A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter

Henry Sinclair-Banks

Based on joint work with Yousef Shakiba and Georg Zetzsche that has been accepted to LICS'25

BASICS Group Seminar

18th June 2025

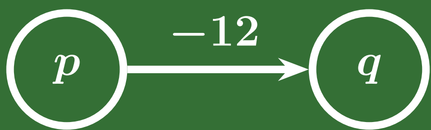
Shanghai Jiao Tong University, Shanghai, China

“Automata with One Counter” is a strange turn of phrase...

Integer Semantics

Transitions have updates in $\mathbb{Z}$.

The counter value is in $\mathbb{Z}$.



$$p(5) \to q(-7)$$

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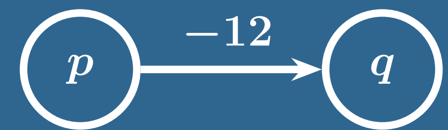

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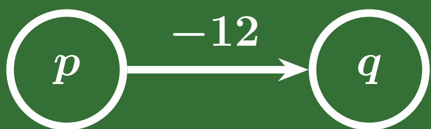

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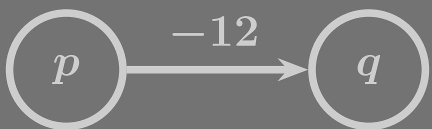

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Motivation

Reachability. From $p(x)$, can you reach $q(y)$?

Coverability. From $p(x)$, can you reach $q(y')$ such that $y' \geq y$?

Theorem. Reachability (with integer semantics) is NP-hard.

Theorem. Coverability (with integer semantics) is in $\mathsf{AC}^1 \subseteq \mathsf{P}$.

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Reachability. From $p(x)$, can you reach $q(y)$?

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What makes reachability hard and coverability easy?

Theorem. Reachability (with integer semantics) is **HARD**

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Generalising Reachability and Coverability

$\textsc{Reach}(\boldsymbol{S})$

Fixed: A semilinear set $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{Z}$.

Input: An integer one-counter automaton $\boldsymbol{\mathcal{A}}$, an initial configuration $\boldsymbol{p(x)}$, and a vector $\vec{\boldsymbol{t}} \in \mathbb{Z}^{\boldsymbol{p}}$.

Question: Does there exist a reachable configuation $\boldsymbol{q(y)}$ such that $(\vec{\boldsymbol{t}}, \boldsymbol{y}) \in \boldsymbol{S}$?

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Examples

Reachability: $S = \{(t, y) : y = t\}$

Coverability: $S = \{(t, y) : y \geq t\}$

Cover and avoid: $S = \{(t_1, t_2, y) : y \geq t_1 \wedge y \neq t_2\}$

Reach an interval: $S = \{(t, y) : t \leq y \leq 2t\}$

Reach an interval or a negative value: $S = \{(t, y) : y \leq 0 \vee t \leq y \leq 2t\}$

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Main Contribution

Theorem. Let $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{Z}$ be a semilinear set.

(1) If $\boldsymbol{S}$ is **dense**, then $\textsc{Reach}(\boldsymbol{S})$ is in AC^1.

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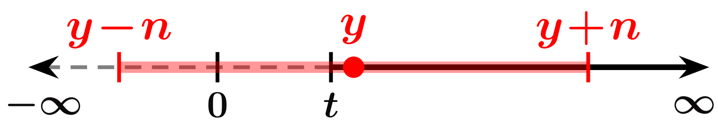
Let $A \subseteq \mathbb{Z}$ be a semilinear set... $\quad$ $\mathit{Density}(A) = \inf_{y \in A} \inf_{n \in \mathbb{N}} \frac{A \cap [y-n, y+n]}{2n+1}$

$A = \{y : y \geq t\}$



Density $= \frac{1}{2}$

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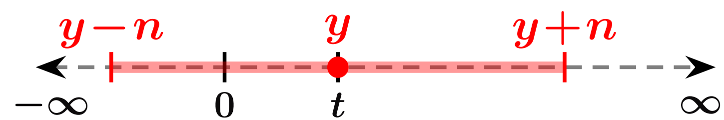

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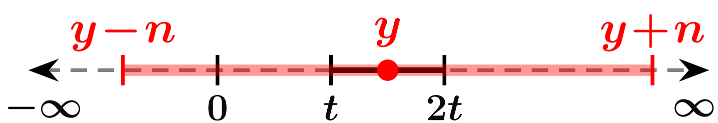

