# A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter




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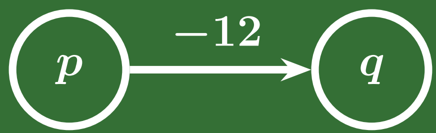
*“Automata with One Counter”* is a strange turn of phrase...

## Integer Semantics

Transitions have updates in $\mathbb{Z}$.

The counter value is in $\mathbb{Z}$.




$$p(5) \to q(-7)$$

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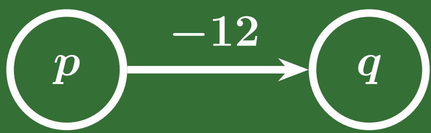


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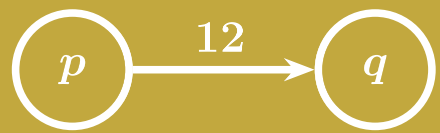


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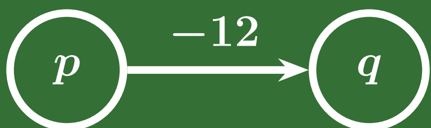


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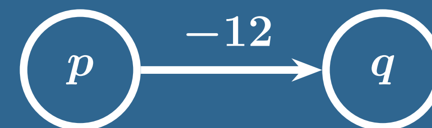


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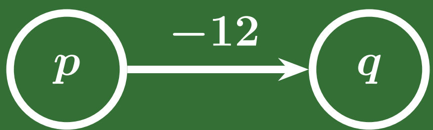


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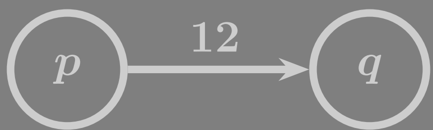


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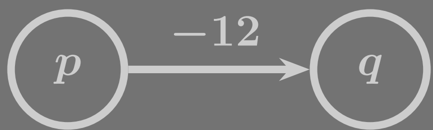


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# Motivation

**Reachability.** From $\boldsymbol{p(x)}$, can you reach $\boldsymbol{q(y)}$?

**Coverability.** From $\boldsymbol{p(x)}$, can you reach $\boldsymbol{q(y')}$ such that $\boldsymbol{y' \geq y}$?

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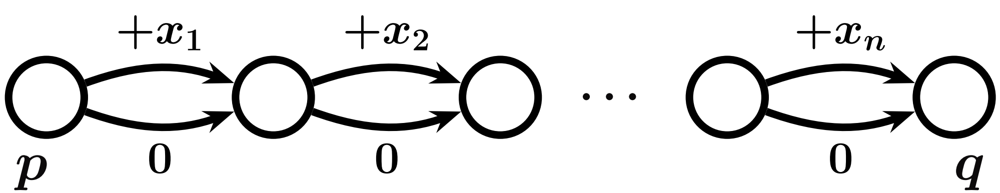
**Theorem.** Reachability (with integer semantics) is NP-hard.

Reduction from subset sum $(x_1, x_2, \ldots, x_n, t)$:




From $p(0)$, can you reach $q(t)$?

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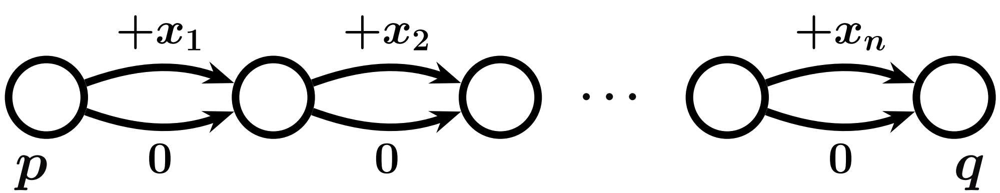


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**Theorem.** Coverability (with integer semantics) is in $AC^1 \subseteq P$.

