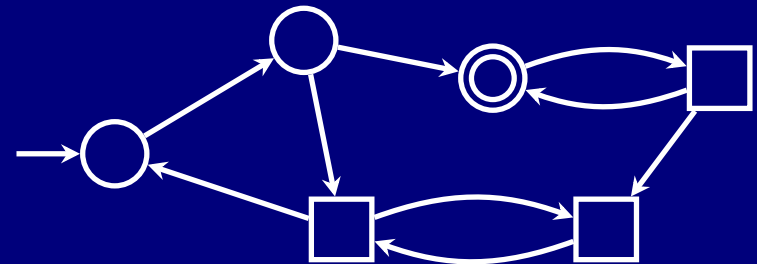
Vector Addition Systems with States

$a(+1, +1)$ $b(-1, 0)$ $c(0, -1)$



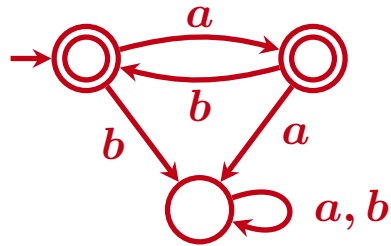
*Acceptance conditions can either specify counters = 0, or ≥ 0 .

Two-player Games

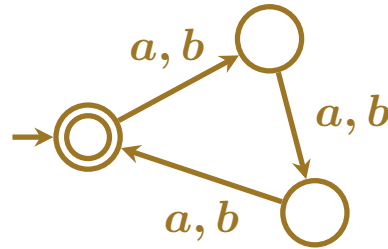


History-Constrained Systems

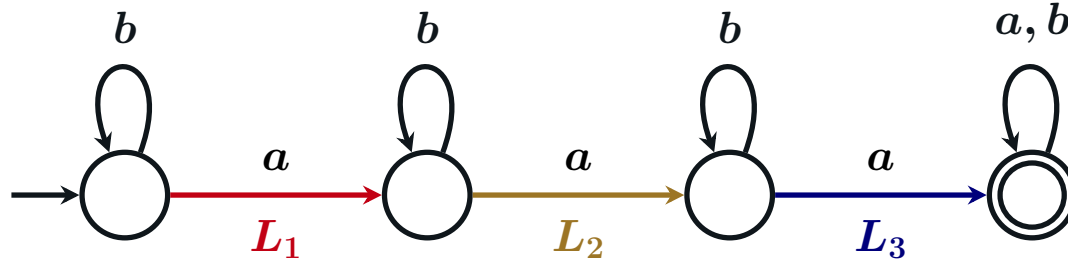
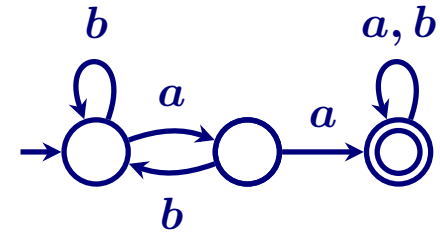
$$L_1 = (ab)^* + a(ba)^*$$