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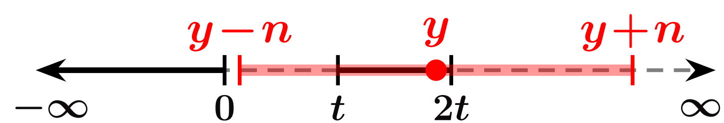

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**simplified version ignoring modulo constraints.*

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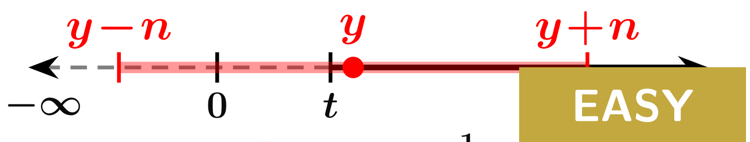

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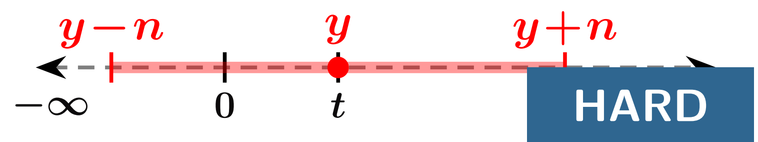

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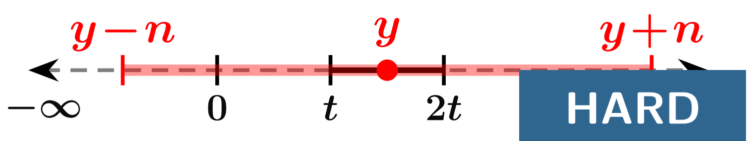

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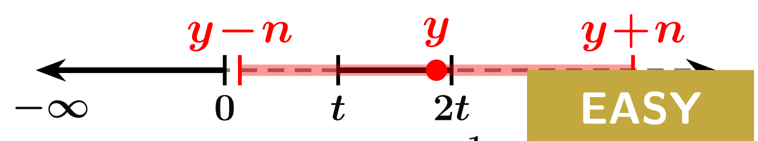

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S is **dense** if $$\inf_{\vec{t} \in \mathbb{Z}^p} \textit{Density}(\{y : (\vec{t}, y) \in S\}) > 0$$

**simplified version ignoring modulo constraints.*

S has **zero density**

$$\Downarrow$$

Reduction from subset sum

$$\Downarrow$$

$\textsc{Reach}(S)$ is NP-hard

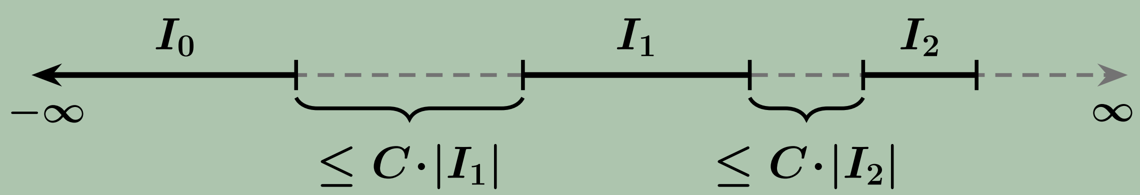
S has **positive density** $\implies$ $\textsc{Reach}(S)$ is in AC^1

Preliminaries:

Remove modulo constraints from S.

Make the integer one-counter automaton acyclic.

Identify *building block* cases $(I_0 \cup I_1 \cup I_2)$ with **positive density**.



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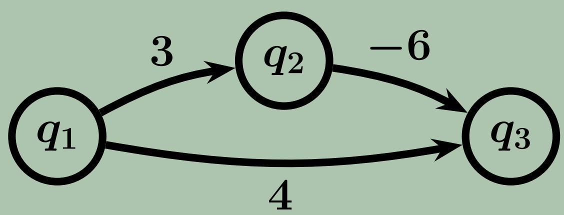

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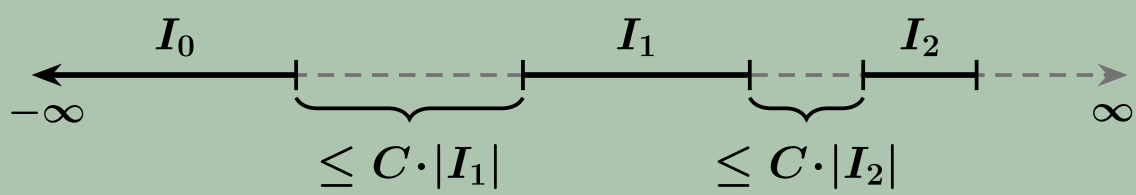