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**Theorem.** Reachability (with integer semantics) is **HARD**

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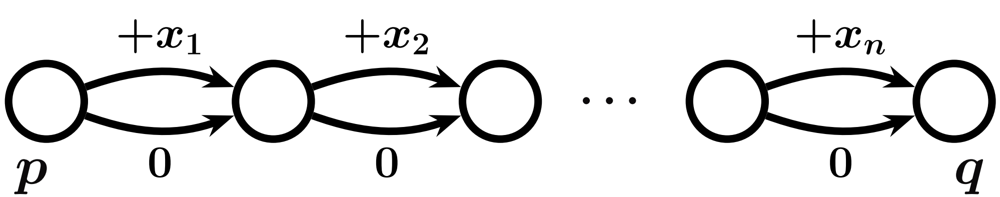


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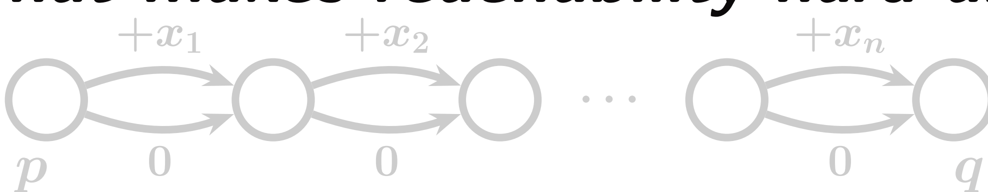
Reduction from subset sum $(x_1, x_2, \ldots, x_n, t)$:

***What makes reachability hard and coverability easy?***

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# Generalising Reachability and Coverability

## $\textsc{Reach}(\boldsymbol{S})$

**Fixed:** A semilinear set $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{Z}$.

**Input:** An integer one-counter automaton $\boldsymbol{\mathcal{A}}$, an initial configuration $\boldsymbol{p(x)}$, and a vector $\vec{\boldsymbol{t}} \in \mathbb{Z}^{\boldsymbol{p}}$.

**Question:** Does there exist a reachable configuation $\boldsymbol{q(y)}$ such that $(\vec{\boldsymbol{t}}, \boldsymbol{y}) \in \boldsymbol{S}$ ?

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## Examples

Reachability: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{y} = \boldsymbol{t}\}$

Coverability: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{y} \geq \boldsymbol{t}\}$

Cover and avoid: $\boldsymbol{S} = \{(\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{y}) : \boldsymbol{y} \geq \boldsymbol{t_1} \wedge \boldsymbol{y} \neq \boldsymbol{t_2}\}$

Reach an interval: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{t} \leq \boldsymbol{y} \leq \boldsymbol{2t}\}$

Reach an interval or a negative value: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{y} \leq \boldsymbol{0} \vee \boldsymbol{t} \leq \boldsymbol{y} \leq \boldsymbol{2t}\}$

# Generalising Reachability and Coverability

## Reach($S$)

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### Examples

- Reachability: $S = \{(t, y) : y = t\}$ **HARD**
- Coverability: $S = \{(t, y) : y \geq t\}$ **EASY**
- Cover and avoid: $S = \{(t_1, t_2, y) : y \geq t_1 \wedge y \neq t_2\}$ **EASY**
- Reach an interval: $S = \{(t, y) : t \leq y \leq 2t\}$ **HARD**
- Reach an interval or a negative value: $S = \{(t, y) : y \leq 0 \vee t \leq y \leq 2t\}$ **EASY**

# Main Contribution

**Theorem.** Let $S \subseteq \mathbb{Z}^p \times \mathbb{Z}$ be a semilinear set.

(1) If $S$ is **dense**, then $\textsc{Reach}(S)$ is in $\mathsf{AC}^1$.

(2) Otherwise, $\textsc{Reach}(S)$ is NP-hard.

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Let $\boldsymbol{A} \subseteq \mathbb{Z}$ be a semilinear set... $\textit{Density}(\boldsymbol{A}) = \inf_{\boldsymbol{y} \in \boldsymbol{A}} \inf_{\boldsymbol{n} \in \mathbb{N}} \frac{\boldsymbol{A} \cap [\boldsymbol{y}-\boldsymbol{n}, \boldsymbol{y}+\boldsymbol{n}]}{2\boldsymbol{n}+1}$

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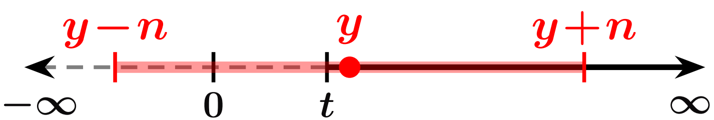


Density $= \frac{1}{2}$

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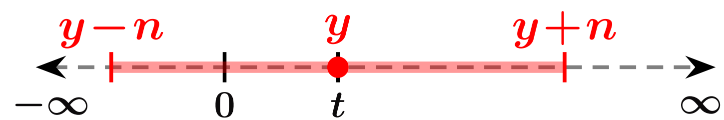