$$L_2 = \{\text{words of length 3}\}$$



$$L_3 = \{\text{words containing } aa\}$$



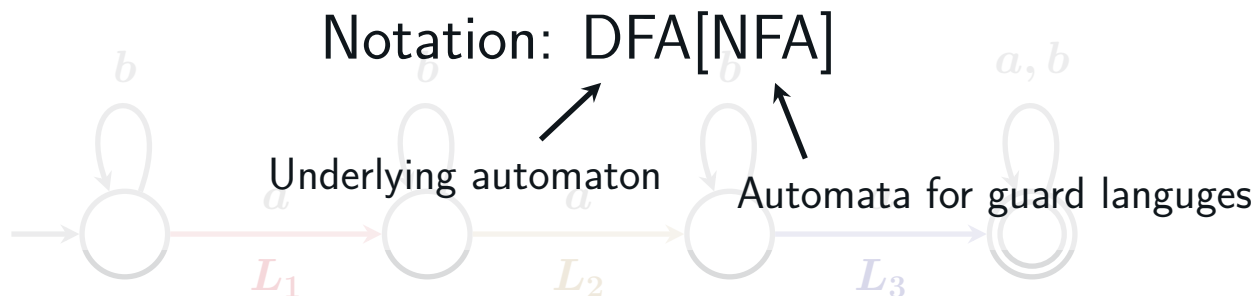
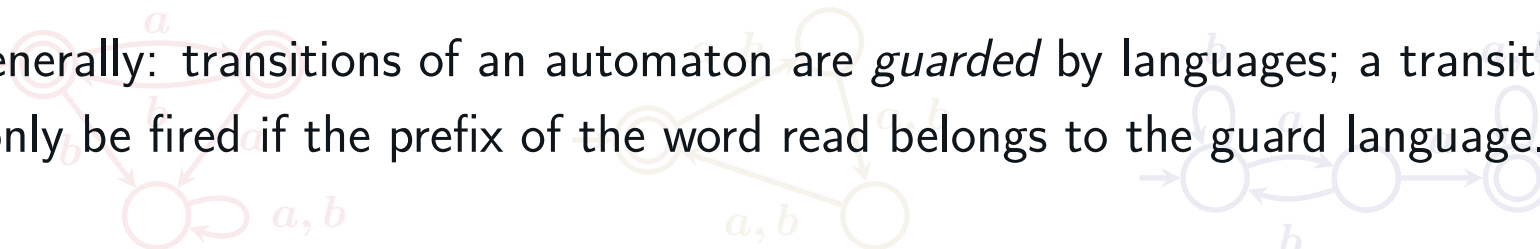
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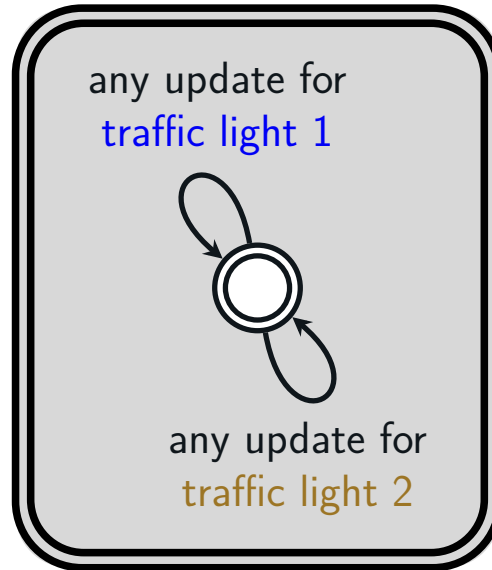
More generally: transitions of an automaton are *guarded* by languages; a transition can only be fired if the prefix of the word read belongs to the guard language.