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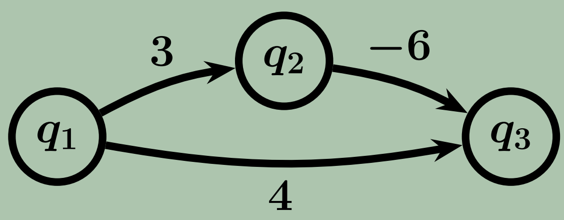

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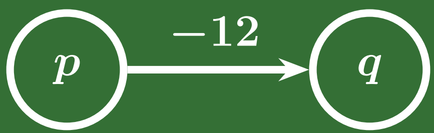

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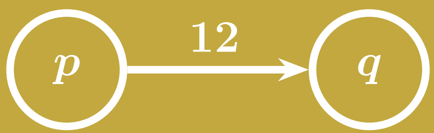

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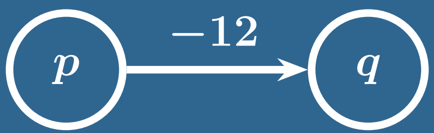

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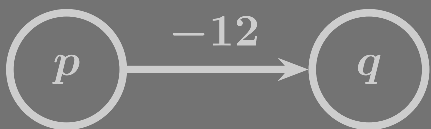

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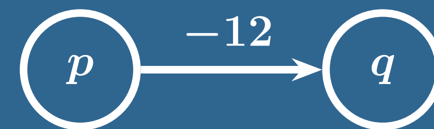

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Which Target Sets Make Reachability in 1-VASS Easy?

Theorem. Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}_{\mathsf{VASS}}(S)$ is in AC^1.

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Upwards closed: $\{10, 11, 12, 13, \ldots\}$.

δ-upwards closed: $\{10, 15, 17, 20, 22, 25, 27, \ldots\}$ is 5-upwards closed.

(δ, M)-upwards closed: $\{10, 17, 20, 22, 27, 30, 32, 35, \ldots\}$ is $(5, 2)$-upwards closed.

Uniformly quasi-upwards closed

There exists $\delta, M \in \mathbb{N}$ such that, for all $\vec{t} \in \mathbb{Z}^p$, $\{y : (\vec{t}, y) \in S\}$ is (δ, M)-upwards closed.

"If S is uniformly quasi-upwards closed, then $\textsc{Reach}_{VASS}(S)$ is in AC^1"

Theorem. Coverability in (binary encoded) 1-VASS is in NC^2.

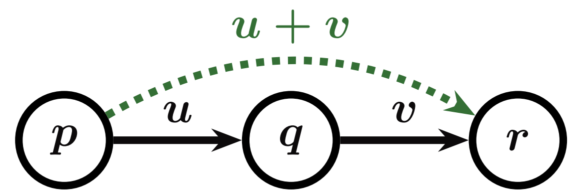
[Almagor, Cohen, Pérez, Shirmohammadi, and Worrell 2020]

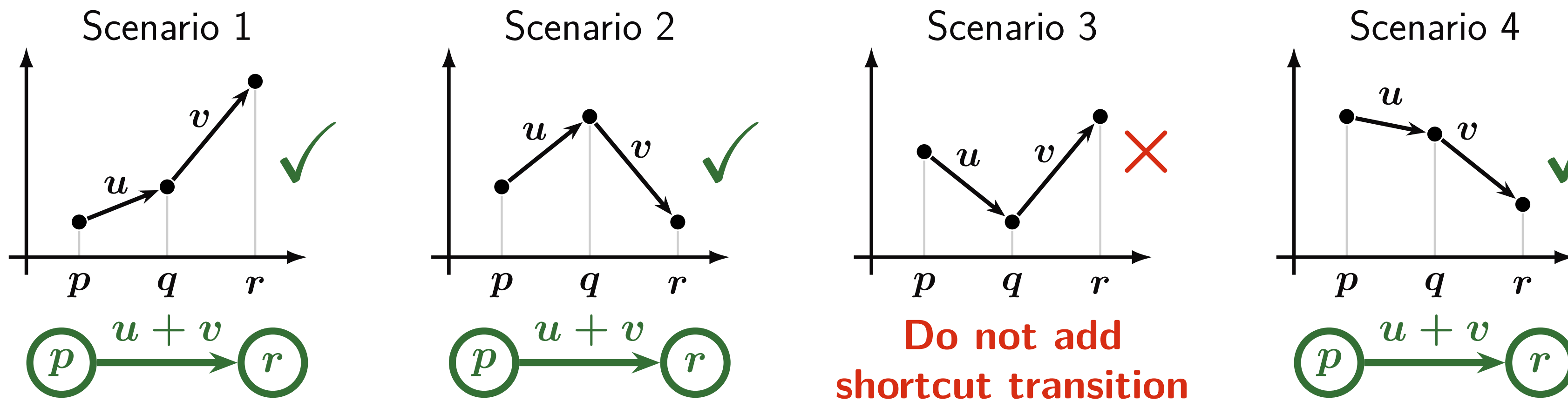
Theorem. Coverability in (binary encoded) 1-VASS is in AC^1. [Shakiba, S., and Zetzsche '25]