Density $= 0$

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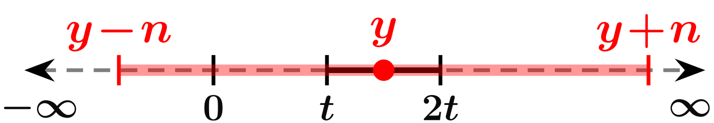


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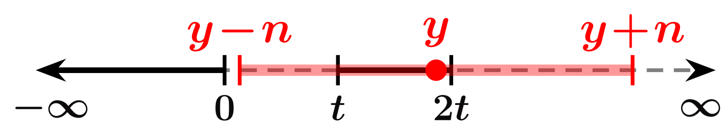


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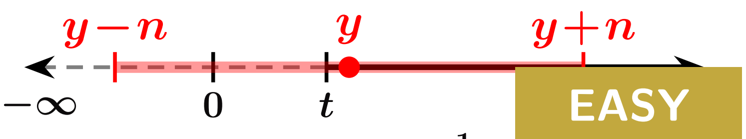


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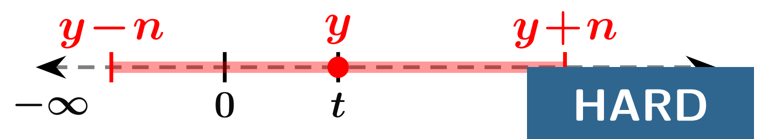


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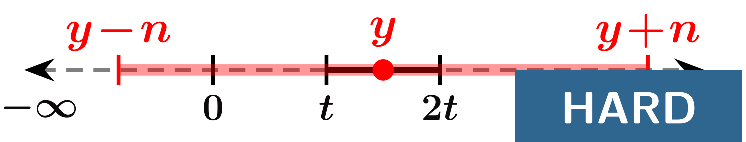


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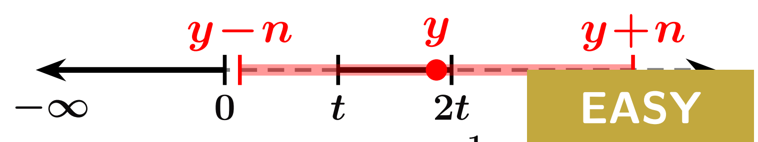


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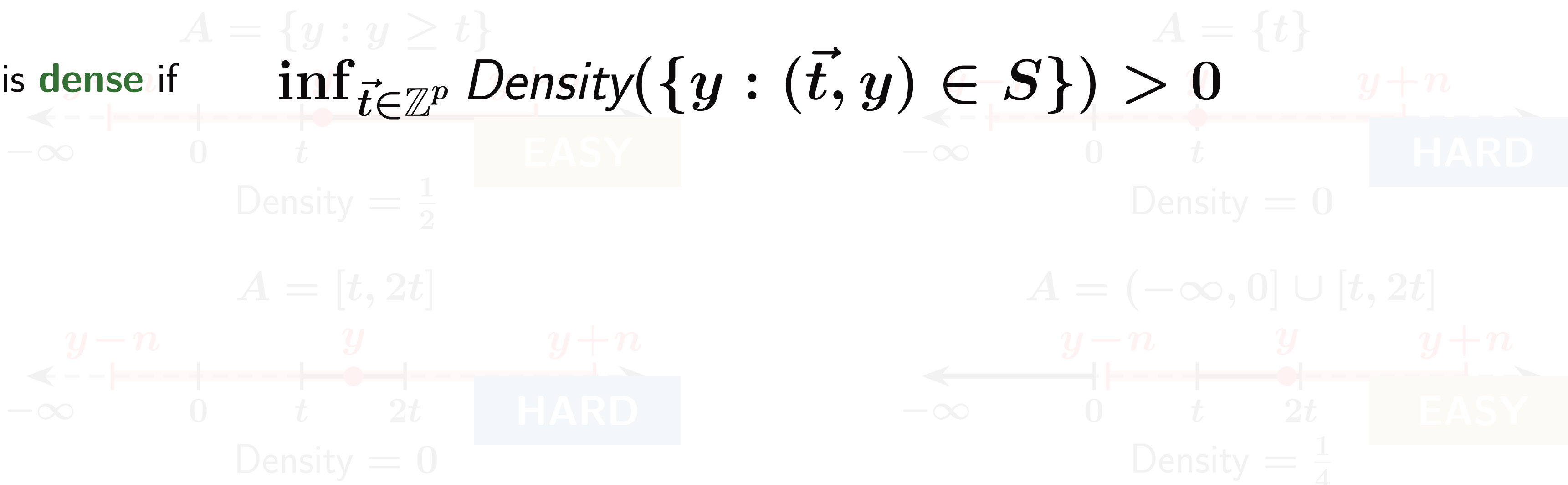
$S$ is **dense** if
$$\inf_{\vec{t} \in \mathbb{Z}^p} \mathit{Density}(\{y : (\vec{t}, y) \in S\}) > 0$$