A Motivating (Toy) Example

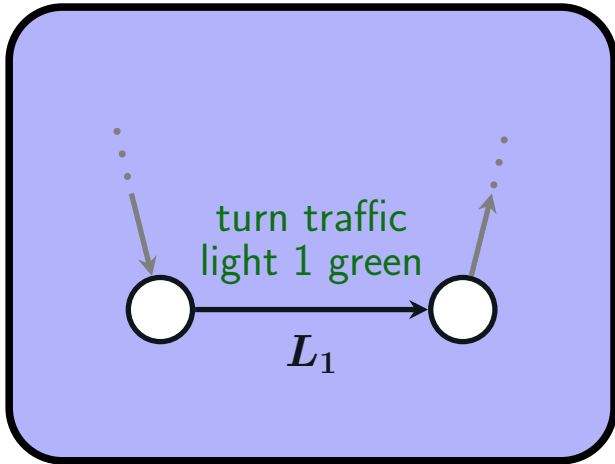


Junction computer

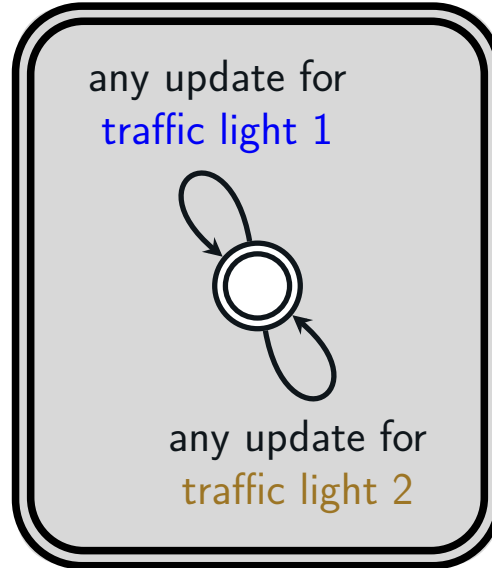


A Motivating (Toy) Example

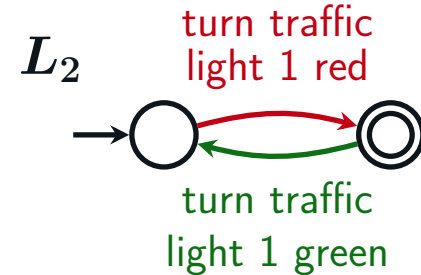
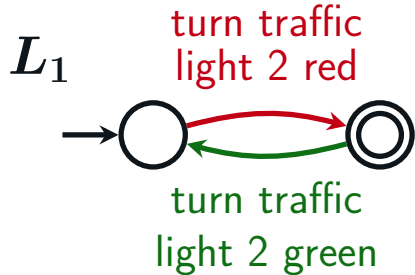
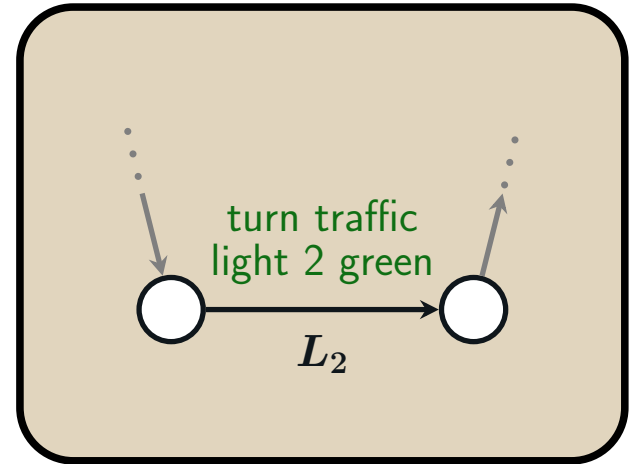
Traffic light 1 controller



Junction computer

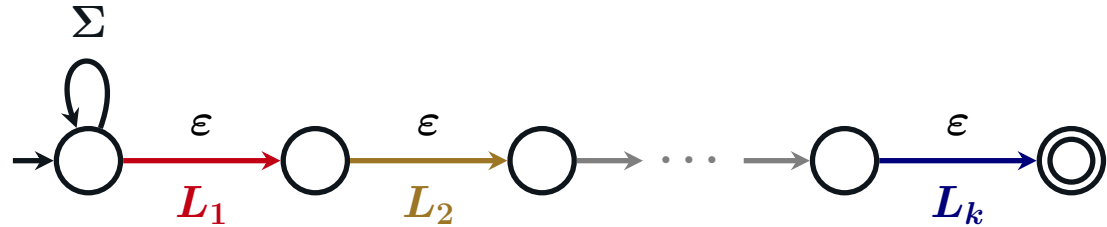


Traffic light 2 controller



Computational Power of HCS

Can at least compute the
intersection of guard languages:



Corollary. Emptiness (reachability) of NFA[PDA] is undecidable.

Corollary. Emptiness (reachability) of NFA[DFA] is PSPACE-hard.

Takeaway message 1: even basic history-constrained systems can capture and exhibit complex system behaviours.

HCS with Regular Guards

Theorem. Given an NFA[NFA] A , one can construct a DFA D such that $L(D) = L(A)$.

Proof idea. (1) Convert each NFA guard into an equivalent DFA.
(2) Create an equivalent NFA by producting the underlying NFA with all guard DFAs.
(3) Convert this NFA into an equivalent DFA.