An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":





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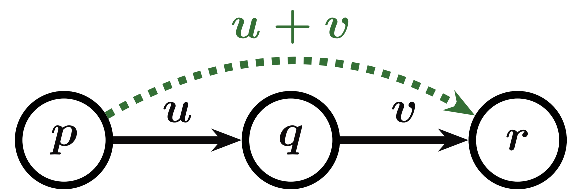

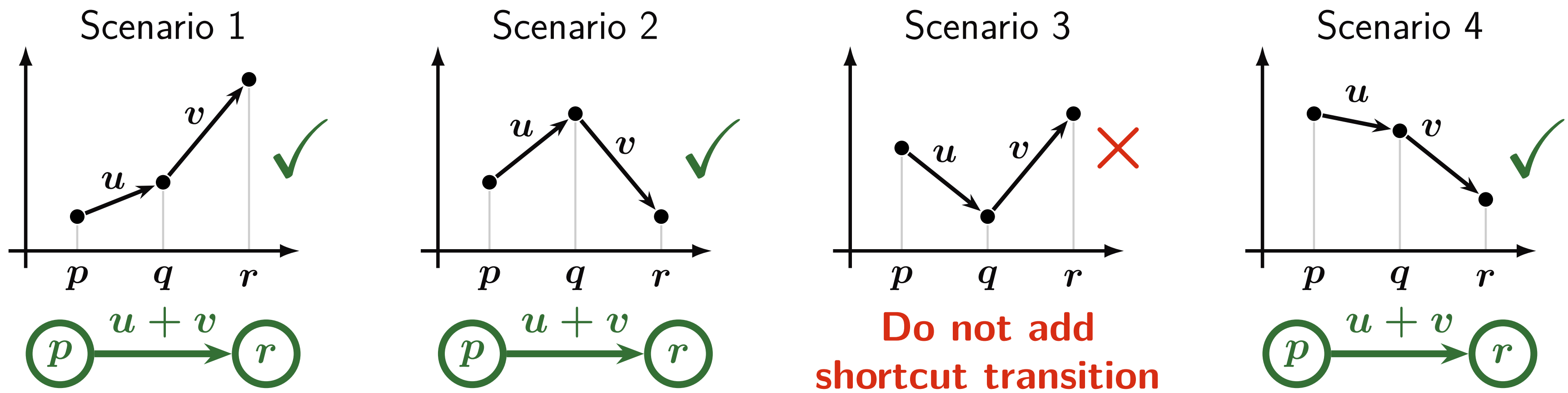

`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

Claim. There is a covering run in $\mathcal{V}_0$ if and only if there is a covering run of **length ≤ 2** in $\mathcal{V}_k$.

Dichotomies for Reaching Semilinear Target Sets

Let $S \subseteq \mathbb{Z}^p \times \mathbb{Z}$ be a semilinear set.

Integer semantics: $\textsc{Reach}_{\mathbb{Z}}(S)$ is … in AC^1 if S is **dense**, NP-hard otherwise.

Natural semantics: $\textsc{Reach}_{\mathbb{N}}(S)$ is … in AC^1 if S is **dense**$^+$, NP-hard otherwise.

$^+$*for a subtly modified definition of density*

VASS semantics: $\textsc{Reach}_{\mathsf{VASS}}(S)$ is … in AC^1 if S is ***uniformly quasi-upward closed***, NP-hard otherwise.

Thank You!

上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

Presented by Henry Sinclair-Banks, University of Warsaw, Poland

BASICS Group Seminar in Shanghai Jiao Tong University, Shanghai, China

Presentation made with Beamer*ikZ*