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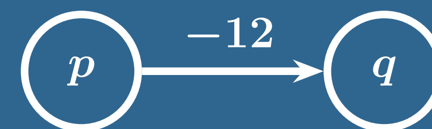


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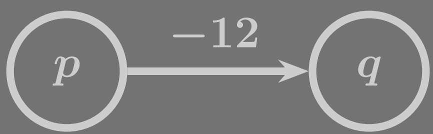


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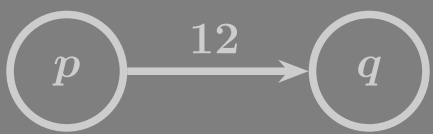


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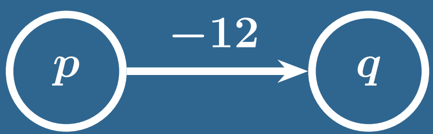


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# Which Target Sets Make Reachability in 1-VASS Easy?

**Theorem.** Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If $S$ is **uniformly quasi-upwards closed**, then $\textsc{Reach}_{\mathsf{VASS}}(S)$ is in $\mathsf{AC}^1$.

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Upwards closed: $\{10, 11, 12, 13, \ldots\}$.

$\delta$-upwards closed: $\{10, 15, 17, 20, 22, 25, 27, \ldots\}$ is 5-upwards closed.

$(\delta, M)$-upwards closed: $\{10, 17, 20, 22, 27, 30, 32, 35, \ldots\}$ is $(5, 2)$-upwards closed.

**Uniformly quasi-upwards closed**

There exists $\delta, M \in \mathbb{N}$ such that, for all $\vec{t} \in \mathbb{Z}^p$, $\{y : (\vec{t}, y) \in S\}$ is $(\delta, M)$-upwards closed.

*“If $S$ is uniformly quasi-upwards closed, then $\textsc{Reach}_{VASS}(S)$ is in $AC^1$”*

**Theorem.** Coverability in (binary encoded) 1-VASS is in $\mathsf{NC}^2$.

[Almagor, Cohen, Pérez, Shirmohammadi, and Worrell 2020]

**Theorem.** Coverability in (binary encoded) 1-VASS is in $\mathsf{AC}^1$. [This paper]

# An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

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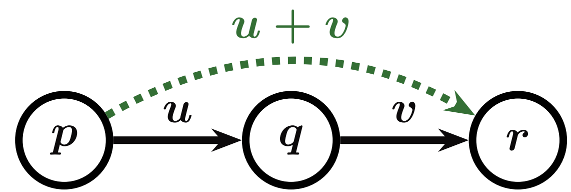