HCS with Regular Guards

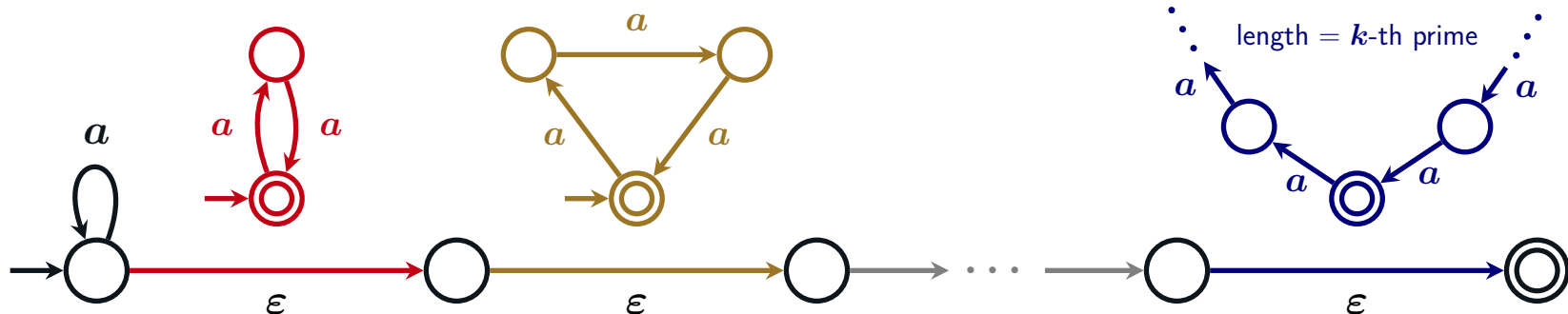
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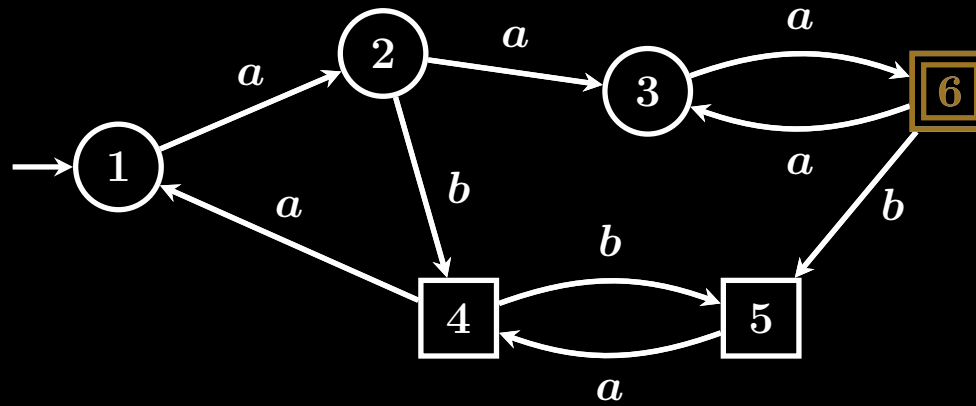
Takeaway message 2: history-constrained systems with regular guards recognise exactly the regular languages (but they can be more succinct).



Two-Player Games on HCS

There are two players: Alice controls transitions taken from \bigcirc

Bob controls transitions taken from \square



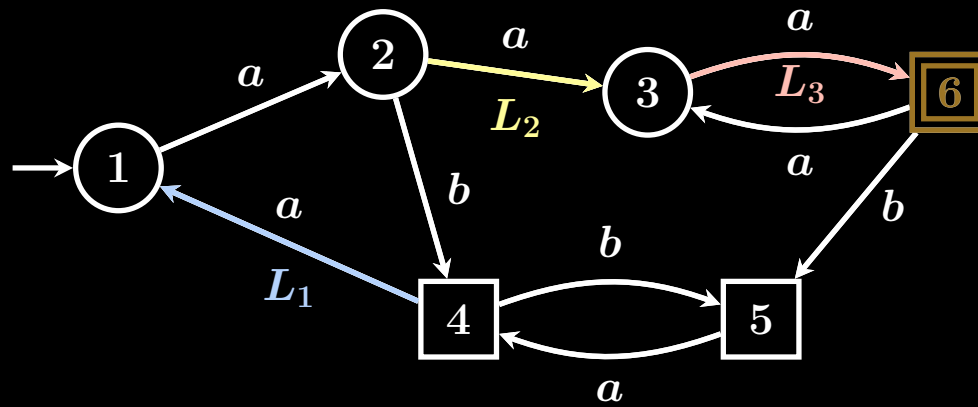
Example play: $(1, a), (2, a), (3, a), (6, a), (3, a), (6, b), (5, a), (4, b), (5, a), (4, b), \dots$

Winning conditions are sets of plays, for example: “visit 6 at least once” is winning for Alice.

Two-Player Games on HCS

There are two players: Alice controls transitions taken from \bigcirc

Bob controls transitions taken from \square



Example play: $(1, a), (2, a), (3, a), (6, a), (3, a), (6, b), (5, a), (4, b), (5, a), (4, b), \dots$

Winning conditions are sets of plays, for example: “visit 6 at least once” is winning for Alice.

Goal: determine whether or not Alice can win a given NFA[NFA] reachability game.

Reachability games on regular HCS

Theorem. Reachability games on NFA[NFA] can be decided in exponential time.

Proof idea. (1) Convert each NFA guard into an equivalent DFA.

(2) Create an equivalent reachability game on a graph by producing the underlying NFA, that is just a graph, with all guard DFAs.

(3) Use the fact that reachability games on graphs are solvable in polynomial time.

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Corollary. Parity, mean-payoff, and energy games on NFA[NFA] can all be decided in exponential time.

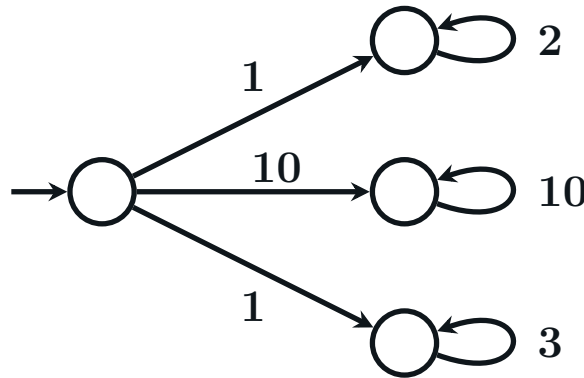
Theorem. Reachability games on NFA[1-VASS] are undecidable.

Takeaway message 3: games on HCS are decidable with regular guards but can easily become undecidable with non-regular guards.

Reachability games on NFA[DFA] are hard!

Theorem. Reachability games on NFA[DFA] are EXPTIME-hard.

Idea: reduce from *countdown games* ... a two-player game played on a graph with positive weights.



This game is winning for Charlie if $B \equiv 0 \pmod{10}$ or $B \equiv 1 \pmod{6}$.

A turn consists of: Charlie first picking weight (of any outgoing transition), then David selects a transition of that weight to take.

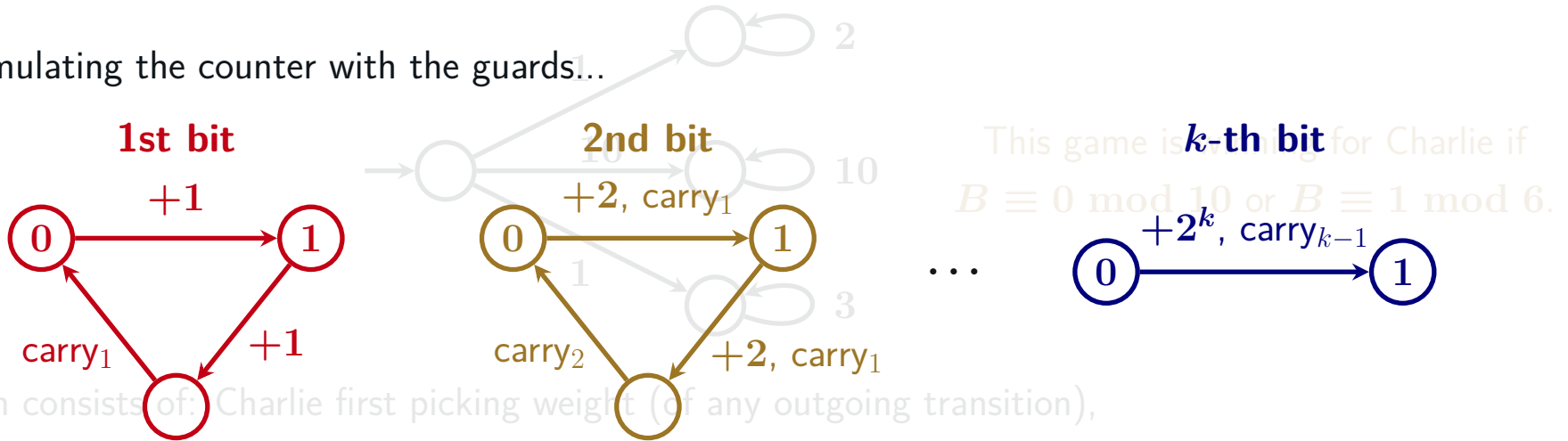
Winning condition: Charlie wants to get the counter to exactly B .

Reachability games on NFA[DFA] are hard!

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Idea: reduce from *countdown games* ... a two-player game played on a graph with positive weights.

By simulating the counter with the guards!..



A turn consists of: Charlie first picking weight (of any outgoing transition), then David selects a transition of that weight to take.