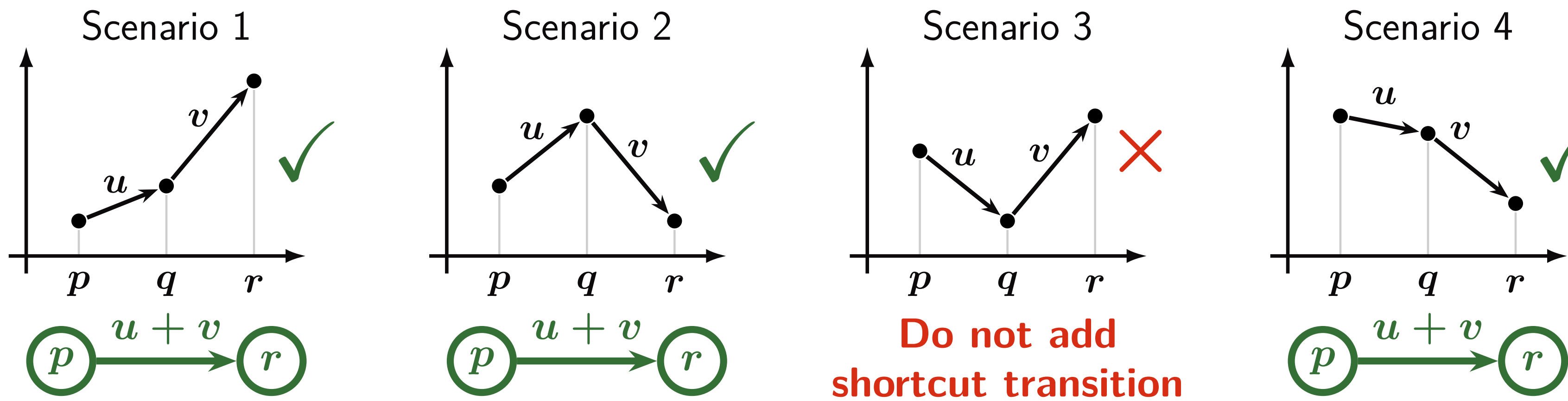

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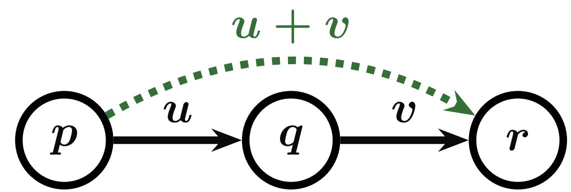


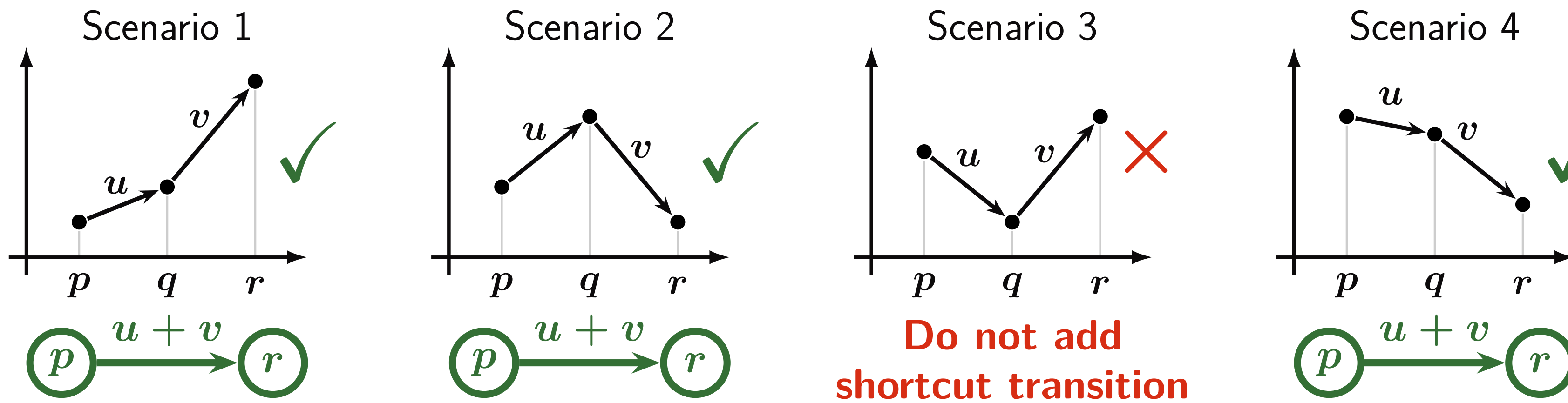


`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

**Claim.** There is a covering run in $\mathcal{V}_0$ if and only if there is a covering run of **length $\leq 2$** in $\mathcal{V}_k$.

# Dichotomies for Reaching Semilinear Target Sets

Let $\boldsymbol{S} \subseteq \mathbb{Z}^p \times \mathbb{Z}$ be a semilinear set.

**Integer semantics:** $\textsc{Reach}_{\mathbb{Z}}(\boldsymbol{S})$ is ... in $\mathsf{AC}^1$ if $\boldsymbol{S}$ is **dense**,
NP-hard otherwise.

**Natural semantics:** $\textsc{Reach}_{\mathbb{N}}(\boldsymbol{S})$ is ... in $\mathsf{AC}^1$ if $\boldsymbol{S}$ is **dense**$^+$,
NP-hard otherwise.

$^+$*for a subtly modified definition of density*

**VASS semantics:** $\textsc{Reach}_{\mathsf{VASS}}(\boldsymbol{S})$ is ... in $\mathsf{AC}^1$ if $\boldsymbol{S}$ is ***uniformly quasi-upward closed***,
NP-hard otherwise.

## Thank You!

*Presented by Henry Sinclair-Banks, University of Warsaw, Poland*

*LICS'25 in National University of Singapore, Singapore*

Presentation made with BeamerikZ