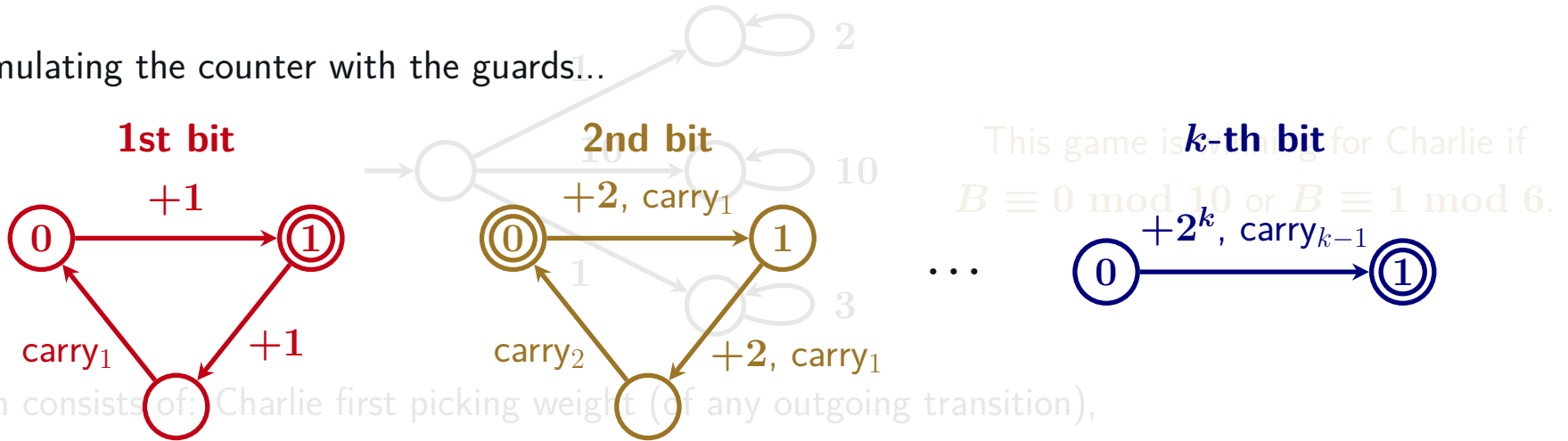
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By simulating the counter with the guards!..



A turn consists of: Charlie first picking weight (of any outgoing transition), then David selects a transition of that weight to take. ϵ

The counter hitting $B = 100 \dots 01$ can be tested: ϵ

Winning condition: Charlie wants to get the counter to exactly B .

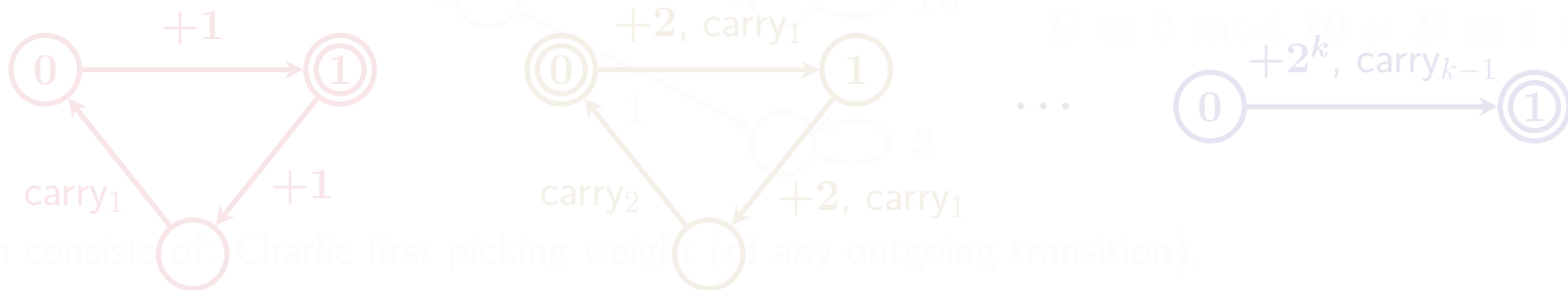
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EXPTIME-complete

Takeaway message 3: games on HCS are ~~decidable~~ with regular guards but can easily become undecidable with non-regular guards.



Winning condition: Charlie wants to get the counter to exactly B .

History-Constrained Systems

Takeaway message 1. Even basic history-constrained systems can capture and exhibit complex system behaviours.

Takeaway message 2. History-constrained systems with regular guards recognise exactly the regular languages (but they can be more succinct).

Takeaway message 3. Games on HCS are EXPTIME-complete with regular guards but can easily become undecidable with non-regular guards.

Bonus theorem. Reachability (emptiness) in NFA[VASS] is: undecidable with reachability acceptance, decidable with coverability acceptance*.

Thank You!

Presented by Henry Sinclair-Banks, MPI-SWS, Kaiserslautern, Germany 

Formal Methods 2026 in Hitotsubashi Hall, Tokyo, Japan 

FM
2026

Presentation made with
BeamerikZ